A STUDY OF MORPHOLOGICAL OPERATORS WITH APPLICATIONS

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Under the guidance of

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A STUDY OF MORPHOLOGICAL OPERATORS WITH APPLICATIONS

Ph.D thesis in the field of Coding theory and Image Processing

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Front Cover: Morphological operation on an image

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CERTIFICATE

Certified that the work presented in this thesis entitled **'A Study of Morphological Operators with Applications'** is based on authentic record of research done by Mr. Ramkumar P.B under my guidance and supervision at the Department of Computer Applications, Cochin University of Science & Technology ,Cochin – 682022 and has not been included in any other work for the award of any degree.

Cochin -22 **Dr. K.V. Pramod**

Date: 9th May 2011 (Supervising Guide)

DECLARATION

I hereby declare that the work presented in this thesis entitled **'A Study of Morphological Operators with Applications'** is based on the original research work done by me under the guidance and supervision of **Dr. K.V. Pramod** ,Head of the Department , Department of Computer Applications, Cochin University of Science & Technology , Cochin and has not been included in any other work for the award of any degree.

Cochin -22 **Ramkumar P.B.**

Date: 9th May 2011

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Preface

In the processing and analysis of images it is important to be able to extract features, describe shapes and recognize patterns. Such tasks refer to geometrical concepts such as size, shape, and orientation. Mathematical morphology uses concepts from set theory, geometry and topology to analyze geometrical structures in an image.

The word 'morphology' originates from the Greek words morfh and logos, meaning 'the study of forms'. The term is encountered in a number of scientific disciplines including biology and geography. In the context of image processing it is the name of a specific methodology designed for the analysis of the geometrical structure in an image. Mathematical morphology was invented in the early 1960s by Georges Matheron and Jean Serra who worked on the automatic analysis of images occurring in mineralogy and petrography. Meanwhile the method has found applications also in several other fields, including medical diagnostics, histology, industrial inspection, computer vision, and character recognition. Mathematical morphology examines the geometrical structure of an image by probing it with small patterns, called 'structuring elements', of varying size and shape, just similar to a blind man explores the world with his stick. This procedure results in nonlinear image operators which are suitable for exploring geometrical and topological structures.

A series of such operators is applied to an image in order to make certain features more clear, distinguishing meaningful information from irrelevant distortions, by reducing it to a sort of caricature (skeletonization). The resulting multi resolution techniques (quadtrees, pyramids, fractal imaging, scale-spaces, etc.) all have their merits and limitations. For example, fractals are great success in image compression but to a much lesser extent for segmentation problems.

In the earliest multi resolution approaches to signal and image processing, the method was to obtain a coarse level signal by sub sampling a fine resolution signal, after linear smoothing, in order to remove high frequencies. A 'detail pyramid' can then be derived by subtracting from each level an interpolated form of the next coarser level. The resulting difference signals (known as detail signals) form a signal decomposition in terms of band pass-filtered copies of the original signal. The human visual system indeed uses a similar kind of decomposition. This tool has been one of the most popular multi resolution schemes used in image processing and computer vision. The emergence of wavelet techniques has boosted the multi resolution approach. Application of wavelets to problems in image processing and computer vision is sometimes hindered by its linearity. Coarsening an image by means of linear operators may not be compatible with a natural coarsening of some image attribute of interest (shape of object, for example), and hence use of linear procedures may be inconsistent in such applications.

Mathematical morphology (nonlinear) is complementary to wavelets (linear). In this it considers images as geometrical objects. It is not like elements of a linear (Hilbert) space. Many of the existing morphological techniques, such as granulometries, skeletons, and alternating sequential filters, are essentially multi resolution techniques. There are relationships between the existing linear (wavelets) and nonlinear (morphological) multi resolution approaches.

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About the Thesis

Chapters in the thesis are organized as follows.

Chapter 1 is Introduction to Mathematical Morphology. In this chapter, Birth of Mathematical Morphology, Image processing using Mathematical Morphology are included.

Dilation and erosion are the elementary operators of Mathematical Morphology, that is, they are building block for a large class of operators.

Application of these operators in image processing is aimed initially to improve the visual quality of the features of interest in digital grayscale images, which will then afterwards be extracted.

Increasingly seeking to get improvement in quality of the extracted feature, the image was binarized through the binary operator with threshold. Image skeletonization is one of the many morphological image processing operations. skeletonization is very often an intermediate step towards object recognition, These operators are widely using in Medical Imaging also.

Chapter 2 is Binary Morphology and Morphological operators. In this chapter, Binary Morphology , Dilation and Erosion ,Opening and Closing ,Properties of Operators are included.

Binary mathematical morphology consists of two basic operations dilation and erosion and several composite relations like closing and opening.

All the images actually process by computer will be digital. That is, they will be defined on an *R* row by *C* column grid of pixels. Typically, all the FG pixels will be black and the BG pixels, white, or *vice-versa*. All pixels are represented by squares. All FG and BG objects or regions are made up of these squares.

In the memory of the computer, all FG pixels are represented by a number, *f*, and all BG pixels by another number, *b*. Typically (*f*, *b*) = (1,0) or (*f*, *b*) = (255,0), or the opposite.

The key process in the dilation operator is the local comparison of a shape, called structuring element, with the object to be transformed. When the structuring element is positioned at a given point and it touches the object, then this point will appear in the result of the transformation, otherwise it will not.

Similarly, In Erosion operation, if, when positioned at a given point, the structuring element is included in the object then this point will appear in the result of the transformation, otherwise not. Other operators can be defined by using these two fundamental operators. These operators also satisfy several important properties. Properties of Dilation operator listed are i) Translation Invariance ii) Distributivity over union iii) Increasing property.

Chapter 3 is Gray Value Morphology and other Morphological Operators. In this chapter, properties of operators defined on gray value images are discussed.

Chapter 4 is Morphological operators defined on a Lattice. In this chapter, Lattice, Properties of Lattices, Operators defined on a Lattice are included.

A lattice is any non-empty poset L in which any two elements x and y have a least upper bound and a greatest lower bound. The operation $\wedge$ (is called meet), and the operation $\vee$ (is called join) are meant for greatest lower bound and least upper bound. A sublattice of L is a subposet of L which is a lattice, that is, which is closed under the operations $\wedge$ and $\vee$ as defined in L.

The operations of meet and join are idempotent, commutative, associative and absorptive.

In Chapter 5, Morphological Slope Transforms, Translation Invariant Systems, Legendre Transform, Slope Transforms, Properties of

Slope Transforms are included. The slope transform is considered as the morphological counterpart of the Fourier transform.

A type of non linear signal transforms that can quantify the slope content of signals and provide a transform domain for morphological systems, is called slope transforms. Slope transforms are based on eigen functions of morphological systems that are lines parameterized by their slope.

The three types of slope transforms are

i) a single valued slope transform for signals processed by erosion systems.
ii) a single valued slope transform for signals processed by dilation systems.
iii) A multi valued transform that results by replacing the suprema and infima of signals with the signal values at stationary points.

All three transforms coincide when we consider continuous-time signals (which are convex or concave and have an invertible derivative) and become equal to the Legendre transform.(irrespective of the difference due to the boundary conditions).

Chapter 6 is Generalized Structure for Mathematical Morphology. In this chapter

Different structures for Morphological Operators, Generalized Structure for Mathematical Morphology, Results in Generalized Structure are included.

In Chapter 7, Partial Self Similarity, Mathematical Morphology and Fractals:- the following are discussed. Fractals and Self similarity, Scaling, Cross section, Partial self similarity, Mathematical Morphology and Fractals. In a Morphological space, K⊆X is called Partial self similar or ***a similar*** if $\exists K_1, K_2, \ldots K_t$ such that K= $\bigcup_{i=1}^{t} K_i$ and for each K_i,∃contraction maps φ_{ijk}, for i=1..... t,r=1 t ,j=1,....t and k=1. $w(i,j)$with $w(i,j)$>0 such that $K_i = \bigcup_{j,k} \varphi_{i,j,k}(K_j)$.Fractals are very useful in medical imaging. Mathematical Morphology and Fractals plays very important role in many image processing applications.

Adjunctions are pairs of operators which satisfy some mathematical property. In mathematical Morphology Dilation and erosion are fundamental operators.

In Chapter 8, Morphological operators termed as adjunctions is discussed. Operators Dilation and erosion form an adjunction between two spaces. These operators are dual operators. In this chapter, Translation Invariance property, Binary Adjunctions, Various Adjunctions in Mathematical Morphology, Generalized Adjunctions are included.

Some definitions and results given in this chapter are listed below.

Dilation: Let $(L, \leq)$ be a complete lattice, with infimum and minimum symbolized by $\wedge$and $\vee$ respectively. A dilation is any operator $\delta : L \to L$that distributes over the supremum, and preserves the least element, $\bigvee_i \delta(X_i) = \delta(\bigvee_i X_i)$ and $\delta(\emptyset) = \emptyset$.

An erosion is any operator $\varepsilon : L \to L$that distributes over the infimum$\bigwedge_i \varepsilon(X_i) = \varepsilon\left(\bigwedge_i X_i\right)$, $\varepsilon(U) = U$.

Dilations and erosions form Galois connections. That is, for all dilation δ there is one and only one erosion εthat satisfies$X \leq \varepsilon(Y) \Leftrightarrow \delta(X) \leq Y$for all $X, Y \in L$.

Similarly, for all erosion there is one and only one dilation satisfying the above connection.

Furthermore, if two operators satisfy the connection, then δ must be a dilation, and ε an erosion. Pairs of erosions and dilations satisfying the above connection are called "adjunctions", and the erosion is said to be the adjoint erosion of the dilation, and vice-versa.

Concluding Remarks and Areas using Morphological operators are given in Chapter 9. Important applications like Image Processing, Signal Processing, Robotics, Medical Imaging and Computer Graphics etc are included.

Chapter 1

Introduction to Mathematical Morphology

CONTENTS

1.1 Introduction

Mathematical Morphology is the analysis of signals in terms of shape. This simply means that morphology works by changing the shape of objects contained within the signal. In the processing and analysis of images it is important to be able to extract features, describe shapes and recognize patterns. Such tasks refer to geometrical concepts such as size, shape, and orientation. Mathematical morphology uses concepts from set theory, geometry and topology to analyze geometrical structures in an image.

