The P_3 Intersection Graph

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Abstract

We define a new graph operator called the P_3 intersection graph, $P_3(G)$- the intersection graph of all induced 3-paths in G. A characterization of graphs G for which $P_3(G)$ is bipartite is given. Forbidden subgraph characterization for $P_3(G)$ having properties of being chordal, H-free, complete are also obtained. For integers a and b with $a > 1$ and $b \geq a-1$, it is shown that there exists a graph G such that $\chi(G) = a$, $\chi(P_3(G)) = b$, where χ is the chromatic number of G. For the domination number $\gamma(G)$, we construct graphs G such that $\gamma(G) = a$ and $\gamma(P_3(G)) = b$ for any two positive numbers $a > 1$ and b. Similar construction for the independence number and radius, diameter relations are also discussed.

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1 Introduction

The study of 'graph operators' and their various properties such as fixedness, convergence and others have been receiving wide attention since Ore's work [3] on the line graph operator.

The H - intersection graph $Int_H(G)$ is the intersection graph of all subgraphs of G that are isomorphic to H, see [4]. If H is K_2 then $Int_H(G)$ is the line graph. Trotter [6] characterized the graphs for which $Int_{K_2}(H)$ is perfect. The K_3 intersection graph is the 3-edge graph provided every edge lies in some triangle where the 3-edge graph is the intersection graph of cliques with at most three vertices or a triangle [5].

For a detailed discussion on other graph operators, the reader may refer to [4].

In [1] Akiyama and Chvátal have characterized the graphs for which $IntP_3(G)$ is perfect. Motivated by this paper, we have defined a new operator, the P_3 intersection graph - $P_3(G)$ as the intersection graph of all induced paths on three vertices in G. We characterize the graphs G such that $P_3(G)$ is bipartite. We obtain forbidden subgraph characterization for $P_3(G)$ being H-free, chordal and complete. Some properties of chromatic number, domination number, independence number, diameter and radius of $P_3(G)$ are also discussed.

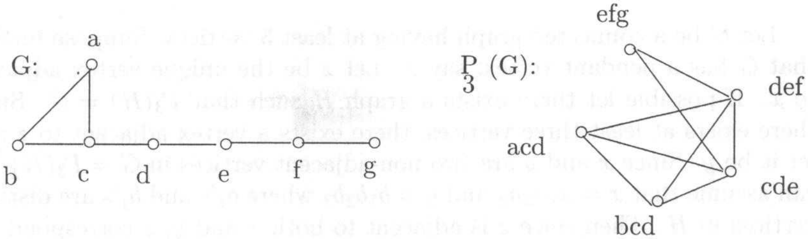
All the graphs considered here are undirected and simple. $P_3(G)$ is the null graph for any graph G which is the union of complete graphs. Hence in this paper we do not consider such graphs. For all other basic concepts and notations not mentioned in this paper we refer [7].

2 P_3 intersection graph

Definition 2.1: Let G be a graph. The P_3 intersection graph of G, $P_3(G)$ has the induced paths on three vertices in G as its vertices and two distinct vertices in $P_3(G)$ are adjacent if the corresponding induced 3-paths in G intersect.

If $a_1 - a_2 - a_3$ is an induced 3-path in G then the corresponding vertex in $P_3(G)$ is denoted by $a_1a_2a_3$. If G is a connected graph of order at most five then $P_3(G)$ is complete.

Example 2.1:



Remark 2.1: In general, the H-intersection graph of a connected graph is not necessarily connected. But, we have

Theorem 2.1: $P_3(G)$ is connected if and only if G has exactly one component containing an induced P_3 .

Proof: Suppose that G contains more than one component containing an induced P_3. Let $a_1 - a_2 - a_3$ and $b_1 - b_2 - b_3$ be any two induced 3-paths in G which lie in distinct components of G. Then by the definition of $P_3(G)$ the corresponding vertices $a_1a_2a_3$ and $b_1b_2b_3$ in $P_3(G)$ cannot be connected by a path and hence $P_3(G)$ is not connected.

