STABILITY OF RANDOM SUMS AND EXTREMES

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CERTIFICATE

Certified that the thesis entitled **'Stability of Random Sums and Extremes'** is a bonafide record of the work done by **Sri. S Satheesh** under my guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin, and that no part of it has been included anywhere previously for the award of any degree or title.

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Notations and Abbreviations Used.

$\bar{F}(.)$ or $R(.)$	- survival function
N_0	- {0,1,2, ...}
R	- the real line (-∞, ∞)
d.f	- distribution function
i.i.d	- independent and identically distributed
ln	- natural logarithm
r.v	- random variable
AM	- Absolutely Monotone
ARMA	- Auto Regressive Moving Average
CF	- Characteristic Function
CM	- Completely Monotone
CMD	- Completely Monotone Derivative
DP	- Discrete Pareto
Geometric(0)	- Geometric law on {0,1,2, ...}
Geometric(1)	- Geometric law on {1,2, ...}
GID	- Geometrically Infinitely Divisible
ID	- Infinitely Divisible
LT	- Laplace Transform
ML/ DML	- Mittag-Leffler/ Discrete Mittag-Leffler
N-ID/ ν-ID	- N- Infinitely Divisible/ ν- Infinitely Divisible
N-Max/ N-Min	- N-Maximum/ N-Minimum
PGF	- Probability Generating Function
SML/ DSML	- Semi Mittag-Leffler/ Discrete Semi Mittag-Leffler
SP/ DSP	- Semi Pareto/ Discrete Semi Pareto
SS/ DSS	- Semi Stable/ Discrete Semi Stable

1. INTRODUCTION

1.1. Introduction

Since the beginning of the eighteenth century, a major point of attention in probability theory had been the normal law pioneered by the 'Doctrine of Chances' of De Moivre in 1718. A considerably long period of time in the early part of the twentieth century was devoted to explore general conditions under which the distribution of sums of independent random variables (r.v) converge to the normal law. This was necessitated due to the importance, sums and arithmetic means of r.vs enjoyed, in many practical problems and the fact that exact distributions of these quantities were hard to find out in many cases. The difficulty in finding exact sampling distributions resulted in considerable problems of computing probabilities concerning the sums that involve a large number of terms. The search for global limit theorems involving normal law that permit calculations of such probabilities with a reasonable degree of accuracy was inspired by De Moiver's result in 1730 for the special case of Bernoulli scheme with $p = q = 1/2$. Laplace later generalized this, in 1812 to the case of arbitrary p different from 0 and 1. Other landmark contributions in the direction of normal limit law are by: Chebychev in 1887, Liapunov in 1901, Lindeberg in 1922 and Feller in 1935.

Based on the realization that the class of limit distributions is not exhausted by far by the normal law, a new trend for searching laws other than normal as the limit of sums of independent r.vs was launched, parallel with the consummation of the central limit theorem. It turned out that the class of these limit distributions coincided with infinitely divisible (ID) laws introduced by de Finetti in 1929. The modern theory of limit distributions of sums of independent r.vs (on $\boldsymbol{R}$) has witnessed significant developments essentially due to the researches of Khintchin, Levy, Kolmogorov, Feller, Gnedenko, Cramer, Doblin, Erdos and Kac.

Functions of sample observations that are of natural interest other than the sum, are the extremes, that is, the minimum and the maximum of the observations. Gumbel (1958) records that interest in the distribution of extremes could be traced back to the works of N. Bernoulli in the early eighteenth century, who was motivated by the application of laws of chance to actuarial and insurance problems. Galambos (1978) states that accurate and general solutions to the problem were implicitly contained in the works of Poisson, who has also contributed to the theory of sums and distribution of rare events.

Extreme value distributions also arise in problems like the study of size effect on material strengths, the reliability of parallel and series systems made up of large number of components, record values and assessing the levels of air pollution. Study of extreme values as we conceive it today, was initiated by Frechet in 1927 followed by Fisher and Tippet in 1928. For the same reason why the asymptotic distributions of the sums occupied much of the interest, asymptotic distributions of extremes also were studied extensively. In 1943 Gnedenko showed that irrespective of the original distribution, the asymptotic distribution (if it exists) of the minimum and of the maximum must be one of the three forms, which is akin to the normal law for the sample mean. A through discussion of these aspects is available in Galambos (1978).

It may be noticed that the theories of sums and extremes are mutually connected. For instance, in the search for asymptotic normality of sums, it is assumed that at least the variance of the population is finite. In such cases the contributions of the extremes to the sum of independent and identically distributed (i.i.d) r.vs is negligible. However, with the investigations of the asymptotic distributions of the sums for which even the expectation is not assumed to be finite (ID laws in general), the relevance and interrelations between the two theories emerged.

1.2. Stability of Sums

ID laws are defined as the class of distributions of r.vs Y that can be represented as

$$Y \stackrel{d}{=} X_1 + \ldots + X_n, \quad \text{for every } n \geq 1 \text{ integer,} \qquad (1.2.1)$$

where X_1, X_2, ... are i.i.d r.vs. Beginning with the analysis of ID laws with finite variance by Kolmogorov in 1932 using the method of characteristic functions (CF), Levy in 1934 and 1937 studied the general theory of ID laws and its one-to-one association with stochastic processes with stationary and independent increments. Khintchin and Feller also did further analytical treatment of CFs of ID laws. A comprehensive account of the work on limit theorems and ID laws is available in Feller (1971) and Gnedenko and Kolmogorov (1954).

The utility of ID laws has been further enhanced by the method of accompanying ID laws introduced by Gnedenko around 1940. The essence of the method is : if we have

$$S_n = X_1 + \dots + X_n \tag{1.2.2}$$

a partial sum of independent r.vs with CFs $f_1, \dots, f_n$, then we can set r.vs Y_n in correspondence with X_n having CFs $g_n = \exp\{f_n - 1\}$ for each n and define

$$\bar{S}_n = Y_1 + \dots + Y_n . \tag{1.2.3}$$

Then the distribution functions (d.f) of (1.2.2) converge to a limit d.f as $n \to \infty$ if and only if (iff) the limit distribution of the sums (1.2.3) exists and further these limits coincide. The method was later advanced by Kruglov in 1975 and is now used to discuss limit distributions of random sums as well, and is referred to as 'transfer theorems'.

Another concept that could be discussed in the framework of the summation scheme already discussed is that of stable laws. If $X_1, \dots, X_n$ are independent observations on X that is not concentrated at the origin, and $S_n = X_1 + \dots + X_n$, then X is stable (in the broad sense) if for each n there exists real constants $b_n > 0$ and a_n such that

$$S_n \stackrel{d}{=} b_n X + a_n . \tag{1.2.4}$$

Feller (1971, p170) has shown that the norming constants should be of the form $b_n = n^{1/\alpha}$, $0<\alpha \leq 2$. The general theory of stable laws was initiated by Levy 1924 who found CFs of all stable laws that correspond to $a_n = 0$. (In fact Cauchy studied stable laws around 1850 with Polya proving that the functions Cauchy studied really corresponded to probability distributions). Stable distributions play an increasing role in many practical problems as a generalization of the normal law. Holtzmark in 1919 itself showed that we could use certain stable laws in modelling electric fields induced at a fixed point in space due to charged particles in random motion. Mendelbrot, Fama, Roll and Du Mouchel advocated the use of stable laws in modelling certain economic phenomena as well; see Du Mouchel (1973).

A variation of equation (1.2.4) can be

$$X \stackrel{d}{=} cX + X_c \tag{1.2.5}$$

where X_c is a r.v independent of X. If such a representation holds good for each $0<c<1$, then X is said to belong to the class-L or to be self-decomposable. Levy and Khintchin studied this class of distributions. The analogy of the description of ARMA processes with equation (1.2.5) makes class-L laws useful in time series modeling; see Gaver and Lewis (1980).

Restricting ourselves to non-negative r.vs and their description in terms of Laplace transforms (LT) we have the following results (Feller (1971, p.448,450)). $\phi(s)$, $s>0$ is the LT of an ID law iff

$$\phi(s) = e^{-\varphi(s)},$$

where $\varphi(o) = 0$, $\varphi(s)$ is non-negative, and has completely monotone derivative (CMD). $\phi(s)$ corresponds to that of a stable(α) law iff

$$\phi(s) = \exp(-s^{\alpha}),\ 0<\alpha \leq 1.$$

These stable(α) r.vs X have the property of summation stability given by:

$$n^{-1/\alpha}\{X_1 + \ldots + X_n\} \stackrel{d}{=} X \text{ for every } n \geq 1 \text{ integer,} \tag{1.2.6}$$

where $\{X_i\}$ are i.i.d as X.

A problem that is of special interest in these cases is when the number of r.vs included in the sum itself is a discrete r.v N, taking non-negative integer values. This is the random summation (N-sum) scheme. Perhaps the most studied is the case of geometric sums (compound geometric laws).

Work on distributions of geometric sums of i.i.d r.vs is seen in many contexts although the relationships between the concepts were not explicitly noticed. One of these is rarefaction and contraction of renewal processes introduced by Renyi (1956). If each renewal epoch t_n of a process is replaced by qt_n, $0<q<1$ (contraction) and erased with probability $p = 1-q$ (rarefaction or p-thinning), we will obtain another renewal process. The point of interest is that the renewal distribution of the resultant process will be a geometric compound of the renewal distribution of the original process with the scale changed. The interconnection between the rarefaction model, geometric compounding model and the well-known damage models introduced by Rao (1965) is explained in Galmbos and Kotz (1977). It may be noticed that when the results in rarefaction and geometric compounding are formulated in terms of limit theorems they reduce to characterizations by the method of limit laws. A comprehensive account of the methods based on geometric sums is available in Kalashnikov (1997).

In the scheme of geometric summation we have the class of geometrically infinitely divisible (GID) laws introduced by Klebanov, et. al (1984). Y is said to be GID if it can be represented as:

$$Y \stackrel{d}{=} X_1 + \dots + X_{N(p)} \text{ for every } p \in (0,1), \tag{1.2.7}$$

where $X_1, X_2, \dots$ are independent copies of a r.v $X^{(p)}$ and $N(p)$ is geometric with mean $1/p$, independent of $X^{(p)}$. Equation (1.2.7) can be thought of as a variation of (1.2.1) and can also be written as:

$$Y \stackrel{d}{=} \varepsilon_p Y + X_p \tag{1.2.8}$$

where ε_p, Y and X_p are independent and

$$P\{\varepsilon_p=1\} = 1 - P\{\varepsilon_p=0\} = 1-p.$$

Y is said to be geometrically strictly stable if for every $0<p<1$ there is a constant $c(p)>0$ such that

$$Y \stackrel{d}{=} c(p)\{Y_1 + \dots + Y_{N(p)}\} \tag{1.2.9}$$

where $\{Y_i\}$ are independent copies of Y that is independent of $N(p)$ as well.

Renyi (1956) proved that Poisson process is the only one that is invariant under rarefaction and contraction applied together, assuming the existence of the mean of the renewal distribution. Introducing semi Mittag-Leffler (SML) laws with LT

$$\phi(s) = 1/[1 + \psi(s)] \tag{1.2.10}$$

where

$$\psi(s) = a\psi(bs) \text{ for all } s>0 \text{ and some } 0 < b < 1 < a$$

satisfying

$$ab^{\alpha} = 1 \text{ for } \alpha \in (0,1],$$

Sandhya (1991b) generalized the result of Renyi by showing that a renewal process is invariant under p-thinning and contraction applied together iff its renewal distribution is SML. In fact the only member of the SML family with finite mean is the exponential law and thus arriving at Renyi's result. She further noticed that if the process is invariant under p-thinning for two different values of p say p_1 and p_2 such that $\ln p_1 / \ln p_2$ is irrational, then the renewal distribution must be Mittag-Leffler (ML(α)), a class of laws introduced by Pillai (1990), with LT

$$\phi(s) = 1/(1 + s^{\alpha}),\ 0 < \alpha \leq 1. \tag{1.2.11}$$

Notice that ML(α) laws are geometrically stable and when $\alpha = 1$ we have the exponential law.

SML and ML laws are respectively semi-α-Laplace and α-Laplace laws ($0<\alpha \leq 2$) (Pillai (1985)), restricted to the half line. A reformulation of Theorem.1 of Pillai (1985) says that a distribution on $\boldsymbol{R}$ is a geometric sum of its own type iff it is semi-α-Laplace as noticed in Sandhya (1991a) and also in Lin (1994). α-Laplace laws are also known as Linnik laws and they are geometrically stable. It was noticed by Sandhya and Satheesh (1996a) that a semi-α-Laplace law belongs to class-L iff it is α-Laplace. For a recent review of geometrically stable laws see Kozubowski and Rachev (1999).

The close association of GID laws (without using this terminology) with p-thinning and Cox and renewal processes was discussed by Yannaros (1988,1989). Yannaros (1988) proved that $\phi(s)$ is the LT of the renewal distribution of a Cox and renewal process iff

$$\phi(s) = 1/[1 + \varphi(s)],$$

$\varphi(o) = 0$, $\varphi(s)$ is non-negative and has CMD. This means that $\phi(s)$ must be GID as observed by Sandhya (1991b). Kingman (1964) also had arrived at an equivalent representation while discussing doubly stochastic Poisson processes. See Fujita (1993) for more on non-negative GID laws.

Generalizing GID laws to the N-sum scheme, N-ID laws has been considered by Sandhya (1991a, 1996) as follows: Y is N-ID if there exists i.i.d sequence of r.vs $\{X_{\theta,i}\}$ and a non-negative integer valued r.v $N(\theta)$ having finite mean $m(\theta) > 1$, independent of the sequence such that

$$Y \stackrel{d}{=} X_{\theta,1} + \dots + X_{\theta,N(\theta)} \qquad (1.2.12)$$

for every θ in the parameter space Θ of θ. Restricting Y and $\{X_{\theta,i}\}$ to be non-negative r.vs having LTs ϕ and ϕ_θ respectively, she formulated the condition for Y to be N-ID as:

$$\phi[\phi_\theta^{-1}(s)] = Q_\theta(s) \text{ for every } \theta \in \Theta \tag{1.2.13}$$

where $Q_\theta(s)$ is a PGF, that of $N(\theta)$. When Y and $\{X_{\theta,i}\}$ (or ϕ and ϕ_θ) are of the same type Y is N-stable and (1.2.12) can be written as:

$$Q_\theta\{\phi[c(\theta)s]\} = \phi(s) \text{ for every } \theta \in \Theta.$$

She noted that this is the Abel's equation (Athreya and Ney (1972),p.10) and ϕ is its solution with respect to Q_θ. She also considered N-semi stable laws.

In more generality, many authors have discussed ν-ID laws starting from (1.2.12) and culminating in the result that a CF $f(t)$ is ν-ID iff

$$f(t) = \phi(-\ln g(t)) \tag{1.2.14}$$

where ϕ is a LT and $g(t)$ is the CF of an ID law. The developments in Gnedenko and Korelev ((1996), p.144) and Klebanov and Rachev (1996) are identical, using a semi-group approach with a transfer theorem, where ϕ is the standard solution of the Poincare equation (same as Abel's equation) given Q_θ. Bunge (1996) has based his arguments on semi-groups and subordinated Levy processes. When $g(t)$ is the CF of a stable law $f(t)$ is ν-stable, as also obtained by Kozubowski and Panorska (1996) invoking the transfer theorem alone. It is important to notice that Gnedenko and Korelev ((1996), p.148) proves that ν-ID laws and only them can be the limit laws for N-sums of identically distributed r.vs as $\theta \downarrow 0$. They also explicitly connects the $N(\theta)$ in the N-sum representation (1.2.12) to the LT ϕ in (1.2.14) by the requirement that

$$\theta N(\theta) \xrightarrow{L} Z \text{ as } \theta \downarrow 0 \tag{1.2.15}$$

and the LT of Z is ϕ, the standard solution of the Poincare equation.

While Sandhya (1991a, 1996) insists N-ID laws to have an N-sum representation, the definition in Gnedenko and Korelev (1996) is asymptotic in spirit; the CF is a function of a LT and is more general. This asymptotic approach is also in tune with the fact that when the exact sampling distribution of the N-sum is not tractable the search will be for the limiting distribution of the N-sum. Klebanov and Rachev (1996) have studied this approximation problem. Notice that the CFs of GID and geometrically stable laws have the form

$$1/\{1-\varphi\},$$

where φ is the natural logarithm of the CF of ID and stable laws. Since $1/(1+s)$ is the LT of the exponential law to which the geometric laws with mean $1/\theta$ converge as $\theta \downarrow 0$, this description is compatible with that of (1.2.15) by Gnedenko and Korelev (1996).