Mathematical morphology is about operations on sets and functions. It is systematized and studied under a new angle, precisely because it is possible to actually perform operations on the computer and see on the screen what happens. The need to simplify a complicated object is the basic impulse behind mathematical morphology. Related to this is the fact that an image may contain a lot of disturbances. Therefore, most images need to be tidied up. Hence another need to process images; it is related to the first, for the border line between dirt and of other kind disturbances is not too clear. Consider Euclidean geometry, and consider cardinalities. The set N of nonnegative integers is infinite, and its cardinality is denoted by card(N) = $\aleph_0$ (Aleph zero). The set of real numbers R has the same cardinality as the set of all subsets of N, thus card(R) = $2^{\aleph_0}$. The points in the Euclidean plane have the same cardinality:

card(R^2) = card(R). But the set of all subsets of the line or the plane has the larger cardinality. There are too many sets in the plane. Consider a large subclass of this huge class, a subclass consisting of nice sets. For instance, the set of all disks has a much smaller cardinality, because three numbers suffice to determine a disk in the plane: its radius and the two coordinates of its center. Similarly, four numbers suffice to specify a rectangle [a1, b1] × [a2, b2] with sides parallel to the axes; a fifth is needed to rotate it. This leads to the idea of simplifying a general, all too wild set, to some reasonable, better-behaved set. Euclidean line containing denumerably many points. Consider a line as the set of solutions in Q2 of an equation $a_1x_1 + a_2x_2 + a_3 = 0$ with integer coefficients. Then two lines which are not parallel intersect in a point with rational coordinates. The cardinality of the set of all subsets of Q_2 is $2^{\aleph_0}$, so there are fewer sets to keep track of than in the real case.

So there are too many subsets in the plane. Consider digital geometry. On a computer screen with, say, 1,024 pixels in a horizontal row and 768 pixels in a vertical column there are 1, 024 × 768 = 786, 432 pixels. On such a screen a rectangle with sides parallel to the axes is the Cartesian product R(a, b) = $[a_1, b_1]_Z \times [a_2, b_2]_Z$ of two intervals.

There are only finitely many binary images. But the number of binary images must be compared with other finite numbers. Thus, although the number of binary images on a computer screen is finite, it is so huge that the conclusion must be the same as in the case of the infinite cardinal: there are too many; it is not possible to search through the whole set; for simplifying this leads, again, to image processing and mathematical morphology, with subsets of Z^2, or, generally, of Z^n, the set of all n-tuples of integers. When consider mathematical morphology both the cases are important. i.e., both the vector space R^n of all n-tuples of real numbers (the addresses of points in space) and the digital space Z^n (the addresses of pixels). R^n and Z^n form an abelian group. Therefore the space, called image carrier, is just an abelian group.

Serra (1982) lists "four principles of quantification." These are about the ways to gather information about the external world. They apply also, but not exclusively, to image analysis.

Serra's first principle is "compatibility under translation." For a mapping, this means that f(A + b) = f(A)+b, which is expressed as $f \circ T_b = T_b \circ f$, where o denotes composition of mappings defined by $(f \circ T_b)(x) = f(T_b(x))$, thus a kind of commutativity, writing Tb for the translation $T_b(A) = A+b$. It means that f commutes with translations. On a finite screen like $\{x \in Z^2; 0 < x1 < 1,024, 0 < x2 < 768\}$ almost nothing can commute with translations. Therefore consider the ideal, infinite, computer screen with sets of addresses equal to Z^2. The principle is equally useful in R^n and Z^n.

Serra's second principle is "compatibility under change of scale." For a mapping this means that it commutes with homotheties (or dilatations).

The third principle is that of "local knowledge." This principle says that in order to know some bounded part of f(A), there is no need to know all of A, only some bounded part of A. Mathematically speaking: for every bounded set Y, there exists a bounded set Z such that $f(A \cap Z) \cap Y = f(A) \cap Y$.

Serra's fourth principle of quantification is that of "semi continuity." It means that if a decreasing sequence (Aj) of closed sets tends to a limit A, thus $A = \cap Aj$, then f(Aj) tends to f(A). Thus if Aj is close to A in some sense and Aj contains A, then f(Aj) must be close to f(A). To express this property as semi continuity, one must define a topology. In this thesis an attempt is made to derive some meaningful results by introducing some topological properties to the theory of morphological operators.

Over the last 10-15 years, the tools of mathematical morphology have become part of the mainstream of image analysis and image processing technologies. The growth of popularity is due to the development of powerful techniques, like granulometries and the pattern spectrum analysis, that provide insights into shapes, and tools like the watershed or connected operators that segment an image. But part of the acceptance in industrial applications is also due to the discovery of fast algorithms that make mathematical morphology competitive with linear operations in terms of computational speed. A breakthrough in the use of mathematical morphology was reached, in 1995, when morphological operators were adopted for the production of segmentation maps in MPEG-4.

J .Serra and George Matheron worked on image analysis .Their work lead to the development of the theory of Mathematical Morphology. Later Petros Maragos contributed to enrich the theory by introducing theory of lattices .Firstly the theory is purely based on set theory and operators are defined for binary cases only .Later ,the theory extended to Gray scale images also .He also gave a representation theory for image processing. Heink J.Heijmans gave an algebraic basis for the theory. Heink J.Heijmans extended the theory to Signal processing also. He also defined the operators for convex structuring elements. Rein Van Den Boomgaard introduced Morphological Scale space operators. In this thesis, an attempt to link some topological concepts to operators is made.

Morphological scale space operators can be linked with Fractals. A general Morphological algebraic structure is also introduced in this thesis. An attempt to characterize morphological convex geometries, using the definition of Moore family is made in this thesis.

The Moore family stands for the family of closed objects. There exist inter-relationships between Moore family, adjunctions and Morphological transforms. Adjunctions are pairs of operators which satisfy, some mathematical property. In mathematical Morphology Dilation and erosion are fundamental operators. These operators form an adjunction between two spaces. These operators are dual operators. All morphological adjunctions can be defined using a general rule .

1.2 Birth of Mathematical Morphology

Mathematical morphology (MM) originates from the study of the geometry of binary porous media such as sandstones. It can be considered as binary in the sense it is made up of two phases: the pores embedded in a matrix. This led Matheron and Serra to introduce in 1967 a set formalism for analyzing binary images.

Mathematical morphology is a non-linear theory of image processing. Its geometry- oriented nature provides an efficient method for analyzing object shape characteristics such as size and connectivity, which are not easily accessed by linear approaches.

Mathematical Morphology (MM) is associated with the names of Georges Matheron and Jean Serra, who developed its main concepts and tools. (Matheron, 1975; Serra, 1982; Serra, 1988), They created a team at the Paris School of Mines. Mathematical Morphology is heavily mathematized. In this respect, it contrasts with different experimental approaches to image processing.

MM stands also as an alternative to another strongly mathematized branch of image processing, the one that bases itself on signal processing and information theory. Main contributors in this area are Wiener, Shannon, Gabor, etc. These classical approaches has a lot of applications in telecommunications. Analysis of the information of an image is not similar to transmitting a signal on a channel. An image should not be considered as a combination of sinusoidal frequencies, nor as the result of a Markov process on individual points .The purpose of image analysis is to find spatial objects. Hence images consist of geometrical shapes with luminance (or colour) profiles. This can be analyzed by their interactions with other shapes and luminance profiles. In this sense the morphological approach is more relevant.

MM has taken concepts and tools from different branches of mathematics like algebra (lattice theory), topology, discrete geometry, integral geometry, geometrical probability, partial differential equations, etc.

1.3 Image Processing using Mathematical Morphology

Mathematical morphology is theoretically based on set theory. It contributes a wide range of operators to image processing, based on a few simple mathematical concepts. MM started by considering binary images and usually referred to as standard mathematical morphology. It also used set-theoretical operations like the relation of inclusion and the operations of union and intersection.

In order to apply it to other types of images, for example grey-level ones (numerical functions), it was necessary to generalize set-theoretical notions. Using the lattice-theory it is generalized. The notions are, the partial order relation between images, for which the operations of supremum (least upper bound) and infimum (greatest lower bound) are defined. Therefore the main structure in MM is that of a complete lattice. All the basic morphological operators are defined by using this framework.. Nowadays, most morphological techniques combine lattice-theoretical and topological methods.

The computer processing of pictures led to digital models of geometry. Azriel Rosenfeld has contributed in this field after having contributed to digital geometry and image processing for 40 years. Mathematical morphology is perfectly adapted to the digital framework.

The operators are particularly useful for the analysis of binary images , boundary detection ,noise removal, image enhancement, shape extraction, skeleton transforms and image segmentation. The advantages of morphological approaches over linear approaches are

1)Direct geometric interpretation,2) Simplicity and 3) Efficiency in hardware implementation.

An image can be represented by a set of pixels. A morphological operation uses two sets of pixels, i.e., two images: the original data image to be analyzed and a structuring element which is a set of pixels constituting a specific shape such as a line, a disk, or a square. A structuring element is characterized by a well- defined shape (such as line, segment, or ball), size, and origin. Its shape can be regarded as a parameter to a morphologicaloperation

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2.1 Introduction

Mathematical Morphology is a tool for extracting image components that are useful for representation and description. It provides a quantitative description of geometrical structures. Morphology is useful to provide boundaries of objects, their skeletons, and their convex hulls. It is also useful for many pre- and post-processing techniques, especially in edge thinning and pruning.

Most morphological operations are based on simple expanding and shrinking operations. Morphological operations preserve the main geometric structures of the object. Only features 'smaller than' the structuring element are affected by transformations.

All other features at 'larger scales' are not degraded. (This is not the case with linear transformations, such as convolution).

The primary application of morphology occurs in binary images, though it is also used on grey level images. It can also be useful on range images. (A range image is one where grey levels represent the distance from the sensor to the objects in the scene rather than the intensity of light reflected from them).

2.2 Preliminaries

2.2.1 Notation and Image Definitions

Types of Images

An image is a mapping denoted as I, from a set, N_P, of pixel coordinates to a set, M, of values such that for every coordinate vector, $\mathbf{p} = (\mathbf{p_1}, \mathbf{p_2})$ in N_P, there is a value I(**p**) drawn from M. N_P is also called the image plane.[1]

Under the above defined mapping a real image maps an n-dimensional Euclidean vector space into the real numbers. Pixel coordinates and pixel values are real.

A discrete image maps an n-dimensional grid of points into the set of real numbers. Coordinates are n-tuples of integers, pixel values are real.

A digital image maps an n-dimensional grid into a finite set of integers. Pixel coordinates and pixel values are integers.