Let G has exactly one component containing an induced P_3. Suppose that $x = a_1a_2a_3$ and $y = b_1b_2b_3$ are any two non-adjacent vertices in $P_3(G)$. If $a_i, i = 1, 2, 3$ and $b_j, j = 1, 2, 3$ are adjacent then $a_1a_2a_3, a_ib_jb_{j+1}$ or $a_ib_jb_{j-1}, b_1b_2b_3$ is a path connecting x and y. If a_i and b_j are not adjacent then let the shortest path connecting a_i, $i = 1, 2, 3$ and b_j, $j = 1, 2, 3$ be $a_i, c_1, c_2, ..., c_n, b_j$. If $n = 1$, then $a_1a_2a_3$, $a_ic_1b_j$, $b_1b_2b_3$ is a path connecting x and y. If $n \geq 2$, then $a_1a_2a_3$, $a_ic_1c_2$, ..., $c_{n-1}c_nb_j$, $b_1b_2b_3$ is a path connecting x and y in $P_3(G)$. Hence any two vertices in $P_3(G)$ are connected by a path and hence $P_3(G)$ is connected.

As to the question whether every graph is the P_3 intersection graph of some graph, we have

Theorem 2.2: The following graphs G cannot be the P_3 intersection graph of any graph.
(1) G is a connected graph having at least 3 vertices and a pendant vertex.
(2) There exists a vertex v in G with degree$(v) = 2$ such that v is adjacent to any two non-adjacent vertices in G.
(3) G is a connected triangle free graph having at least three vertices.

Proof:

Let G be a connected graph having at least 3 vertices. Suppose further that G has a pendant vertex, say x. Let z be the unique vertex adjacent to x. If possible let there exists a graph H such that $P_3(H) = G$. Since there exists at least three vertices, there exists a vertex adjacent to z and let it be y. Since x and y are two non-adjacent vertices in $G = P_3(H)$, we can assume that $x = a_1a_2a_3$ and $y = b_1b_2b_3$ where a_i's and b_j's are distinct vertices in H. Then since z is adjacent to both x and y, z corresponds to a 3-path in H which must contain at least one a_i and b_j. So z must be of the form a_ib_jc or a_icb_j or ca_ib_j.

Let $z = a_ib_jc$. If $i = 1$, then $a_2 - a_1 - b_j$ is a 3-path. But if this is an induced path, then x cannot remain as a pendant vertex. So $a_2 - b_j$ is an edge in H. Then $a_3 - a_2 - b_j$ is a 3-path. But if this is an induced path then x cannot remain as a pendant vertex. So $a_3 - b_j$ is an edge in H. Then $a_1 - b_j - a_3$ is an induced 3-path. If the corresponding vertex $a_1b_ja_3$ is different from z, then it is adjacent to x is a contradiction to the fact that x is a pendant vertex. If $a_1b_ja_3 = z$, then we can show that there exists an induced 3-path with a_1 and two b_l's, $l = 1, 2, 3$ as its vertices. The corresponding vertex which is different from z is adjacent to x which will also lead to a contradiction. So G cannot be the P_3-graph of any graph. The case is similar when $i = 2, 3$ also. The proof is similar when $z = a_icb_j$ and $z = ca_ib_j$.

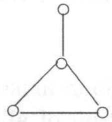
Suppose now that G has a vertex v with degree$(v) = 2$ and let $G = P_3(H)$. Let v be adjacent to v_1 and v_2 where v_1 and v_2 are non-adjacent vertices. Let $v_1 = a_1a_2a_3$ and $v_2 = b_1b_2b_3$ where a_i's and b_j's are distinct vertices in H. Then v must be of the form a_ib_jc or a_icb_j or ca_ib_j. So as in the proof given above, we can show that there exists a vertex adjacent to v which is different from both v_1 and v_2 which is a contradiction to the fact that degree$(v) = 2$.

Finally, let G be a connected triangle free graph. If possible assume that $G = P_3(H)$. Since G has at least three vertices it contains a vertex z which is adjacent to two non-adjacent vertices x and y. Let $x = a_1a_2a_3$ and $y = b_1b_2b_3$, where a_i's and b_j's are distinct vertices in H. Then z must be of the form a_ib_jc or a_icb_j or ca_ib_j. Using the similar arguments as in the above proofs, we can show that there exists a vertex which is adjacent to both x and z which is a contradiction to the fact that G is triangle free.

Lemma 2.1: If G is a connected graph having at least five vertices, then $P_3(G)$ has at least three vertices.