Again, when $N(\theta)$ is geometric with mean $1/\theta$, the week limit T of the geometric sum as $\theta \downarrow 0$ also satisfies

$$h_\theta\{ T_1 + \dots + T_{N(\theta)} \} \stackrel{d}{=} T, \text{ for every } \theta \in \Theta , \tag{1.2.16}$$

where $T_1 , T_2 , \dots$ are independent copies of T. But it is observed that the class of distributions for $N(\theta)$ satisfying (1.2.16) is obtained under very tight conditions on $N(\theta)$, (Kozubowski and Panorska (1998)). It is however, not known whether the geometric law is the only one for $N(\theta)$, in the asymptotic set up (1.2.15) of ν-stable laws that also satisfy (1.2.16). Another point of interest here is whether a discussion of N-sum stability for a particular value of θ is possible other than the geometric and whether it could be extended to include discrete distributions for X as well. Observe that when X has a continuous distribution $N(\theta)$ must be positive under (1.2.16).

A notion that is closely associated with stable laws is domains of attraction. In the geometric summation scheme Sandhya (1991a) has developed attraction and partial attraction and characterized geometrically stable and semi-α-Laplace laws, see also Sandhya and Pillai (1999). Employing transfer theorem, Mohan, et. al (1993) were able to improve some of these results. Further results on these lines are available in Ramachandran (1997). In the case of ν-stable laws attraction was developed in Gnedenko and Korelev (1996) and Klebanov and Rachev (1996).

It is known that geometrically stable laws (Linnik and ML laws belong to this class) successfully compete with stable laws in modelling financial asset returns, Kozubowski and Rachev (1994). Random summation schemes appear in applied problems in many fields, see Gnedenko and Korelev (1996) and Kalashnikov (1997). Since ν-stable laws approximate random sums, they have varied practical applications in many fields.

Though ID laws have found fruitful utility in both theory and applications it was a bit hard to verify whether or not a given distribution is

ID. Goldie (1967) achieved a breakthrough in this direction for non-negative continuous r.vs. He showed that distributions that are mixtures of exponential laws are ID. Steutel (1969) went further by showing that densities of mixtures of exponential laws are completely monotone (CM) and hence all CM densities are ID. This result brought in the notion of CM functions to the realm of probability densities. Shanbhag and Sreehari (1977, 1979) extended Goldie's result to include gamma mixtures and log-convex densities, which were shown to be ID and discussed their relation to class-L. By the method of Goldie, Thorin (1977) observed that Pareto laws are ID and that they are self-decomposable also thus bringing in the notion of generalized gamma convolutions that are in class-L. Pillai and Sandhya (1990) strengthened the Goldie-Steutel result by showing that CM densities are characteristic of mixtures of exponentials and further they are GID. Thorin proved this result independently in the context of insurance mathematics where they have found a lot of applications, see Grandall (1991). Pillai and Sandhya (1990) also gave a density that is GID but not CM. Some more results on mixtures of exponential laws are available in Sandhya and Satheesh (1996b, 1997). Bondesson (1990) discusses recent results on generalized gamma convolutions and CM functions.

In the discrete set up the notion of ID laws had been discussed by Feller (1968, p.290) as compound Poisson laws, independent of the general theory. Steutel and van Harn (1979) introduced discrete analogue of self-decomposable laws by the PGF $Q(s)$ satisfying for every $0<c<1$

$$Q(s) = Q(1-c+cs)\, Q_c(s) \tag{1.2.17}$$

where $Q_c(s)$ is a PGF. However how this can be arrived at from (1.2.5) is not mentioned therein to justify the definition. They even recorded that a formal analogue of class-L laws led them to compound Poisson laws (that is discrete ID laws). Subsequently they also discussed discrete stable(α) laws. But they arrived at it by considering the discrete analogue of

$$(s+t)^{1/\alpha}\, X = s^{1/\alpha} X_1 + t^{1/\alpha} X_2\,, \text{ for all } s, t>0, \tag{1.2.18}$$

where X is a stable(α) law and X_1 , X_2 independent copies of it, and proved that the corresponding PGF is of the form

$$\exp\{-\lambda(1-s)^{\alpha}\},\ \lambda>0 \text{ and } 0<\alpha\leq 1. \tag{1.2.19}$$

The method was to replace cX in the continuous set up by $c\circ X = \sum_{j=1}^{X} B_j$, where

$$P\{B_j=1\} = 1-P\{B_j=0\} = c\,. \tag{1.2.20}$$

Rao and Shanbhag (1994, p.160) arrived at this PGF from (1.2.18) from a different approach. Christoph and Schreiber (1998) also studied discrete stable(α) laws. But it is not clearly mentioned in these works why and / or how the replacement of cX by $c\circ X$ is justified, and whether discrete stable(α) laws are in fact stable under ordinary summation as given in equation (1.2.6).

A work related to discrete analogue of ML(α) law is by Pillai and Jayakumar (1995). They derived many properties of it similar to that of ML laws but their stability under geometric summation has not been mentioned.

The discrete version of CM densities as CM probability sequences was discussed in Satheesh and Sandhya (1997) and was shown to be characteristic of mixtures of geometric laws.

1.3. Stability of Extremes

In the study of distribution of extremes when the number of observations is random we speak of random extremes. The case of geometrically distributed sample size was perhaps the first one studied. It may be noted that study of geometric minimums has been suggested in 1976, as a model for income distribution by Arnold and Laguna. In 1982, Pakes used a system involving repeated geometric maxima to model entrepreneurial earnings.

A r.v Y is said to be stable under geometric maximum (max) if

$$cY \stackrel{d}{=} \text{Max}\{Y_1, \dots, Y_{N(p)}\} \text{ for some } c>0, \tag{1.3.1}$$

where $N(p)$ is geometric with mean $1/p$ independent of Y and $\{Y_i\}$ are independent copies of Y. One can conceive stability of geometric minimums (min) on a similar line. The study of stability of distributions of geometric max and geometric min was initiated in Arnold, et. al (1986) though they did not use the term stability. They discussed schemes involving geometric max and min one after the other, which could arise as models in competition for employment and characterized Pareto-III and logistic laws in the contexts.

A formal definition and discussion of the problem of stability of random (N)-max and N-min began with Voorn (1987, 1989). He required a condition similar to (1.3.1) to be satisfied for some positive integer valued r.v N, and characterized logistic, log-logistic, their extended versions (for Y), and the geometric law and its extended version (for N) in the set up. Further, he assumed that the distribution of Y must be absolutely continuous and a sequence of distributions for N, that takes the value 1 with probability tending to one. Now, is it possible to extend these notions to include discrete distributions also? Introducing semi Pareto family of laws, whose survival function has the structure of the LT of SML laws (1.2.10) with $\alpha > 0$, Pillai (1991) characterized it by geometric-max stability while Pillai and Sandhya (1996) by geometric-min stability. Bivariate and multivariate extensions of these laws were studied by Balakrishna and Jayakumar (1997). Exploiting the close association of the geometric-min structure with minification processes, Yeh, et. al (1988), Pillai (1991) and Balakrishna and Jayakumar (1997) have developed logistic, Pareto and semi Pareto processes as well.

A closely related problem is the parameterization scheme of Marshall and Olkin (1997). For a survival function $\bar{F}(x)$, $x \in \boldsymbol{R}$, they defined another survival function

$$G(x,a) = \frac{a\bar{F}(x)}{1-(1-a)\bar{F}(x)}, \quad x \in \boldsymbol{R}, \text{ and } a>0, \tag{1.3.2}$$

and showed that this family is geometric max and min stable. They attributed this remarkable property (partially) to the fact that geometric law is closed under its own compounding. However, they neither gave an analytic proof supporting this claim nor provided the conditions under which the claim can be true. They further stated that one cannot expect stability of extremes w.r.t a sample size distribution other than the geometric.

It is worth mentioning here that characterizations of distributions using N-sum and N-min together have been considered in the literature. The stability relation of interest here is:

$$cS_N \stackrel{d}{=} N\,\mathrm{Min}\{X_1, \ldots, X_N\}, \text{ for some } c>0. \tag{1.3.3}$$

For some works on this see Kakosyan et al. (1984), and Mohan (1992).

In the following section we present certain problems that have emerged as a result of the discussion made in this and previous sections.

1.4. Certain Problems Identified in the Present Study

In the present study we conceive N-sum stability and N-max & N-min stability in the following way.

Let $X, X_1, X_2, \ldots$ be non-degenerate non-negative i.i.d r.vs with a common LT $\phi(s)$ and N a positive integer valued r.v independent of X with probability generating function (PGF) $Q(t)$. Set

$$S_N = X_1 + X_2 + \ldots + X_N$$

which we refer to as the N-sum of X_i's. When $cS_N \stackrel{d}{=} X$ for some $c>0$, or equivalently for all $s>0$

$$Q(\phi(cs)) = \phi(s), \tag{1.4.1}$$

we say that the distribution of X is stable under the operation of summation with respect to (w.r.t) the r.v N, that is, X is N-sum stable.

Let $X, X_1, X_2, \ldots.$ be non-degenerate i.i.d r.v with a common distribution function (d.f) F and N be a positive integer valued r.v with PGF $Q(s)$ independent of X. Set

$$V = \text{Max}\,(X_1, \ldots., X_N) \quad \text{and} \quad U = \text{Min}\,(X_1, \ldots., X_N).$$

Then F is maximum stable w.r.t N (F is N-max stable) if there exists some constants $a \in \boldsymbol{R}$ and $b>0$ such that

$$b^{-1}(V-a) \stackrel{d}{=} X. \tag{1.4.2}$$

An equivalent representation of (1.4.2) in terms of the PGF $Q(s)$ of N is,

$$Q(F(x)) = F\left(\frac{x-a}{b}\right) \text{ for all } x \in \boldsymbol{R}. \tag{1.4.3}$$

When the support of $F(x)$ is $[0,\infty)$, invoking Lemma.2.2 of Voorn (1987) we have $a = 0$ and $b > 1$, and thus F is N-max stable if:

$$Q(F(x)) = F(cx)\,, \text{ for all } x \geq 0 \text{ and some } c \in (0,1)\,. \tag{1.4.4}$$

Similarly F is minimum stabile w.r.t N (F is N-min stable) if:

$$Q[\bar{F}(cx)] = \bar{F}(x) \quad \text{for all} \quad x \geq 0 \text{ and some } c > 0, \tag{1.4.5}$$

where $\bar{F}(x) = 1 - F(x)$.

Notice that our description of N-sum stability and N-extreme stability demands stability only for some $c > 0$.

In the N-sum scheme, an interesting problem is to identify N that imparts N-sum stability of a known X. This could give some idea regarding the mechanism that generates the random sum. Cinlar and Agnew (1968), Balakrishna and Nair (1997) do characterize the geometric law while Sandhya (1996) the ML law. But it appears that a general method is not available in the literature. We present a method here motivated by the equation (1.2.13). Bunge (1996) has come to our notice recently, where examples (on similar lines) illustrating the generation of PGFs by LTs are given, but not as a method to identify N. Also, distributions other than geometric have not been studied extensively. Whether the geometrically stable law is the only one that satisfies both the asymptotic setup (1.2.15) and the N-sum setup (1.2.16) of ν-stable laws is also worth looking in to.

It is well known that the geometric law on $\{0,1,....\}$ shares many properties of the exponential law. It is easy to see that this geometric law is N-sum stable w.r.t the positive geometric law. But a comprehensive treatment of N-sum stability of discrete distributions is not seen in the literature. One possible reason is that we do not have a discrete analogue of the notion of distributions of the same type, which is at the heart of the concept of stability. A hint of course, is there in Steutel and van Harn (1979). Developing such a notion may also justify their definition of discrete class-L laws. This is also important, as there are many methods to verify whether a distribution is a member of class-L or not, see eg. Lukacs (1970), Pillai and Sabu George (1984), Shanbhag and Sreehari (1977, 1979), Ismail and Kelkar (1979), Pillai and Satheesh (1992), Sandhya and Satheesh (1996a) and Jurek (1997).

In the discussion of stability of N-extremes a glaring omission that appears is that stability of exponential laws was not considered. Also, as in the case of N-sum stability, identifying N that imparts N-max and N-min stability for a given $F(x)$ is a relevant problem. After developing a method here, motivated by the analogous problem in N-sum stability, we have come to know the work of Sreehari (1995), which in fact is more general. The assumption of absolute continuity of $F(x)$ is made in all the works beginning from Voorn (1987). Thus extending the notion to the discrete set up is another problem that we will study in the present work. Also, other than the extended geometric law of Voorn, no distribution other than the geometric has been discussed in this context.

The semi Pareto family of laws was characterized among non-negative continuous distributions by geometric-max stability in Pillai (1991) and by geometric-min stability in Pillai and Sandhya (1996). From these two characterizations it is clear that among distributions with non-negative support geometric-max stability implies geometric-min stability and vice-versa as both identify the same family. A natural curiosity thus is whether we can prove this without referring to the family of semi Pareto laws and also whether it is true in general for d.fs with support $\boldsymbol{R}$. Another question is whether it is unique of the geometric law. This is also relevant in the Marshall-Olkin parameterization scheme since they have not given an analytic proof supporting their claim regarding the uniqueness of the geometric law in the context and also have not specified the conditions under which their claim is true.

Thus in Chapter.2 we will discuss the problem of identification of N given X, and distributions other than geometric for N in the N-sum stability scheme. The distribution identified thus is the Harris law and we derive the most general distribution that is N-sum stable w.r.t this law. We also show that the Harris stable law satisfies both the asymptotic setup (1.2.15) and the N-sum setup (1.2.16) of ν-stable laws.

In the attempt to extend the notion N-sum stability to the domain of discrete laws, we justify the Steutel-van Harn definition of discrete class-L laws and the replacement of cX by $c \circ X$. This is done in Chapter.3 by developing the discrete analogue of distributions of the same type. The justification shows that discrete analogue of class-L laws are (naturally) in discrete class-L. Also discrete counterpart of stability of geometric sums and the results corresponding to those in Chapter.2 are discussed here.

Stability of N-extremes of exponential laws, a distribution other than the geometric in the set up, namely the Sibuya law, and a method to identify the distribution of N in N-extreme stability are discussed in Chapter.4. The notion is then extended to include discrete distributions as well by developing distributions of the same type for discrete laws that is shown to be different from the one in the previous chapter.

In Chapter.5 the uniqueness of the geometric law for N in N-extreme stability is studied culminating in a conjecture characterizing the geometric law. Its relevance in the Marshall-Olkin scheme is then discussed and we complement their reasoning.

1.5. Some Basic Concepts Required

We have already mentioned that the notion of distributions of the same type is at the heart of the concept of stability. Two r.vs X & Y with d.fs F & G and CFs ϕ_X & ϕ_Y are of the same type if there exist some constants $a \in \boldsymbol{R}$ and $b > 0$ such that

$$Y = (X - a)/b \text{ or}$$

$$G(x) = F(bx+a) \quad \text{for all } x \in \boldsymbol{R} \text{ or}$$

$$\phi_Y(t) = e^{-i(a/b)t}\, \phi_X(t/b) \quad \text{for all } t \in \boldsymbol{R}.$$

Now we describe certain classes of functions that will be used in our discussion frequently. This is the class of continuous functions satisfying

$$\psi(u) = a\, \psi(bu) \text{ for all } u \in \boldsymbol{R}, \text{ and some } a,\, b > 0, \tag{1.5.1}$$

with $\psi(o) = 0$. Such functions are discussed in the context of CFs of semi stable laws and regression equations in Kagan, et. al ((1973), p. 9, 163, 323, 324)), integrated Cauchy functional equations and LTs in Pillai and Anil (1996) and certain variations in Dubuc (1990) and Biggins and Bingham (1991) in the context of branching processes. It has been proved that for (1.5.1) to hold the condition $0<b<1<a$ is necessary and that there must exist a unique $\alpha > 0$ such that $ab^{\alpha} = 1$. When

$$\phi(u) = \exp\{-\psi(u)\}$$

is a CF, α has to be restricted to (0,2] and $\alpha \in (0,1]$ when $\phi(u)$ is a LT. On the other hand when $\phi(u)$ is considered as a survival function (Pillai (1991)),

α need only be positive. It is also known that (Pillai and Anil (1996)) $\psi(u)$ has a representation in terms of a periodic function as:

$$\psi(u) = h(u)\,|u|^{\alpha}$$

where $h(u)$ is periodic in $\ln(u)$ with period $-\ln(b)$. Based on this representation we also have; when (1.5.1) is satisfied for two values of b, say b_1 and b_2 such that $\ln b_1/\ln b_2$ is irrational then

$$\psi(u) = \lambda|u|^{\alpha}$$

for some positive constant λ. For examples of $\psi(u)$ with the component $h(u)$, as a CF, LT and d.f see Pillai (1985), Jayakumar and Pillai (1993) and Pillai (1991) respectively. In the present thesis the symbol $\psi(u)$ will be used exclusively for functions satisfying (1.5.1).