A binary image has only 2 values. That is, M= $\{m_{fg}, m_{bg}\}$, where m_{fg}, is called the foreground value and m_{bg} is called the background value.

The foreground value is $m_{fg} = 0$, and the background is $m_{bg} = -\infty$. Other possibilities are $\{m_{fg}, m_{bg}\} = \{0,\infty\}, \{0,1\}, \{1,0\}, \{0,255\}$, and $\{255,0\}$.

2.2.2 Definition

The foreground of binary image I is

$$FG\{I\} = \{I(\mathbf{p}), p = (p_1, p_2) \epsilon N_p / I(p) = m_{fg}.$$

The background is the complement of the foreground and vice-versa.

2.2.3 Definition

The support of a binary image, I, is

$$\mathrm{Supp(I)} = \{ \mathbf{p} = (p_1, p_2) \epsilon N_p / I(\mathbf{p}) = m_{fg}.$$

That is, the support of a binary image is the set of foreground pixel locations within the image plane.

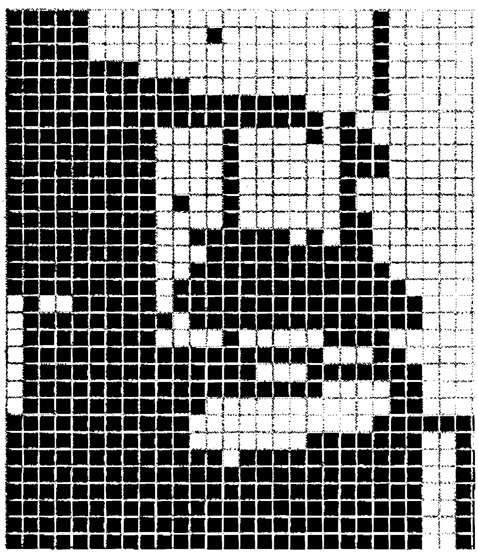
The complement of the support is, therefore, the set of background pixel locations within the image plane.

$$\{\mathrm{Supp(I)}\}^c = \{ \mathbf{p} = (p_1, p_2) \epsilon N_p / I(\mathbf{p}) = m_{fg}.$$

All the images actually process by computer will be digital. That is, they will be defined on a R row by C column grid of pixels. Typically, all the FG pixels will be black and the BG pixels, white, or vice-versa.

All pixels are represented by squares. All FG and BG objects or regions are made up of these squares.

A Binary Image



In the memory of the computer, all FG pixels are represented by a number, f, and all BG pixels by another number, b. Typically (f, b) = (1,0) or (f, b) = (255,0), or the opposite.

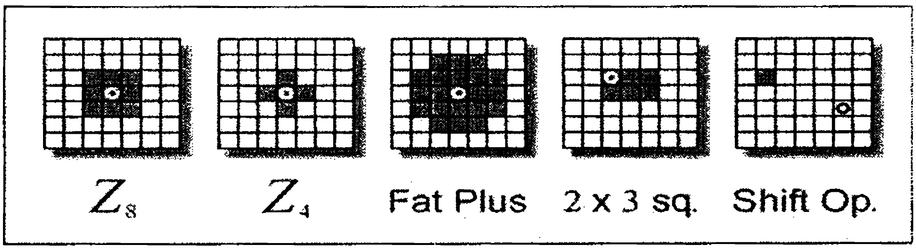
2.3 Structuring Element

2.3.1 Definition

The processing consist of the interaction between an image A (the object of interest) and a structuring set B, called the *structuring element*. It can be of any shape, size.

2.3.2 Example

In the figure, the location of the structuring element's origin is marked as circles which can be placed anywhere relative to its support.



The image and structuring element sets need not be restricted to sets in the 2D plane, but could be defined in 1, 2, 3 (or higher) dimensions.

Note: The structuring element is to mathematical morphology what the convolution kernel is to linear filter theory

2.4 Binary Operations

2.4.1 Definition

Given two sets ***A*** and ***B*** , the Minkowski addition is defined as

$$A \oplus B = \bigcup_{\beta \in B} (A + \beta)$$

2.4.2 Definition

Minkowski subtraction is defined as

$$A \ominus B = \bigcap_{\beta \in B} (A + \beta)$$

.

2.4.3 Definition

Let I be an image and *Z* a Structuring Element. Z + **p** means that *Z* is moved so that its origin coincides with location **p** in N_P.Z + **p** is the *translate* of *Z* to location **p** in N_P.

2.4.5 Definition

The set of locations in the image denoted by Z+ **p** is called the *Z-neighborhood* of **p** in I denoted N {I, *Z*}(**p**).

The complement of *A* is denoted A^c, and the difference of two sets *A* and *B* is denoted by *A* - *B*.

2.4.6 Definition

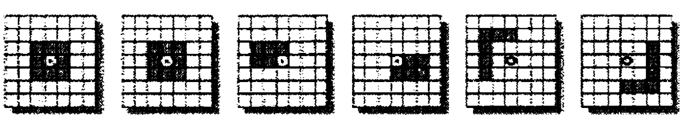
Let A and B be subsets of Z^2. The *translation* of A by x is denoted A_x and is defined as $A_x = \{c : c = a + x, \text{for } a \in A\}$.

2.4.7 Definition

Let B be a Structuring Element and let S be the square of pixel locations which contains the set$\{(p_1, p_2), (-p_1, -p_2) / (p_1, p_2) \in \text{Supp}(B)\}$, then reflection of B is $\check{B}(\beta,\delta) = B(-\beta,-\delta) \forall (\beta,\delta) \in S$.[1],[2],[3].Or, in other words, the *reflection* of B is defined as

$$\check{B} = \{x : x = -b, \text{for } b \in B\}.$$

2.4.8 Examples



Z_1 $\check{Z}_1$ Z_2 $\check{Z}_2$ Z_3 $\check{Z}_3$

2.5 Binary Morphology

2.5.1 Binary representation

An image is defined as an (amplitude) function of two, real (coordinate) variables (x, y) or two, discrete variables $[m, n]$.

An image consists of a set (or collection) of either continuous or discrete coordinates. The set corresponds to the points or pixels that belong to the objects in the image.

2.5.2 Example

This is illustrated in Figure which contains two objects or sets A and B. In the coordinate system, consider the pixel values to be binary.[1]

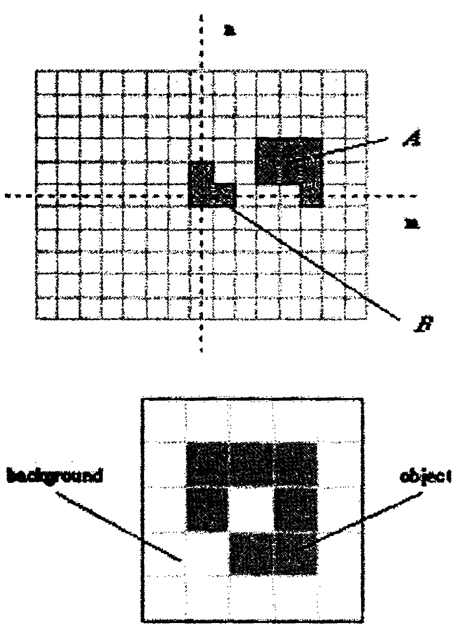


Figure: A binary image containing two object sets A and B.

The object A consists of pixels which share some common property:

Object B in Figure consists of {[0, 0], [1,0], [0,1]}.

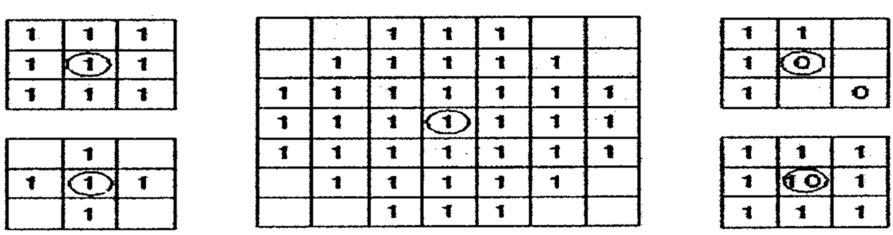
2.5.3 Description of Image regions

The basis of mathematical morphology is the description of image regions as sets [1]. For a binary image, consider the "on" (1) pixels to all comprise a set of values from the "universe" of pixels in the image. An image A, we mean the set of "on" (1) pixels in that image.

The "off" (0) pixels are thus the set compliment of the set of on pixels. By Ac, we mean the compliment of A, or the off (0) pixels.

The background of A is given by A^c (the *complement* of A) which is defined as those elements that are not in A:

2.5.4 Example



Examples of structuring elements

2.6 Dilation and Erosion

Morphology uses 'Set Theory' as the foundation for many functions [1].

The simplest functions to implement are 'Dilation' and 'Erosion'

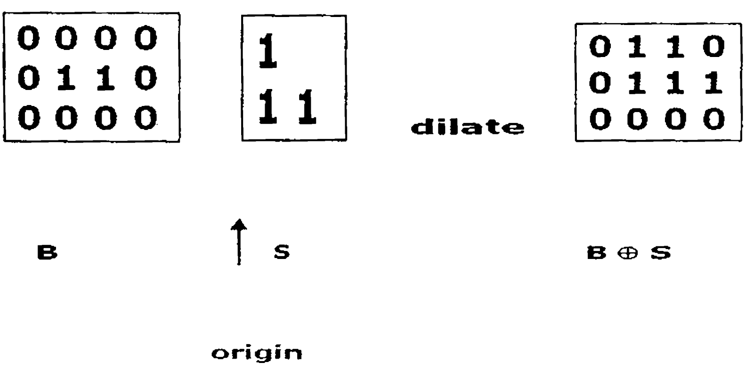
2.6.1 Definition Dilation of the object A by the structuring element B is

given by $A \oplus B = \{x : \hat{B}_x \cap A \neq \emptyset\}$.

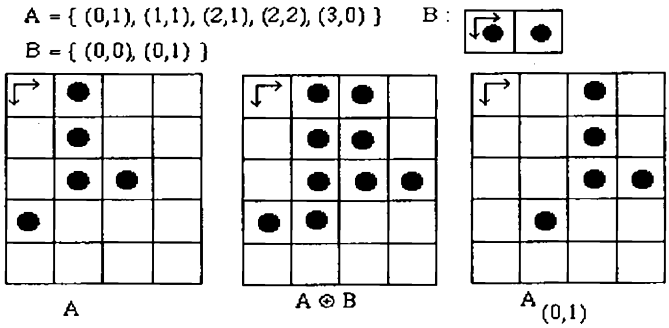
Usually A will be the signal or image being operated on A and B will be

the Structuring Element'

2.6.2 Example 1 Dilate (B,S) takes binary image B, places the origin of structuring element S over each 1-pixel, and Ors the structuring element S into the output image at the corresponding position.



