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Bivariate semi-Pareto distributions and processes

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A bivariate semi-Pareto distribution is introduced and characterized using geometric minimization. Autoregressive minification models for bivariate random vectors with bivariate semi-Pareto and bivariate Pareto distributions are also discussed. Multivariate generalizations of the distributions and the processes are briefly indicated.

Key Words: AUTOREGRESSIVE PROCESS, GEOMETRIC MINIMIZATION, MINIFICATION PROCESSES, PARETO AND SEMI-PARETO DISTRIBUTIONS.

1. Introduction

It is well known that one of the popular distributions used to fit heavy tailed data is the Pareto distribution. For details see Arnold (1983). Some characterizations of the Pareto type III distribution based on geometric minimization and maximizations are studied by Arnold , Robertson and Yeh (1986). Recently Yeh, Arnold and Robertson (1988) have defined an auto-regressive minification process with Pareto type III marginals. Pillai (1991) generalized this process in terms of semi-Pareto random variables (r.v.s). These processes possess all the properties of a linear first order autoregressive (AR(1)) process. In fact these models are used to model non-Gaussian time series. The non-Gaussian time series models with various marginals are studied extensively in the literature (See Adke and Balakrishna (1992), Jayakumar and Pillai (1993) and Pillai and Jayakumar (1994)).

Suppose that we have a set of data on a bivariate random vector whose marginals show a tendency to follow heavy tailed distributions. Hutchinson (1979) explains the applications of such distributions in biological study. For applications of the bivariate Pareto distributions in reliability see Sankaran and Nair (1993). Note that the observations made on these systems at different time points are not independent. As a remedy we may assume that the observations are generated by a bivariate Markov model. One way of defining bivariate Markov sequences is by linear models as in the case of bivariate exponential autoregressive processes of Block, et al. (1988) and Dewald et al. (1989). Yeh et al.(1988) defined Pareto processes and discussed the applications of their model in income analysis. Here we discuss a bivariate extension of this model and try to generalize this model. This ended in obtaining the bivariate semi-Pareto distribution as the stationary solution of the bivariate minification sequence that we define.

In this paper we discuss different aspects of the bivariate semi-Pareto and a particular bivariate Pareto distributions. We characterize these distributions using geometric minimization. Further, we also study the properties of autoregressive minification processes with bivariate semi-Pareto and bivariate Pareto random vectors.

In Section 2 we define a bivariate semi-Pareto and a Pareto distribution and study their properties using geometric minimization. The AR(1) minification models for random vectors with the above distributions are discussed in Section 3. The second order properties of the distributions and the processes are described in Section 4. In section 5 we briefly indicate the multivariate extensions of the distributions and the process defined in Section 2 and 3.

2 Characterizations of bivariate Semi-Pareto Distribution

A random vector (X,Y) is said to have the bivariate semi-Pareto distribution with parameters α_1, α_2, p and we denote it by $(X,Y) \stackrel{D}{=} BSP(\alpha_1,\alpha_2,p)$ if its survival function is of the form

$$\bar{F}(x,y) = P(X>x,Y>y) = 1/\{1+\psi(x,y))\}, \tag{2.1}$$

where $\psi(x,y)$ satisfies the functional equation

$$\psi(x,y) = (1/p)\ \psi(p^{1/\alpha_1} x, p^{1/\alpha_2} y), \tag{2.2}$$

$o<p<1; \alpha_1,\alpha_2>0;\ x,y\geq 0.$

<u>Lemma 2.1</u>: The solution of the functional equation (2.2) is given by

$$\psi(x,y) = x^{\alpha_1} h_1(x) + y^{\alpha_2} h_2(y), \tag{2.3}$$

where $h_1(x)$ and $h_2(y)$ are the periodic functions in log x and log y with periods $\frac{2\pi\alpha_1}{-\log p}$ and $\frac{2\pi\alpha_2}{-\log p}$ respectively.

A proof of this lemma can be found in Kangan, Linnik and Rao (1963), pp 163. □

As an example, if we take

$h_i(x) = \exp\{\beta \cos(\alpha_i \log x)\}$, i=1,2, then we can see that it satisfies (2.2) with $p=e^{-2\pi}$.

