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Note

The $\langle t \rangle$-property of some classes of graphs

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Abstract

In this note, the $\langle t \rangle$-properties of five classes of graphs are studied. We prove that the classes of cographs and clique perfect graphs without isolated vertices satisfy the $\langle 2 \rangle$-property and the $\langle 3 \rangle$-property, but do not satisfy the $\langle t \rangle$-property for $t \geq 4$. The $\langle t \rangle$-properties of the planar graphs and the perfect graphs are also studied. We obtain a necessary and sufficient condition for the trestled graph of index k to satisfy the $\langle 2 \rangle$-property.

Keywords: Clique transversal number; $\langle t \rangle$-property

1. Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A complete of a graph G is a complete subgraph of G and a clique of a graph G is a maximal complete of G. A subset V' of V is called a clique transversal if it intersects with every clique of G. The clique transversal number $\tau_c(G)$ of a graph G is the minimum cardinality of a clique transversal of G [13]. For details, the reader may refer to [1,6,12].

The order n of G is an obvious upper bound for the clique transversal number. In an attempt to find graphs which admit a better upper bound, Tuza [13] introduced the concept of the $\langle t \rangle$-property. A class $\mathcal{G}$ of graphs satisfies the $\langle t \rangle$-property if $\tau_c(G) \leq \frac{n}{t}$ for every $G \in \mathcal{G}_t = \{G \in \mathcal{G}$: every edge of G is contained in a $K_t \subseteq G\}$. Note that the $\langle t \rangle$-property does not imply the $\langle t-1 \rangle$-property.

It is known [7] that every chordal graph satisfies the $\langle 2 \rangle$-property. In [13], it is proved that the $\langle 3 \rangle$-property holds for chordal graphs; split graphs have the $\langle 4 \rangle$-property, but do not have the $\langle 5 \rangle$-property and hence the chordal graphs also do not have the $\langle 5 \rangle$-property. It is proved [9] that the $\langle 4 \rangle$-property does not hold for chordal graphs.

Motivated by the open problems mentioned in [7], we studied the $\langle t \rangle$-property for the cographs, the clique perfect graphs, the perfect graphs, the planar graphs and the trestled graphs of index k. The cographs are a subclass of the perfect graphs [10] and also of the clique perfect graphs [12].

The $\langle t \rangle$-properties of the various classes of graphs which we studied in this paper are summarized in the following table.

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Class	Satisfy $\langle t\rangle$-property	Do not satisfy $\langle t\rangle$-property
Cographs	2, 3	≥ 4
Clique perfect graphs	2, 3	≥ 4
Planar graphs	–	2, 3, 4
Perfect graphs	–	≥ 2

All graph theoretic terminology and notation not mentioned here are from [2].

2. The $\langle t\rangle$-property

2.1. Cographs and clique perfect graphs

A graph which does not have P_4- the path on four vertices - as an induced subgraph is called a cograph. The join of two graphs G and H is defined as the graph with $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv,$ where $u \in V(G)$ and $v \in V(H)\}$.

Cographs [5] can also be recursively defined as follows:

(1) K_1 is a cograph;
(2) if G is a cograph, so is its complement $\overline{G}$; and
(3) if G and H are cographs, so is their join $G \vee H$.

A clique independent set is a subset of pairwise disjoint cliques of G. The clique independence number $\alpha_c(G)$ of a graph G is the maximum cardinality of a clique independent set of G. Clearly, $\alpha_c(G)$ is a lower bound for $\tau_c(G)$. A graph for which this lower bound is attained for all its induced subgraphs also is called a clique perfect graph [3,11]. The class of cographs is clique perfect [12]. A characterization of clique perfect graphs by means of a list of minimal forbidden subgraphs is still an open problem.

Lemma 1. *If $G = G_1 \vee G_2$ then $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$.*

Proof. Any clique in G is of the form $H_1 \vee H_2$ where H_1 is a clique in G_1 and H_2 is a clique in G_2. If V' is a clique transversal of G_1 (or G_2), then any clique of G which contains a clique of G_1 (or G_2) is covered by V' and hence V' is a clique transversal of G also.