A function ϕ on $(0,\infty)$ is completely monotone (CM) if it possesses derivatives $\phi^{(n)}$ of all orders and

$$(-1)^n\phi^{(n)}(s) \geq 0 \text{ for all } s > 0. \qquad (1.5.2)$$

Bernstein's theorem for CM functions states that a function ϕ on $[0,\infty)$ is the LT of a probability distribution, iff it is CM and $\phi(o) = 1$ (Feller (1971), p. 439). Clearly ϕ is non-negative.

Bernstein's theorem for absolutely monotone (AM) functions states that for a continuous function u on $[0,1]$ the following two properties are equivalent (Feller (1971), p.223).

$$u(x) = p_0 + p_1 x + p_2 x^2 + \ldots, \quad p_j \geq 0 \text{ and}$$

$$u^{(n)}(x) \geq 0. \tag{1.5.3}$$

Here (1.5.3) describes AM functions on (0,1). When $u(1) = 1$, $u(x)$ is a PGF.

Given a finite or infinite numerical sequence $\{a_i\}$ the differencing operator Δ is defined by

$$\Delta a_i = a_{i+1} - a_i .$$

The successive differences of order r is inductively obtained as

$$\Delta^r = \Delta\Delta^{r-1},\ r = 1, 2, \ldots .$$

In this notation sequences $\{c_k\}$ are called CM if

$$(-1)^r \Delta^r c_k \geq 0, \quad r = 0,1,2, \ldots, \tag{1.5.4}$$

where $\Delta^o c_k = c_k$. Hausdorff's theorem for CM sequences (Feller (1971), p.226) states that the moment sequence $\{c_k\}$ of a probability distribution on [0,1] form a CM sequence with $c_o = 1$. Conversely, an arbitrary CM sequence $\{c_k\}$ subject to $c_o = 1$ coincides with the moment sequence of a unique probability distribution on [0,1].

In the sequel, a geometric law on {0,1, ...} with parameter p will be denoted by geometric(0,p) and the one on {1,2, ...} by geometric(1,p) (with or without the parameter p being specified). This difference in notation is necessitated, as we require both the distributions in our discussion.

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2. STABILITY OF RANDOM SUMS OF CONTINUOUS VARIABLES

2.1. Introduction

Let us recall from Section.1.4 equation (1.4.1) that the distribution of X with LT ϕ is stable under the operation of summation w.r.t an independent positive integer valued r.v N with PGF Q if;

$$Q[\phi(cs)] = \phi(s) \text{ for all } s>0 \text{ and some } c>0. \qquad (2.1.1)$$

Often N is described as the compounding r.v. We will refer to this as X is N-sum stable. We had also noticed (page.18) that a problem that does not appear to have been discussed in a general framework in the literature is the identification of the distribution of N given that of X. For example, it is known (Sandhya 1991b) that X is a geometric(1) sum of its own type iff X is semi Mittag-Leffler(a,b) (SML(a,b)). Here the sufficiency part says that a SML(a,b) r.v is N-sum stable when N is geometric(1). Now the question is whether N must necessarily be geometric(1) for the N-sum stability of SML(a,b) r.v. This is also important as it could give some idea regarding the random mechanism generating the random sum. Certain known results in this regard are: When X is exponential (univariate) Cinlar and Agnew (1968) showed that N must be geometric(1) under N-sum stability. Again when X is bivariate exponential Balakrishna and Nair (1997) showed that N must be

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This chapter is based on Satheesh, Nair and Sandhya (1999).

geometric(1). When X has a Mittag-Leffler (ML(α)) law Sandhya (1996) showed that N must be geometric(1). Another point that we have noticed in this context is that most of the discussions concentrate around geometric(1) law for N. Now, can we discuss other distributions for N instead of the geometric(1)? In this chapter we focus attention on the problems of identifying the distribution of N that imparts N-sum stability of X (given) and discuss distributions for N other than geometric(1).

A method based on Sandhya (1991*a*, 1996), for the identification of the distribution of N given that of X, which is applicable when X has a non-negative continuous distribution, is discussed in the present chapter. We begin Section.2.2 with the relevant methodology and follow it with a Lemma on PGFs that is useful in fixing the range of the parameters in the model. This leads to characterizations of various distributions for N given X such as geometric(1), Harris(a,k) and the degenerate laws and these are established in Section.2.3. When N follows Harris(a,k) law the distribution of X can be identified in a more general set up as shown in Section.2.4.

2.2. Identifying N in N-sum Stability

From Sandhya (1991a, 1996) we have the following result.

A necessary and sufficient condition that a r.v X with LT ϕ is N-ID is that there exists a LT ϕ_m such that $\phi[\phi_m^{-1}]$ is a PGF for every m in the parameter space of m.

Now, in the N-sum stability scheme the key equation is;

$$Q[\phi(cs)] = \phi(s), \text{ for all } s>0 \text{ and some } c>0. \quad (2.2.1)$$

Setting $\phi_c(s) = \phi(cs)$ we have from (2.2.1) that

$$Q[\phi_c(s)] = \phi(s) \text{ for all } s>0 \text{ and some } c>0.$$

Since LTs are decreasing in $(0,\infty)$ the inverse function ϕ^{-1} exists and is unique. Therefore by taking

$s = \phi_c^{-1}(t)$ for $0< t <1$, we have;

$$Q(t) = \phi[\phi_c^{-1}(t)] \text{ for some } c>0. \quad (2.2.2)$$

Thus the PGF Q of the compounding law is identified uniquely from the LT ϕ under the stability condition (2.2.1). One may as well conceive (2.2.2) as a definition of N-sum stability. Also, (2.2.2) suggests that c appears as a parameter in the PGF. The range of the values of c and other parameters in ϕ in the above scheme remains to be determined. For this purpose, we establish the following Lemma.

Lemma.2.2.1 If $Q(s)$ is a PGF, then $Q(s^u)$ is a PGF iff $u>0$ is an integer.

Proof. $Q(s^u)$ is a PGF iff it can be represented as a power series in s (Feller, (1971), p.223). When $u>0$ is not an integer this is not possible. □

Remark.2.2.1 Defining $p_i = P\{N = i\}$, i = 1,2,3, ...we have from (2.2.1);

$$\phi(s) = \phi(cs)\ \{p_1 + p_2\,\phi(cs) + \ldots\} \text{ for all } s>0 \text{ and some } c>0.$$

Since $0< \phi(s) <1$ and $\phi(s)$ is a decreasing function of s one must have $\phi(cs) > \phi(s)$ and hence $cs<s$. This implies $0<c<1$. This result appears to be true intuitively also as otherwise the distribution of the sum can grow out of proportion if the contribution of the individual components is not restricted. We will refer to c as the scale parameter in the N-sum scheme.

Lemma.2.2.2 In the set up of equation (2.2.1), $\phi(s)$ cannot have a real zero.

Proof. Notice that $Q(s) = 0$ only when $s = 0$, as Q is the PGF of a positive r.v. Now if $\phi(s)$ has a real zero say τ, then $\phi(c\tau) = 0$ and $\phi(c^n\tau) = 0$ for every $n>1$ integer, on iteration. Since $c \in (0,1)$ this will imply that

$$\underset{n \to \infty}{Lt}\ \phi(c^n\tau) = \phi(o) = 0,$$

also by the continuity of $\phi(s)$, which is a contradiction. Hence the proof. □

This argument is true even when ϕ is a CF. This property also suggests the possibility of them being ID laws. We now demonstrate our method of identifying the distribution of N that imparts stability of X (known) in the N-sum scheme in the next section.

2.3. Characterization of N given X

In the context of branching processes Harris(1948) showed that the gamma$(1,1/k)$ law with LT

$$\phi(s) = 1/(1 + s)^{-1/k},\ k>0$$

is N-sum stable when the PGF of N is of the form

$$Q(t) = t/\{a - (a-1)t^k\}^{1/k},\ k>0 \text{ integer},\ a>1. \qquad (2.3.1)$$

Notice that (2.3.1) reduces to the PGF of the geometric(1) law when $k = 1$. We refer to the PGF (2.3.1) as that of Harris(a,k) law. Now, we show that N must necessarily be Harris(a,k) for Gamma$(1,\beta)$ to be N-sum stable.

Theorem.2.3.1 Gamma($1,\beta$) law is N-sum stable iff N is Harris(a,k), $\beta = 1/k$, k>0 integer, and $c = 1/a$.

Proof. We have:

$$\phi_c(s) = (1+cs)^{-\beta} \quad \text{and} \quad \phi_c^{-1}(t) = \frac{1}{c}\frac{1-t^{1/\beta}}{t^{1/\beta}}.$$

Hence,

$$\phi[\phi_c^{-1}(t)] = t \,/\, \left\{\frac{1}{c} - \left(\frac{1}{c} - 1\right)t^{1/\beta}\right\}^{\beta}$$

$$= \left\{\frac{ct^{1/\beta}}{1-(1-c)t^{1/\beta}}\right\}^{\beta}. \qquad (2.3.2)$$

Since $t\,\{c/[1-(1-c)t]\}^{\beta}$ is a PGF for $0<c<1$ and $\beta>0$, (2.3.2) is a PGF only when $1/\beta = k$ a positive integer, by invoking Lemma.2.2.1. Hence N is Harris(a,k) where $a = 1/c$.

Conversely, as shown by Harris (1948) with

$\phi_c(s) = 1/(1+cs)^{-1/k}$, $c>0$, $k>0$ integer and

$Q(t) = t/\{a-(a-1)t^k\}^{1/k}$, $k>0$ integer, $a>1$,

we have:

$$Q[\phi(cs)] = \frac{\phi(cs)}{\{a-(a-1)\phi^k(cs)\}}$$

$$= \frac{(1+cs)^{-1/k}}{\{a(1+cs)-(a-1)\}^{1/k}/(1+cs)^{-1/k}}$$

$$= (1+acs)^{-1/k}$$

$= \phi(s)$, when $c = 1/a$,

as required to be shown. □

Note. The above result also shows that the gamma$(1,\beta)$ law can not be N-sum stable unless $1/\beta$ is an integer.

SML(a,b), $0<b<1<a$ laws are defined (Sandhya (1991b)) by the LT

$$\phi(s) = [1+\psi(s)]^{-1}, \tag{2.3.3}$$

where $\psi(s)$ satisfies (1.5.1). It has been shown by her (in the context of renewal processes invariant under p-thinning) that a positive r.v X is a geometric(1) sum of its own type iff X is SML(a,b). Now we prove that for the N-sum stability of SML(a,b) laws N must be geometric(1).

Theorem.2.3.2 SML(a,b) law is N-sum stable iff N is geometric(1) with parameter $p = b^{\alpha}$ and $c = b$.

Proof. We have

$$\phi(s) = [1+\psi(s)]^{-1} = [1+ a\psi(bs)]^{-1}.$$

Now,

$$\phi_c(s) = [1+\psi(cs)]^{-1} \quad \text{and} \quad \phi_c^{-1}(t) = \frac{1}{c}\psi^{-1}\left(\frac{1-t}{t}\right).$$

Hence,

$$\phi[\phi_c^{-1}(t)] = \left\{1 + a\psi\left[b\frac{1}{c}\psi^{-1}(v)\right]\right\}^{-1}, \quad \text{where} \quad v = \frac{1-t}{t}$$

$$= 1/(1+av), \text{ when } c = b$$

$$= \frac{pt}{1-(1-p)t}, \quad p = 1/a.$$

Since $ab^{\alpha} = 1$ we have $p = b^{\alpha} = c^{\alpha}$.

Converse says that SML(a,b) laws are geometric(1)-sum stable, which is known from Sandhya(1991b). □

Noticing that when $\psi(s) = s^{\alpha}$, SML(a,b) law reduces to the corresponding ML(α) law of Pillai (1990) with LT

$$\phi(s) = [1+ s^{\alpha}]^{-1}$$

we have the following result of Sandhya (1996) as a corollary.

Corollary.2.3.3 ML(α) law is N-sum stable iff N is geometric(1) with parameter $p = c^{\alpha}$, $0<c<1$.

Putting $\alpha = 1$ ML(α) law reduces to the exponential law and we have:

Corollary.2.3.4 Exponential law is N-sum stable iff N is geometric(1) with parameter $p = c$, which is the result of Cinlar and Agnew (1968).

Now we prove the following Theorem.

Theorem.2.3.5 A SS(a,b) law is N-sum stable iff N is degenerate at $a = k >1$ an integer, and $c = b = k^{1/\alpha}$.

Proof. The LT of a semi stable (SS(a,b)) law is,

$$\phi(s) = \exp\{-\psi(s)\}, \tag{2.3.4}$$

where $\psi(s)$ satisfies (1.5.1).

When $\phi_c(s) = \exp\{-\psi(cs)\}$ we have

$$\phi_c^{-1}(t) = \frac{1}{c}\psi^{-1}\left(\ln\frac{1}{t}\right).$$

In particular, $c = b$ gives $\phi[\phi_c^{-1}(t)] = t^a$ which is a PGF only when a>1 is an integer say k . Since $ab^\alpha = 1$, we must have $kc^\alpha = 1$, $0<\alpha \leq 1$ and hence N is degenerate at $k = c^{-\alpha}$.

Conversely, the LT of a SS(k,b) law can be written as,

$$\exp\{-\psi(s)\} = \exp\{-k\psi(bs)\} = [\exp\{-\psi(bs)\}]^k.$$

Hence the theorem is proved. □

The stable(α) law (Feller, 1971, p.448) has the LT

$$\phi(s) = \exp\{-s^\alpha\}$$

It is easy to see that this is a particular case of the LT of SS(a,b) law when $\psi(s) = s^\alpha$, and we have:

Theorem.2.3.6 A stable(α) law is N-sum stable iff $P\{N=k\} = 1$ for any (arbitrary) $k \geq 1$ integer and $c = k^{-1/\alpha}$.

Proof. We have $\phi[\phi_c^{-1}(t)] = t^a$ where $a = c^{-\alpha}$ and this is a PGF only when $c^{-\alpha} > 1$ is an integer say k. Hence N is degenerate at k and $c = k^{-1/\alpha}$.

Conversely, with $c = k^{-1/\alpha}$ the LT ϕ of a stable(α) law satisfies $\phi(s) = [\phi(cs)]^k$ for any $k \geq 1$ integer. Noticing that k is not determined by $\phi(s)$ as in the case of SS(a,b) laws the theorem is proved. □

Feller (1971, p.448) has shown that if X is stable(α), then

$$X \stackrel{d}{=} n^{-1/\alpha}\,[X_1 + \ldots. + X_n] \qquad \text{for every } n \geq 1 \text{ integer.}$$

Here we have shown that when X is stable(α) a fixed sum alone imparts stability in the N-sum scheme and $c = n^{-1/\alpha}$ for every n, the number of components in the sum.

Notice that for a given SS(k,b) law, k>1 integer, the number of components in the sum is determined by the parameter k and it cannot vary. Contrary to this, in the case of stable(α) laws if we vary the number of components in the sum from k to n, a corresponding change in the scale parameter to $n^{-1/\alpha}$ will guarantee the stability. Hence if a semi stable law shows stability w.r.t every k then it must be stable(α). This conclusion is supported also by the fact that if this is true for every positive integer n, then there exists two integers say, n_1 and n_2 such that $ln\, n_1 / ln\, n_2$ is irrational and consequently $\psi(s) = \lambda s^{\alpha}$ for a positive constant λ (see the description of equation (1.5.1)). Intuitively it thus appears that the condition

$$X \stackrel{d}{=} N^{-1/\alpha}\{X_1 + \ldots. + X_N\}$$

where N is a positive integer valued r.v, characterizes the stable(α) law. This has been proved in Ramachandran and Lau (1991, p.88).

The conclusions in the above results are arrived at by us using Lemma.2.2.1, while those in the examples of generating PGFs by LTs (not to identify N in N-sum stability) in Bunge (1996) are based on certain aspects of convolution semi-groups of PGFs.

Next we look for more general laws that are N-sum stable w.r.t Harris(a,k).

2.4. Stability w.r.t Harris(a,k) Law

Perhaps the first known and most quoted result in random summation stability is that exponential laws are stable w.r.t geometric(1) summation. Stability w.r.t geometric(1) summation has since been shown to hold for ML(α) and SML(a,b) laws that are generalizations of the exponential law. Motivated by this here we look for generalizations of gamma$(1,\beta)$ laws that are N-sum stable w.r.t Harris(a,k). In this connection we prove:

Theorem.2.4.1 A LT $\phi(s)$ is N-sum stable w.r.t. Harris(a,k) law, iff

$$\phi(s) = [1+\psi(s)]^{-1/k},$$

where $\psi(s)$ satisfies (1.5.1) with $b = c$, 'a' being as in Harris(a,k).