Example 2



2.6.3 Basic effect of Dilation

Gradually enlarge the boundaries of regions of foreground pixels on a binary image.

Dilation, in general, causes objects to grow in size

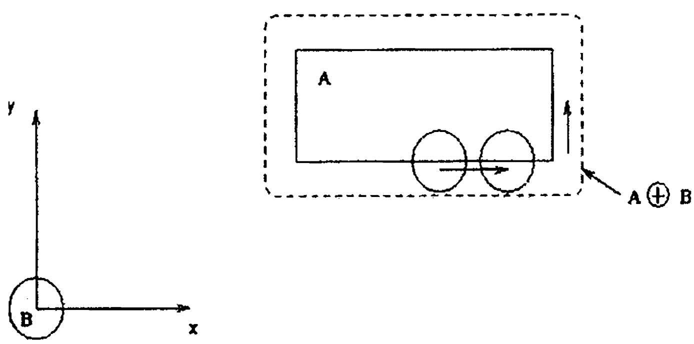


Figure: *A* is dilated by the structuring element *B*.

2.6.4 Example Binary Signal

Figure shows how dilation works on a 1D binary signal.[1] The structuring element shown in Figure uses the value of the elements immediately to the right and left of the current element (the structuring element in this case looks for ones on the input sequence)[3]. Any shape or size-structuring element can be used, where an element with the value of 0 indicates that the corresponding element in set A is not to be used, and a value of 1 indicates that it is to be used.

For example, the structuring element shown could be considered to have 0's on the extreme left and right, as the corresponding inputs would be ignored.

In the figure, Structuring element (B) with shaded showing the origin. A is an input signal and C ,an output signal. Set the output to be the intersection .Slide the structuring element along A. Get the intersection for the new position. Repeat this until all elements have been done.

1	1	1	0	1	0	0	0	1	1	0	1

Input signal (A)

1	1	1

Structuring element (B)

1	1	1	1	0	1	1	1	1	1

Output signal (C)

Figure: Example of how dilation works

The output is given by (1) and will be set to one unless the input is the inverse of the structuring element. For example, '000' would cause the output to be zero. The output is placed at the origin of the structuring element as shown.

From Figure, it can be seen that dilation operation completely removes any runs of zeros less than the length of the structuring element (this is only for this type of structuring element though). Longer runs of zeros are shortened at their extremities.

2.6.5 Example

Let $X = \{(1,0),(1,1,),(1,2),(2,2),(0,3),(0,4)\} \, and \; B = \{(0,0),(1,0)\}$

Then

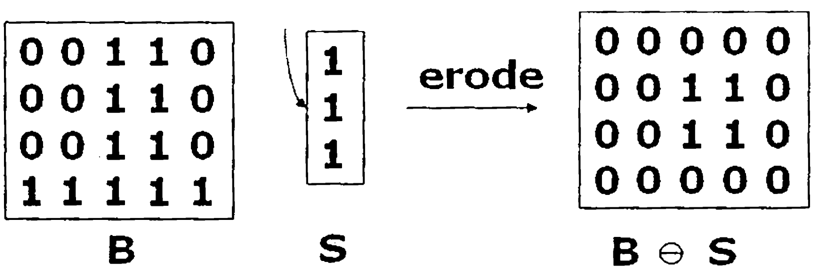
$X \oplus B = \{(1,0),(1,1),(1,2),(2,2),(0,3),(0,4),(2,0),(2,1),(2,2),(3,2),(1,3),(1,4)\}$

2.6.6 Definition Erosion

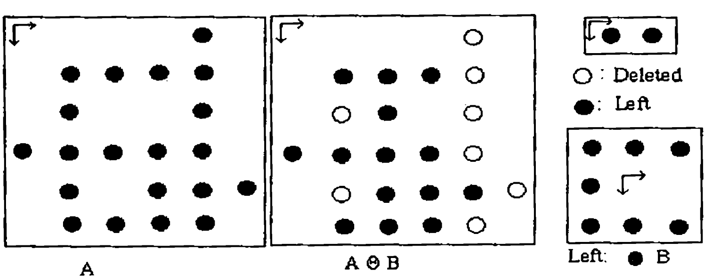
The opposite of dilation is known as erosion. Erosion of the object A by a structuring element B is given by $A \ominus B = \{x : B_x \subseteq A\}$.

Erosion of A by B is the set of points x such that B translated by x is contained in A.

2.6.7 Example 1 Erode(B,S) takes a binary image B, places the origin of structuring element S over every pixel position, and ORs a binary 1 into that position of the output image only if every position of S (with a 1) covers a 1 in B[1].

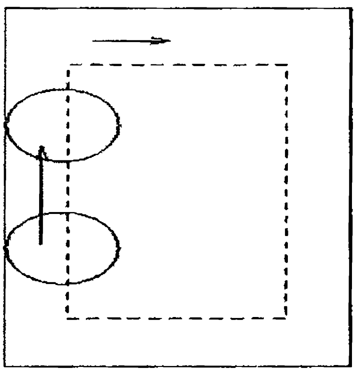


Example 2



2.6.8 Effect of Erosion *Erosion* causes objects to shrink. The amount and the way that they grow or shrink depend upon the choice of the structuring element.

Basic effect: Erode away the boundaries of regions of foreground pixels (*i.e.* white pixels, typically).



2.6.9 Example Binary Signal

Figure shows how erosion works on a 1D binary signal [1],[2]. This works in exactly the same way as dilation. For the output to be a one, all of the inputs must be the same as the structuring element. Thus, erosion will remove runs of ones that are shorter than the structuring element.

Structuring element (B) with shaded showing the origin.

Set the output to be the translation of B contained in A.

Slide the structuring Element along. Get the intersection for the new position. Repeat this until all elements have been done.

Figure: Example of how erosion works

1	1	1	0	1	0	0	0	1	1	0	1	Input signal (A)

1	1	1	Structuring element (B)

1	0	0	0	0	0	0	0	0	0	Out put signal C

2.7 Opening and Closing

Two very important transformations are *opening* and *closing*. Dilation expands an image object and erosion shrinks it. Opening, generally smoothes a contour in an image, breaking narrow isthmuses and eliminating thin protrusions. Closing tends to narrow smooth sections of contours, fusing narrow breaks and long thin gulfs, eliminating small holes, and filling gaps in contours.

2.7.1 Definition Opening

The opening of A by B, denoted by $A \circ B$, is given by the erosion by B, followed by the dilation by B, that is $A \circ B = (A \ominus B) \oplus B$.

2.7.2 Binary Opening Example

This simply erodes the signal and then dilates the result as shown in Figure. As can be seen, the zeros are opened up. Any ones that are shorter than the structuring element are removed, but the rest of the signal is left unchanged.

1	1	1	0	1	0	0	0	1	1	0	1	Input signal (A)

1	1	1	Structuring element

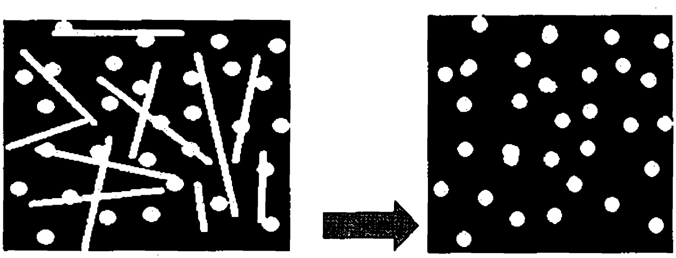
1	1	0	0	0	0	0	0	0	0	0	0	Output of erosion ($A \ominus B$)

1	1	1	0	0	0	0	0	0	0	0	0	Output of dilation ($(A \ominus B) \oplus B$)

Figure: Example of how an opening works

2.7.3 Example:

Opening Separate out the circles from the lines

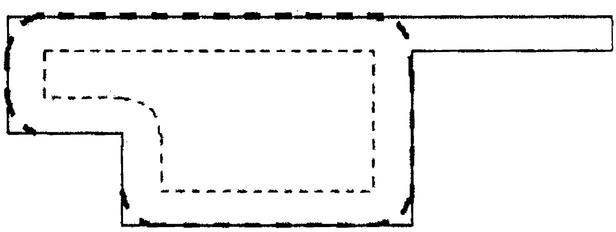


A mixture of circle and lines.

The lines have been almost completely removed while the circles remain almost completely unaffected.

Opening is the compound operation of erosion followed by dilation (with the same structuring element)

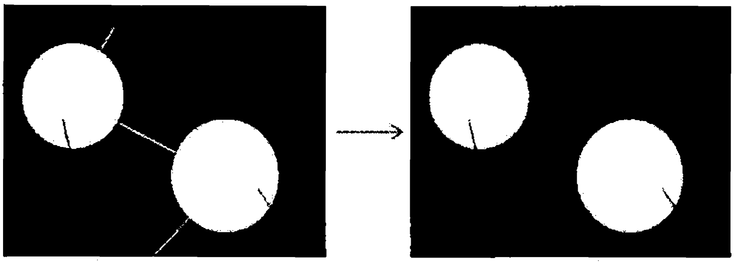
2.7.4 Example Figure The opening (given by the dark dashed lines) of *A* (given by the solid lines. The structuring element *B* is a disc. The internal dashed structure is *A* eroded by *B*. [3]



Opening is like 'rounding from the inside': the opening of *A* by *B* is obtained by taking the union of all translates of *B* that fit inside *A*. Parts of *A* that are smaller than *B* are removed. Thus $A \circ B = \bigcup \{B_x : B_x \subseteq A\}$.

Opening an image is achieved by first eroding an image and then dilating it. Opening removes any narrow "connections" between two regions.

Example:



Morphological Opening

2.7.5 The Basic Effect

Somewhat like erosion in that it tends to remove some of the foreground (bright) pixels from the edges of regions of foreground pixels. To preserve *foreground* regions that has a similar shape to structuring element.

