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Information Sciences 113 (1999) 293–300

INFORMATION SCIENCES
AN INTERNATIONAL JOURNAL

A characterization of fuzzy trees

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Received 19 October 1997; received in revised form 29 May 1998; accepted 20 August 1998

Abstract

In this paper some properties of fuzzy bridges and fuzzy cutnodes are studied. A characterization of fuzzy trees is obtained using these concepts. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

The theory of fuzzy sets finds its origin in the pioneering paper of Zadeh [11]. Since then, this philosophy of "gray mathematics" [6] had tremendous impact on logic, information theory, etc. and finds its applications in many branches of engineering and technology [5].

A fuzzy subset [9] of a nonempty set S is a mapping $\sigma : S \to [0,1]$. A fuzzy relation on S is a fuzzy subset of $S \times S$. If μ and ν are fuzzy relations, then $\mu \circ \nu(u,w) = \operatorname{Sup}\{\mu(u,v) \wedge \nu(v,w) : v \in S\}$ and $\mu^k(u,v) = \operatorname{Sup}\{\mu(u,u_1) \wedge \mu(u_1,u_2) \wedge \cdots \wedge \mu(u_{k-1},v) : u_1, u_2, \ldots, u_{k-1} \in S\}$, where '$\wedge$' stands for minimum.

The theory of fuzzy graphs was independently developed by Rosenfeld [9] and Yeh and Bang [10] in 1975. A fuzzy graph is a pair $G : (\sigma, \mu)$, where σ is a fuzzy subset of S and μ is a fuzzy relation on S such that $\mu(u,v) \leqslant \sigma(u) \wedge \sigma(v)$ for all u, v in S. A fuzzy graph $H : (\tau, \nu)$ is called a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) \leqslant \sigma(u)$ and $\nu(u,v) \leqslant \mu(u,v)$ for all u, v. Further, H is a spanning subgraph if $\tau(u) = \sigma(u)$ for all u. A path p of length n is a sequence of distinct nodes $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, 3, \ldots, n$ and the weight of the

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PII: S0020-0255(98)10066-X

weakest arc is defined as its strength. If $u_0 = u_n$ and $n \geqslant 3$ then ρ is called a cycle. Also, $\mathrm{Sup}\{\mu^k(u,v) : k = 1,2,3\ldots\}$ gives the strength of connectedness between any two nodes u and v, denoted by $\mu^\infty(u,v)$. A fuzzy graph $G : (\sigma,\mu)$ is connected if $\mu^\infty(u,v) > 0$ for all u,v.

Recently, automorphisms of fuzzy graphs [3], fuzzy interval graphs [4], fuzzy line graphs [7], cycles and cocycles of fuzzy graphs [8], etc., have also been studied.

In this paper some properties of fuzzy bridges and fuzzy cutnodes are studied and a characterization of fuzzy trees is obtained using them.

Throughout, we assume that S is finite, μ is reflexive and symmetric [9]. In all the examples σ can be chosen in any manner satisfying the definition of a fuzzy graph. Also, we denote the underlying crisp graph by $G^* : (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in S : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in S \times S : \mu(u,v) > 0\}$.

2. Fuzzy bridges and fuzzy cutnodes

Definition 1 [9] . An arc (u,v) is a fuzzy bridge of $G : (\sigma,\mu)$ if deletion of (u,v) reduces the strength of connectedness between some pair of nodes.

Equivalently, (u,v) is a fuzzy bridge if and only if there exist x,y such that (u,v) is an arc of every strongest x–y path.

Definition 2[9]. A node is a fuzzy cutnode of $G : (\sigma,\mu)$ if removal of it reduces the strength of connectedness between some other pair of nodes.

Equivalently, w is a fuzzy cutnode if and only if there exist u,v distinct from w such that w is on every strongest u–v path.