Proof: Let G be a connected graph having at least five vertices. Let x and y be two non-adjacent vertices of G. Let the shortest path connecting x and y be $x, v_1, v_2, ...v_n, y$. If $n \geq 3$ then $P_3(G)$ clearly contains at least three vertices. If $n = 2$ then since G is a connected graph having at least five vertices, the fifth vertex must be adjacent to at least one of x, v_1, v_2, y. Then there exists at least three induced 3-paths in G and hence $P_3(G)$ contains at least three vertices. If $n = 1$, there exists at least two more vertices in G and they must be connected to x, v_1, y. In any case there exists at least three induced 3-paths in G and hence $P_3(G)$ contains at least 3 vertices.

Notation: The graph obtained by deleting any edge of K_n is denoted by $K_n - \{e\}$.

The graph  is called the 'paw'.

Theorem 2.3: Let G be a connected graph. Then $P_3(G)$ is bipartite if and only if G is P_3, P_4, $K_4 - \{e\}$ or paw.

Proof: Let $P_3(G)$ be bipartite. Then $P_3(G)$ cannot contain triangles. So by Theorem 2.2 (3), the only bipartite graphs are K_1 and K_2. Again by Lemma 2.1, G can have at most four vertices. Since we are considering only the connected graphs, the theorem follows.

3 Forbidden subgraph characterizations

A graph H is a forbidden subgraph for a property P of a graph G, if G does not contain an induced subgraph isomorphic to H. A forbidden subgraph H for the property P is a vertex minimal forbidden subgraph if no induced subgraph of H is a forbidden subgraph for the property P. A graph G is H-free if G does not contain H as an induced subgraph. Many classes of H-free graphs are discussed in [2]. A property P of a graph G is vertex hereditary if every induced subgraph of G also has the property P.

Theorem 3.1: If G is a P_3 intersection graph then $K_{1,4}$ is a forbidden subgraph for G.

Proof: Suppose that $G = P_3(H)$ contains $K_{1,4}$ as an induced subgraph. Let v be the central vertex and v_1, v_2, v_3, v_4 be the neighbors of v in $K_{1,4}$ in G. Then v corresponds to an induced 3-path in H which intersects with all the four distinct 3-paths corresponding to v_1, v_2, v_3 and v_4 which is not possible. Hence $K_{1,4}$ is a forbidden subgraph for the P_3 intersection graph.

Theorem 3.2: Let $\varphi = \{G : P_3(G) \text{ is } H\text{-free}\}$ where H is any graph. Then the property P, $G \in \varphi$ is vertex hereditary.

Proof: Let $G \in \varphi$. Suppose that $G - \{v\} \notin \varphi$. So $P_3(G - \{v\})$ contains H as an induced subgraph. Then this H will be induced in $P_3(G)$ also, which is a contradiction to the fact that $G \in \varphi$.

Corollary 3.1: The collection φ has only a finite class of vertex minimal forbidden subgraphs.

Proof: The property $G \in \varphi$ is vertex hereditary. So φ must have vertex minimal forbidden subgraphs. Let F be the collection of all such vertex minimal forbidden subgraphs. Let $G_1 \in F$. Then $P_3(G_1)$ contains H as an induced subgraph. So, corresponding to a vertex in H there exists an induced 3-path in G_1. So, number of vertices in G_1 covered by these 3-paths cannot exceed $3n$ where n is the number of vertices in H. If G_1 contains more than $3n$ vertices, then there exists a vertex v in G_1 such that any induced 3-path containing v does not determine a vertex of H in $P_3(G_1)$. Then $G_1 - \{v\}$ is also forbidden for φ which is a contradiction to the vertex minimality of G_1. Hence the number of vertices of G_1 is bounded by $3n$ and hence φ has only a finite class of vertex minimal forbidden subgraphs.

Theorem 3.3: Let $\Im = \{G : P_3(G) \text{ is chordal}\}$. The property P, $G \in \Im$ is vertex hereditary.

Proof: Let $G \in \Im$. Suppose that $G - \{v\} \notin \Im$. That is, $P_3(G - \{v\})$ is not chordal which implies that $P_3(G - \{v\})$ contains an induced C_n, $n \geq 4$. Then these C_n's are induced cycles in $P_3(G)$ which is a contradiction to the fact that $P_3(G)$ is chordal.

Corollary 3.2: The collection $\Im$ has an infinite class of vertex minimal forbidden subgraphs.