2.7.6 Closing

The opposite of opening is 'Closing' defined by

$$A \bullet B = (A \oplus B) \ominus B.$$

Closing is the dual operation of opening and is denoted by $A \bullet B$. It is produced by the dilation of *A* by *B*, followed by the erosion by *B*:

2.7.7 Example

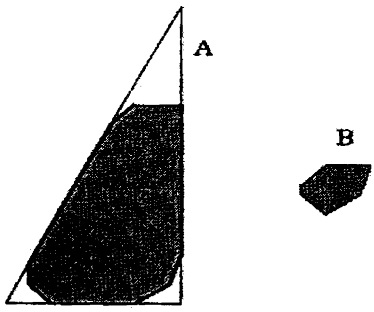
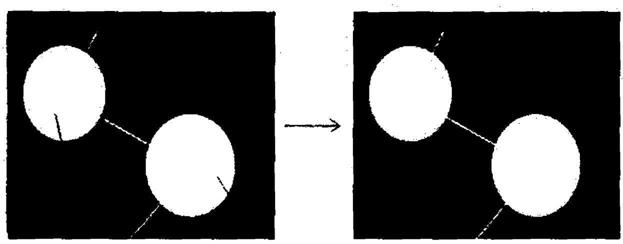


Figure: The closing of *A* by the structuring element *B*.

Closing an image is done by first dilating the image and then eroding it. The order is the reverse of opening. Closing fills up any narrow black regions in the image.

Example:



Morphological Closing

Closing is the compound operation of dilation followed by erosion (with the same structuring element)

Closing is one of the two important operators from mathematical morphology.

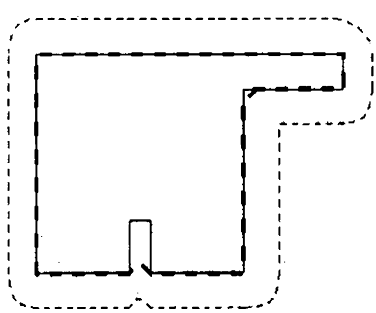
Closing is similar in some ways to dilation in that it tends to enlarge the boundaries of foreground (bright) regions in an image [1].

2.7.8 Example Binary closing

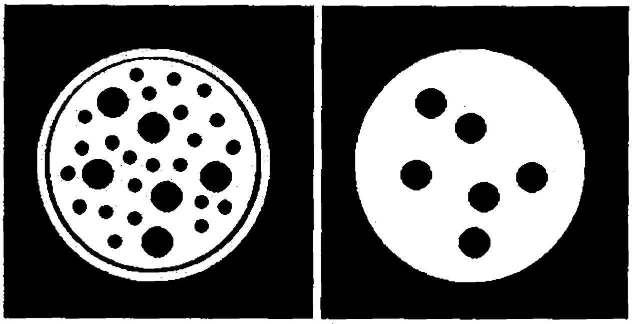
1	1	1	0	1	0	0	0	1	1	0	1	Input signal (A)
1	1	1										Structuring element
1	1	1	1	1	1	0	1	1	1	1	1	Output of dilation $(A \oplus B)$
1	1	1	1	1	0	0	0	1	1	1	1	Output of erosion $(A \oplus B)\Theta B$

Figure : Example of how a closing works

2.7.9 Example: Figure shows how this works. It can be seen that this closes gaps in the signal in the same way as opening opened up gaps.



2.7.10 Example :



Original image Result of a closing with a 22 pixel diameter disk.

If it is desired to remove the small holes while retaining the large holes, then we can simply perform a closing with a disk-shaped structuring element with a diameter larger than the smaller holes, but smaller than the larger holes.

Just as with dilation and erosion, opening and closing are dual operations. [12] That is $(A \bullet B)^c = (A^c \circ B^c)$.

$(A \circ B) \circ B = A \circ B$ and $A \bullet B = (A \oplus B) \ominus B$

The *opening* operation can separate objects that are connected in a binary image. The *closing* operation can fill in small holes. Both operations generate a certain amount of smoothing on an object contour given a "smooth" structuring element. The *opening* smoothes from the inside of the object contour and the *closing* smoothes from the outside of the object contour. [1],[12]

2.8 Properties of Operators

2.8.1 Dilation

Dilation has several interesting properties [1],[3], which make it useful for image processing.

a) Translation invariant:

This means that the result of A dilated with B translated is the same as A translated dilated with B as given by: $(A \oplus B)_x = A_x \oplus B$

b) Order invariant:

This simply means that if several dilations are to be done, then the order in which they are done is irrelevant. The result will be same irrespective.

$(A \oplus B) \oplus C = A \oplus (B \oplus C)$

c) Increasing operator

This means that if a set, A, is a subset of another set, B, then the dilation of A by C is still a subset of B dilated by C:

$(A \subseteq B) \Rightarrow (A \oplus C) \subseteq (B \oplus C)$

d) Scale invariant

This means that the input and structuring element can be scaled, then dilated and will give the same as scaling the dilated output:

$rA \oplus rB = r(A \oplus B)$ where r is a scale factor.

e) **Commutative -** $A \oplus B = B \oplus A$

f) **Associative -** $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

g) **Translation Invariance -** $A \oplus (B \oplus x) = (A \oplus B) \oplus x$

2.8.2 Erosion

Erosion, like dilation also contains properties that are useful for image processing:[2,3,4,5]

a) **Translation invariant**

This means that the result of A eroded with B translated is the same as A translated eroded with B as given by: $(A \ominus B)_x = A_x \ominus B$

b) **Order invariant**

This simply means that if several erosion are to be done, then the order in which they are done is irrelevant. The result will be same irrespective.

$(A \ominus B) \ominus C = A \ominus (B \ominus C)$

c) **Increasing operator**

This means that if a set, A, is a subset of another set, B, then the erosion of A by C is still a subset of B eroded by C:

$(A \ominus B) \ominus C = A \ominus C$ contained in $B \ominus C$

d) Scale invariant

This means that the input and structuring element can be scaled, then eroded and will give the same as scaling the dilated output:

$rA \ominus rB = r(A \ominus B)$ where r is a scale factor.

Dilation and erosion are duals of each other with respect to set complementation and reflection. That is, $(A \ominus B)^c = A^c \oplus \hat{B}$.

e) **Non-Commutative** – $A \odot B \neq B \odot A$

2.8.3 The decomposition theorems

a) Dilation - $A \oplus (B \cup C) = (A \oplus B) \cup C = (B \cup C) \oplus A$

b) Erosion - $A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C)$

c) Erosion - $(A \ominus B) \ominus C = A \ominus (B \oplus C)$

d) Multiple Dilations - $nB = \underbrace{(B \oplus B \oplus B \oplus \cdots \oplus B)}_{n \text{ times}}$

2.8.4 The opening and closing operation satisfies the following properties

a) $A \circ B$ is a subset of A. [12]

b) If C is a subset of D, then $C \circ B$ is a subset of $D \circ B$.

c) **Idempotency.** $(A \circ B) \circ B = A \circ B$

d) Similarly, A is a subset of $A \bullet B$.

e) If C is a subset of D, then $C \bullet B$ is a subset of $D \bullet B$.

f) **Idempotency.** $(A \bullet B) \bullet B = A \bullet B$

It means that any application of the operation more than once will have no further effect on the result.

g) Opening is anti extensive, i.e., $A \circ B \subseteq A$, whereas the closing is extensive, i.e., $A \subseteq A \bullet B$.

h) Opening and closing satisfy the duality $A \bullet B = (A^c \circ B^s)^c$.

i) **Duality Relationships**

1) Erosion in terms of dilation: $I - Z = [I^C \oplus \breve{Z}]^C$

2) Dilation in terms of erosion $I \oplus Z = [I^C - \breve{Z}]^C$

3) Opening in terms of closing $I \circ Z = [I^C \bullet Z]^C$

4) Closing in terms of opening $I \bullet Z = [I^C \circ Z]^C$

2.9 References

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Chapter 3

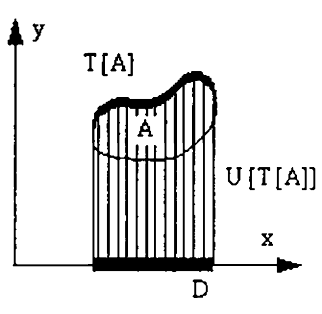
Gray – value Morphology and other Morphological operators

CONTENTS

3.1 Gray-value Morphological processing

Grayscale morphology is a multidimensional generalization of the binary operations. Binary morphology is defined in terms of set-inclusion of pixel sets. So is the grayscale case, but the pixel sets are of higher dimension.

3.1.1 Set Inclusion in Grayscale Images



For morphological operations on gray level images sets like E^N is using. The first (N-1) coordinates conventionally form the spatial domain and the last coordinate is for the surface. For gray level images N=3, the first two coordinates of an element in a set are the (x, y) in the image and the third is the gray level.

Concepts such as top or top-surface of a set and the shadow (umbra) of a surface are used in the definitions of the operations.

Let A be $\subseteq E^N$. The domain of A is defined as:

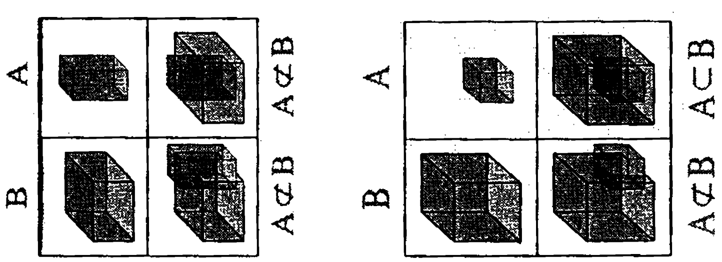
$$D = \{ \mathbf{x} \in E^{N-1} \mid \text{there is a } y \in E, (\mathbf{x}, y) \in A \}$$

The top or top-surface of A is a function T[A] : D -> E: is defined as

$$T[A](\mathbf{x}) = \text{Max} \{ y \mid (\mathbf{x}, y) \in A \}$$

In grayscale morphology, set inclusion depends on the implicit 3D structure of a 2D image.

3.1.2 Example



3.1.3 Grayscale Structuring Elements

A grayscale structuring element is a small image that delineates a volume at each pixel [p, I(p)] throughout the image volume

3.1.4 Definition: Gray-level dilation, $D_G(*)$, is given by: [2]

Dilation -

$$D_G(A,B) = max_{[j,k]\in B}\{a[m-j,n-k]+b[j,k]\}$$

3.1.5 Definition: Gray-level erosion, $E_G(*)$, is given by:

Erosion -

$$E_G(A,B) = min_{[j,k]\in B}\{a[m+j,n+k]-b[j,k]\}$$

3.2 Other Operators

The definitions of higher order operations such as gray-level opening and gray-level closing are given below.

