SYNOPSIS OF THE Ph.D. THESIS ENTITLED

ON APPROXIMATION METHODS FOR UNBOUNDED OPERATORS

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Introduction

Many of the practical problems that occur in Physics, Chemistry, Engineering Sciences, Economics etc. lead to problems of solving linear operator equations or computing singular values or spectral values of linear operators. And many of the operators that arise in this way are unbounded. Functional Analysis plays a pivotal role in supplying tools for attacking these problems. An attempt is made in this thesis to concentrate on the solutions of operator equations and approximation numbers (generalization of singular values) of linear operators. Naturally, the operators in consideration are unbounded.

Though there are effective techniques and efficient algorithms for the analysis and computation of solutions of operator equations, there are still plenty of equations whose actual solutions defy all computational endeavours. In respect of those cases, the obvious interest is to make approximations of the actual solutions.

One of the goals of Numerical Functional Analysis is to investigate the problem of approximating solutions of operator equations. Significant work has been done in this direction by many eminent mathematicians for equations involving special types of operators such as Toeplitz operators, differential operators and integral operators. The basics of finding solutions of equations involving certain kinds of bounded operators can be seen in [2], [4], [7] and [9].

The present study focuses on developing a general theory on approximate solutions of unbounded operator equations, in arbitrary Hilbert spaces. The treatment is entirely different from that of bounded operators. The usual convergence may not be expected in the case of unbounded operators. A variant of the classical notion of resolvent convergence is used for the approximation.

Yet another area the present study focuses is to discuss certain approximation numbers of unbounded operators. Approximation numbers are generalization of the classical singular values (of compact operators). If A is a compact operator on a Hilbert space H, then A^*A is a compact self-adjoint (in fact, positive) operator. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ be the sequence of (non-zero) eigen values of $(A^*A)^{1/2}$ (counting multiplicity). $s_k(A) = \lambda_k$ is the k^{th} singular value of A. It is true that $s_k(A) = \inf\{\|A - F\| / F \in \mathcal{B}(H),\ rank\ F \leq k-1\}$, where $\mathcal{B}(H)$ denotes the class of bounded operators on H. The same definition works for bounded operators as well. In the general setting, $s_k(A)$ are called approximation numbers. Inspired with this, several other approximation numbers are introduced for bounded operators by many mathematicians.

The notion of approximation numbers was further extended to unbounded operators by M.N.N. Namboodiri and A.V. Chithra [14]. One among them is the relative approximation numbers. And a few other similar approximation numbers have been introduced in this study.

Summary of the Thesis

The thesis consists of four chapters. Chapter 1 presents some basic definitions and results in Functional Analysis and Operator Theory which are used in the subsequent chapters.

In chapter 2, approximation methods for unbounded self-adjoint operators are analyzed. A variant of the classical notion of resolvent convergence is used for the purpose of approximating the solution x of the equation

$$Ax = y, \tag{1}$$

where y is a given element of a separable Hilbert space H and A is a self-adjoint unbounded operator in H. Computation of the actual solution x is not possible in many of the cases. In those cases the obvious interest will be finding approximate solutions x_n of (1). So, the problem is reduced to finding a sequence of 'uniquely solvable' (invertible) operators A_n defined on certain finite-dimensional subspaces D_n of the domain $D(A)$ of A such that (A_n) converges to A in some sense, and a sequence (y_n) in H with $y_n \in R(A_n)$, the range of A_n, such that (y_n) converges to y in H satisfying the following:

If x_n is the unique solution of the equation

$$A_n x_n = y_n (n = 1, 2, 3, \ldots), \tag{2}$$

then (x_n) converges to the unique solution x of (1). We call (A_n) an approximation method or approximating sequence for A if (A_n) converges to A in some sense and we call (A_n) an applicable sequence or applicable method for A if (x_n) converges to x in H.