Proof: The property $G \in \Im$ is vertex hereditary. So $\Im$ must have vertex minimal forbidden subgraphs. If G contains C_n, $n \geq 6$ as an induced subgraph, then $P_3(G)$ contains C_n, $n \geq 4$ and hence cannot be chordal. Also $P_3(C_n - \{v\})$, $n \geq 6$ is chordal. So C_n, $n \geq 6$ are vertex minimal forbidden subgraphs for $\Im$. Thus there exists an infinite class of vertex minimal forbidden subgraphs for $\Im$.

Theorem 3.4: Let $\Psi = \{G : P_3(G) \text{ is complete }\}$. The property $G \in \Psi$ is vertex hereditary.

Proof: Let $G \in \Psi$. Suppose that $G - \{v\} \notin \Psi$ for some v. Then there exist at least two non-adjacent vertices in $P_3(G - \{v\})$. These vertices will remain as non-adjacent vertices in $P_3(G)$ also, which is a contradiction to the fact that $P_3(G)$ is complete.

Corollary 3.3: Any vertex minimal forbidden subgraph for Ψ has exactly six vertices.

Proof: The property $G \in \Psi$ is vertex hereditary. So it has vertex minimal forbidden subgraphs. $P_3(G)$ is complete for any graph having at most five vertices. So, a forbidden subgraph must have at least six vertices. Let G_1 be any vertex minimal forbidden subgraph for Ψ. Since G_1 is a forbidden subgraph for $P_3(G)$ being complete, it must have at least two disjoint 3-paths, $a_1 - a_2 - a_3$ and $b_1 - b_2 - b_3$. If some a_i is adjacent to any of these b_j's then these six vertices are enough to induce a vertex minimal forbidden subgraph. Now, let no a_i be adjacent to any of the b_j's. Then, since G_1 is connected there must exist some path connecting a_i's and b_j's. Let the shortest such path be $a_i - c_1 - c_2 - ... - c_k - b_j$. Then we can find two disjoint induced 3-paths, $a_1 - a_2 - a_3$; and the other containing c_1. These six vertices are enough to induce a vertex minimal forbidden subgraph.

Remark 3.1: Ψ has only a finite collection of vertex minimal forbidden subgraphs.

4 Chromatic Number

In this section $\omega(G)$ denotes the clique number of G which is defined as the order of the largest complete subgraph of G. The chromatic number $\chi(G)$ is the minimum colors required for a proper coloring of vertices of G.

Lemma 4.1: For a connected graph G, $\omega(P_3(G)) \geq \omega(G) - 1$.

Proof: Let $\omega(G) = k$. Since G is non-complete and connected, there exists a vertex u adjacent to at least one vertex of the k-clique in G. If u is joined to t vertices of this k-clique then there are $t(k-t)$ induced 3-paths in G where u is common to all these induced 3-paths. So $\omega(P_3(G)) \geq t(k-t)$. Now, if $t(k-t) < k-1$ then $k < (t+1)$ which is a contradiction to the fact that $\omega(G) = k$. So $\omega(P_3(G)) \geq k-1$.

Theorem 4.1: For a connected graph G, $\chi(P_3(G)) \geq \chi(G) - 1$. The equality holds if and only if G is either $K_n - \{e\}$ or a complete graph with a pendant vertex attached to it.

Proof: Let $\chi(G) = k$. Then there exists a vertex v in G with color k such that its neighbors $v_1, v_2, ..., v_{k-1}$ have distinct colors $1, 2, ..., k-1$ respectively.

If these vertices form a k-clique then $\omega(G) \geq k$. So $\chi(P_3(G)) \geq \omega(P_3(G)) \geq k-1$, by lemma 4.1.

If these vertices do not form a k clique then let 'm' be the size of maximal clique in the subgraph induced by these vertices. Clearly v is a vertex in this m-clique. Then among the k vertices, there are $k-m$ vertices adjacent to v which are not in the m-clique. Let v_i be such a vertex. Then this v_i can be adjacent to at most $m-1$ vertices of the m-clique. In any case we can find at least k distinct induced 3-paths having a common vertex. The corresponding k vertices in $P_3(G)$ will form a k-clique and hence $\chi(P_3(G)) \geq k$.