In particular, if we choose $h_1(x)=h_2(y)=1$, the BSP(α_1,α_2,p) reduces to a bivariate Pareto distribution with survival function

$$\bar{F}(x,y) = 1/\{1+x^{\alpha_1} + y^{\alpha_2}\}, \qquad x\geq 0, y\geq 0, \alpha_1>0, \alpha_2>0. \tag{2.4}$$

Now we study some of the characterization properties of BSP(α_1,α_2,p) distributions via geometric minimization. Let $\{(X_i,Y_i), i\geq 1\}$ be a sequence of independent identically distributed (i.i.d) random vector with common survival function (2.1) and N be a geometric random variable with parameter p and

$$P[N=n] = pq^{n-1}, \quad n=1,2,\ldots, \quad 0<p<1, \quad q=1-p. \tag{2.5}$$

Further assume that N is independent of X_i, Y_i.

Define

$$U_N = \min_{1\leq i\leq N} X_i \text{ and } V_N = \min_{1\leq i\leq N} Y_i. \tag{2.6}$$

Theorem 2.1: Let $\{(X_i,Y_i), i\geq 1\}$ be a sequence of i.i.d bivariate non-negative random vectors with common survival function $\bar{F}(x,y)$ and N be a geometric random variable as in (2.5), which is independent of (X_i,Y_i) for all $i\geq 1$. The random vectors $(p^{-1/\alpha_1} U_N, p^{-1/\alpha_2} V_N)$ and (X_1, Y_1) are identically distributed if and only if (X_i, Y_i) have the BSP(α_1, α_2, p) distribution.

Proof: Consider

$$\bar{H}(x,y) = \Pr\left[p^{-1/\alpha_1} U_N > x, p^{-1/\alpha_2} V_N > y\right]$$

$$= \sum_{n=1}^{\infty} \left[\bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2})\right]^n pq^{n-1}.$$

That is,

$$\bar{H}(x,y) = \frac{p\ \bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2})}{1-q\bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2})}. \tag{2.7}$$

Now if $\bar{F}(x,y)$ is as in (2.1) and (2.2), the equation (2.7) becomes

$$\bar{H}(x,y) = \frac{1}{1+\psi(x,y)} = \bar{F}(x,y).$$

This proves the sufficiency part of the theorem.

Conversely, suppose that $\bar{H}(x,y)=\bar{F}(x,y)$. Note that any survival function $\bar{F}(x,y)$ can be represented as

$$\bar{F}(x,y) = \frac{1}{1+\phi(x,y)}, \tag{2.8}$$

where $\phi(x,y)$ is a monotonically increasing function in both x and y $(x \geq 0,\ y \geq 0)$ and $\lim_{x\to 0} \lim_{y\to 0} \phi(x,y) = 0$ and $\lim_{x\to\infty} \lim_{y\to\infty} \phi(x,y) = \infty$. Using the representation (2.8) in (2.7) with $\bar{H}(x,y)=\bar{F}(x,y)$, we get the equation,

$$\phi(x,y) = \frac{1}{p}\ \phi(xp^{1/\alpha_1}, yp^{1/\alpha_2}).$$

This is the functional equation (2.2) satisfied by BSP(α_1,α_2,p) with $\phi(.,.)$ in the place of $\psi(.,.)$. Hence the proof is complete.

Let $\{N_k,\ k\geq 1\}$ be a sequence of geometric random variables with parameters p_k, $0\leq p_k<1$. Define

$$\bar{F}_k(x,y) = \Pr[U_{N_{k-1}}>x, V_{N_{k-1}}>y],\ k=2,3,\ldots$$

$$= \frac{p_{k-1}\bar{F}_{k-1}(x,y)}{1-(1-p_{k-1})\bar{F}_{k-1}(x,y)}. \tag{2.9}$$

Here we refer $\bar{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of iid random vectors with $\bar{F}_{k-1}$ as the common survival function.