Proof. To prove the only if part, let $\phi(s)$ be a LT that is N-sum stable w.r.t Harris(a,k). Setting

$\psi(s) = [1-\phi^k(s)] / \phi^k(s)$, we have,

$$\phi(s) = [1+\psi(s)]^{-1/k}.$$

By the assumption that $\phi(s)$ is N-sum stable w.r.t Harris(a,k), we have,

$$[1+\psi(s)]^{-1/k} = \frac{[1+\psi(cs)]^{-1/k}}{\{a-(a-1)/[1+\psi(cs)]\}^{1/k}}$$

$$= [1+a\psi(cs)]^{-1/k}.$$

Hence $\psi(s) = a\psi(cs)$ for all $s>0$ and some $0<c<1<a$.

Conversely, the LT $\phi(s) = [1+\psi(s)]^{-1/k}$, where $\psi(s)$ satisfies (1.5.1), is stable under summation w.r.t the Harris(a,k) law which follows from the above discussion. This completes the proof. □

Remark.2.4.2 Again from the description of equation (1.5.1), if $\phi(s)$ is N-sum stable w.r.t Harris(a_1,k) and Harris(a_2,k) such that $\ln a_1/\ln a_2$ is irrational, then

$$\phi(s) = [1+\lambda s^{\alpha}]^{-1/k},\ \lambda>0,\ 0<\alpha \le 1.$$

In other words, this means that the above LT is Harris(a,k) stable for every $a>1$. Klebanov and Rachev (1996) states that the standard solution of the Poincare equation corresponding to the PGF of the Harris(a,k) law is,

$$\phi(s) = [1+ks]^{-1/k},\ k>0, \text{ integer.}$$

Equivalently this means that in the asymptotic set up of (1.2.15) we have,

$$\frac{1}{a}N(\frac{1}{a}) \xrightarrow{L} Z \text{ as } a \uparrow \infty,$$

where $N(1/a)$ denote the Harris(a,k) r.v and Z the gamma$(k,1/k)$. Hence the corresponding ν-stable law has the LT

$$\phi(s) = [1+ks^{\alpha}]^{-1/k} ,\ k>0,\ 0<\alpha \le 1.$$

Thus we have the case of a non-geometric(1) law (the Harris(a,k)) w.r.t which both the descriptions in (1.2.15) and (1.2.16) are satisfied (see p.12).

Since SML(a,b) laws are ID we can have a generalization of gamma$(1,\beta)$ law and prove the following that generalizes Theorem.2.3.2 using the method described in Section.2.2.

Theorem.2.4.3 The LT $\phi(s) = [1+\psi(s)]^{-\beta}$, $\beta > 0$ where $\psi(s)$ satisfies (1.5.1), is N-sum stable iff N is Harris(a,k), $\beta = 1/k$, k a positive integer and $c = b$.

Characteristic functions (CF) $\phi(u)$ having the structure of SML(a,b) laws with $\alpha \in (0,2]$, define the class of semi-α-Laplace laws of Pillai (1985). As semi-α-Laplace laws are also ID we have the following generalization of Theorem.2.4.1 to include distributions on $\boldsymbol{R}$.

Theorem.2.4.4 A distribution on $\boldsymbol{R}$ with CF $\phi(u)$ is N-sum stable w.r.t Harris(a,k) law iff $\phi(u) = [1+\psi(u)]^{-1/k}$, where $\psi(u)$ satisfies (1.5.1) with $\alpha \in (0,2]$, $b = c$ and 'a' being as in Harris(a,k).

Proof. To prove the only if part, let $\phi(u)$ be a CF that is N-sum stable w.r.t Harris(a,k). Now, as in the proof of Theorem.2.4.1, setting

$$\psi(u) = [1-\phi^k(u)] / \phi^k(u), \text{ we have } \phi(u) = [1+\psi(u)]^{-1/k}.$$

By the assumption that $\phi(u)$ is N-sum stable w.r.t Harris(a,k), we have,

$$[1+\psi(u)]^{-1/k} = [1+ a\psi(cu)]^{-1/k}.$$

Hence $\psi(u) = a\psi(cu)$ for all $u \in \boldsymbol{R}$ and some $0<c<1<a$.

Conversely, the CF $\phi(u) = [1+\psi(u)]^{-1/k}$, where $\psi(u)$ satisfies (1.5.1), is stable under summation w.r.t the Harris(a,k) law which follows from the above discussion. This completes the proof. □

This theorem also generalizes the result (Sandhya, 1991a) and Theorem.3 of Lin (1994) that a distribution on $\boldsymbol{R}$ is a geometric(1) sum of its own type iff it is semi-α-Laplace, which is a reformulation of Theorem.1 of

Pillai (1985). A condition analogous to the one in Remark.2.4.2 yields the CF $\phi(u) = [1+\lambda|u|^{\alpha}]^{-1/k}$, which is a generalization of α-Laplace laws.

In ν-ID laws a point highlighted by Gnedenko and Korelev (1996, p.148) is that if the LT ϕ in (1.2.14) is ID then the corresponding ν-ID law is ID in the classical scheme of ordinary summation of r,vs. In the N-sum scheme also a similar result holds. That is, if N is an ID r.v then the corresponding N-sum is also ID (see problem.19 in Feller (1971), p.464). It will be interesting to know whether the converses, that is, the infinite divisibility of ϕ or N is necessary in the contexts. The case with regard to ϕ appears to be open, while the answer w.r.t the case of N is negative, as shown by the following example.

Example.2.4.1 It is well-known that an exponential law can be represented as a geometric(1) sum of exponential laws of the same type and it is ID. But the geometric(1) law is not ID as it has no atom at the origin.

The above example also disproves the conclusion that N must be ID for the N-sum to be ID in Sandhya (1996).

Since for a CF the inverse may not be unique our method is applicable only for distributions with support $[0,\infty)$. Instead of LTs for X which usually presumes X to be non-negative and continuous, we may take X to be non-negative and integer valued. But to develop stability of random sums for discrete laws we need extend the notion of distributions of the same type to non-negative integer valued r.vs. This is done in the next chapter that has another motivation as well.

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3. STABILITY OF RANDOM SUMS OF LATTICE VARIABLES

3.1. Introduction

Discussions in literature on the properties of distributions using CFs or LTs do not clearly specify whether such a framework can be made use of in the study of lattice distributions as well. In particular it is not clearly established whether the concept of random sums and distributions of the same type naturally carry over to the lattice domain. Certain works on discrete distributions that opens up such a possibility are: Steutel and van Harn (1979) and Pillai and Jayakumar (1995). See also page 19 in Chapter.1.

In this chapter we address the question whether we can extend N-sum stability to non-negative lattice (lattice for short) r.vs, that is, r.vs whose support is the set $N_0 = \{0,1,2, \ldots.\}$. A notion that is at the heart of the concept of stability of distributions is, distributions of the same type. Since such a notion is not available for lattice laws our first aim is to extend this notion to the lattice domain, a hint of course, is there in Steutel and van Harn (1979). Another important achievement of such an extension other than the study of random sums in the lattice domain could be that the Steutel - van Harn definition of discrete class-L laws would be a natural consequence of the definition of class-L laws. Notice that they have not made it clear and

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This chapter is based on Satheesh and Nair (1999).

justified why and/or how their definition should be the discrete analogue, though their definition of the PGF of discrete stable(α) laws closely resembles the LT of stable(α) laws. Further they record that a formal analogue led them to ID lattice laws in the form of compound Poisson laws. Such a justification seems to be very important for the following reasons as well: (i) There are more than one-way to verify whether a distribution is in class-L or not. Therefore, a justification of the definition of discrete class-L laws as the discrete version of their continuous counterpart will naturally prove their membership in the discrete class-L, and (ii) Such a membership will enable us to develop the corresponding auto regressive processes. According to MacDonald and Zucchini (1997, p.29) the usual construction of a binomial auto regressive process has not been possible as it is not a member of discrete class-L. These problems will be our main concern in this chapter. Two other works discussing discrete stable(α) laws viz. Rao and Shanbhag (1994, p.160) and Christoph and Schreiber (1998), do not discuss 'stability' of these distributions under ordinary summation, the very property justifying the terminology. Similarly, though discrete ML(α) laws have been developed, the key property of it being stable under geometric(1) summation has not been mentioned in Pillai and Jayakumar (1995). These then are the main motivations for the study presented here.

In Section.3.2 we prove a Lemma that is basic to the main developments in this chapter and opens up the possibility of many more extensions. It enables us to construct PGFs on $N_0 = \{0,1,2 \ldots.\}$ as lattice analogues of LTs, discuss lattice distributions of the same type and thus describe N-sum stability for lattice laws, done subsequently. As it turns out, these lattice laws are mixtures of Poisson laws. We then show that a formal lattice analogue of class-L laws does lead to the Steutel-van Harn definition. The concept of lattice distributions of the same type is developed in Section.3.3 and the role of Bernoulli laws in the context justifying the replacement of cX by $c\circ X$. Following this, generalizations of Bernoulli and Poisson laws and those of some known results are presented in Section.3.4. Next, in Section.3.5 we define and study discrete semi Mittag-Leffler laws and its subclass, viz. discrete Mittag-Leffler laws in various compound geometric(1) setups. Some of the results in this section are the discrete versions of Lin (1994). In Section.3.6 we define N-sum stability of lattice laws and discuss the problem of identifying N in the context. Finally we present an example wherein we have N-sum stability when N has an atom at the origin, which is not possible in the continuous setup.

3.2. Basic Results

We begin with the following Lemma that justifies our method of construction of lattice analogue of continuous distributions on $[0,\infty)$.

Lemma.3.2.1 If $\phi(s)$ is a LT, then $Q(s) = \phi(1-s)$, $0<s<1$ is a PGF. Conversely, if $Q(s)$ is a PGF and $Q(1-s)$ is completely monotone (CM) for all $s>0$, then $\phi(s) = Q(1-s)$ is a LT.

Proof. First notice that by the construction, the norming requirement for LTs and PGFs are naturally satisfied as $\phi(o) = 1 \Leftrightarrow Q(1) = 1$. Again, since $\phi(s)$ is a LT, it is CM for all $s>0$ and hence $Q(s)$ is absolutely monotone (AM) for all $0<s<1$. Thus by Feller (1971, 223) $Q(s)$ is a PGF. Conversely, if $Q(s)$ is PGF it is AM for all $0<s<1$and hence $\phi(s) = Q(1\text{-}s)$ is CM for all $0<s<1$. But for $\phi(s)$ to be a LT it must be CM for all $s>0$, Feller (1971, p.439). Hence the proof is complete. □

Example.3.2.1 Consider the degenerate law with LT

$$\phi(s) = \exp\{-\lambda s\},\ \lambda>0.$$

The PGF constructed from this LT by Lemma.3.2.1 will be of the form

$$Q(s) = \exp\{-\lambda(1-s)\}$$

which is the Poisson(λ) law. In this case beginning with the PGF $Q(s)$ of a Poisson law the function $Q(1-s)$ constructed is a LT as it is well defined in the domain $s>0$ and CM there.

More generally from the stable(α,λ) law with LT

$$\phi(s) = \exp\{-\lambda s^{\alpha}\},\ \lambda>0,\ 0<\alpha \leq 1,$$

we get the PGF as

$$Q(s) = \exp\{-\lambda(1-s)^{\alpha}\},\ \lambda>0,\ 0<\alpha \leq 1,$$

and vice-versa. This is a simpler way of arriving at the discrete stable PGF of Steutel and van Harn (1979). □

Example.3.2.2 Consider the exponential law with LT

$$\phi(s) = 1/\,(1 + \lambda s),\ \lambda>0.$$

The PGF constructed from this LT is of the form

$$Q(s) = 1/\,\{1 + \lambda(1-s)\}$$

which is the geometric law with mean $\lambda = p/q$, and $p+q = 1$. Here also, beginning with the PGF $Q(s)$ of a geometric law the function $Q(1-s)$ constructed is a LT as it is well defined in the domain $s>0$ and is CM there.

In general, from the ML(α,λ) law with LT

$$\phi(s) = 1/(1 + \lambda s^{\alpha}),\ \lambda>0,\ 0<\alpha \le 1,$$

we get the PGF as

$$Q(s) = 1/(1 + \lambda(1-s)^{\alpha}\},\ \lambda>0,\ 0<\alpha \le 1,$$

and vice-versa. This is another way of looking at the discrete ML(α) of Pillai and Jayakumar (1995). □

The following example shows that there are situations that do not guarantee that $Q(1\text{-}s)$ is CM for all $s>0$.

Example.3.2.3 Starting from the PGF of the Bernoulli law $[1\text{-}b(1\text{-}s)]$, $0<b<1$ it follows that $[1\text{-}b(1\text{-}s)^{\alpha}]$, $0<\alpha \le 1$ is also a PGF which we refer to as that of an α-Bernoulli law. Setting

$$Q(s) = 1 - b(1-s)^{\alpha},\ \ 0<s<1 \text{ and } 0<\alpha \le 1,\ 0<b<1$$

$Q(1-s) = 1 - bs^{\alpha}$ which is CM for $0<s<1$. But when $s>1$, $Q(1-s)$ could be negative and thus is not CM for all $s>0$. Hence $Q(1-s)$ is not a LT. □

Remark.3.2.2 Notice that $Q(1-s) = \phi(s)$ is CM for $s \in (0,1)$ and is known in the literature as alternate PGF, see MacDonald and Zucchini (1997, p.21).

Next, we give a formal justification of the Steutel-van Harn definition of discrete class-L laws.

Theorem.3.2.3 A PGF $Q(s)$ is in discrete class-L if, for each $0<\alpha<1$, there exists another PGF $Q_\alpha(s)$ such that

$$Q(s) = Q(1-\alpha+\alpha s)\, Q_\alpha(s). \tag{3.2.1}$$

Proof. Class-L laws on $[0,\infty)$ are defined by LTs $\phi(s)$, satisfying,

$$\phi(s) = \phi(\alpha s)\, \phi_\alpha(s) \qquad \text{for each } 0<\alpha<1, \tag{3.2.2}$$

where $\phi_\alpha(s)$ is another LT. Setting

$$Q(s) = \phi(1-s) \text{ and } Q_\alpha(s) = \phi_\alpha(1-s),\ 0<s<1,$$

this is reflected in Q and Q_α as; for each $0<\alpha<1$,

$$Q(1\text{-}s) = Q(1\text{-}\alpha s)\, Q_\alpha(1\text{-}s), \tag{3.2.3}$$

Setting $1\text{-}s = u$ in (3.2.3) we get

$$Q(u) = Q(1-\alpha+\alpha\, u)\;\; Q_\alpha(u),$$

justifying the definition of Steutel and van Harn (1979). □

Corollary.3.2.4 If a LT $\phi(s)$ is in class-L then the PGF $Q(s) = \phi(1-s)$, $0<s<1$, is in discrete class-L. Conversely if $Q(s)$ is in discrete class-L and $\phi(s) = Q(1-s)$ is a LT, then $\phi(s)$ is in class-L.

The importance of Theorem.3.2.3 and Corollary.3.2.4 is that discrete analogue of a class-L law is naturally a member of discrete class-L, without verifying the Steutel-van Harn condition (3.2.1). For example, Pillai and Sabu George (1984) proved that ML(α) laws are in class-L by invoking a result from Lukacs (1970) to get the desired decomposition. Sandhya and Satheesh (1996) showed that semi-α-Laplace laws (and hence also SML laws) in general do not belong to class-L unless it is α-Laplace (or ML(α)) using a limiting argument for the component ϕ_α . Shanbhag and Sreehari (1977) showed that inverse Gaussian laws are in class-L based on a property of the corresponding Levy spectral measure while Pillai and Satheesh (1992) reaches the same conclusion using a method of Ismail and Kelkar (1979). Different approaches to class-L laws are available in Jurek (1997). The point is that, whether the corresponding discrete analogue (as derived by Lemma.3.2.1) is in class-L or not, is now quite straight forward, while it is not that easy to verify this just by the definition of Steutel and van Harn.

In Example.3.2.1 we saw that the discrete analogue of stable(α) laws is given by the PGF,

$$Q(s) = \exp\{-\lambda(1-s)^{\alpha}\},\ \lambda>0,\ 0<\alpha\leq 1 \tag{3.2.4}$$

and when $\alpha = 1$ the Poisson law results as the discrete analogue. Thus (3.2.4) can be considered as a generalization of Poisson laws that we may refer to it as α-Poisson. Further properties of these distributions will be discussed in Section.3.4.

Setting $\alpha = 1$ and randomizing λ in (3.2.4) with a distribution having LT ϕ we have $Q(s) = \phi(1-s)$ and thus:

Theorem.3.2.5 Every PGF $Q(s) = \phi(1-s)$, where $\phi(s)$ is a LT, is a mixture of Poisson laws.