3.2.1 Definition Opening

$$O_G(A,B) = D_G(E_G(A,B),B)$$

3.2.2 Definition Closing

$$C_G(A,B) = -O_G(-A,-B)$$

The important properties such as idempotence, translation invariance, increasing in A, are also applicable to gray level morphological processing.

Complexity of gray level morphological processing is significantly reduced through the use of symmetric structuring elements. It is denoted by $b[j, k] = b[-j,-k]$.

The most common of these is based on the use of B = *constant* = 0. For this important case and using again the domain $[j, k] \subset B$, the definitions above reduce to:

3.2.3 Dilation using symmetric structuring elements:

Dilation -

$$D_G(A,B) = max_{[j,k]\in B}\{a[m-j,n-k]\} = max_B(A)$$ [2],[3]

3.2.4 Effects of Grayscale Dilation

Generally brighten the image

Bright regions surrounded by dark regions grow in size, and dark regions surrounded by bright regions shrink in size.

3.2.5 Erosion using symmetric structuring elements:

Erosion -

$$E_G(A,B) = min_{[j,k]\in B}\{a[m-j,n-k]\} = min_B(A)$$

3.2.6 Effects of Grayscale Erosion

Generally darken the image

Bright regions surrounded by dark regions shrink in size, and dark regions surrounded by bright regions grow in size.

3.2.7 Definition Opening

$O_G(A,B) = max_B(min_B(A))$

3.2.8 Definition Closing

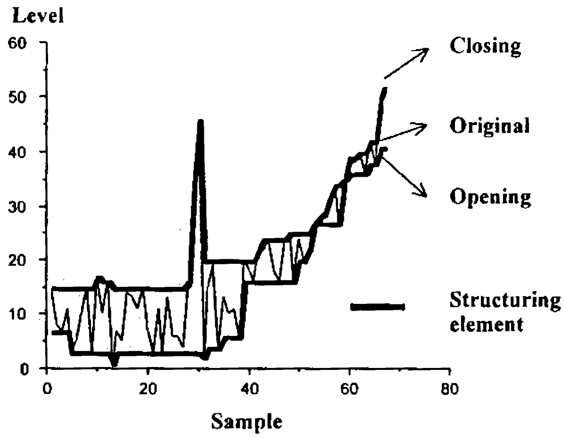
$C_G(A,B) = min_B(max_B(A))$

3.2.9 Example

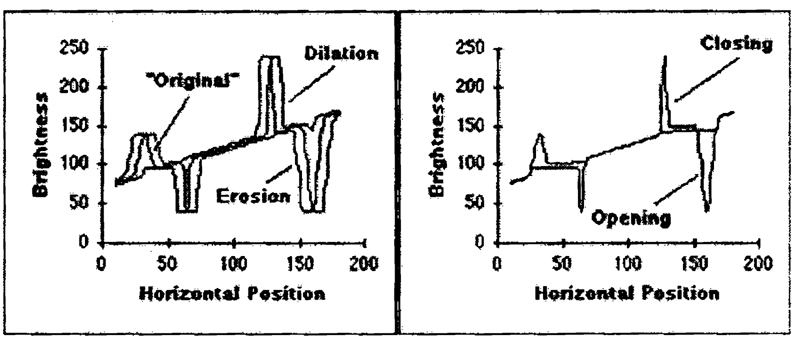
The adjunction opening [2] and closing create a ***simpler function*** than the original. They smooth in a nonlinear way.

The ***opening*** (closing) removes ***positive*** (negative) ***peaks*** that are thinner than the structuring element.

The opening (closing) remains below (above) the original function.



The remarkable conclusion is that the *maximum filter* and the *minimum filter*, are gray-level dilation and gray-level erosion for the specific structuring element given by the shape of the filter window with the gray value "0" *inside* the window. Examples of these operations on a simple one-dimensional signal are shown in Figure.



Effect of 15 x 1 *dilation* and *erosion* **b)** Effect of 15 x 1 *opening* and *closing*

3.2.10 Definition Morphological gradient

The morphological gradient [2],[3] is the difference between the dilation and the erosion of the image.

This gradient is used to find boundaries or edges in an image

$$Gradient(A,B) = \frac{1}{2}\left((D_G(A,B)) - (E_G\ (A,B))\right)$$

$$= \frac{1}{2}\left((\max(A) - \min(A))\right)$$

3.2.11 Definition Morphological Laplacian

The morphologically-based Laplacian filter [2] ,[3] is defined by:

$$Laplacian\ (A,B) = \frac{1}{2}\left((D_G(A,B) - A) - (A - E_G\ (A,B))\right)$$

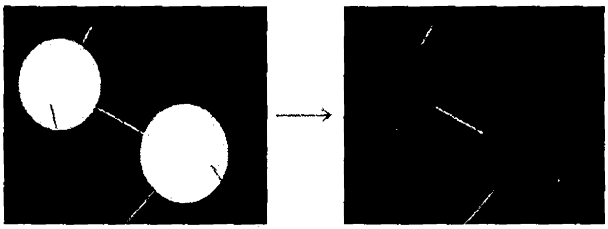
$$= \frac{1}{2}\left((D_G(A,B)) + (E_G\ (A,B)) - 2A\right)$$

$$= \frac{1}{2}\left((\max(A) + \min(A) - 2A)\right)$$

3.2.12 Definition Top hat

The top hat is the difference of the source image and the opening of the source image. It highlights the narrow pathways between different regions.

Example

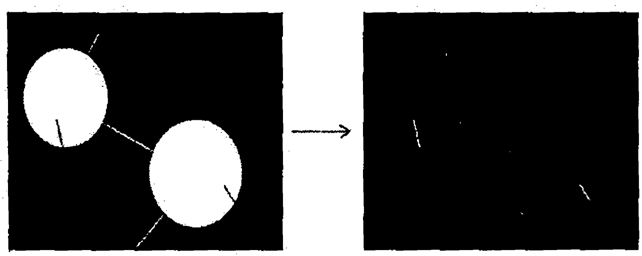


Morphological Top hat

3.2.13Definition Black hat

The black hat is the difference between the closing of an image and the image itself. This highlights the narrow black regions in the image.

Example



Morphological Black hat

3.3 Applications of morphological operations

Erosion and dilation can be used in a variety of ways, in parallel and series, to give other transformations including thickening, thinning, skeletonization and many others.

The morphological filter $(A \circ B) \bullet B$ can be used to eliminate `salt and pepper' noise. Salt and pepper noise is random, uniformly distributed small noisy elements often found corrupting real images. It will appear as black dots or small blobs on a white background, and white dots or small blobs on the black object. The background noise is eliminated at the erosion stage, under the assumption that all noise components are physically smaller than the structuring element *B*. Erosion on its own will increase the size of the noise components on the object. However, these are eliminated at the closing operation.

3.3.1 The *boundary* of a set *A*, denoted ∂A, can be obtained by first eroding *A* with *B*, where *B* is a suitable structuring element, and then performing the set difference between *A* and its erosion. That is $\partial A = A - (A \ominus B)$. Typically, *B* would be a matrix of 1s.

3.3.2. Region filling algorithm

Region filling [1], [2], [3] can be accomplished iteratively using dilations, complementation, and intersections.

Step 1: Let A be an image containing a subset whose elements are 8-connected [1] boundary points of a region.

Step 2: Let p be a point inside the boundary.

Step 3: The objective is to fill the entire region with 1s. Assume that all non-boundary points are labeled 0, assign 1 to p,

Step 4: Construct $X_k = (X_{k-1} \oplus B) \cap A^c$, for $k = 1, 2, \ldots$

Where $X_0 = p$, and B is the `cross' structuring element shown in figure .

Step 5: When $X_k = X_{k-1}$, the algorithm terminates.

The set union of X_k and A contains the filled set and its boundary.

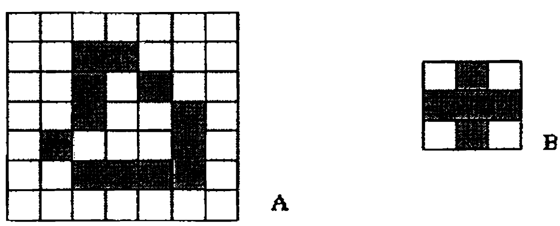


Figure : The region in A is filled using the structuring element B.

3.3.3 Algorithm for extracting connected components in an image

Connected components [1], [2]can also be extracted using morphological operations.

Step 1: Let Y represents a connected component in an image A and a point p in Y is known.

Step 2: Iterate-

$X_k = (X_{k-1} \oplus B) \cap A, \text{for } k = 1, 2, \ldots$ where $X_0 = p$ and B is a matrix of 1s.

Step 3 :

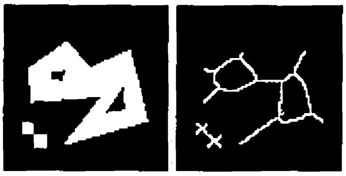
If $X_k = X_{k-1}$ the algorithm has converged and let $Y = X_k$.

3.3.4 Definition

Thinning and Thickening operators are defined as follows:

Thinnig: $X \bigcirc B = X \setminus (X \otimes B)$

Example:



Thickenning : $X \odot B = X \cup (X \otimes B)$

3.3.5 Definition

An important step in representing the structural shape of a planar region is to reduce it to a graph. This is very commonly used in robot path planning. This reduction is most commonly achieved by reducing the region to its *skeleton.*

The **skeleton** [1], [2] of a region is defined by the medial axis transformation MAT. The MAT of a region *R* with border *B* is defined as follows: for each point *p* in *R*, find its closest neighbour in *B*.

If *p* has more than one such closest neighbour, then *p* belongs to the medial axis (or skeleton) of *R*. Closest depends on the metric used. Figure shows some examples with the usual Euclidean metric.

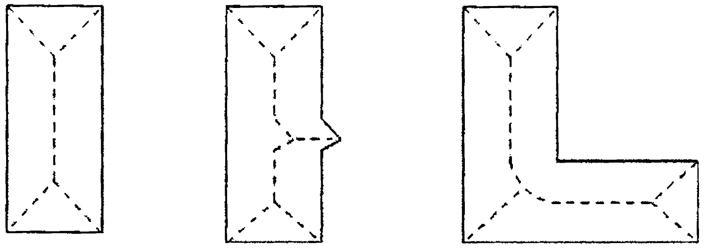


Figure: The skeletons of three simple regions

The skeleton of a set can be expressed in terms of erosions and openings.