<u>Theorem 2.2</u>: Let$\{(X_i,Y_i),\ i\geq 1\}$ be a sequence of iid non-negative random vectors with common survival function $\bar{F}(x,y)$. Define $\bar{F}_1=\bar{F}$ and $\bar{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of iid random vectors with common survival function $\bar{F}_{k-1}$, k=2,3,... . Then

$$\bar{F}_k\left(\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right) = \bar{F}(x,y) \tag{2.10}$$

if and only if (X_1,Y_1) has BSP(α_1,α_2,p) distribution.

<u>Proof</u>: By definition, the survival function $\bar{F}_k$ satisfies the equation (2.9). As in (2.8) we can write

$$\bar{F}_k(x,y) = \frac{1}{1+\phi_k(x,y)},\quad k=1,2,\ldots$$

Substituting this in (2.9), we get

$$\phi_k(x,y) = \frac{1}{p_{k-1}}\ \phi_{k-1}(x,y),\ k=2,3,\ldots\ .$$

Recursively using this relation, we have

$$\phi_k(x,y) = \frac{1}{\prod_{j=1}^{k-1} p_j} \phi_1(x,y), \text{ since } F_1=F \text{ implies } \phi_1=\phi.$$

This implies

$$\phi_k\left[\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right]$$

$$= \frac{1}{\prod_{j=1}^{k-1} p_j} \phi_1\left[\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right]. \quad (2.11)$$

This gives us (2.10) if we replace ϕ_k by ψ_k and if we assume that ψ_1 satisfies (2.2).

Conversely, assume that (2.10) is true. By the hypothesis of the theorem we have (2.11). Thus (2.10) and (2.11) together lead to the equation,

$$\left[1 + \frac{1}{\prod_{j=1}^{k-1} p_j} \phi_1\left[\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right]\right]^{-1}$$

$$= \bar{F}(x,y) = \frac{1}{1+\phi(x,y)}.$$

This implies that

$$\phi(x,y) = \frac{1}{\prod_{j=1}^{k-1} p_j} \phi\left[\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right]$$

which is same as (2.2).

Hence the proof is complete.

We have already noted that the bivariate Pareto distribution (2.4) is a special case of BSP(α_1,α_2,p). Let us denote the distribution having survival function (2.4) by BP(α_1,α_2). Now we prove some characterization results for

$BP(\alpha_1,\alpha_2)$. Suppose that the survival function $\bar{F}(x,y)$ is of the form (2.8). Let $\mathcal{F}_{\alpha_1,\alpha_2}$ be a family of all distributions $F(x,y)$ with the property that

$$\lim_{x\to 0^+}\lim_{y\to 0^+}\frac{\phi(x,y)}{(x^{\alpha_1}+y^{\alpha_2})} = 1, \qquad (2.12)$$

where $\phi(x,y)$ is as in (2.8)

Theorem 2.3: Let $\{(X_i,Y_i),\ i\geq 1\}$ and N be as defined in Theorem 2.1 with common distribution F of (X_i,Y_i) belong to $\mathcal{F}_{\alpha_1,\alpha_2}$ and $0<p<1$. Then $\left[p^{-1/\alpha_1}U_N, p^{-1/\alpha_2}V_N\right]$ and (X_1,Y_1) are identically distributed if and only if F is $BP(\alpha_1,\alpha_2)$.

Proof: As before we have (from(2.7))

$$\bar{H}(x,y) = Pr\left[p^{-1/\alpha_1}U_N > x,\ p^{-1/\alpha_2}V_N > y\right] = \frac{p\ \bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2})}{1-q\bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2})}.$$

Now the sufficient part is straight forward.