Note. If $Q(s)$ is the PGF, the mixing distribution is the one with LT, $\phi(s) = Q(1-s)$.

Examples. From equation (3.2.4) since $Q(1-s)$ is the LT of a stable(α) law we can see that α-Poisson laws are stable(α) mixtures of Poisson laws (that is, randomizing λ with a stable(α,β) law) with LT $\exp\{-\beta s^{\alpha}\}$. Again, when the mixing law is ML(α) with LT $1/(1+s^{\alpha})$ we get the discrete ML(α) laws. Also the PGF in Example.3.2.3 is not a mixture of Poisson laws. □

3.3. Discrete Analogue of Distributions of the Same Type.

It is known that distributions of the same type in the continuous case do not have an analogy in the lattice case. Here we arrive at it in terms of PGFs using the construction in Lemma.3.2.1 and then specialize it to the case $0<c<1$ to define and study the concept of D-type. In the case of continuous distributions, two LTs ϕ_1 and ϕ_2 are of the same type if $\phi_1(s) = \phi_2(cs)$ for all $s>0$ and some $c>0$. Hence:

Definition.3.3.1 Two PGFs $Q_1(s) = \phi_1(1\text{-}s)$ and $Q_2(s) = \phi_2(1\text{-}s)$, where ϕ_1 and ϕ_2 are LTs, are of the same type iff $\phi_1(1\text{-}s) = \phi_2(c(1\text{-}s))$, for all $0<s<1$ and some $c>0$.

Clearly this definition applies to PGFs derived from LTs, while Example.3.2.3 (including the Bernoulli law) suggest that this is not the case always. Further for these two distributions the range $c<1$ alone is safely applicable in Definition.3.3.1. Exploring this range of c showed that it has some nice implications as is seen below which motivates us to coin the nomenclature D-type in the next definition. Also, this is the range of c in the context of distribution of the same types in random summation schemes to be discussed in Sections 3.4, 3.5 and 3.6 (see also Remark.2.2.1). Hence for a different perspective set $Q_1(1\text{-}s) = \phi_1(s)$ and $Q_2(1\text{-}s) = \phi_2(s)$, where ϕ_1 and ϕ_2 are CM in $0<s<1$, but not necessarily LTs. Now the condition $\phi_1(s) = \phi_2(cs)$ is reflected in Q_1 and Q_2 as,

$$Q_1(1\text{-}s) = Q_2(1\text{-}cs), \text{ for all } 0<s<1 \text{ and some } 0<c<1.$$

Equivalently, putting $u = 1\text{-}s$,

$$Q_1(u) = Q_2(1\text{-}c+cu), \text{ for all } 0<u<1 \text{ and some } 0<c<1.$$

This justifies the following definition of the notion of distributions of the same type for discrete laws (D-type).

Definition.3.3.2 Two PGFs $Q_1(s)$ and $Q_2(s)$ are of the same D-type iff $Q_1(1\text{-}s) = Q_2(1\text{-}cs)$, for all $0<s<1$, or equivalently, $Q_1(u) = Q_2(1\text{-}c+cu)$ for all $0<u<1$, and some $0<c<1$.

Remark.3.3.1 Writing the Definition.3.3.2 using the structure of the corresponding CM functions ϕ_1 and ϕ_2, we have, $\phi_1(1\text{-}u) = \phi_2[1\text{-} (1\text{-}c\text{+}cu)] = \phi_2(c(1\text{-}u))$ as it should be in the light of Definition.3.3.1.

Remark.3.3.2 The relevance of Definition.3.3.2 is that the PGF need not necessarily be derived from a LT ϕ by the relation $Q(s) = \phi(1\text{-}s)$. The PGF in Example.3.2.3 illustrates this fact. However, Definition.3.3.1 is still relevant that will be highlighted towards the end of this section as a note on page 56.

The next two results give the probabilistic implication and the stochastic representation suggested by Definition.3.3.2.

Theorem.3.3.1 Two PGFs $Q_1(s)$ and $Q_2(s)$ are of the same D-type iff Q_1 is a Q_2 compounded Bernoulli law with Bernoulli probability c.

Theorem.3.3.2 Two non-negative lattice variables X and Y are of the same D-type iff $X = \sum_{i=1}^{Y} Z_i$ for some i.i.d Bernoulli variable $\{Z_i\}$ independent of Y, the Bernoulli probability being c.

These two results also justify the replacement of cX in the continuous set up by $c\circ X$ to obtain the corresponding lattice analogue as done in Steutel and van Harn (1979). (Also see equations (1.2.18) and (1.2.20) and the discussions on page 14). As another role of Bernoulli law we prove:

Theorem.3.3.3 Every PGF $Q_1(s) = \phi_1(1\text{-}s)$, where ϕ_1 is a LT, is a compound of Bernoulli law.

Proof. We have;

$$Q_1(s) = \phi_1(1\text{-}s),\ 0<s<1$$

$$= \phi_1[ab(1\text{-}s)],\ 0<b<1,\ ab = 1$$

$$= \phi_1\{a[1\text{-}(1\text{-}b + bs)]\}$$

$$= Q_2(1\text{-}b + bs),$$

where $Q_2(s) = \phi_1[a(1\text{-}s)]$. This completes the proof. □

Clearly Q_1 and Q_2 are of the same type according to Definition.3.3.1. (w.r.t a) and D-type as well (w.r.t b). Also the description of the PGF, $Q_2(s) = \phi_1[a(1\text{-}s)]$, for $a>0$ and possibly $a>1$ above is justified because ϕ_1 is a LT. The following example illustrates the concepts discussed in this section.

Example.3.3.1 Let $Q_1(s) = 1/\{1+ \lambda(1-s)^\alpha\}$, $\lambda>0$, $0<\alpha \le 1$. This PGF has been considered in Example.3.2.2. Now choose a and $0<b<1$ such that $ab = \lambda$. Then,

$$Q_1(s) = 1/\{1+ ab(1\text{-}s)^\alpha\} = \phi_1\{ab(1\text{-}s)\}$$

$$= 1/\{1+ a[1\text{-}(1\text{-}b^{1/\alpha} + b^{1/\alpha}s)]^\alpha\}$$

$$= Q_2(1\text{-}\ b^{1/\alpha} + b^{1/\alpha}s),$$

where $Q_2(s) = 1/\{1+ a(1\text{-}s)^\alpha\}$

$$= \phi_1\{a(1\text{-}s)\}.$$ □

Note. This is a situation where Definition.3.3.1 is still relevant as it takes care of the full range of the parameter λ. Similar is the case with α-Poisson laws.

The following example shows that the conclusion of Theorem.3.3.3 can hold good even when the PGF is not derived from a LT by Lemma.3.2.1.

Example.3.3.2 Let $Q_1(s) = 1\text{-}\lambda(1\text{-}s)^{\nu}$, $0<\lambda<1$ and $0<\nu<1$.

Choose b such that $0<\lambda<b<1$ and write $ab = \lambda$ so that $0<a<1$ also holds true. Now,

$$Q_1(s) = 1 - ab(1\text{-}s)^{\nu}$$

$$= 1 - b[a^{1/\nu}(1\text{-}s)]^{\nu}$$

$$= 1 - b\,[1 - (1 - a^{1/\nu} + a^{1/\nu}\,s)]^{\nu}$$

$$= Q_2(1\text{-}\,a^{1/\nu} + a^{1/\nu}\,s), \quad \text{where}$$

$$Q_2(s) = 1 - b(1\text{-}s)^{\nu}\,. \qquad \square$$

Remark.3.3.3 The point stressed here is that when $Q_1(s) = \phi_1(1\text{-}s)$ where ϕ_1 is a LT, the choice $0<b<1$ alone is to be assured and the value of a being greater than or less than unity is immaterial. But in the case of PGFs not constructed from LTs as in Lemma.3.2.1 one should take care that both the factors a and b are less than unity. eg. in the above example if $b< \lambda$, then $\lambda = ab$ would imply that $a>1$ and hence speaking about a Bernoulli probability of $a^{1/\nu}>1$ is meaningless.

3.4. Generalizations of some Lattice laws

From the PGF of the α-Bernoulli law in Example.3.2.3 it follows that

$$Q_n(s) = [1\text{-}p(1\text{-}s)^{\alpha}]^{n},\ 0<p<1,\ 0< \alpha \leq 1,\ n = 1, 2, \ldots$$

is another PGF which we refer to as that of the corresponding α-binomial law. Its limiting case, as $n \to\infty, p\downarrow 0$ such that $np = \lambda$, a constant is

$Q(s) = \exp\{-\lambda(1-s)^{\alpha}\}$ which is the PGF of α-Poisson law.

As a generalization of geometric(0) laws Jayakumar and Pillai (1992) considered the discrete Mittag-Leffler (DML(α)) law with PGF

$$Q(s) = [1+\lambda(1-s)^{\alpha}]^{-1},\ \lambda>0,\ 0<\alpha\leq 1.$$

Clearly,

$Q_n(s) = \left\{1+\frac{\lambda}{n}(1-s)^{\alpha}\right\}^{-n}$ is also a PGF.

Since

$$\lim_{n\to\infty} Q_n(s) = \exp\{-\lambda(1-s)^{\alpha}\},$$

we have proved,

Theorem.3.4.1 Corresponding to each α-Poisson law we have a sequence of DML(α) laws converging weakly to it.

Definition.3.4.1 In analogy with the continuous case, a PGF $Q(s)$ is said to be discrete semi stable (DSS(a,b,α)), if $Q(s) = \exp\{-\psi(1-s)\}$, where

$$\psi(1-s) = a\psi[b(1-s)] \tag{3.4.1}$$

for all $0<s<1$ and some $0<b<1<a$ satisfying $ab^{\alpha}=1$ for a unique $0<\alpha\leq 1$.

Notice that the class of functions $\psi(1-s)$ belongs to the one described in equation (1.5.1) with the domain restricted to $0<s<1$.

Remark.3.4.1 From the inequality $0<\alpha = \frac{-\ln(1/a)}{-\ln b} \leq 1$, a solution to $0<\alpha \leq 1$ of $ab^{\alpha}=1$ exist iff $ab \leq 1$. This imposes certain restrictions in the choice of Bernoulli probabilities b, as will be seen in subsequent deliberations.

Theorem.3.4.2 The sum of n i.i.d discrete variables is distributed as the same D-type as the components, iff it is DSS(n,b,α).

Proof. We have;

$$\begin{aligned} Q(s) &= \exp\{-\psi(1-s)\} \\ &= \exp\{-n\psi[b(1-s)]\} \\ &= [\exp\{-\psi[b(1-s)]\}]^{n}. \end{aligned}$$

This proves the if part. The only if part follows by setting $-\ln Q(s) = \psi(1-s)$ and retracing the above steps. □

Theorem.3.4.3 If a discrete r.v can be expressed as the sum of n_1 and n_2 independent variables of the same D-type such that $\ln n_1/ \ln n_2$ is irrational, then the variable is α-Poisson.

Proof. When the DSS(n,b,α) law can assume, for the same $0<\alpha \leq 1$ two different values for n, say n_1 and n_2 such that their logarithms are in irrational ratio, them $\psi(1-s) = \lambda(1-s)^{\alpha}$, for some $\lambda>0$ constant. See the discussion following equation (1.5.1). Hence the variable is α-Poisson. □

Theorem.3.4.4 Let $\{Y_j\}$ be a sequence of i.i.d Bernoulli variables with parameter b. Let M be a non-negative lattice variable independent of $\{Y_j\}$ with PGF $Q(s)$ satisfying

$$\lim_{s\to 1}\frac{-\ln Q(s)}{(1-s)^{\alpha}} = \lambda, \text{ a positive constant and } 0<\alpha\leq 1. \qquad (3.4.2)$$

Define $X = \sum_{j=1}^{M} Y_j$. Let X_1 and X_2 are independent copies of X. Then M is identically distributed as $X_1 + X_2$ iff M is α-Poisson and $b^{\alpha} = ½$.

Proof. Let $Q(s) = \phi(1\text{-}s)$.

We have

$$\phi(1\text{-}s) = [\phi(b(1\text{-}s))]^2.$$

Setting

$$\psi(1\text{-}s) = -\ln \phi(1\text{-}s),$$

$$\exp\{-\psi(1\text{-}s)\} = \exp\{-2\ \psi[b(1\text{-}s)]\}$$

Solving we find $b^{\alpha} = ½$ (because $ab^{\alpha} = 1$). Writing

$$P(1\text{-}s) = \frac{\psi(1-s)}{(1-s)^{\alpha}},$$

$$\exp\{-(1\text{-}s)^{\alpha}P(1\text{-}s)\} = \exp\{-2.\ ½(1\text{-}s)^{\alpha}\ P[b(1\text{-}s)]\}, \text{ or;}$$

$$P(1\text{-}s) = P[b(1\text{-}s)],\ 0<b<1$$

$$= P[b^{n}\ (1\text{-}s)], \text{ on iteration for every } n \geq 1 \text{ integer.}$$

Since $0<b<1$ by virtue of the condition (3.4.2) satisfied by $Q(s)$, $P(1\text{-}s) = \lambda >0$ and hence $Q(s) = \exp\{-\lambda(1\text{-}s)^{\alpha}\}$.

Conversely, when M is α-Poisson, the PGF of $X_1 + X_2$ is

$[\exp\{-\lambda[b(1-s)]^{\alpha}\}]^2$.

When $b^{\alpha} = ½$ this equals $\exp\{-\lambda(1-s)^{\alpha}\}$ and the proof is complete. □

Corollary.3.4.1 In the setup of Theorem.3.4.4 let $\{X_i\}$ be independent copies of X, then $M \overset{d}{=} \sum_{i=1}^{n} X_i$ iff M is α-Poisson and $b^{\alpha} = \frac{1}{n}$.

Note. The choice of the Bernoulli probability is restricted by $b \le \frac{1}{n}$ or ½ as the case may be (c.f Remark.3.4.1).

Theorem.3.4.5 In Theorem 3.4.4 if $\{Y_j\}$ is a sequence of Bernoulli r.vs with parameter ½, then both M and X are Poisson.

Proof. Proceeding as in the proof of Theorem.3.4.4 we have $(½)^{\alpha} = ½$. Hence $\alpha = 1$ implying M to be Poisson. Now by virtue of Raikov's theorem X also is Poisson. □

Theorem.3.4.6 Under the assumptions of Theorem.3.4.4, the condition $\sum_{i=1}^{m} M_i \overset{d}{=} \sum_{i=1}^{n} X_i$, where M_i's are independent copies of M, characterizes the α-Poisson law. In this case, the condition $n>m$ should be satisfied.

Proof. The condition implies

$$\exp\{-m\psi(1-s)\} = \exp\{-n\psi[b(1-s)]\}$$

$$\Rightarrow \quad m\,\psi(1-s) = n\psi[b(1-s)]$$

$$\Rightarrow \quad \text{or} \quad \psi(1-s) = \frac{n}{m}\psi[b(1-s)].$$

since $\frac{n}{m}>1$ must be satisfied we should have $n>m$. Proof of $\psi(1-s) = \lambda(1-s)^{\alpha}$ is as in the proof of Theorem.3.4.4. □

3.5. Discrete Semi Mittag-Leffler laws and Geometric(1)-Sums

Here in analogy with the continuous case we define discrete semi Mittag-Leffler (DSML) and prove certain results under geometric(1)-sums.

Definition.3.5.1 A r.v X is said to have a discrete semi Mittag-Leffler (DSML(a,b,α)) distribution if its PGF is given by

$$Q(s) = [1+\psi(1-s)]^{-1}, \text{ where}$$

$$\psi(1-s) = a\psi[b(1-s)]$$

for all $0<s<1$ and some $0<b<1<a$ satisfying $ab^{\alpha} = 1$ for a unique $0<\alpha \leq 1$.

Definition.3.5.2 A r.v X is said to have a DML(λ,α) distribution if its PGF is given by $Q(s) = [1+\lambda(1-s)^{\alpha}]^{-1}$, $\lambda>0$, $0< \alpha \leq 1$ (see Example.3.2.2)

In Definition.3.5.1, $\psi(1-s)$ belongs to the class of functions described in equation (1.5.1). Now we will see the discrete version of the characterization of SML laws discussed in Sandhya (1991) (See equation (2.3.3) and the discussion following it on page-36).

Theorem.3.5.1 A discrete r. v X is a geometric$(1,p)$ sum of its own D-type variables, iff it is DSML$(\frac{1}{p}, b, \alpha)$.