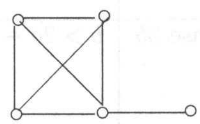
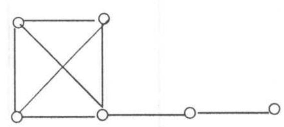
Hence the equality holds only when there is a k-clique in G. Since G is connected and non-complete, there exists a vertex u_1 which is adjacent to some of the v_i's in the k-clique. If u_1 is adjacent to t vertices of the k-clique where $2 \leq t \leq k-2$, then there exists at least k distinct induced 3-paths having a common vertex. Hence, in this case $\chi(P_3(G)) > k-1$. So u_1 can be adjacent with either 1 or $k-1$ vertices of the k-clique. If there exists one more vertex in G other than these $k+1$ vertices, then also we can find at least k induced 3-paths having a common vertex and hence

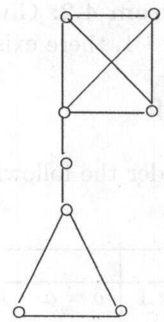
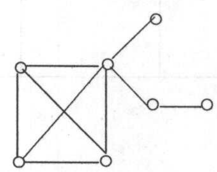
$\chi(P_3(G)) \geq k$. So when $\chi((P_3(G))) = \chi(G) - 1$, there exists exactly $k+1$ vertices such that u_1 is adjacent to 1 or $k-1$ vertices of the k-clique. If u_1 is adjacent to only one vertex of the k-clique, then the graph is a complete graph with a pendant vertex attached to it and if u_1 is adjacent to $k-1$ vertices of the k-clique, then the graph is $K_{k+1} - \{e\}$ and hence the result.

Theorem 4.2: Given any two positive numbers a and b where $a > 1$ and $b \geq a - 1$, there exists a graph G such that $\chi(G) = a$ and $\chi(P_3(G)) = b$.

Proof:

Consider the following cases, all of which have $P_3(G) = K_b$:

		Construction	Illustration
Case 1	$b = a - 1$	Attach a pendant vertex to any one vertex of K_a	$a = 4; b = 3$ 
Case 2	$b = a$	Consider the graph G of case 1. Then attach a single vertex to the pendant vertex of G.	$a = 4; b = 4$ 

Case 3	$b > a$	Construction	Illustration
Subcase $3a$	$b \leq 2a - 1$	Consider the graph G of case 1. Any one vertex of K_{b-a+1} is joined to the pendant vertex.	$a = 4; b = 6$ 
Subcase $3b$	$b > 2a - 1$	Let k be the maximum integer satisfying the equation ${}^{k}C_2 + (a-1)k = b$. Join k pendant vertices to the same vertex of K_a. Replace any one of these pendant vertices by $K_{b-[\ {}^{k}C_2+(a-1)k]}$	$a = 4; b = 9$ 

5 Some other Graph parameters

In this section we consider two other graph parameters, the domination number and the independence number. A subset S of the vertex set $V(G)$ is a dominating set if every vertex belongs to S or has a neighbor in S. The domination number of a graph G, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of vertices in G. The independence number of a graph G, denoted by $\alpha(G)$ is the maximum cardinality of an independent set of vertices in G.

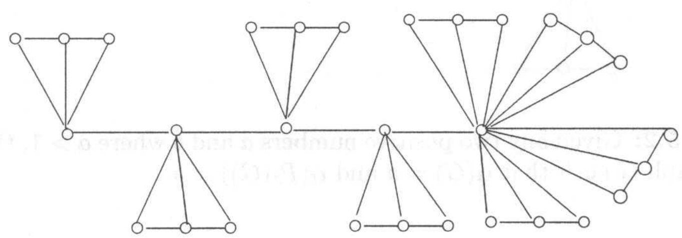
Theorem 5.1: Given any two positive numbers a and b where $a > 1$ there exists a graph G such that $\gamma(G) = a$ and $\gamma(P_3(G)) = b$.

Proof: Consider the following cases.

Case 1: Suppose $a < b$.

Consider a path $v_1v_2...v_a$. To each $v_i, i = 1, 2, ..., a-1$ join an induced 3-path - $w_{i1} - w_{i2} - w_{i3}$. To v_a join $2(b - a + 1)$ disjoint induced 3-paths. This is the required graph G. Clearly $\gamma(G) = a$. Consider the $a-1$ vertices in $P_3(G)$ which are of the form $w_{i1}v_iv_{i+1}, i = 1, 2, ..., a-1$. In $P_3(G)$, these vertices will dominate all the vertices except the vertices corresponding to the $2(b - a + 1)$ disjoint paths joined to v_a. These $2(b - a + 1)$ vertices can be dominated exactly by $b - a + 1$ vertices which are of the form $u_iv_au_j$ where u_i and u_j are vertices in any two of the disjoint induced P_3's joined to v_a. The above described collection of $a - 1$ vertices together with these $b - a + 1$ vertices will form a minimum dominating set for $P_3(G)$. Hence $\gamma(P_3(G)) = (b - a + 1) + (a - 1) = b$.