In order to prove the necessary part assume that (2.7) holds with $\bar{F}(x,y)$ in the place of $\bar{H}(x,y)$. In terms of $\phi(x,y)$, (2.7) leads to the equation

$$\phi(x,y) = \frac{1}{p}\,\phi(xp^{1/\alpha_1}, yp^{1/\alpha_2}). \qquad (2.13)$$

Repeated use of this relation gives us

$$\phi(x,y) = \frac{1}{p^k}\,\phi(xp^{k/\alpha_1}, yp^{k/\alpha_2}) \text{ for any k, integer.}$$

Now rewriting the right hand side expression and taking limit as $k\to\infty$, we have

$$\phi(x,y) = (x^{\alpha_1} + y^{\alpha_2}) \lim_{k\to\infty} \left[\frac{\phi(xp^{1/\alpha_1}, yp^{1/\alpha_2})}{(xp^{k/\alpha_1})^{\alpha_1} + (yp^{k/\alpha_2})^{\alpha_2}} \right]$$

$$= (x^{\alpha_1} + y^{\alpha_2}), \text{ follows by (2.12).}$$

Thus we have $\bar{F}(x,y)$ given by (2.4). This completes the proof.

Corollary 2.1: Let $\{(X_i,Y_i),\ i\geq 1\}$ be a sequence of iid non-negative random vectors with the common distribution function F satisfying the condition (2.12) and F_k be the distribution function of geometric (p_{k-1}) minimum of i.i.d random vectors with F_{k-1} as the common distribution functions, k=2,3,... . If $\prod_{j=1}^{k-1} p_j \to 0$ as $k\to\infty$, then

$$F_k\left(\left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_1} x,\ \left(\prod_{j=1}^{k-1} p_j\right)^{1/\alpha_2} y\right) \xrightarrow{D} BP(\alpha_1,\alpha_2).$$

Proof of this corollary follows from the proofs of Theorems 2.2, 2.3 and the condition (2.12).

3. Bivariate Semi-Pareto AR(1) model.

In this section we study the properties of first order autoregressive (AR(1)) models with minification structures in bivariate semi-Pareto and Pareto random vectors. The univariate AR(1) models with Pareto and semi Pareto marginals are studied by Yeh, et al (1988) and Pillai (1991) respectively. We define a bivariate minification process, $\{(X_n,Y_n),\ n\geq 0\}$ as follows.

Let $\{(\varepsilon_n,\eta_n),\ n\geq 1\}$ be a sequence of i.i.d bivariate non-negative extended real random vectors and define

$$X_n = \min(p^{-1/\alpha_1} X_{n-1}, \varepsilon_n)$$

and (3.1)

$$Y_n = \min(p^{-1/\alpha_2} Y_{n-1}, \eta_n),\ n\geq 1,\ 0\leq p<1,\ \alpha_1,\alpha_2>0.$$

Assume that (X_0,Y_0) is independent of (ε_i,η_i). Then it easily follows that $\{(X_n,Y_n),\ n\geq 0\}$ is a bivariate Markov sequence.

As ε_n and η_n are extended real random variables we assume that either both are infinity with probability p or both are finite with probability 1-p and hence we can represent them as

$$(\varepsilon_n,\eta_n) = \begin{cases} (+\infty,+\infty) & \text{with probability } p \\ (\xi_n,\lambda_n) & \text{with probability } (1-p), 0<p<1, \end{cases} \tag{3.2}$$

where ξ_n and λ_n are real-valued random variables.

<u>Theorem 3.1</u>: Assuming that $(X_0,Y_0) \overset{D}{=} (\xi_1,\lambda_1)$, the process $\{(X_n,Y_n),\ n\geq 0\}$ defined by (3.1) and (3.2) is stationary if and only if (ξ_n,λ_n) has a BSP(α_1,α_2,p) distribution.

<u>Proof</u>: Definition of the model implies that

$$\bar{G}_n(x,y) = P[X_n>x, Y_n>y]$$

$$= \bar{G}_{n-1}(xp^{1/\alpha_1}, yp^{1/\alpha_2})\{p+(1-p)\bar{F}(x,y)\}, \tag{3.3}$$

where $\bar{F}(x,y)$ is the survival function of (ξ_1,λ_1).