Now, let V' be a clique transversal of G. If possible assume that V' does not cover cliques of G_1 and G_2. Let H_1 and H_2 be the cliques of G_1 and G_2 respectively which are not covered by V'. Then $H_1 \vee H_2$ is a clique of G which is not covered by V', which is a contradiction. Hence V' contains a clique transversal of G_1 or G_2.

Therefore, $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$.

Lemma 2. *The class of all cographs without isolated vertices does not satisfy the $\langle t\rangle$-property for $t \geq 4$.*

Proof. The proof is by construction.

Case 1: $t = 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_2) \vee (3K_1 \cup K_2)$ and $G_2 = (3K_1 \cup K_2)$. Then $n = 15, t = 4$ and $\tau_c(G) = 4$ which implies that $\frac{n}{t} < \tau_c(G)$.

Case 2: $t > 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_{t-3}) \vee (3K_1 \cup K_{t-3})$ and $G_2 = (3K_2 \cup K_{t-2})$.

Then $n(G) = 3t + 4$ and $\tau_c(G) = 4$.

Every edge in G_1 lies in a complete of size t for $t \geq 4$ in G since G_1 contains a clique of size $2t - 6$. Every edge in G_2 lies in a complete of size t in G since G_2 contains a clique of size $t - 2$. An edge with one end vertex in G_1 and the other end vertex in G_2 lies in a complete of size t since every vertex in G_1 lies in a complete of size $t - 2$ and every vertex of G_2 lies in a complete of size 2. Hence G is a cograph in which every edge lies in a clique of size t.

Also, $\frac{n}{t} = 3 + \frac{4}{t}$

Therefore, $\frac{n}{t} < \tau_c(G)$ for $t > 4$.

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Theorem 3. *The class of clique perfect graphs without isolated vertices satisfies the $\langle t\rangle$-property for $t = 2$ and 3 and does not satisfy the $\langle t\rangle$-property for $t \geq 4$.*

Proof. Let G be a clique perfect graph in which every edge lies in a complete of size t. G being clique perfect, $\tau_c(G) = \alpha_c(G)$.
Case 1: $t = 2$

Since G is without isolated vertices $\alpha_c(G) \leq \frac{n}{2}$. So $\tau_c(G) = \alpha_c(G) \leq \frac{n}{2}$ and hence the class of clique perfect graphs satisfies the $\langle 2\rangle$-property.
Case 2: $t = 3$

Every edge of G lies in a clique of size 3. So, the size of the smallest clique of G is 3. Therefore, $\alpha_c(G) \leq \frac{n}{3}$ and $\tau_c(G) = \alpha_c(G) \leq \frac{n}{3}$.
Case 3: $t \geq 4$

The class of cographs is a subclass of clique perfect graphs. So by Lemma 2, the claim follows.

Corollary 4. *The class of cographs without isolated vertices satisfies the $\langle t\rangle$-property for $t = 2$ and 3. Moreover, for the class of connected cographs without isolated vertices, $\tau_c(G)$ is maximum if and only if G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. Since the class of cographs is a subclass of clique perfect graphs, it satisfies the $\langle t\rangle$-property for $t = 2$ and 3.

Since the class of cographs satisfy the $\langle 2\rangle$-property and $\tau_c(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n}{2}$, the maximum value of $\tau_c(G)$ is $\frac{n}{2}$. Conversely, let G be a connected cograph with $\tau_c(G) = \frac{n}{2}$. Since G is a connected cograph, $G = G_1 \vee G_2$. Therefore, $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$. But, $\tau_c(G_1)$ and $\tau_c(G_2)$ cannot exceed the numbers of vertices in G_1 and G_2 respectively and hence the number of vertices in G_1 and G_2 must be $\frac{n}{2}$. Again, since $\tau_c(G) = \frac{n}{2}$ all these vertices must be isolated. Therefore, $G = K_{\frac{n}{2},\frac{n}{2}}$.