To illustrate this, consider $a = 5; b = 6$. The corresponding graph is,

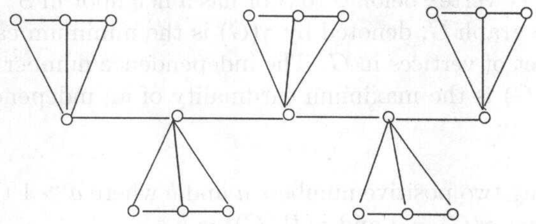


Case 2: Suppose $a = b$.

Consider a path $v_1v_2...v_a$. To each $v_i, i = 1, 2, ..., a$ join an induced 3-path, $w_{i1}w_{i2}w_{i3}$. This is the required graph G. Clearly $\gamma(G) = a$. In

$P_3(G)$, vertices of the form $w_{i1}v_iw_{i3}, i = 1, 2, ..., a$ is a minimum dominating set. Hence $\gamma(P_3(G)) = a = b$.

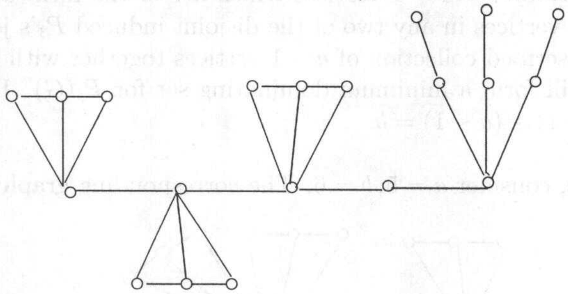
To illustrate this, consider $a = 5; b = 5$. The corresponding graph is,



Case 3: Suppose $a > b$.

Consider a path $v_1v_2...v_{b+1}$. To each $v_i, i = 1, 2, ..., b-1$ join an induced 3-path, $w_{i1}w_{i2}w_{i3}$. To v_{b+1}, attach $a - b + 1$ disjoint K_2's. This is the required graph G. Clearly $\gamma(G) = (b-1) + (a-b+1) = a$. In $P_3(G)$ the $(b-1)$ vertices which are of the form $w_{i1}v_iv_{i+1}, i = 1, 2, ..., b-1$ and $v_bc_1c_2$ where c_1, c_2 are the vertices in any K_2 attached to v_b will dominate all the vertices. Clearly this is the minimum number of vertices in any dominating set of $P_3(G)$. Hence $\gamma(P_3(G)) = b$.

To illustrate this, consider $a = 6; b = 4$. The corresponding graph is,



Theorem 5.2: Given any two positive numbers a and b where $a > 1$, there exists a graph G such that $\alpha(G) = a$ and $\alpha(P_3(G)) = b$.

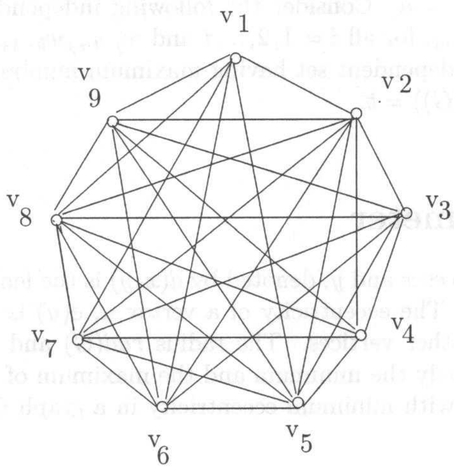
Proof:

Consider the following cases.

Case 1: Suppose $a < b$.

Consider a complete graph on $3b$ vertices labelled as $v_1, v_2, ..., v_{3b}$. From this graph, edges of the form $v_{3k-2} - v_{3k}, k = 1, 2, ..., b$ and edges whose both end vertices are of the form $v_{3k+1}, k = 0, 1, ..., a-1$ are deleted. This is the required graph G. Clearly $\alpha(G) = a$ where a maximum independent set is $\{v_1, v_4, ...v_{3a-2}\}$. Also $\alpha(P_3(G)) = b$ where a maximum independent set is $\{v_{3k-2}v_{3k-1}v_{3k}\}, k = 1, 2, ..., b$.