Assume that $\{(X_n,Y_n) n\geq 0\}$ is stationary and $(X_0,Y_0) \overset{D}{=} (\xi_1,\lambda_1)$. Then for n=1, (3.3) gives us

$$\bar{F}(xp^{1/\alpha_1}, yp^{1/\alpha_2}) = \frac{\bar{F}(x,y)}{p+(1-p)\bar{F}(x,y)} . \tag{3.4}$$

As in (2.8) if we write

$\bar{F}(x,y) = \frac{1}{1+\psi(x,y)}$, the equation (3.4) leads to the relation

$$\psi(x,y) = \frac{1}{p}\,\psi(xp^{1/\alpha_1}, yp^{1/\alpha_2}).$$

That is, $\bar{F}(x,y)$ is of the form (2.1) and hence by (3.3), (X_1,Y_1) is a BSP(α_1,α_2,p) distributed random vector. Then by induction argument we have $\{(X_n,Y_n),\ n\geq 0\}$ is a BSP(α_1,α_2,p) Markov sequence.

Conversely, suppose that (ξ_n,λ_n) has BSP(α_1,α_2,p) distribution for every $n\geq 1$ with $(X_0,Y_0)\overset{D}{=}(\xi_1,\lambda_1)$. In this case for n=1, from (3.3) and (2.2) we get

$$\bar{G}_1(x,y) = \frac{1}{1+\psi(x,y)} .$$

That is, (X_1,Y_1) has BSP(α_1,α_2,p) distribution. Now by (3.3) and an easy induction argument it follows that (X_n,Y_n) has BSP(α_1,α_2,p) distribution for every $n\geq 0$. That is, $\{(X_n,Y_n),\ n\geq 0\}$ is a stationary BSP(α_1,α_2,p) sequence.

<u>Corollary 3.1</u>: Let (X_0,Y_0) be an arbitrary random vector with survival function $G_0(u,v)$ such that $\lim_{u\to 0}\lim_{v\to 0}\bar{G}(u,v) = 1$ and $\{(\xi_n,\lambda_n),\ n\geq 1\}$ be a sequence of i.i.d BSP(α_1,α_2,p) random vectors. Then the bivariate sequence $\{(X_n,Y_n),\ n\geq 0\}$ defined by (3.1) and (3.2) converges in distribution to BSP(α_1,α_2,p) as $n\to\infty$.

<u>Proof</u>: The definition of the model and the relations (3.3), (2.1) and (2.2) together imply that

$$\bar{G}_n(x,y) = \bar{G}_0(xp^{n/\alpha_1}, yp^{n/\alpha_2}) \left\{ \frac{1+p^n \psi(x,y)}{1+\psi(x,y)} \right\}$$

$$\rightarrow \frac{1}{1+\psi(x,y)}, \text{ as } n\rightarrow\infty,$$

where $\psi(x,y)$ is as in (2.2). Hence the corollary is proved.

Remark 3.1: Recall that the bivariate Pareto distribution (2.4) is a special case of BSP(α_1,α_2,p). If we assume that $\{(\xi_n,\lambda_n),\ n\geq 1\}$ is a sequence of i.i.d BP(α_1,α_2) random vectors in Theorem 3.1, then $\{(X_n,Y_n),\ n\geq 0\}$ defined by (3.1) and (3.2) becomes a stationary BP(α_1,α_2) sequence. We refer such a sequence by ARBP(1) sequence.

4. Second order properties:

The implicite nature of BSP(α_1,α_2,p) distribution does not allow us to obtain exact expressions of its moments. However, we obtain the moments of BP(α_1,α_2) distribution and the auto-correlation matrix of ARBP(1) process when they exist.

Suppose that (X,Y) has the survival function (2.4). Then its $(r,s)^{th}$ moment vector is given by

$$(\mu_r, m_s) = E(X^r, X^s) = (E(X^r), E(X^s))$$

$$= \left[\Gamma(1+\frac{r}{\alpha_1})\Gamma(1-\frac{r}{\alpha_1}), \Gamma(1+\frac{s}{\alpha_2})\Gamma(1-\frac{s}{\alpha_2})\right]$$

provided $r<\alpha_1$ and $s<\alpha_2$. If $\alpha_i>2$, $i=1,2$ then the variance covariance matrix of (X,Y) is