In this context, filtrations are used to define resolvent convergence. Let H be a separable Hilbert space. By a filtration of H we mean an increasing sequence (H_n) of finite dimensional subspaces of H such that $\overline{\cup H_n} = H$. In the second section, some relevant results on resolvent convergence are extended in our setting. Applicability is discussed in the third section. The relation between applicability and resolvent convergence is established. Stability plays a vital role here. (A_n) is said to be stable if A_n is invertible for every $n \geq n_0$, for some n_o, and sup $\{\|A_n^{-1}\| / n \geq n_0\} < \infty$.

We consolidate the results established in this section as follows:

1. *If (A_n) is stable, then, (A_n) is an applicable method for A if and only if $A_n \to A$ in the strong resolvent sense.*

2. *If $A_n \to A$ in the strong resolvent sense, then, (A_n) is an applicable method for A if and only if (A_n) is stable.*

We discuss stability in the fourth section. The following result is established in this section:

Let A be a self-adjoint operator in H. Let A_n be self-adjoint and invertible *on H_n, for each n, where (A_n) is a filtration of H. Suppose $A_n \to A$ in the strong resolvent sense. If (A_n) is stable, then A is injective.*

The fifth section deals with the finite section method. Let (H_n) be a filtration of H contained in $D(A)$, the domain of A, and let P_n be the orthogonal projection on H with range H_n. The sequence (A_n), where $A_n = P_nAP_n/_{Hn}$ is called the finite section method.

We prove a few results which give some sufficient conditions for the resolvent convergence of (A_n) to A.

Let us state the main results of this section :

1. *If $A(H_n) \subseteq H_n$, then $A_n \to A$ in the strong resolvent sense.*
2. *Let $\Gamma(A)$ be the graph of A and let $G = \{(x, Ax)/x \in \cup H_n\}$. If G is dense in $\Gamma(A)$, then $A_n \to A$ in the strong resolvent sense.*

We deduce some useful corollaries from this result, including the following one:

$A_n \to A$ in the strong resolvent sense if $AP_nx \to Ax$ for every $x \in D(A)$.

Finally we prove that for self-adjoint operators with $A(H_n) \subseteq H_n$, the series expansion for e^{itA} is valid in the dense set $\cup H_n$ of the domain $D(A)$ of A.

In Chapter 3, the truncation method for unbounded matrices is analyzed. By an unbounded matrix we mean an infinite matrix such that the operator it induces in l^2 with respect to the standard orthonormal basis is unbounded. We discuss how and when the corner truncations of a given unbounded self-adjoint matrix A can be used to approximate the solution of the equation $Ax = y$, for a given y in l^2. Some of the results established in the previous chapter are applied here to investigate the resolvent convergence of the truncations to the given matrix. Also, an attempt is made to discuss unbounded Toeplitz matrices. For self-adjoint Toeplitz matrices $A = T(a)$ with $a \in L^2(\mathbb{T})$ ($\mathbb{T}$ being the unit circle), we have proved the following result:

Let $D' = D(A) \cap l^1$ *and* $G' = \{(x, Ax)/x \in D'\}$ *. Suppose* G' *is dense in* $\Gamma(A)$*, the graph of* A *. Then the truncations of* A *converge to* A *in the strong resolvent sense.*

Chapter 4 is devoted to analyzing certain approximation numbers of unbounded operators. We consider unbounded operators which are bounded relative to some other operator. Various kinds of relative approximation numbers are discussed in this chapter. The first section deals with relative approximation numbers and the second section deals with approximation number sets. The following connection between the eigenspectrum and the first approximation number set of a closed unbounded operator is established:

$|e(A)| \subseteq \tilde{s}_1(A)$ *, where* $|e(A)|$ *denotes the set* $\{|k|/k \in e(A)\}$ *. Here* $e(A)$ *is the eigenspectrum of* A *and* $\tilde{s}_1(A)$ *is the first approximation number set of* A *. Fourth section deals with generalized approximation numbers and fifth section deals with square approximation numbers. As a final result, we prove that the square approximation numbers* $\tau^*_{k,T}(A_n)$ *converge to* $\tau^*_{k,T}(A)$ *as* $n \to \infty$ *, where* A_n *are the finite sections of* A *, namely* P_nAP_n *. Here,* P_n *are the orthogonal projections on* H *with some finite dimensional subspaces* H_n *as range.*

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