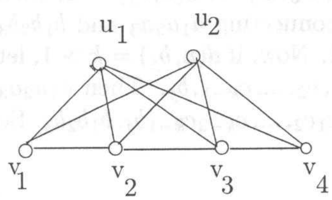
To illustrate this, consider $a = 2; b = 3$. The corresponding graph is,



Case 2: Suppose $a = b$.

Consider $G = (K_a)^c \vee P_{2a}$. Clearly $\alpha(G) = a$. $\alpha(P_3(G)) = a$ where the maximum independent set is $\{v_i u_i v_{a+i}\}, i = 1, 2, ..., a$ where v_i and v_{a+i} are vertices in P_{2a} and u_i is a vertex in $(K_a)^c$.

To illustrate this, consider $a = 2; b = 2$. The corresponding graph is,



Case 3: Suppose $a > b$.

Subcase 3a: Let $a \geq 2b$

Consider $G = K_{a,b}$ with the partition $\{u_1, u_2, ..., u_a\}$ and $\{v_1, v_2, ..., v_b\}$. Clearly $\alpha(G) = a$. Since a maximum independent set in $P_3(G)$ is $\{u_i v_i u_{b+i}\}, i = 1, 2, ..., b$, $\alpha(P_3(G)) = b$.

Subcase 3b: Let $a < 2b$

Let $t = \lfloor a/2 \rfloor$. Let $G = K_{a,b} \vee P_{2(b-t)}$. Let the partition of $K_{a,b}$ be $\{u_1, u_2, ..., u_a\}$ and $\{v_1, v_2, ..., v_b\}$ and let the vertices in the path be $w_1, w_2, ..., w_{2(b-t)}$. Then $\alpha(G) = a$. Consider the following independent set of vertices in $P_3(G)$: $u_i v_i u_{t+i}$, for all $i = 1, 2, ..., t$ and $w_j v_{t+j} w_{b-t+j}$, j = 1, 2, ..., $b - k$. This is an independent set having maximum number of vertices in $P_3(G)$. Hence $\alpha(P_3(G)) = b$.

6 Radius and Diameter

The distance between two vertices x and y, denoted by $d(x, y)$ is the length of a shortest $x - y$ path in G. The eccentricity of a vertex u, $e(u)$ is the maximum of its distances to other vertices. The radius $rad(G)$ and the diameter $diam(G)$ are respectively the minimum and the maximum of the vertex eccentricities. A vertex with minimum eccentricity in a graph G is called a center of G.

Theorem 6.1: For a connected graph G, $rad(P_3(G)) \leq rad(G) + 1$. The equality holds only when $rad(G) = 1$. Further if $rad(G) \geq 4$ then $rad(P_3(G)) < rad(G)$.

Proof: Let u be a center of G. So $d(u, v) \leq rad(G)$ for all $v \in V(G)$. Since G is not a complete graph, there exists an induced 3-path having u as a vertex in it. Let the corresponding vertex in $P_3(G)$ be $a_1 a_2 a_3$ where u is some a_i. Let $b_1 b_2 b_3$ be any other vertex in $P_3(G)$. If $d(u, b_j) = 1$, then $a_1 a_2 a_3, u b_j b_{j+1}$ or $u b_j b_{j-1}, b_1 b_2 b_3$ is a path connecting $a_1 a_2 a_3$ and $b_1 b_2 b_3$ and hence $d(a_1 a_2 a_3, b_1 b_2 b_3) \leq 2 = d(u, b_j) + 1$. Now, if $d(u, b_j) = k > 1$, let a shortest path connecting u and b_j be $u, c_1, c_2, ..., c_{k-1}, b_j$. Then $a_1 a_2 a_3$ and $b_1 b_2 b_3$ are connected by a path $a_1 a_2 a_3, u c_1 c_2, ..., c_{k-2} c_{k-1} b_j, b_1 b_2 b_3$. So $d(a_1 a_2 a_3, b_1 b_2 b_3) \leq k = d(u, b_j)$.

So $d(a_1 a_2 a_3, b_1 b_2 b_3) \leq d(u, b_j) + 1 \leq rad(G) + 1$, since $d(u, b_j) \leq rad(G)$. Hence $e(a_1 a_2 a_3) \leq rad(G) + 1$. Therefore $rad(P_3(G)) \leq rad(G) + 1$.