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = ((\sigma_{ij})),\ i,j=1,2,$$

where $\sigma_{ij} = \Gamma(1 - \frac{1}{\alpha_i})\Gamma(1 - \frac{1}{\alpha_j})\Gamma(1-\frac{1}{\alpha_i}-\frac{1}{\alpha_j})$

$$-\left[\Gamma(1+\frac{1}{\alpha_i})\Gamma(1 - \frac{1}{\alpha_i})\right]\left[\Gamma(1+\frac{1}{\alpha_j})\Gamma(1 - \frac{1}{\alpha_j})\right]. \qquad (4.1)$$

Now we discuss the covariance structure of ARBP(1) process with stationary distribution (2.4) (See remark 3.1). We define the autocovariance matrix of a bivariate process $\{(X_n,Y_n),\ n\geq 0\}$ by

$$\Gamma(h) = \begin{bmatrix} cov(X_n,X_{n+h}) & cov(X_n,Y_{n+h}) \\ cov(X_{n+h},Y_n) & cov(Y_n,Y_{n+h}) \end{bmatrix}.$$

In the rest of this section we assume that $\alpha_i>2$, $i=1,2$.

The definition of our model allows us to write,

$$X_{n+h} = \min\left\{ p^{-\frac{h}{\alpha_1}} X_n,\ p^{-\frac{h-1}{\alpha_1}} \varepsilon_{n+1}, \dots,\ p^{-1/\alpha_1}\varepsilon_{n+h-1}, \varepsilon_{n+h}\right\}$$

and (4.2)

$$Y_{n+h} = \min\left\{ p^{-\frac{h}{\alpha_2}} Y_n,\ p^{-\frac{h-1}{\alpha_2}} \eta_{n+1}, \dots,\ p^{-1/\alpha_2}\eta_{n+h-1}, \eta_{n+h}\right\}.$$

These relations will help us in evaluating the above covariance functions. Observe that

$$\Pr[X_{n+h}\leq x|X_n=y] = \begin{cases} [(1-p^h)x^{\alpha_1}]/(1+x^{\alpha_1}) & \text{if } x\leq yp^{-h/\alpha_1} \\ 1 & \text{if } x>yp^{-h/\alpha_1} \end{cases} \qquad (4.3)$$

Now

$$E(X_{n+h}X_n) = E[X_nE(X_{n+h}|X_n)],$$

where the conditional expectation can be evaluated using (4.2) as

$$E(X_{n+h}|X_n=y)=(1-p^h)\alpha_1 \int_0^{yp^{-h/\alpha_1}} \frac{x^{\alpha_1}}{(1+x^{\alpha_1})^2}\,dx + yp^{-h/\alpha_1}\left[\frac{1+y^{\alpha_1}}{1+y^{\alpha_1}p^{-h}}\right].$$

Therefore,

$$\mathrm{Cov}(X_n,X_{n+h})=(1-p^h)E\left\{X_n\int_0^{p^{-h}X_n^{\alpha_1}} \frac{u^{1/\alpha_1}}{(1+u)^2}\,du\right\}$$

$$+ p^{-h/\alpha_2}B\left[1+\frac{2}{\alpha_1},1-\frac{2}{\alpha_1}\right]F(1,1+\frac{2}{\alpha_1};2;1-p^{-h})$$

$$- \left[\Gamma(1+\frac{1}{\alpha_1})\Gamma(1-\frac{1}{\alpha_1})\right]^2$$

$$= \gamma_{xx}(h;\alpha_1,\alpha_1), \text{ say},$$

where $F(\alpha,\beta;\gamma,z) = [1/B(\beta,\gamma-\beta)] \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}dt.$

Similar computations show that

$$\mathrm{Cov}(Y_n,Y_{n+h}) = \gamma_{yy}(h;\alpha_2,\alpha_2).$$

In order to compute the other elements of $\Gamma(h)$, we consider,

$$\Pr[X_n>x,Y_{n+h}>y] = \frac{1+p^h y^{\alpha_2}}{(1+x^{\alpha_1}+p^h y^{\alpha_2})(1+y^{\alpha_2})},\quad x\geq 0, y\geq 0.$$