Proof. If $Q(s)$ is the PGF of X, a DSML(a,b) r.v, then

$$
\begin{aligned}
Q(s) &= \frac{1}{1+\psi(1-s)} \\
&= \frac{1}{1+a\psi[b(1-s)]} \\
&= \frac{p}{p+\psi[b(1-s)]}, \quad p=\tfrac{1}{a} \\
&= \frac{p/[1+\psi(b(1-s))]}{1-q/[1+\psi(b(1-s))]},
\end{aligned}
$$

which proves our assertion in both directions. □

Theorem.3.5.2 If a discrete r.v X is a geometric$(1,p)$ sum of its own D-type for two values of p, say p_1 and p_2 such that $\ln p_1 / \ln p_2$ is irrational, then it is DML(λ,α).

Proof. $\ln p_1/\ln p_2$ is irrational implies $\ln a_1/\ln a_2$ is irrational where $a_i = 1/p_i$, i=1,2. Hence $\psi(1\text{-}s) = \lambda(1\text{-}s)^{\alpha}$ for $\lambda>0$ (as in Theorem 3.4.3) and X is DML(λ, α). □

Note. Sandhya and Satheesh (1996) showed that a semi-α-Laplace law is in class-L if and only if it is α-Laplace. Restricting the support to $[0,\infty)$ and then to the non-negative lattice we have the following two results invoking Corollary 3.2.4.

Theorem.3.5.3 A SML(a,b,α) law is in class-L iff it is ML(α).

Theorem.3.5.4 A DSML(a,b,α) law is discrete class-L iff it is DML(α).

Theorem.3.5.5 Consider a sequence $\{Y_j\}$ of i.i.d Bernoulli variables with parameter $b = p^{1/\alpha}$, $0<p<1$, $0<\alpha \le 1$. Let M be a non-negative lattice variable independent of $\{Y_j\}$, with PGF $Q(s)$ such that

$$\lim_{s \to 1} \frac{1-Q(s)}{(1-s)^{\alpha}} = \lambda > 0, 0<\alpha \le 1 \tag{3.5.1}$$

and put $X = \sum_{j=1}^{M} Y_j$. Let $\{X_i\}$ be a sequence of independent copies of X and define $S_N = \sum_{i=1}^{N} X_i$, where N is a geometric$(1,p)$ variable independent of $\{X_i\}$. Then as $p \downarrow 0$, S_N converges in law to a DML variable.

Proof. The PGF of S_N is $Q_p(s) = \dfrac{pQ[1-b(1-s)]}{1-qQ[1-b(1-s)]}$

$$= \frac{Q[1-b(1-s)]}{Q[1-b(1-s)] + p^{-1}\{1-Q[1-b(1-s)]\}}$$

Now, $$\frac{1-Q[1-b(1-s)]}{p} = \frac{1-Q[1-p^{1/\alpha}(1-s)]}{[p^{1/\alpha}(1-s)]^{\alpha}}(1-s)^{\alpha}.$$

Under the condition (3.5.1) the R.H.S converges to $\lambda(1-s)^{\alpha}$ as $p \downarrow 0$. Hence

$$\lim_{p \to 0} Q_p(s) = [1+\lambda(1-s)^{\alpha}]^{-1},$$

proving the result. □

Theorem.3.5.6 In the set up of Theorem 3.5.5, let $\{Y_j\}$ be Bernoulli with parameter b. Then S_N is identically distributed as M iff $b^{\alpha} = p$ for a unique $0<\alpha \le 1$ and M is DML(λ,α).

Proof. We have $Q(s) = \dfrac{pQ[1-b+bs]}{1-qQ[1-b+bs]}$.

Setting $Q(s) = \phi(1-s)$ and $\psi(1-s) = \dfrac{1}{\phi(1-s)} - 1$, we have;

$$\frac{1}{1+\psi(1-s)} = \frac{p/[1+\psi(b(1-s))]}{1-q/[1+\psi(b(1-s))]]}$$

$$= \frac{1}{1+a\psi(b(1-s))}, \quad a = \tfrac{1}{p}.$$

Hence $Q(s)$ corresponds to a DSML($\frac{1}{p}$,b,α), with α defined by $b^{\alpha} = p$ for a unique $0<\alpha \leq 1$. Writing $P(1-s) = \dfrac{\psi(1-s)}{(1-s)^{\alpha}}$, we have:

$$\frac{1}{1+(1-s)^{\alpha}P(1-s)} = \frac{1}{1+ab^{\alpha}(1-s)^{\alpha}P(b(1-s))}.$$

As $ab^{\alpha} = 1$, we have $P(1-s) = P(b(1-s))$. On iteration $P(1-s) = P(b^{n}(1-s))$ for each positive integer n. Since $0<b<1$, his means $P(1-s) = \lambda>0$ under the condition (3.5.1). Hence $[1+\lambda(1-s)^{\alpha}]^{-1}$. The converse is contained in Theorem.3.5.1. Hence the proof. □

Note. Notice that the geometric(1) parameter p and the Bernoulli parameter b are related by $b^{\alpha} = p$ and the choice is under the restriction $b \leq p$ (c.f. Remark.3.4.1).

Theorem.3.5.7 In the setup of Theorem.3.5.6, let $\{M_i\}$ be a sequence of independent copies of M, and N_0 be a geometric($1,p_0$) variable independent of

M and $p_0 \neq p$. Then $\sum_{i=1}^{N_0} M_i$ and $\sum_{i=1}^{N} X_i$ and identically distributed iff $p < p_0$, $b^{\alpha} = p/p_0$ and M is DML(λ,α).

Proof. The condition is equivalent to

$$p_0\, Q(s) / (1 - q_0\, Q(s)) = [p\, Q(1\text{-}b\text{+}bs)] / [1\text{-}[q\, Q(1\text{-}b\text{+}bs)]]$$

Setting $\phi(1\text{-}s)$, $\psi(1\text{-}s)$ and $P(1\text{-}s)$ as in Theorem 3.5.6 we have;

$$p_0^{-1}\, \psi(1\text{-}s) = p^{-1}\, \psi[\mathrm{b}(1\text{-}s)].$$

Thus $0<\alpha \leq 1$ is uniquely defined by $b^{\alpha} = p/p_0$ implying $p<p_0$ as $0<b<1$. Further,

$$p\,(1\text{-}s)^{\alpha}\, P(1\text{-}s) = p_0\, b^{\alpha}\,(1\text{-}s)^{\alpha}\, P[b(1\text{-}s)] \text{ implies } P(1\text{-}s) = P[b(1\text{-}s)].$$

Now, proceeding as in the proof of Theorem.3.5.6 we see that $Q(s) = [1+ \lambda(1\text{-}s)^{\alpha}]^{-1}$ under the condition (3.5.1) thus completing the proof. □

3.6. Stability of Random Sums - Lattice Case

Since PGFs are increasing in (0,1), their inverses exist uniquely. Thus corresponding to equations (2.2.1) and (2.2.2) with $Q(s)$ the PGF of N, $P(s)$ that of X and $P_c(s) = P(1\text{-}c\text{+}cs)$ that of $c \circ X = \sum_{i=1}^{X} Z_i$, we have:

$$Q[P(1\text{-}c\text{+}cs)] = P(s) \text{ for all } s\in(0,1) \text{ and some } 0<c<1, \tag{3.6.1}$$

and when s equals $P_c^{-1}(t)$, $0<t<1$,

$$Q(t) = P[\,P_c^{-1}(t)\,], \text{ for some } 0<c<1. \tag{3.6.2}$$

Thus, equation (3.6.1) defines stability of N-sums and (3.6.2) a method to identify the distribution of N that imparts N-sum stability for lattice r.vs.

The discrete analogue of stable(α) laws (with LT $\exp\{-\lambda s^{\alpha}\}$) is

$$P(s) = \exp\{-\lambda(1-s)^{\alpha}\},\ \lambda>0,\ 0<\alpha \leq 1, \qquad (3.6.3)$$

and hence $P_c(s) = \exp\{-\lambda[c^{\alpha}(1-s)]^{\alpha}\}$.

Since

$$P_c^{-1}(t) = 1-\left[\frac{1}{\lambda c^{\alpha}}\ln\left(\frac{1}{t}\right)\right]^{1/\alpha}, \qquad P[P_c^{-1}(t)] = t^{1/c^{\alpha}}$$

which is a PGF only when $1/c^{\alpha} = k>0$ integer, and we have proved;

Theorem.3.6.1 A discrete stable(α) law (or the α-Poisson law) is N-sum stable iff $P\{N=k\} = 1$ for any (arbitrary) $k>0$ integer and $c = k^{-1/\alpha}$.

Thus the discrete stable(α) law is stable under ordinary summation quite like the stable(α) (continuous) law, an important property in the summation scheme not stressed in the literature. Steutel and van Harn (1979) defines discrete stable(α) laws by PGFs satisfying

$$Q(s) = Q[1-c(1-s)]\ Q[1-(1-c^{\alpha})^{1/\alpha}(1-s)],\ 0<c<1 \qquad (3.6.4)$$

and arrived at the form in (3.6.3). Rao and Shanbhag (1994, p.160) arrives at (3.6.3) from (3.6.4) using a different approach. Here we have obtained (3.6.3) directly from the LT of the continuous case and further shown that the PGF thus obtained is 'stable' under ordinary summation (alone). Steutel and van Harn (1979) describes 'discrete stability' of the r.vs in the following sense:

$$X = c \circ X_1 + (1 - c^{\alpha})^{1/\alpha} \circ X_2 \qquad (3.6.5)$$

where $c \circ X = \sum_{i=1}^{X} Z_i$, $P(Z_i=0) = 1-c$, all r.vs being independent. We may also notice that (3.6.5) is equivalent to (3.6.4) by invoking Lemma.3.2.1.

Invoking Lemma.3.2.1 we have the discrete analogues of gamma, ML their generalizations (denoted with a prefix D) and results analogous to those obtained in the Chapter.2 with proofs on similar lines. Considering the importance of the discrete setup we only state the following general results omitting the proofs. Here $\psi(1\text{-}s)$ satisfies (1.5.1).

Theorem.3.6.2 The PGF $Q(s) = [1+\psi(1\text{-}s)]^{-\beta}$, is N-sum stable iff N is Harris(a,k), $\beta = 1/k$, k>0 integer and $c = b$.

Theorem.3.6.3 The DSS(a,b) law with PGF $\exp\{-\psi(1\text{-}s)\}$ is N-sum stable iff N is degenerate at a, the parameter a>1 is an integer say k and $c = k^{-1/\alpha}$.

Theorem.3.6.4 A PGF $Q(s)$ is N-sum stable w.r.t Harris(a,k) law iff $Q(s) = [1+\psi(1\text{-}s)]^{-1/k}$, $b = c$ and 'a' is that in Harris (a,k).

Corollary.3.6.5 A PGF Q(s) is a geometric$(1,p)$ sum of its own D-type iff it is DSML(a,b), $a = 1/p$ and $c = b$. (See Theorem.3.5.1 also).

If we consider, in the context of p-thinning, a renewal counting process this corollary means that, a renewal counting process is invariant under p-thinning iff it's renewal distribution is DSML(a,b). This is the discrete analogue of the characterization of SML renewal processes by Sandhya (1991) in the context of p-thinning.

A Curious Example. Considering the following PGFs

$$P(s) = 1 - \delta\,(1-s)^{\nu},\ 0<\delta<1,\ 0<\nu\leq 1 \ \text{ and } \ Q(s) = 1-\lambda(1-s),\ 0<\lambda<1,$$

$$Q[P(s)] \ = 1 - \lambda\delta\,(1-s)^{\nu},$$

which is again of the same D-type as $P(s)$. Thus in the discrete set up $P\{N=0\}>0$ is possible which is clearly impossible in the continuous set up.

It is also worth noticing that in the case of the above example the problem of identification of N as in equation (3.6.2) works. That is,

$$1 - \lambda\delta\,\{1 - P^{-1}(s)\}^{\nu} = Q(s).$$

In this chapter we have thus extended the notion of N-sum stability to distributions on the non-negative lattice.

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4. STABILITY OF RANDOM EXTREMES

4.1. Introduction

Let us recall from Section.1.4, equations (1.4.4) and (1.4.5) that a r.v X with d.f $F(x)$ having support $[0,\infty)$, is maximum stable w.r.t an independent r.v N with PGF $Q(s)$ ($F(x)$ is N-max stable) if

$$Q(F(x)) = F(cx) \text{ for all } x \geq 0 \text{ and some } 0<c<1. \tag{4.1.1}$$

Similarly $F(x)$ is minimum stable w.r.t N ($F(x)$ is N-min stable) if

$$Q[\bar{F}(cx)] = \bar{F}(x) \text{ for all } x \geq 0 \text{ and some } c > 0. \tag{4.1.2}$$

Here $\bar{F}(x) = 1 - F(x)$. Also N-extremes refer to both N-max and / or N-min.

We had also noticed (p.20) that the following problems are worth studying. (i) Stability of N-extremes of exponential laws, (ii) Discussion of distributions other than geometric(1) or its generalizations for N, (iii) Identifying N that imparts N-max and/or N-min stability for a given $F(x)$, and (iv) Extending these notions to lattice laws, which means that we describe distributions of the same type in the lattice set up so that equations analogous to (4.1.1) and (4.1.2) hold. In this chapter we focus attention on these problems. Also for notational convenience we may use R instead of $\bar{F}$.

Sreehari (1995) has considered the connection between the LT of N and the d.f $F(x)$ (in fact a generalization of N-max stability). The method

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This chapter is based on Satheesh and Nair (2000a) and (2000b).

connecting the PGF of N and the d.f $F(x)$, presented in the following sections, was done independently and motivated by our study of stability of random sums. It has the following advantages over the discussions in Sreehari (1995). We are using PGFs consequent of which (i) we can use Lemma.2.2.1 to exploit the connection between the parameters of N and $F(x)$ in a transparent manner and the range of the parameters can be found out and (ii) the discussion is directly relevant in the context of stability of extremes with random (N) sample size.

In Section.4.2 we discuss the problems (i) and (ii) giving rise to a non-geometric(1) law for N viz. the Sibuya(ν) law, and also characterize the general distribution that is max stable w.r.t Sibuya. Identifying N in N-extreme stability is taken up in Section.4.3. Showing that we can suitably define lattice analogue of distributions of the same type we extend the notions of N-extreme stability to the lattice domain that is done in Section.4.4. The chapter ends with an example showing that in the case of lattice laws distributions of the same type in the contexts of N-sum and N-extremes are different. We end this section by defining the semi Pareto law of Pillai (1991) that has a main role in the discussions to follow.

Definition.4.1.1 A d.f $F(x)$, $x \geq 0$, is said to follow semi Pareto(p,α) law if,

$$F(x) = 1 - [1 + \psi(x)]^{-1}, \ \psi(o) = 0$$

and $\psi(x)$ satisfies the functional equation (a variation of equation (1.5.1))

$$\psi(x) = \frac{1}{p}\psi(p^{1/\alpha} x) \text{ for all } x > 0 \text{ and some } 0 < p < 1, \text{ and } \alpha > 0. \qquad (4.1.3)$$

4.2. Stability of Extremes - Continuous Case

We know that (Voorn (1987)) under N-max stability $c\in(0, 1)$. Since $\bar{F}(x)$ is a decreasing function, under N-min stability also we have $c\in(0, 1)$. The value of c will be discussed in the sequel. Now we have the following result concerning the stability of exponential law.

Theorem.4.2.1 If $F(x) = 1- e^{-x}$, $x>0$, then it is N-max stable iff N follows Sibuya(v) distribution with PGF

$$Q(s) = 1-(1-s)^{v},\ 0<v<1,$$

and $c = v$. It is N-min stable iff N is degenerate at k, an integer greater than unity and $c = 1/k$.

Proof. When N follows Sibuya(v) law,

$$Q[F(x)] = 1 - e^{-vx} = F(vx).$$

Conversely,

$$1 - e^{-cx} = 1 - [1-(1 - e^{-x})]^{c} \text{ shows that}$$

$$Q(s) = 1 - (1 - s)^{c} \text{ and hence } N \text{ is Sibuya with } v = c.$$

For N-min stability we have,

$$\bar{F}(cx) = e^{-cx} \text{ and } e^{-x} = [e^{-cx}]^{1/c},$$

proving that N must be degenerate at $c^{-1} = k$, an integer greater than unity. Conversely, when $Q(s) = s^{k}$, $k = 1/c$, we have

$$Q(e^{-cx}) = e^{-x}$$

and the proof is complete. □

Theorem.4.2.1 suggests the conditions for N-max stability of exponential laws and also a distribution for N other than geometric(1,p). Next, we identify the most general form of $F(x)$ that is max stable w.r.t Sibuya(ν) and min stable w.r.t degenerate.

Theorem.4.2.2 A distribution is max stable w.r.t. Sibuya(ν) law iff it has semi Weibull(p,α) law with d.f.

$$F(x) = 1 - e^{-\psi(x)},\ x \geq 0,$$

where $\psi(x)$ satisfies the functional equation (4.1.3). Further $p = \nu = c^{\alpha}$.