Theorem 1 [9]. *The following statements are equivalent.*
1. *(u,v) is a fuzzy bridge.*
2. *(u,v) is not a weakest arc of any cycle.*

Remark 1. Let $G : (\sigma,\mu)$ be a fuzzy graph such that $G^* : (\sigma^*,\mu^*)$ is a cycle and let $t = \min\{\mu(u,v) : \mu(u,v) > 0\}$. Then all arcs (u,v) such that $\mu(u,v) > t$ are fuzzy bridges of G.

Theorem 2. *Let $G : (\sigma,\mu)$ be fuzzy graph such that $G^* : (\sigma^*,\mu^*)$ is a cycle. Then a node is a fuzzy cutnode of G if and only if it is a common node of two fuzzy bridges.*

Proof. Let w be a fuzzy cutnode of G. Then there exist u and v, other than w, such that w is on every strongest u–v path. Now $G^* : (\sigma^*,\mu^*)$ being a cycle, there exits only one strongest u–v path containing w and by Remark 1, all its arcs are fuzzy bridges. Thus w, is a common node of two fuzzy bridges. Conversely, let

w be a common node of two fuzzy bridges (u, w) and (w, v). Then both (u, w) and (w, v) are not the weakest arcs of G (Theorem 1). Also the path from u to v not containing the arcs (u, w) and (w, v) has strength less than $\mu(u, w) \wedge \mu(w, v)$. Thus the strongest u–v path is the path u, w, v and $\mu^{\infty}(u, v) = \mu(u, w) \wedge \mu(w, v)$. Hence w is a fuzzy cutnode □

Theorem 3. *If w is a common node of at least two fuzzy bridges, then w is a fuzzy cutnode.*

Proof. Let (u_1, w) and (w, u_2) be two fuzzy bridges. Then there exist some u, v such that (u_1, w) is on every strongest u–v path. If w is distinct from u and v it follows that w is a fuzzy cutnode. Next, suppose one of v, u is w so that (u_1, w) is on every strongest u–w path or (w, u_2) is on every strongest w–v path. If possible let w be not a fuzzy cutnode. Then between every two nodes there exist, at least one strongest path not containing w. In particular, there exist at least one strongest path ρ, joining u_1 and u_2, not containing w. This path together with (u_1, w) and (w, u_2) forms a cycle.

Case 1. If u_1, w, u_2 is not a strongest path, then clearly one of (u_1, w), (w, u_2) or both become the weakest arcs of the cycle which contradicts that (u_1, w) and (w, u_2) are fuzzy bridges.

Case 2. If u_1, w, u_2 is also a strongest path joining u_1 to u_2, then $\mu^{\infty}(u_1, u_2) = \mu(u_1, w) \wedge \mu(w, u_2)$, the strength of ρ. Thus arcs of ρ are at least as strong as $\mu(u_1, w)$ and $\mu(w, u_2)$ which implies that (u_1, w), (w, u_2) or both are the weakest arcs of the cycle, which again is a contradiction. □

Remark 2. The condition in the above theorem is not necessary. In Fig. 1, w is a fuzzy cutnode: (u, w) and (v, x) are the only fuzzy bridges.

Remark 3. In the following fuzzy graph (Fig. 2), (u_1, u_2) and (u_3, u_4) are the fuzzy bridges and no node is a fuzzy cutnode. This is a significant difference from the crisp graph theory.

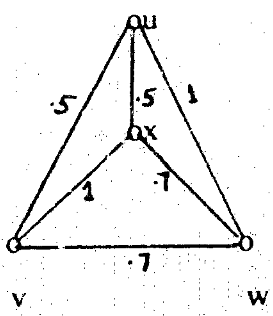

Fig. 1.

Theorem 4. *If (u, v) is a fuzzy bridge, then $\mu^\infty(u, v) = \mu(u, v)$.*

Proof. Suppose that (u, v) is fuzzy bridge and that $\mu^\infty(u, v)$ exceeds $\mu(u, v)$. Then there exists a strongest u–v path with strength greater than $\mu(u, v)$ and all arcs of this strongest path have strength greater than $\mu(u, v)$. Now, this path together with the arc (u, v) forms a cycle in which (u, v) is the weakest arc, contradicting that (u, v) is a fuzzy bridge. □

Remark 4. The converse of the above theorem is not true. The condition for the converse to be true is discussed in Theorem 9.