Thus, it can be shown that $S(A) = \bigcup_{k=0}^{K} S_k(A)$,

where $S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]$,

3.3.6 Reconstruction using Skelton:

B is a structuring element, $(A \ominus kB)$ indicates *k* successive erosions of *A*, and *K* is the last iterative step before *A* erodes to an empty set.

Thus *A* can be reconstructed from its skeleton subsets $S_k(A)$ using the equation

$$A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB),$$

where $S_k(A) \oplus kB$ represents *k* successive dilations of $S_k(A)$.

3.4 Vincent's decomposition theorem

3.4.1 Definition

A *convex* set (in R^2) is one for which the straight line joining any two points in the set consists of points that are also in the set.

3.4.2 Definition

A set is *bounded* if each of its elements has a finite magnitude, in this case distance to the origin of the coordinate system.

3.4.3 Definition

A set is *symmetric* if $\boldsymbol{B} = -\boldsymbol{B}$. The sets N_4 and N_8 in Figure are examples of convex, bounded, symmetric sets.

3.4.5 Theorem

Vincent's theorem, [1]when applied to an image consisting of discrete pixels, states that for a bounded, symmetric structuring element $\boldsymbol{B}$ that contains no holes and contains its own center, : $[0,0] \in B$

$D(A,B) = A \oplus B = A \cup (\partial A \oplus B)$ where ∂A is the contour of the object. That is, ∂A is the set of pixels that have a background pixel as a neighbor.

3.4.6 Implication of the theorem

The implication of this theorem is that it is not necessary to process all the pixels in an object in order to compute a dilation or (using eq.) an erosion but have to process the boundary pixels. This also holds for all operations that can be derived from dilations and erosions.

3.5 Boolean Convolution

3.5.1 Definition

An arbitrary binary image object (or structuring element) *A* can be represented as: [1]

$$A \leftrightarrow \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} a[j,k].\delta[m-j,n-k]$$

where Σand * are the Boolean operations *OR* and *AND* , *a*[*j*,*k*] is a characteristic function that takes on the Boolean values "1" and "0" as follows:

$$a[j,k] = \begin{Bmatrix} 1\ for\ a \in A \\ 0\ for\ a \in A \end{Bmatrix}$$

and d[*m*, *n*] is a Boolean version of the Dirac delta function that takes on the Boolean values "1" and "0" as follows:

$$\delta[j,k] = \begin{Bmatrix} 1\ for\ j = k = 0 \\ 0\ , otherwise \end{Bmatrix}$$

3.5.2 Definition

Dilation for binary images can therefore be written as:

$$D(A,B) = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a[j,k] \bullet b[m-j,n-k] = a \otimes b$$

which, because Boolean OR and AND are commutative, can also be written as

$$D(A,B) = \sum_{k=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a[m-j,n-k] \bullet b[j,k] = b \otimes a = D(B,A)$$

3.5.3 Definition

Using De Morgan's theorem:

$$\overline{(a+b)} = \bar{a} \bullet \bar{b} \quad \text{and} \quad \overline{(a \bullet b)} = \bar{a} + \bar{b}$$

on eq. together with eq. , erosion can be written as:

$$E(A,B) = \prod_{k=-\infty}^{+\infty} \prod_{j=-\infty}^{+\infty} \left(a[m-j,n-k] + \bar{b}[-j,-k]\right)$$

Thus, dilation and erosion on binary images can be viewed as a form of convolution over a Boolean algebra.

When convolution is considered, choice of the boundary conditions for an image is essential. The two most common choices are that either everything outside the binary image is "0" or everything outside the binary image is "1".

3.6 References

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Chapter 4

Morphological operators defined on lattices

CONTENTS

4.1 Lattices

4.1.1 Definition Binary Relation

A binary relation from set A to set B is a subset R of $A \times B$. Thus for any pair (x,y) in $A \times B$, x is related to y by R, written $xR\,y$, if and only if $(x,y) \in R$. [9]

4.1.2 Definition

Let R be a binary relation on a set A.

- R is reflexive if for all $x \in A$, xRx.
- R is symmetric if for all $x,y \in A$, if xRy, then yRx.
- R is transitive if for all $x,y,z \in A$, if xRy and yRz, then xRz.
- R is antisymmetric if for all $x,y \in A$, if xRy and yRx, then $x=y$.

▶ R is a partial order relation if R is reflexive, antisymmetric and transitive.

4.1.3 Definition

Let R be a partial order relation on set A.

Two elements $a, b \in A$ are comparable if either aRb or bRa, i.e. either $a \preceq b$ or $b \preceq a$.

4.1.4 Definition

Let R be a partial order relation on set A. If all elements of A are comparable with each other, then the partially ordered set A (w.r.t. R) is said to be a totally ordered set.

4.1.5 Definition

Let R be a partial order relation on set A. An element $a \in A$ is a maximal element of A if $b \preceq a$ holds for every $b \in A$ whenever b and a are comparable.

4.1.6 Definition

Let R be a partial order relation on set A. An element $a \in A$ is a greatest element of A if $b \preceq a$ holds for all $b \in A$.

4.1.7 Definition

Let R be a partial order relation on set A. An element $a \in A$ is a minimal element of A if $a \preceq b$ holds for every $b \in A$ whenever b and a are comparable.

4.1.8 Definition

Let R be a partial order relation on set A. An element $a \in A$ is a least element of A if $a \preceq b$ holds for all $b \in A$.

4.1.9 Example

Let A be the set of all subsets of set $\{a, b, c\}$. The ``subset'' relation $\subseteq$ on A, i.e. $\forall u, v \in A$, $u \preceq v$ or uRv , iff $u \subseteq v$, is a partial order relation.

4.1.10 Definition

A *lattice* (L,≥) is an ordered set in which any two elements X1 and X2 have both a sup. (X1 ∨X2) and an inf (X1 ∧ X2).

The lattice is complete if any family of element Xi has both a supremum and infimum [9].

Complete lattices are partially ordered sets, where every subset has an infimum and a supremum. In particular, it contains a least element and a greatest element.

4.2 Properties of Lattices

4.2.1 Idempotency

$x \wedge x = x, x \vee x = x$

4.2.2 Commutativity

$x \wedge y = \inf(x, y) = \inf(y, x) = y \wedge x.$

$x \vee y = \sup(x, y) = \sup(y, x) = y \vee x.$

4.2.3 Associativity

$x \vee (y \wedge z) = (x \vee y) \wedge z, x \wedge (y \vee z) = (x \wedge y) \vee z$

4.2.4 Absorption

$x \vee (x \wedge y) = x, x \wedge (x \quad y) = x$

4.2.5 Definition

An algebraic lattice $(L, \vee, \wedge)$ is a non empty set L with two binary operations $\wedge$ (meet) and $\vee$ (join), which satisfy the following conditions for all x, y, z [9]

L1. $x \wedge y = y \wedge x, x \vee y = y \vee x$. (Commutative)

L2 .$x \vee (y \wedge z) = (x \vee y) \wedge z, x \wedge (y \vee z) = (x \wedge y) \vee z$ (Associative)

L3. $x \vee (x \wedge y) = x, x \wedge (x \vee y) = x$ (Absorption)

L4. (Idempotent) $x \wedge x = x, x \vee x = x$

4.2.6 Definition

Let (L, $\vee$, $\wedge$) be a lattice and let S be subset of L. The substructure (S, $\vee$, $\wedge$) is a sub lattice of (L, $\vee$, $\wedge$) if and only if S is closed under both operations $\vee$ and $\wedge$.

4.2.7 Definition

A lattice L is said to be modular if for all x, y, z , $x \leq z$ then

$x \vee (y \wedge z) = (x \vee y) \wedge z$.

4.2.8 Definition

A lattice (L, $\vee$, $\wedge$) is called a distributive lattice if for any a, b, c ε L,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

4.2.6 Definition Compliment

In a lattice L with universal bounds 0 & 1,an element $x \in L$ is said to have a compliment $x^c \in L$ if $x \vee x^c \in L$ and $x \wedge x^c = 0$

4.3 Morphological Operators defined on a Lattice

4.3.1 Definition Dilation

Let $(L, \leq)$ be a complete lattice, with infimum and minimum symbolized by $\wedge$and $\vee$, respectively.[1],[2].[11]

A dilation is any operator $\delta : L \rightarrow L$ that distributes over the supremum and preserves the least element. $\bigvee_i \delta(X_i) = \delta\left(\bigvee_i X_i\right)$, $\delta(\emptyset) = \emptyset$.

4.3.2 Definition Erosion

An erosion is any operator $\varepsilon : L \rightarrow L$ that distributes over the infimum

$\bigwedge_i \varepsilon(X_i) = \varepsilon\left(\bigwedge_i X_i\right)$, $\varepsilon(U) = U$.

4.3.3 Galois connections

Dilations and erosions form Galois connections. That is, for all dilation δ there is one and only one erosion ε that satisfies

$X \leq \varepsilon(Y) \Leftrightarrow \delta(X) \leq Y$ for all $X, Y \in L$.

Similarly, for all erosion there is one and only one dilation satisfying the above connection.

Furthermore, if two operators satisfy the connection, then δ must be a dilation , and ε an erosion.

4.3.4 Definition Adjunctions

Pairs of erosions and dilations satisfying the above connection are called "adjunctions", and the erosion is said to be the adjoint erosion of the dilation, and vice-versa.

4.3.5 Opening and Closing

For all adjunction (ε, δ), the morphological opening $\gamma : L \rightarrow L$ and morphological closing $\phi : L \rightarrow L$ are defined as follows:[2]

$\gamma = \delta\varepsilon$, and $\phi = \varepsilon\delta$.

The morphological opening and closing are particular cases of algebraic opening (or simply opening) and algebraic closing(or simply closing). Algebraic openings are operators in *L* that are idempotent, increasing, and anti-extensive. Algebraic closings are operators in *L* that are idempotent, increasing, and extensive.

4.3.6 Particular cases

Binary morphology is a particular case of lattice morphology, where *L* is the power set of *E* (Euclidean space or grid), that is, *L* is the set of all subsets of *E*, and $\leq$ is the set inclusion. In this case, the infimum is set intersection, and the supremum is set union.