Proof. Setting $\psi(x) = -\ln \overline{F}(x)$, we have

$$Q[F(x)] = 1 - [1 - F(x)]^{\nu}$$

$$= 1 - e^{-\nu\psi(x)}.$$

Under max stability w.r.t Sibuya(ν) we have

$$1 - e^{-\nu\psi(x)} = 1 - e^{-\psi(cx)} \text{ or}$$

$$\psi(x) = \frac{1}{\nu}\,\psi(cx) \text{ for all } x > 0.$$

Hence $F(x)$ has semi Weibull(p,α) law with $p = \nu = c^{\alpha}$.

Conversely, if $F(x)$ has semi Weibull(p,α) law and N has a Sibuya(ν) law with $p = \nu = c^{\alpha}$, then

$$Q[F(x)] = 1 - e^{-\nu\psi(x)} = 1 - e^{-\psi(cx)}.$$

This completes the proof. □

Note. Notice that a Sibuya(ν)-sum of Bernoulli law results in ν-Bernoulli law of example 3.2.3.

$F(x) = 1 - [1+ \psi(x)]^{-1/k}$, $k > 0$ integer.

Under min stability of $F(x)$ w.r.t Harris(a,k) law we have

$$\frac{[1+\psi(cx)]^{-1/k}}{\{a-(a-1)/[1+\psi(cx)]\}^{1/k}} = \frac{1}{[1+\psi(x)]^{1/k}}.$$

But L.H.S equals $[1 + a\psi(cx)]^{-1/k}$ and hence $F(x)$ follows a generalized semi Pareto law with $p = \frac{1}{a} = c^{\alpha}$ and $\beta = \frac{1}{k}$.

Conversely, with

$F(x) = 1 - [1+ \psi(x)]^{-\beta}$, $x > 0$ and $\beta > 0$

and N having a Harris(a,k) law with $a = 1/p$ and $\beta = \frac{1}{k}$ we have

$Q[R(cx)] = R(x)$

which completes the proof. □

Notice that when $\beta = 1$ the generalized semi Pareto law becomes the semi Pareto law of Pillai (1991). Also notice that the extended geometric(1) law with parameters λ and k of Voorn (1987) is a reparametrization of Harris(a,k) law by setting $1 - \lambda^{k} = 1/a$.

4.3. Identifying N that imparts stability for a given $F(x)$

Results (except Theorem.4.2.1) in the previous Section characterizes the distribution of $F(x)$ for a given N. The sufficiency part in these results state that certain $F(x)$ is N-max (min) stable for the given N. Here we look for the stronger assertion that for a certain $F(x)$ this N is necessary as well. That is, characterizing the distribution of N for a given $F(x)$. Here again we assume that $F(x)$ is absolutely continuous so that $F^{-1}(x)$ exists and is unique.

From (4.1.1) we have for all $x \geq 0$ and some $0 < c < 1$,

$$Q[F(x)] = F(cx) = F_c(x).$$

In particular, when $x = F^{-1}(s)$ for $0 < s < 1$, we have

$$F_c[F^{-1}(s)] = Q(s) \text{ under N-max stability.} \qquad (4.3.1)$$

Similarly, from (4.1.2) using R for $\overline{F}$, when $x = R_c^{-1}(s)$ for $0<s<1$,

$$R[R_c^{-1}(s)] = Q(s) \text{ under N-min stability.} \qquad (4.3.2)$$

(4.3.1) and (4.3.2) can be viewed as definitions of N-max and N-min stability of $F(x)$. We now reproduce Lemma.2.2.1 here (with a different proof) to identify the range of parameters in the distributions of N and $F(x)$.

Lemma.4.3.1 If $Q(s)$ is a PGF, then $Q(s^t)$ is a PGF iff $t > 0$ is an integer.

Proof: Let N be the r.v with PGF $Q(s)$. Let X be a r.v independent of N such that $P\{X=k\} = 1$, k being a positive integer so that its PGF is $P(s) = s^k$. If $X_1, X_2, \ldots$ are independent and identically distributed as X, then $S_N = X_1 + \ldots + X_N$ has the PGF $Q(P(s)) = Q(s^k)$. Conversely, consider the function $Q(s^t)$, where $n<t<n+1$, $n=0, 1, 2, \ldots$.. The first $(n+1)$ derivatives of $Q(s^t)$ are positive and the derivatives from $(n+2)^{nd}$ onwards are alternatively positive and negative. Thus the function $Q(s^t)$ is not absolutely monotone when t is not a positive integer and hence cannot be a PGF.

Note. The converses of the following results are identical to the sufficiency part of the results in the previous section.

Theorem.4.3.2 The semi Weibull(p,α) law is N-max stable iff N follows Sibuya(p) and N-min stable iff N is degenerate at $\frac{1}{p}>1$ integer. In both the cases $c=p^{1/\alpha}$.

Proof. We have,

$$F(x) = 1 - e^{-\psi(x)} = 1 - \exp\left\{-\frac{1}{p}\psi(p^{1/\alpha}x)\right\} \text{ and}$$

$$F^{-1}(s) = p^{-1/\alpha}\,\psi^{-1}\{\ln(1-s)^{-p}\}.$$

Hence

$$F_c\{F^{-1}(s)\} = 1-(1-s)^p \text{ when } c=p^{1/\alpha}.$$

Therefore when $F(x)$ is semi Weibull(p,α) N must have Sibuya(p) distribution under N-max stability .

Now,

$$R_c^{-1}(s) = \frac{1}{c}\,\psi^{-1}[\log(\tfrac{1}{s})] \text{ so that}$$

$$R[R_c^{-1}(s)] = s^{1/p} \text{ when } c=p^{1/\alpha}.$$

But $s^{1/p}$ is a PGF only when $\frac{1}{p}>1$ is an integer. Therefore when $F(x)$ is semi Weibull(p,α) N must have a degenerate distribution under N-min stability. Converses of both the statements are easy, hence the proof. □

Theorem.4.3.3 The generalized semi Pareto(p,α,β) law of Theorem.4.2.4 is N-min stable iff $\beta=\frac{1}{k}$, $c=p^{1/\alpha}$ and N follows Harris(a, k) law, with $a=\frac{1}{p}$.

Proof. We have,

$$R(x) = \left[1 + \frac{1}{p}\psi(p^{1/\alpha}x)\right]^{-\beta} \text{ and}$$

$$R_c^{-1}(s) = \frac{1}{c}\ \psi^{-1}(u), \text{ where } u = \frac{1-s^{1/\beta}}{s^{1/\beta}}.$$

When $c = p^{1/\alpha}$ we have

$$R[R_c^{-1}(s)] = \frac{s}{[a-(a-1)s^{1/\beta}]^{\beta}} \text{ where } a = \frac{1}{p}.$$

This is a PGF only when $\beta^{-1} = k$ a positive integer and hence N follows Harris(a,k), thus completing the proof, as the converse is easy. □

Corollary.4.3.4 Putting k =1 we have: A semi Pareto(p,α) law is N-min stable iff N has geometric(1,p) distribution, $c = p^{1/\alpha}$.

Next we identify N under N-max stability of the extended log-logistic law of Voorn (1987) where he had shown that the extended log-logistic law is max stable w.r.t the extended geometric(1) law under the assumption that the sequence of distributions for N takes the value one with probability tending to one. However, in this discussion, we do not make any such assumptions on the distributions of N and we prove:

Theorem.4.3.5 $F(x) = [1+x^{-\alpha}]^{-1/k}$, $k \geq 1$ integer and $\alpha > 0$ is N-max stable iff N follows Harris$(c^{-\alpha},k)$.

Proof. We have

$$F^{-1}(s) = [s^{-k} - 1]^{-1/\alpha} \text{ and}$$

$$F_c[F^{-1}(s)] = [1 + c^{-\alpha}(s^{-k}-1)]^{-1/k}$$

$$= \frac{s}{[c^{-\alpha} - (c^{-\alpha} - 1)s^k]^{1/k}}$$

which is the PGF of Harris($c^{-\alpha}$,k) law. As the converse is easy we have proved the assertion. □

4.4. Random Extreme Stability for Non-negative Lattice Distributions

As in the case of Chapter.3, the main requirement for extending the notions of stability of extremes to the lattice domain is to be able to conceive distributions of the same type in the context in a manner analogous to its continuous counterpart. Here we show that we can have scale families of lattice distributions that fit in to the scheme of things. Satheesh and Sandhya (1997) observed that the d.f of a mixture of geometric(0) laws has the general form

$$F(k) = P\{X < k\} = 1 - m(k),\ k = 0, 1, \ldots, \tag{4.4.1}$$

where $\{m(k)\}$ is the moment sequence of the mixing distribution. Further $\{m(k)\}$ is also the sequence of realizations of a LT $m(s)$, $s > 0$, at the non-negative integral values of s. Since $m(\alpha s)$ also is a LT for a constant $\alpha > 0$, we can define another d.f. by

$$G(k) = P\{Y < k\} = 1 - m(\alpha k),\ \ k = 0, 1, \ldots. \tag{4.4.2}$$

Writing $F_c(k) = 1 - m(ck)$, for c > 0, we have:

$$G(k) = F_\alpha(k) \text{ for all } k = 0, 1, \ldots. \tag{4.4.3}$$

Thus we have $\{F_c(k) : c>0\}$, a parametric family of d.fs (of non-negative lattice laws) and $F(k)$ is a member of it. The existence of such a family of d.fs justifies the following definition.

Definition.4.4.1 Two lattice laws $F(k)$ and $G(k)$ are of the same type if equation (4.4.3) is satisfied for some $\alpha > 0$.

Further, by analogy with the continuous case (equations (4.1.1) and (4.1.2)) we can now propose:

Definition.4.4.2 Let $F(k)$ be the d.f of a non-negative lattice r.v X, and N a positive integer valued r.v. independent of X with PGF $Q(s)$. Then $F(k)$ is N-max stable if

$$Q[F(k)] = F_c(k) \text{ for all } k = 0,1,2, \dots \text{ and some } c \in (0,1),$$

and $F(k)$ is N-min stable if

$$Q[\bar{F}_c(k)] = \bar{F}(k) \text{ for all } k = 0,1,2, \dots \text{ and some } c \in (0,1),$$

where $\bar{F}(k) = \text{P}\{X \geq k\}$.

Note. Notice that Definition.4.4.1 appears different from Definition.3.3.2 but is quite similar to the one in the continuous case. We will clarify this point at the end of this chapter.

The following properties of mixtures of geometric(0) laws suggest their potential for applications in different contexts (see Satheesh and Sandhya (1997)). They are log-convex, compound geometric(1) (and hence GID), infinitely divisible and have decreasing hazard rate.

Sandhya and Satheesh (1996) observed that if $F(x)$ is a mixture of exponential laws then

$$F(x) = 1 - \phi(x), x \geq 0 \quad (4.4.4)$$

where $\phi(x)$ is the LT of the mixing distribution. Notice the similarity of (4.4.4) with (4.4.1) the only difference being in the support of the distributions. Further in the definitions of semi Weibull, semi Pareto, and generalized semi Pareto laws the restriction of α to $0 < \alpha \leq 1$ make them mixtures of exponential laws. This is because under the restriction their survival functions are LTs (by virtue of the description of equation (1.5.1)). Accordingly we can conceive discrete analogs of semi Weibull, semi Pareto, and generalized semi Pareto laws as mixtures of geometric(0) laws. Now let us define discrete semi Pareto laws.

Definition.4.4.3 A r.v X has discrete semi Pareto law (DSP(a, b, α)) if its d.f, in the support of {0, 1, 2, ...}, has the form

$$P\{X < k\} = F(k) = 1 - \frac{1}{1 + \psi(k)}, k = 0, 1, 2, \ldots.$$

where $\psi(k)$ satisfies $\psi(k) = a\psi(bk)$ for all $k = 0,1,2, \ldots.$ and for some $0<b<1<a$, satisfying $ab^{\alpha} = 1$ for a unique $\alpha \in (0,1]$ (the condition is same as that in equation (1.5.1) or (4.1.3)).

A r.v X in the same support has discrete Pareto distribution (DP(λ, α)) if its d.f is

$$P\{X<k\} = F(k) = 1 - \frac{1}{1 + \lambda k^{\alpha}}, \lambda > 0, \text{ and } 0<\alpha \leq 1.$$

The survival sequence of DSP(a,b,α) is the sequence of realizations of the LT of a semi Mittag-Leffler law introduced in Sandhya (1991) and that of DP(λ,α) corresponds to the realizations of the LT of a Mittag-Leffler law discussed in Pillai (1990). Here the construction conforms to that of a mixture of geometric(0) laws studied in Satheesh and Sandhya (1997). The mixing distribution in these cases are that of $Y = \exp\{-X\}$, X being semi Mittag-Leffler and Mittag-Leffler respectively.

Theorem.4.4.1 A non-negative lattice distribution is geometric(1,p)-max (min) stable iff it is DSP(a, b,α) with $a = 1/p$ and $b^{\alpha} = p$.

Proof. (i) For geometric(1,p)-max stability of $F(k)$ we should have, for all integer $k \geq 0$,

$$\frac{pF(k)}{1-qF(k)} = F_c(k), \text{ for some } c > 0 \tag{4.4.5}$$

writing $F(k) = 1 - \dfrac{1}{1+\psi(k)}$, this is equivalent to

$$\frac{p\psi(k)}{1+p\psi(k)} = \frac{\psi(k)}{1+\psi(ck)}, \text{ or } \psi(k) = \frac{1}{p}\,\psi(ck).$$

This means that $F(k)$ is DSP(a,b,α) with $a = 1/p$, $b = c$ and $0<\alpha \leq 1$ is defined by $b^{\alpha} = p$.

Conversely, suppose that $F(k)$ is DSP (a,b,α) with $a = 1/\text{p}$, $b^{\alpha} = p$. Then,

$$\frac{p\psi(k)}{1+p\psi(k)} = \frac{pa\psi(bk)}{1+pa\psi(bk)} = \frac{\psi(bk)}{1+\psi(bk)}$$

and hence (4.4.5) is satisfied with $c = b = p^{1/\alpha}$.

(ii) For geometric(1,p)-min stability we consider the requirement for all non-negative integer k,

$$\frac{p\bar{F}_c(k)}{1-q\bar{F}_c(k)} = \bar{F}(k), \text{ for some } c>0 \tag{4.4.6}$$

writing

$$F(k) = 1 - \frac{1}{1+\psi(k)}, \text{ we should have}$$

$$\frac{p/[1+\psi(ck)]}{1-q/[1+\psi(ck)]} = \frac{1}{1+\psi(k)}, \text{ or } \frac{1}{p}\psi(ck) = \psi(k).$$

Hence $F(k)$ is DSP(a,b,α) with $a = 1/p$, $b = c$ and $0 < \alpha \le 1$ is defined by $b^{\alpha} = p$. Conversely, by retracing the steps we see that if $F(k)$ is DSP(a,b,α) with $a = 1/p$, $b^{\alpha} = p$, (4.4.6) is satisfied with $c = b$. This completes the proof. □

Corollary.4.4.2 The only non-negative lattice distribution that is geometric(1,p)-max (min) stable for two values of p, say p_1 and p_2 such that $\ln p_1/\ln p_2$ is irrational is DP(λ,α) .

Proof. By Theorem 4.4.1 we know that the distribution must be DSP(a,b,α). Further if $\psi(k) = a\psi(bk)$ for two different values of a say a_1 and a_2 such that $\ln a_1/\ln a_2$ is irrational, then by the description of equation (1.5.1) $\psi(k) = \lambda k^{\alpha}$ for some $\lambda>0$ constant and the result follows. □

We may conceive the definitions of semi Weibull and generalized semi Pareto laws in an analogous manner as is done in Definition.4.4.3 and accordingly the following results. Proofs being similar to those in Section.4.2, only statements of the theorems are presented.

Theorem.4.4.3 If $F(k)$ is geometric$(0,p)$ then it is N-max stable iff N has Sibuya(ν) distribution and $c = \nu$. It is N-min stable iff N is degenerate at $k > 1$ integer and $c = 1/k$.

Theorem.4.4.4 A non-negative lattice law is max stable w.r.t Sibuya(ν) iff it is discrete semi Weibull (p, α), $0 < \alpha < 1$, where $p = \nu = c^{\alpha}$.

Theorem.4.4.5 A non-negative lattice law is min stable w.r.t a degenerate law at $k > 1$ integer, iff it is discrete semi Weibull(p, α), $0 < \alpha < 1$, where $p = \frac{1}{k} = c^{\alpha}$.