3. Fuzzy trees

Definition 3 [9]. A connected fuzzy graph $G : (\sigma, \mu)$ is a fuzzy tree if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree, where for all arcs (u, v) not in F, $\mu(u, v) < \nu^\infty(u, v)$.

Equivalently, there is a path in F between u and v whose strength exceeds $\mu(u, v)$.

Lemma 1 [9]. *If (τ, ν) is a fuzzy subgraph of (σ, μ), then for all u, v, $\nu^\infty(u, v) \leqslant \mu^\infty(u, v)$.*

Theorem 5. *If $G : (\sigma, \mu)$ is a fuzzy tree and $G^* : (\sigma^*, \mu^*)$ is not a tree, then there exists at least one arc (u, v) in μ^* for which $\mu(u, v) < \mu^\infty(u, v)$.*

Proof. If G is a fuzzy tree, then by definition there exists a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree and $\mu(u, v) < \nu^\infty(u, v)$ for all arcs (u, v) not in F. Also $\nu^\infty(u, v) \leqslant \mu^\infty(u, v)$ by Lemma 1. Thus $\mu(u, v) < \mu^\infty(u, v)$ for all (u, v) not in F and by hypothesis there exist at least on arc (u, v) not in F, which completes the proof. □

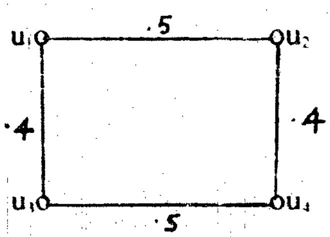

Fig. 2.

Definition 4 [3]. A complete fuzzy graph is a fuzzy graph $G:(\sigma,\mu)$ such that $\mu(u,v)=\sigma(u)\wedge\sigma(v)$ for all u and v.

Lemma 2 [3]. *If G is a complete fuzzy graph, then* $\mu^{\infty}(u,v)=\mu(u,v)$.

Lemma 3 [3]. *A Complete fuzzy graph has no fuzzy cutnodes.*

Remark 5. The converse of lemma 2 is not true (Fig. 3). Also, a complete fuzzy graph may have a fuzzy bridge (Fig. 4).

Theorem 6. *If* $G:(\sigma,\mu)$ *is a fuzzy tree, then G is not complete.*

Proof: If possible let G be a complete fuzzy graph. Then $\mu^{\infty}(u,v)=\mu(u,v)$ for all u,v [lemma 2]. Now G being a fuzzy tree, $\mu(u,v)<v^{\infty}(u,v)$ for all (u,v) not in F. Thus $\mu^{\infty}(u,v)<v^{\infty}(u,v)$, contradicting lemma 1. □

Theorem 7 [9]. *If G is a fuzzy tree, then arcs of F are the fuzzy bridges of G.*

Theorem 8. *If G is a fuzzy tree, then internal nodes of F are the fuzzy cutnodes of G.*

Proof. Let w be any node in G which is not an end node of F. Then by Theorem 7, it is the common node of at least two arcs in F which are fuzzy bridges of G and by Theorem 3, w is a fuzzy cutnode. Also, if w is an end node of F, then w is not a fuzzy cutnode; for, if so, there exist u,v distinct from w such that w is on every strongest u–v path and one such path certainly lies in F. But w being an end node of F, this is not possible. □

Corollary: *A fuzzy cutnode of a fuzzy tree is the common node of at least two fuzzy bridges.*

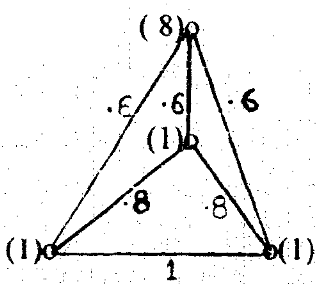

Fig. 3.