Now, let $rad(P_3(G)) = rad(G)+1$. We have proved that if $d(u, b_j) > 1$, then $d(a_1a_2a_3, b_1b_2b_3) \leq d(u, b_j) \leq rad(G)$. So $e(a_1a_2a_3) \leq rad(G)$ and hence $rad(P_3(G)) \leq rad(G)$. So the equality does not hold when $rad(G) > 1$.

Consider the case when $rad(G) \geq 4$. Consider $a_1a_2a_3$ where u is some a_i and let $b_1b_2b_3$ be any other vertex in $P_3(G)$. Let $d(u, b_j) = k$ and a shortest path connecting a_i and b_j be $a_i, c_1, c_2..., c_{k-1}, b_j$. Then $a_1a_2a_3$ and $b_1b_2b_3$ are connected by a path $a_1a_2a_3, uc_1c_2, c_2c_3c_4, ..., c_{k-2}c_{k-1}b_j, b_1b_2b_3$. So if $k \leq 3$, then $d(a_1a_2a_3, b_1b_2b_3) \leq 3$ and if $k \geq 4$, then $d(a_1a_2a_3, b_1b_2b_3) < k$. So $e(a_1a_2a_3) < k \leq rad(G)$. Hence $rad(P_3(G)) < rad(G)$.

Remark 6.1: The condition $rad(G) = 1$ is not sufficient for the equality $rad(P_3(G)) = rad(G) + 1$.

Eg:- $G = K_{1,n}, n \geq 3$, then $rad(G) = rad(P_3(G)) = 1$

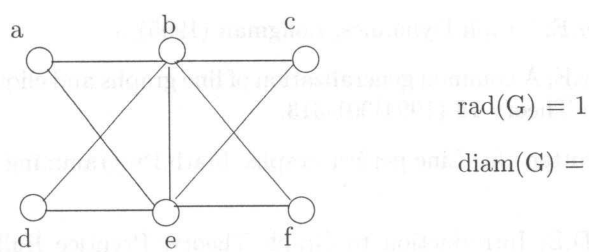
Theorem 6.2: For a connected graph G, $diam(P_3(G)) \leq diam(G)$. Further if $diam(G) \geq 4$ then $diam(P_3(G)) < diam(G)$.

Proof: Since G is not a complete graph $diam(G) > 1$. By the similar arguments as in the above proof, we can prove that for any two vertices $a_1a_2a_3$ and $b_1b_2b_3$ in $P_3(G)$, $d(a_1a_2a_3, b_1b_2b_3) \leq d(a_i, b_j) \leq diam(G)$. So $diam(P_3(G)) \leq diam(G)$.

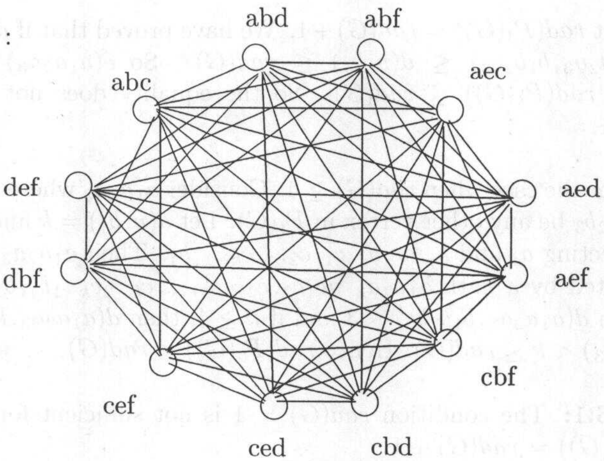
Let $diam(G) \geq 4$. Let $a_1a_2a_3$ and $b_1b_2b_3$ be any two vertices in $P_3(G)$ such that $d(a_1a_2a_3, b_1b_2b_3) = diam(P_3(G))$. Using the similar arguments as in the above proof, we can show that $d(a_1a_2a_3, b_1b_2b_3) < d(a_i, b_j) \leq diam(G)$. Hence $diam(P_3(G)) < diam(G)$.

Remark 6.2: The inequalities $rad(P_3(G)) \leq rad(G) + 1$ and $diam(P_3(G)) \leq diam(G)$ are strict.

G:



$P_3(G)$:



$rad(P_3(G)) = 2$ $\quad diam(P_3(G)) = 2$

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