Using this it can be shown that

$$\mathrm{Cov}(X_n,Y_{n+h}) = (1-p^h)p^h\,\Gamma(1+\frac{1}{\alpha_1})\Gamma(2-\frac{1}{\alpha_1})B\left[1+\frac{1}{\alpha_2},2-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right]$$

$$F(2-\frac{1}{\alpha_1},1+\frac{1}{\alpha_2};3-\frac{1}{\alpha_1};1-p^h)$$

$$+ (1-p^h)\Gamma(1+\frac{1}{\alpha_1})\Gamma(1-\frac{1}{\alpha_1}) B\left[1+\frac{1}{\alpha_2}, 2-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right]$$

$$F(1-\frac{1}{\alpha_1}, 1+\frac{1}{\alpha_2}; 3-\frac{1}{\alpha_1}; 1-p^h)$$

$$+ p^{-h/\alpha_2} p^h \Gamma(1+\frac{1}{\alpha_2})\Gamma(1+\frac{1}{\alpha_1})\Gamma(1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2})$$

$$- \Gamma(1+\frac{1}{\alpha_1})\Gamma(1-\frac{1}{\alpha_1}) \Gamma(1+\frac{1}{\alpha_2})\Gamma(1-\frac{1}{\alpha_2})$$

$$= \gamma_{xy}(h;\alpha_1,\alpha_2), \text{ say}$$

Proceeding as above we also have

$$\mathrm{Cov}(X_{n+h}, Y_n) = \gamma_{xy}(h;\alpha_2,\alpha_1).$$

Thus the autocovariance matrix of the stationary ARBP(1) process is given by

$$\Gamma(h) = \begin{bmatrix} \gamma_{xx}(h;\alpha_1,\alpha_1) & \gamma_{xy}(h;\alpha_1,\alpha_2) \\ \gamma_{xy}(h;\alpha_2,\alpha_1) & \gamma_{yy}(h;\alpha_2,\alpha_2) \end{bmatrix}.$$

The expressions of $\gamma_{xx}(1;\alpha_1,\alpha_1)$ and $\gamma_{yy}(1;\alpha_2,\alpha_2)$ can also be obtained from Yeh et al. (1988) for a proper choice of the parametrs in equation (2.4)

5. Multivariate generalization

In this section we provide a brief discussion of the multivariate extension of the models studied in Sections 2 and 3. The random vector $(X_1, X_2, \ldots, X_k)$ is said to have a k-variate semi-Pareto distribution with parameters α_1, α_2, α_3, ..., α_k and p if

$$\Pr[X_1>x_1, X_2>x_2, \ldots, X_k>x_k] = \frac{1}{1+\psi(x_1, x_2, \ldots, x_k)} \qquad (5.1)$$

such that

$$\psi(x_1,x_2,\ldots,x_k) = \frac{1}{p}\psi(p^{1/\alpha_1}x_1, p^{1/\alpha_2}x_2,\ldots,p^{1/\alpha_k}x_k). \quad (5.2)$$

The solution of equation (5.2) is given by

$$\psi(x_1,x_2,\ldots,x_k) = \sum_{i=1}^{k} x_i^{\alpha_i}\, h_i(x_i),$$

where $h_i(x_i)$, $i=1,2,\ldots,k$ are periodic functions in $\log x_i$ with period $\frac{2\pi\alpha_i}{-\ln p}$ (cf. Kagan, Linnik and Rao (1963),p 163).

If $h_i(x_i)\equiv 1$, for $i=1,2,\ldots,k$, then we get

$$P[X_1>x_1,X_2>x_2,\ldots,X_k>x_k] = \frac{1}{1+x_1^{\alpha_1}+x_2^{\alpha_2}+\ldots+x_k^{\alpha_k}},$$

which is the survival function of k-variate Pareto random vector.

If we have n independent copies of $(X_1,X_2,\ldots,X_k)$, we can define the componentwise geometric minima of random variables and then it is straight forward to prove the multivariate extension of the Theorems 2.1 and 2.3. It is also possible to define a stationary k-variate Pareto process $\{(X_{1n},X_{2n},\ldots,X_{kn}),\ n\geq 1\}$ by extending the definitions (3.1) and (3.2). However, we skip the details as the computations are straight forward.

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