Corollary 5. *For the class of clique perfect graphs without isolated vertices, $\tau_c(G)$ is maximum if and only if there exists a perfect matching in G in which no edge lies in a triangle.*

Proof. The class of clique perfect graphs without isolated vertices satisfies the $\langle 2\rangle$-property. Therefore, the maximum value that $\tau_c(G)$ can obtain is $\frac{n}{2}$. Let G be a clique perfect graph with $\tau_c(G) = \frac{n}{2}$. G being clique perfect, $\alpha_c(G) = \tau_c(G) = \frac{n}{2}$. Since each clique must have at least two vertices and there are $\frac{n}{2}$ independent cliques, the cliques are of size exactly 2. Again, this independent set of $\frac{n}{2}$ cliques forms a perfect matching of G and a clique being maximal complete, the edges of this perfect matching do not lie in triangles.

Conversely, if there exists a perfect matching in which no edge lies in a triangle, the edges of this perfect matching form an independent set of cliques with cardinality $\frac{n}{2}$. Therefore, $\alpha_c(G) \geq \frac{n}{2}$. But, $\alpha_c(G) \leq \tau_c(G) \leq \frac{n}{2}$ and therefore $\tau_c(G) = \frac{n}{2}$.

2.2. *Planar graphs*

It is known that a graph G is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem 6. *The class of planar graphs does not satisfy the $\langle t\rangle$-property for $t = 2$, 3 and 4 and $\mathcal{G}_t$ is empty for $t \geq 5$.*

Proof. Every odd cycle is a planar graph and $\tau_c(C_{2k+1}) = k + 1 > \frac{2k+1}{2}$. Clearly, odd cycles belong to $\mathcal{G}_2$ and hence the class of planar graphs does not satisfy the $\langle 2\rangle$-property.

The graph in Fig. 1 is planar and every edge lies in a triangle. Here, $n = 8$ and the clique transversal number is 3 which is greater than $\frac{n}{3}$ and hence planar graphs do not satisfy the $\langle 3\rangle$-property. Also, the graph in Fig. 2 is planar and every edge lies in a K_4. Here, $n = 15$ and the clique transversal number is 4 which is greater than $\frac{n}{4}$ and hence planar graphs do not satisfy the $\langle 4\rangle$-property.

Since K_5 is a forbidden subgraph for planar graphs, there is no planar graph G such that all its edges lie in a K_t for $t \geq 5$. Hence, the theorem.

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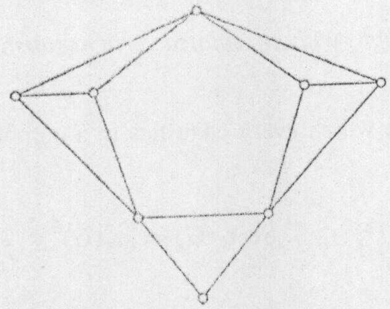

Fig. 1.

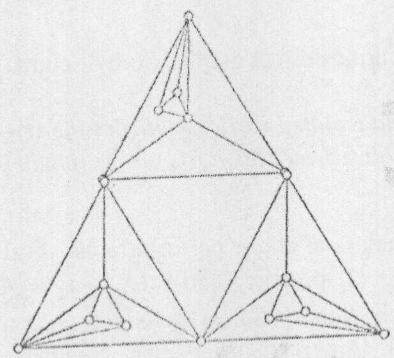

Fig. 2.

2.3. *Perfect graphs*

A graph G is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of G, where $\chi(H)$ is the chromatic number Q2
and $\omega(H)$ is the clique number of H [10]. By the celebrated strong perfect graph theorem [4], a graph is perfect if and only if it has no odd hole or odd anti-hole as an induced subgraph.

Theorem 7. *The class of perfect graphs does not satisfy the $\langle t \rangle$-property for any $t \geq 2$.*

Proof. Let G be the cycle of length $3k$, say $v_1 v_2, \ldots, v_{3k} v_1$ where $k > 2$ is odd, in which the vertices $v_1, v_4, \ldots, v_{3k-2}$ are all adjacent to each other. Then G is perfect and $\tau_c(G) = \lceil \frac{3k}{2} \rceil > \frac{3k}{2}$, since $3k$ is odd. Therefore the class of perfect graphs does not satisfy the $\langle 2 \rangle$-property.