Theorem.4.4.6 A non-negative lattice law is min stable w.r.t a Harris(a, k) law iff it is discrete generalized semi Pareto(p, α, β) distribution, $0 < \alpha < 1$, where $p = 1/a = c^{\alpha}$ and $\beta = 1/k$.

Having conceived the idea of distributions of the same type for lattice laws in the context of stability of N-extremes it is interesting to know whether this is equivalent to the Definition.3.3.2 of distributions of the same D-type (for lattice laws) in the context of stability of N-sums. Notice that in the case of continuous distributions both are equivalent though we are using the description in terms of d.fs for stability of N-extremes and that in terms of CFs (or LTs) for stability of N-sums.

Recall that two lattice laws $F(k)$ and $G(k)$ (with PGFs $Q_1(s)$ and $Q_2(s)$) are of the same type in terms of d.fs, if equation (4.4.3) is satisfied for some $\alpha > 0$. They are of the same D-type if $Q_1(1\text{-}s) = Q_2(1\text{-}cs)$, for all $0<s<1$, or equivalently, $Q_1(u) = Q_2(1\text{-}c\text{+}cu)$ for all $0<u<1$, and some $0<c<1$. Now consider the following example.

Example.4.4.1 Let X has a geometric(0,p) law. Then its d.f is

$$F(k) = 1 - q^k,\ k = 0,1,2, \dots \text{ and } q = 1 - p$$

and PGF is

$$Q_X(s) = p/(1 - qs).$$

Now in accordance with Definition.4.4.1 consider the r.v Y with d.f

$$G(k) = 1 - q^{ck} \text{ for some } 0<c<1.$$

Setting $q = \frac{1}{4}$ and $c = \frac{1}{2}$, we have $q^c = \sqrt{(\frac{1}{4})} = \frac{1}{2}$. Further;

$$Q_X(s) = 3/(4 - s) \text{ and } Q_Y(s) = 1/(2 - s) \text{ and}$$

$$Q_X(1 - \tfrac{1}{2} + s/2) = 6/(7 - s) \text{ and } Q_Y(\tfrac{1}{2} + s/2) = 2/(3 - s).$$

Thus neither $Q_X(s) = Q_Y(\frac{1}{2} + s/2)$ nor $Q_Y(s) = Q_X(\frac{1}{2} + s/2)$ considering both the possibilities. Hence the two definitions are not equivalent.

In the next chapter we will consider the uniqueness of geometric(1) laws in the context of stability of extremes motivated by another look at the characterizations of semi Pareto laws by Pillai (1991) amd Pillai and Sandhya (1996). The discussion also has relevance in a parameterization scheme introduced by Marshall and Olkin.

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5. STABILITY OF GEOMETRIC EXTREMES

5.1. Introduction

As we had mentioned in the introduction (p.20) possible reasons for the uniqueness of the geometric(1) law in the context of stability of random extremes of distributions are explored in this chapter. This is relevant in the parameterization scheme of Marshall and Olkin (1997) also. The study is motivated by the following considerations.

The semi Pareto family of laws was characterized among continuous distributions on $[0,\infty)$ by geometric(1)-max stability in Pillai (1991) and by geometric(1)-min stability in Pillai and Sandhya (1996). From these two characterizations it is clear that among distributions with non-negative support geometric(1)-max stability implies geometric(1)-min stability and vice-versa as both identify the same family. A natural curiosity thus is whether we can prove this without referring to the family of semi Pareto laws and also whether it is true in general for d.fs with support $\boldsymbol{R}$. As the requirement is that the distributions of N-max and N-min should be of the same type as that of $F(x)$, for brevity we put $F_t(x) = F(a+bx)$, $b>0$ and $a \in \boldsymbol{R}$.

Proving that this indeed is true, leads to the question whether it is unique of the geometric(1) law. Finding that this is not so, we narrow down our search in an attempt to characterize the geometric(1) law and arrive at a conjecture, and these are done in Section.5.2. In Section.5.3 we discuss the

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This chapter is based on Satheesh and Nair (2000b and 2000c)

Marshall - Olkin parameterization scheme which is similar in structure to the geometric(1) minimums, and show why their argument (without an analytic proof) regarding the uniqueness of the geometric(1) law in the context is not complete. Our conjecture comes closer in justifying the geometric(1) law in the situation and supplement their argument. We wind up with some concluding remarks in Section.5.4.

5.2. Uniqueness of the Geometric(1) law

Theorem.5.2.1 $F(x)$, $x \in R$ is geometric(1,p)-max stable iff it is geometric(1,p)- min stable. Importantly, the geometric(1) law is the same.

Proof. We have:

$$\frac{pF(x)}{[1-(1-p)F(x)]} = F_t(x) \tag{5.2.1}$$

$$\Leftrightarrow \quad 1 - \frac{pF(x)}{[1-(1-p)F(x)]} = \bar{F}_t(x) \tag{5.2.2}$$

$$\Leftrightarrow \quad \frac{\bar{F}(x)}{p+(1-p)\bar{F}(x)} = \bar{F}_t(x) \tag{5.2.3}$$

$$\Leftrightarrow \quad \bar{F}(x) = \frac{p\bar{F}_t(x)}{1-(1-p)\bar{F}_t(x)} \tag{5.2.4}$$

Where (5.2.1) represents geometric(1)-max stability (with parameter p) and (5.2.4) geometric(1)-min stability (with parameter p) and thus completing the proof. □

The curiosity now is whether N-max and N-min stability of $F(x)$ with respect to the same N implies N is geometric(1). A closer look at the above proof reveals a property of the PGF $Q(s)$ (here that of the geometric(1)) in this context. The L.H.S of (5.2.2) when written in terms of $\bar{F}(x)$ and equated to the L.H.S of (5.2.3) shows that $Q(s)$ satisfied

$$1 - Q(1-s) = Q^{-1}(s) \tag{5.2.5}$$

and hence a subsequent inversion resulted in N-min stability. Equation (5.2.5) can be equivalently written as

$$Q[1 - Q(1-s)] = s \tag{5.2.6}$$

Now the question is whether the geometric(1) PGF is the unique solution of (5.2.6). The examples in Shaked (1975) show that it is not. e.g.,

Example.5.2.1 If $F(x)$ is N-max stable where N has the PGF $1-(1-s^m)^{1/m}$, $m>1$ integer, then

$$1 - \{1 - [F(x)]^m\}^{1/m} = F_t(x)$$

$$\Leftrightarrow \quad 1 - [F(x)]^m = [\bar{F}_t(x)]^m$$

$$\Leftrightarrow \quad F(x) = \{1 - [\bar{F}_t(x)]^m\}^{1/m}$$

$$\Leftrightarrow \quad \bar{F}(x) = 1 - \{1 - [\bar{F}_t(x)]^m\}^{1/m}.$$

Hence $F(x)$ is N-min stable as well. Clearly the converse is also true. □

Shaked (1975) has also solved the functional equation (5.2.6) under the assumption that $Q(s)$ is single valued and meromorphic in the complex plane, to characterize the geometric(1) PGF. Now, can we restrict our search more realistically so as to characterize the geometric(1) law?

Notice that the L.H.S of (5.2.3) which specifies the distribution of the geometric(1)-max can also be written as:

$$\frac{\lambda \bar{F}(x)}{1-(1-\lambda)\bar{F}(x)} \quad , \quad \lambda = 1/p. \tag{5.2.7}$$

Now taking geometric(1)-min of these geometric(1)-maxs w.r.t an independent geometric(1) law with parameter q the resulting distribution is specified by,

$$\frac{q\lambda \bar{F}(x)}{1-(1-q\lambda)\bar{F}(x)} \, . \tag{5.2.8}$$

This has been possible only because the L.H.S of (5.2.3) could be written as (5.2.7) and the PGFs of independent geometric(1) laws is closed under its own compounding. That is $Q_u(s)$, denoting the PGF of N with parameter $u > 0$, satisfied

$$Q_u^{-1}(s) = Q_\lambda(s), \quad \lambda = 1/u \text{ and} \tag{5.2.9}$$

$$Q_u[Q_v(s)] = Q_{uv}(s) \quad \text{for all } |s|<1 \text{ and } \quad u, v > 0. \tag{5.2.10}$$

Remark.5.2.2 Another consequence of (5.2.9) and (5.2.10) along with (5.2.6) is that the structure of (5.2.7) is retained even if we take geometric(1)-max instead of geometric(1)-min of (5.2.7). This is because we can retrace (5.2.7), (5.2.3), (5.2.2) and (5.2.1) in that order, and then take geometric(1)-max and come back to the form of (5.2.7). Again the question is, whether the PGFs satisfying (5.2.6), (5.2.9) and (5.2.10) is unique of the geometric(1) PGFs.

The PGF of the Harris(a,k) law

$$\frac{s}{\{a-(a-1)s^k\}^{1/k}}, \quad k>0 \text{ integer and } a>1 \tag{5.2.11}$$

satisfy (5.2.9) and (5.2.10) (We prove this in Theorem.5.3.1) but is not a solution of (5.2.6). While none of the examples in Shaked (1975) which are solutions of (5.2.6), satisfy (5.2.9) and (5.2.10). Thus it appears that the only PGF that satisfies (5.2.6), (5.2.9) and (5.2.10) is that of the geometric(1) law. We frame this as a conjecture for want of an analytic proof.

Conjecture.5.2.3 A PGF $Q_u(s)$, $u>0$ satisfies (5.2.6), (5.2.9) and (5.2.10) iff it is that of the geometric(1) law with mean $1/u$.

The following result shows that (5.2.10) is stronger than (5.2.9).

Lemma.5.2.4 If a one-to-one function $Q_u(s)$, $u>0$ satisfies (5.2.10) then $Q_u^{-1}(s) = Q_\lambda(s)$, $\lambda = 1/u$ and $Q_1(s) = s$ for all s.

Proof. We have $Q_u[Q_v(s)] = Q_{uv}(s)$.

When $s = Q_{1/v}(s)$,

$$Q_u[Q_v[Q_{1/v}(s)]] = Q_u(s),$$

which shows that

$$Q_{1/v}(s) = Q_v^{-1}(s) \text{ and } Q_1(s) = s \text{ for all } s.$$ □

5.3. The Marshall-Olkin Parameterization Scheme

Recently Marshall and Olkin (1997) introduced a parameterization scheme for a survival function $\bar{F}(x)$, $x \in \boldsymbol{R}$ by defining another survival function

$$\overline{G}(x,a) = \frac{a\overline{F}(x)}{1-(1-a)\overline{F}(x)}, \quad x\in \boldsymbol{R}, \ a>0 \qquad (5.3.1)$$

and showed that this family is geometric(1)-extreme (that is both geometric(1)-max and geometric(1)-min) stable. They attributed this property, partially to the fact that geometric(1) laws are closed under its own compounding and concluded that the random minimum stability cannot be expected if the geometric(1) distribution is replaced by another distribution. But Example.5.2.1 suggests otherwise. Here we demonstrate that in the place of geometric(1) distribution another distribution possessing a similar property (closure under compounding) can be used to generate a parameterization scheme that is more general than that of Marshall and Olkin (1997) and that random minimum stability holds in this case. Noticing that the scheme (5.3.1) closely resembles the geometric(1)-minimum problem, we generalize it by considering the Harris(a,k) law.

We know that (the fact stressed by Marshall and Olkin) a geometric($1,p_1$) sum of independent and identically distributed geometric($1,p_2$) variables has a geometric($1,p_1p_2$) distribution. Similarly for the Harris(a,j) we can verify by direct computation using PGFs that a similar property holds (c.f the discussion following equation (5.2.11)).

Theorem.5.3.1 Let $P_u(s) = \dfrac{s}{[u-(u-1)s^j]^{1/j}}$ and $Q_v(s) = \dfrac{s}{[v-(v-1)s^j]^{1/j}}$.

Then, $P_u(Q_v(s)) = \dfrac{s}{[uv-(uv-1)s^j]^{1/j}}$. Also $P_u^{-1}(s) = P_\lambda(s)$, $\lambda = 1/u$.

Notice that the parameter j must be the same for both the PGFs and it is invariant in the compound as well. Now we introduce two parameterization schemes on the lines of Marshall and Olkin (1997) as suggested by the PGF (5.2.11) of the Harris law.

In Theorem.5.3.1 let $P_u(s)$ be the PGF of a r.v. N and $Q_v(s)$ be that of M. Let $\{X_i\}$ be independent copies of a r.v. X with d.f $F(x)$, $x \in \boldsymbol{R}$. Let N, M and X are mutually independent. Put $U = \text{Min}(X_1, \ldots, X_N)$. Then,

$$P\{U > x\} = \frac{\bar{F}(x)}{[u-(u-1)[\bar{F}(x)]^j]^{1/j}}, \; x \in \boldsymbol{R}, j > 0 \text{ integer and } u > 1. \qquad (5.3.2)$$

Theorem.5.3.2 The family of distributions of the form (5.3.2) is M-min stable.

Proof. Let U_1, U_2, ... are independent copies of U and N_1, N_2, ... Independent copies of N. Then (as in the proof of Proposition.5.1 in Marshall and Olkin (1997)) we have,

$$\begin{aligned} \text{Min}(U_1, \ldots, U_M) &= \text{Min}(X_{11}, \ldots, X_{1N_1}, \ldots, X_{M1}, \ldots, X_{MN_M}) \\ &= \text{Min}(X_1, \ldots, X_{N_1+\ldots+N_M}) \end{aligned}$$

by re-indexing X_{ij}. Now by virtue of Theorem 5.3.1, $N_1 + \ldots + N_M$ has a Harris(uv,j) distribution. Hence the distribution of Min $(U_1, \ldots, U_M)$ is specified by a survival function of the form (5.3.2) with uv instead of u. Hence the result is proved. □

Again setting $V = \text{Max}(X_1, \ldots, X_N)$ we have:

$$P\{V < x\} = \frac{F(x)}{\{u-(u-1)[F(x)]^j\}^{1/j}}, \; x \in \boldsymbol{R}, j > 0 \text{ integer and } u > 1. \qquad (5.3.3)$$

Now proceeding as in the proof of Theorem.5.3.2 we have:

Theorem.5.3.3 The family of distributions of the form (5.3.3) is M-max stable.

Note. Since the PGF of the Harris law is not a solution of (5.2.6) the families (5.3.2) and (5.3.3) do not have Harris-extreme stability.

Expressions in (5.3.2) and (5.3.3) can be thought of as two other parameterization schemes that are more general than (5.3.1) and may be useful in lending more flexibility to the d.f. $F(x)$ in modelling. In fact, Propositions 5.1 and 5.2 in Marshall and Olkin (1997) follow as particular cases of Theorems, 5.3.2 and 5.3.3, respectively by setting $j=1$. The following points are also worth noting:

(i) Every d.f F generates an M-min and M-max stable family of laws.

(ii) Denoting these families by $U_M(F)$ and $V_M(F)$ we can also observe that:

(a) If $G \in U_M(F)$, then $U_M(G) = U_M(F)$ and

(b) If $G \in V_M(F)$, then $V_M(G) = V_M(F)$.

By Theorem 5.3.1 we have a positive integer valued r.v. which is closed under its own compounding - a property hitherto discussed only in the case of a geometric(1) variable in this context. Also we have minimum and maximum stability with respect to a non-geometric(1) r.v. These results bring to light a generalization of the geometric(1) law. We do not have a parameterization scheme on the lines of Marshall and Olkin (1997) having the corresponding N-extreme stability property, though we do have a non-geometric(1,p) variable that is closed under its own compounding.

5.4. Concluding Remarks

Shaked (1975) arrived at (5.2.6) from the requirement that N-min of N-maxs must be stable (though he doesn't use these terms). Arnold, et. al (1986) also considered geometric(1)-mins and geometric(1)-maxs applied one after the other (in any order) and observe the similarity in the functional forms (slightly different from that in (5.2.8) and similar to the L.H.S of (5.2.3) here) of their survival functions and characterize Pareto-type-III laws under stability. Marshall and Olkin (1997) in their parameterization scheme that is similar in form to (5.2.7) with $\lambda>0$, had observed that it is geometric(1)-extreme stable. They had attributed this property (partially) to the fact that geometric(1) laws are closed under its own compounding, that is equivalent to (5.2.10). Here we notice that this is not the only reason, by introducing two parameterization schemes on similar lines making use of the Harris law (5.2.11), which shares the property (5.2.10), and showing that the schemes are not Harris-extreme stable. Thus the reason given by Marshall and Olkin, that explains only part of the picture, is complemented by our observation (c.f Remark.5.2.2) that the structure of (5.2.7) is retained because the geometric(1) PGF is a solution of (5.2.6).

Finally, a complete proof of the conjecture will also show that among d.fs $F(x)$ with non-negative support, N-extreme stability will imply that N is geometric(1) and consequently $F(x)$ is semi Pareto. This will be a simultaneous characterization of the geometric(1) and the semi Pareto laws.

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