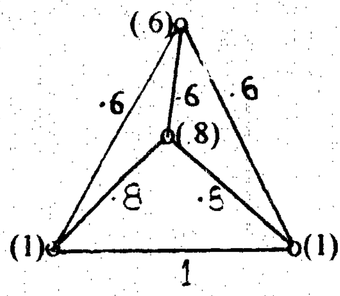

Fig. 4.

4. Main result

Theorem 9. *$G:(\sigma,\mu)$ is a fuzzy tree if and only if the following are equivalent.*
(1) *(u,v) is a fuzzy bridge.*
(2) *$\mu^\infty(u,v)=\mu(u,v)$.*

Proof. Let $G:(\sigma,\mu)$ be a fuzzy tree and let (u,v) be a fuzzy bridge. Then $\mu^\infty(u,v)=\mu(u,v)$ (Theorem 4). Now, let (u,v) be an arc in G such that $\mu^\infty(u,v)=\mu(u,v)$. If G^* is a tree, then clearly (u,v) is a fuzzy bridge; otherwise, it follows from theorem 5 that (u,v) is in F and (u,v) is a fuzzy bridge (Theorem 7).

Conversely, assume that (1) $\Longleftrightarrow$ (2). Construct a maximum spanning tree $T:(\sigma,\nu)$ for G [2]. If (u,v) is in T, by an algorithm in [2], $\mu^\infty(u,v)=\mu(u,v)$ and hence (u,v) is a fuzzy bridge. Now, these are the only fuzzy bridges of G; for, if possible let (u',v') be a fuzzy bridge of G which is not in T. Consider a cycle C consisting of (u',v') and the unique u'–v' path in T. Now arcs of this u'–v' path being fuzzy bridges they are not weakest arcs of C and hence (u',v') must be the weakest arc of C and hence cannot be a fuzzy bridge (Theorem 1).

Moreover, for all arcs (u',v') not in T, we have $\mu(u',v')<\nu^\infty(u',v')$; for, if possible let $\mu(u',v')\geqslant\nu^\infty(u',v')$. But $\mu(u',v')<\mu^\infty(u',v')$ (strict inequality holds, since (u',v') is not a fuzzy bridge). So, $\nu^\infty(u',v')<\mu^\infty(u',v')$ which gives a contradiction, since $\nu^\infty(u',v')$ is the strength of the unique u'–v' path in T and by an algorithm in [2], $\mu^\infty(u',v')=\nu^\infty(u',v')$. Thus T is the required spanning subgraph F, which is a tree and hence G is a fuzzy tree. □

Remark 6. For a fuzzy tree G, the spanning subgraph F is unique (Theorem 7). It follows from the proof of the above theorem that F is nothing but the maximum spanning tree T of G.

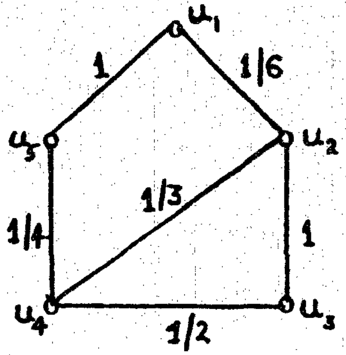

Fig. 5.

Theorem 10. *A fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree.*

Remark 7. For a fuzzy graph which is not a fuzzy tree there is at least one arc in T which is not a fuzzy bridge and arcs not in T are not fuzzy bridges of G. This observation leads to the following theorem.

Theorem 11. *If $G : (\sigma, \mu)$ is a fuzzy graph with $\sigma^* = S$ and $|S| = p$ then G has at most $p - 1$ fuzzy bridges.*

Theorem 12. *Let $G(\sigma, \mu)$ be a fuzzy graph and let T be a maximum spanning tree of G. Then end nodes of T are not fuzzy cut nodes of G.*

Corollary: *Every fuzzy graph has at least two nodes which are not fuzzy cut nodes.*

However, there are fuzzy graphs with diametrical nodes, nodes which have maximum eccentricity [1], as fuzzy cutnodes, distinct from crisp graph theory. See u_3 and u_5 of Fig. 5.

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