Now, the class of perfect graphs does not satisfy the $\langle 3 \rangle$-property since $\overline{C_8}$ is a perfect graph in which every edge lies in a triangle and $\tau_c(\overline{C_8}) = 3 > \frac{8}{3}$.

Since the cographs are a subclass of perfect graphs [5], by Lemma 2, the class of perfect graphs also does not satisfy the $\langle t \rangle$-property for $t \geq 4$.

2.4. *Trestled graph of index k*

For a graph G, $T_k(G)$ the trestled graph of index k is the graph obtained from G by adding k copies of K_2 for each edge uv of G and joining u and v to the respective end vertices of each K_2 [8]. The vertex cover number of a graph G, denoted by $\beta(G)$, is the minimum number of vertices required to cover all the edges of G.

Lemma 8. *For any graph G without isolated vertices, $\tau_c(T_k(G)) = km + \beta(G)$.*

Proof. We shall prove the theorem for the case $k = 1$. The cliques of $T_1(G)$ are precisely the cliques of G together with the three K_2s formed corresponding to each edge of G.

Let $V' = \{v_1, v_2, \ldots, v_\beta\}$ be a vertex cover of G. Corresponding to each edge uv of G choose the vertex which corresponds to u of the corresponding K_2, if u is not present in V'. If u is present in v', then, choose the vertex corresponding to v, irrespective of whether v is present in V' or not. Let this new collection together with V' be V''. Then V'' is a clique transversal of $T_1(G)$ of cardinality $m + \beta(G)$. Therefore, $\tau_c(T_1(G)) \leq m + \beta(G)$.

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Let $V' = \{v_1, v_2, \ldots, v_t\}$, where $t = \tau_c(T_1(G))$, be a clique transversal of $T_1(G)$. Let uv be an edge in G and let $u'v'$ be the K_2 introduced in $T_1(G)$ corresponding to this K_2. At least one vertex from $\{u', v'\}$, say u', must be present in V', since V' is a clique transversal and $u'v'$ is a clique of $T_1(G)$. Remove u' from V'. If V' contains v' also then replace v' by v. If $v' \notin V'$ then $v \in V'$, since V' is a clique transversal and vv' is a clique of $T_1(G)$. In either case, one vertex v of the edge uv is present in the new collection. Repeat the process for each edge in G to get V''. Clearly, V'' is a vertex cover of G with cardinality $\tau_c(T_1(G)) - m$. Hence, $\beta(G) \le \tau_c(T_1(G)) - m$. Thus, $\tau_c(T_1(G)) = m + \beta(G)$.

By a similar argument we can prove that $\tau_c(T_k(G)) = km + \beta(G)$.

Notation. For a given class $\mathcal{G}$ of graphs, let $T_k(\mathcal{G}) = \{T_k(G) : G \in \mathcal{G}\}$.

Theorem 9. *The class $T_k(\mathcal{G})$ satisfies the $\langle 2 \rangle$-property if and only if $\beta(G) \le \frac{n}{2}\ \forall G \in \mathcal{G}$ and $T_k(\mathcal{G})_t$ is empty for $t \ge 3$.*

Proof. Let $G \in \mathcal{G}$. $n(T_k(G)) = n + 2km$ and by Lemma 8, $\tau_c(T_k(G)) = km + \beta(G)$. Therefore,

$$\tau_c(T_k(G)) \le \frac{n(T_k(G))}{2} \langle = \rangle km + \beta(G) \le \frac{n + 2km}{2} \langle = \rangle \beta(G) \le \frac{n}{2}.$$

Hence, $T_k(\mathcal{G})$ satisfies $\langle 2 \rangle$-property if and only if $\beta(G) \le \frac{n}{2}\ \forall G \in \mathcal{G}$.

If G contains at least one edge then $T_k(G)$ has a clique of size 2 and hence $T_k(\mathcal{G})_t$ is empty for $t \ge 3$.

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