Similarly, grayscale morphology is another particular case, [2] where *L* is the set of functions mapping *E* into $\mathbb{R} \cup \{\infty, -\infty\}$, and $\leq$, $\vee$, and $\wedge$, are the point-wise order, supremum, and infimum, respectively. That is, is *f* and *g* are functions in *L*, then $f \leq g$ if and only if $f(x) \leq g(x), \forall x \in E$; the infimum $f \wedge g$ is given by

$(f \wedge g)(x) = f(x) \wedge g(x)$; and the supremum $f \vee g$ is given by $(f \vee g)(x) = f(x) \vee g(x)$.[1]

4.3.7 Definition Morphological filter

A morphological filter is an increasing and idempotent transformation of a complete lattice into itself.[2]

Image filters are useful for removing image components. Morphological dilation is used to smooth small dark regions. Morphological erosion is used to smooth small light regions. Alternating sequential filters are a combination of iterative morphological filters with increasing size of structuring elements.

Let X denote a binary image and let B denote a binary structuring element. The Alternating Filter is defined as an opening followed by a closing or a closing followed by an opening .ie. $AF_B(X) = (X \circ B) \bullet B$ or $AF_B(X) = (X \bullet B) \circ B$.

An Alternating Sequential Filter is an iterative application of $AF_B(X)$ with increasing size of structuring elements denoted as $AF_{B_N}(X) AF_{B_{N-1}}(X) ...... AF_{B_1}(X)$ where N is an integer and $B_N, B_{N-1}, ... B_1$ are structuring elements with decreasing sizes. $B_N = B_{N-1} \oplus B_1, N \geq 2$

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CONTENTS

5.1Introduction

Fourier transforms are among the most useful linear signal transformations for quantifying the frequency content of signals. It is useful for analyzing their processing by linear time – invariant systems .They enable the analysis and design of linear time invariant systems (LTI)in the frequency domain.

Slope transforms are non linear signal transforms that can quantify the slope content of signals and provide a transform domain for morphological systems. They are based on eigen functions of morphological systems that are lines parameterized by their slope.

The operation of tangential dilation, describes the touching of differentiable surfaces. It generalizes the classical dilation, but is invertible. It is shown that line segments are eigen functions of this dilation, and are parallel-transported, and that curvature is additive. The slope transform is a re-representation of morphology onto the morphological eigen functions. As such, the slope transform provides for tangential morphology the same analytical power as the Fourier transform provides for linear signal processing. Under the slope transform dilation becomes addition (just as under a Fourier transform, convolution becomes multiplication).

The slope transform has emerged as a transform which has similar properties with respect to morphological signal processing. Fourier transform does this with respect to linear signal processing .Main property of slope transform is that it transforms a supremal convolution (morphological dilation) into an addition ,which is similar to the concept in Fourier transform transforms. In Fourier transform a linear convolution changed into a multiplication.

There is an important difference between the Fourier transform and its morphological counterpart ,the slope transform. The Fourier transform is invertible but the slope transform only has an adjoint in the sense of adjunctions. This means that the 'inverse' of the slope – transformed signal is not the original signal but only an approximation within the sub collection of convex or concave signals. Hence convex analysis plays an important role in the study of the slope transform. Concepts from the theory of convex sets and functions, such as the Legendre transform ,the (Young – Fenchel) conjugate ,the support function ,the guage function ,and set polarity are listed. The complete lattices considered in this chapter are either lattices of sets or of functions.

5.2 Translation Invariant Systems

In convex analysis and optimization the nonlinear signal operation $\oplus$ is usually called supremal convolution. A dual operation is the so called infimal convolution given by $(f \square g)(x) = \wedge_{y \in R^d} f(x-y) + g(y)$.

$\square$ is closely related to the morphological erosion Θ, because $f \Theta g = f \square (-\breve{g})$ where $\breve{g}$ is the reflection of g given by $\breve{g}(x) = g(-x)$.Denote $\oplus$ and $\square$ as the supremal and infimal convolution, respectively, to distinguish them from the concept of a dilation and erosion operator on a lattice.

5.2.1 Definition DTI system

A mapping Δ which sends a signal f to a transformed signal $\Delta(f)$ is called dilation translation invariant(DTI) system if

i) it is a dilation, ie, $\Delta(\vee_i f_i) = \vee_i (\Delta f_i)$,

ii) if it is translation- invariant, ie. $\Delta(f_y + c) = \Delta(f)_y + c$ for any shift y and any real constant c .

A system is DTI if it is a horizontally shift-invariant and obeys the morphological supremum superposition principle

$\Delta\left[\bigvee_{i\in I} f_i(x)+c_i\right]=\bigvee_{i\in I}\left[\Delta(f_i(x)+c_i\right]$, where{fi}is any signal collection and $c_i \in R$.

Many important aspects of a DTI system[1],[11] can be determined in the time or spatial domain .

The morphological lower impulse $q_\wedge$ given by $q_\wedge(x) := \begin{cases} 0, x=0 \\ -\infty, x \neq 0. \end{cases}$

Let the corresponding output of the D T I system Δ when the input is the lower impulse be its lower impulse response: $g := \Delta(q_\wedge)$.

This uniquely characterizes a DTI system in the time domain, because any DTI system is equivalent to a supremal convolution(also called 'morphological dilation') by its lower impulse response: $\Delta(f) = f \oplus g$

5.2.2 Definition ETI system

A signal operator $\varepsilon : f \mapsto \varepsilon(f)$ is called an erosion translation invariant (ETI) system [11] if

i) it is horizontally shift –invariant

ii) obeys the morphological infimum superposition principle $\varepsilon\left[\bigwedge_{i\in I} f_i(x)+c_i\right]=\bigwedge_{i\in I}\left[\varepsilon(f_i)(x)+c_i\right]$ where $c_i \in R$.

Let the upper impulse response h of an ETI system ε as its response $h:=\varepsilon(q\vee)$ to the upper impulse $h=-\breve{g}$.

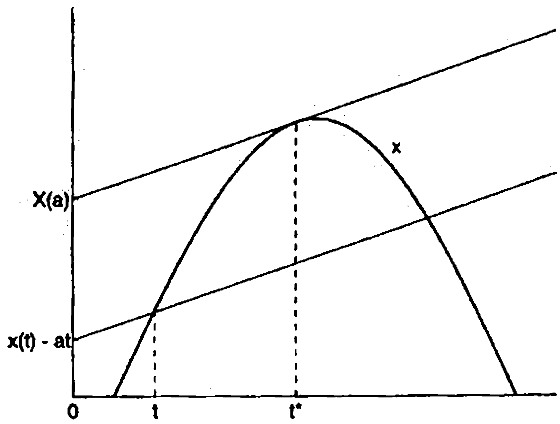
$q_\vee(x):=\begin{cases}0, x=0\\ +\infty, x\neq 0.\end{cases}$ then it follows that $\varepsilon(f)=f\square h$.

5.3 Legendre Transform

Let the signal $x(t)$ be concave and have an invertible derivative $x'=\dfrac{dx}{dt}$.The Legendre transform of x is based on the following concept:

Imagine that ,a graph of x ,not as a set of points $(t,x(t))$ but as the lower envelope of all its tangent lines .The tangent at a point $(t,x(t))$ on the graph has slope and intercept equal to $X=x(t)-\alpha(t)$

$\therefore X_L(\alpha)=x[(x')^{-1}(\alpha)]-\alpha(x')^{-1}(\alpha)$ where f^{-1} denotes the inverse. The function X_L of the tangents intercept versus the slope is the Legendre transform [1],[11] of x and $x(t)=X_L[(X_L')^{-1}(-t))+t(X_L')^{-1}(-t)]$.



;. 2. Concave signal x, its tangent with slope α, and a line parallel to tangent.

Note:

If the signal x is convex, then the signal is viewed as the upper envelope of its tangent lines.

5.4 Slope transforms

5.4.1 Definition Upper Slope Transform

For any signal $x: R \to \bar{R}$ its upper slope transform is the function $X_\vee : R \to \bar{R}$ with $X_\vee(\alpha) = \bigvee_{t\in R} x(t) - \alpha t, \alpha \in R$ .The mapping between the signal and its transform is denoted by $A_\vee : x \to X_\vee$.

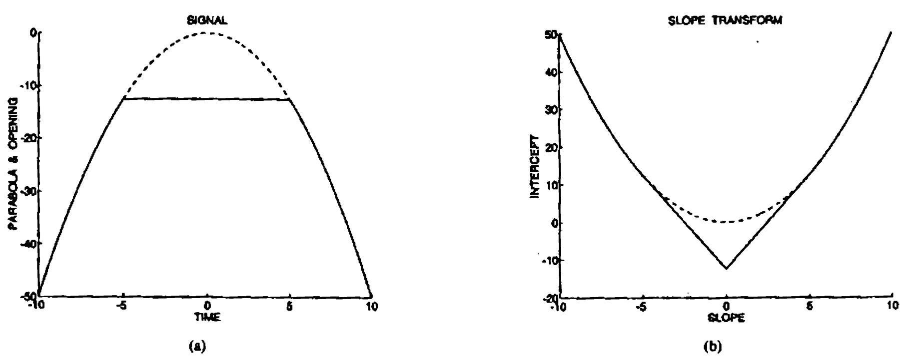
If there is one to one correspondence between the signal and its transform, then it is denoted by $x(t) \overset{A_\vee}{\to} X_\vee(\alpha)$

For continuous-time signals that are convex or concave and have an invertible derivative, all three transforms coincide and become equal to the Legendre transform.(irrespective of the difference due to the boundary conditions).

The morphological signal operators are parallel or serial inter connections of morphological dilation and erosions, respectively, defined as

$$(f+g)(x) = \vee_{y \in R^d} f(x-y) + g(y)$$
$$and (f+g)(x) = \wedge_{y \in R^d} f(x+y) - g(y)$$

where $\vee$ denotes supremum and $\wedge$ denotes infimum.



. (a) Original parabola signal $x(t) = -t^2/2$ (in dashed line) and its morphological opening (in solid line) by a flat structuring element [−5, 5]. pper slope transform of the parabola (in dashed line) and of its opening (in solid line).

5.4.2 Upper Slope Transform and Concave function

Let $x(t)$ is concave and has an invertible derivative. For each real α, the intercept of the line passing from the point $(t, x(t))$ in the signals graph with slope α is equal to $x(t) - \alpha t$.

Let α be fixed and t is varying. Then there exist a time instant t^* for which the intercept attains its maximum value. This happens when the line becomes tangent to the graph, then $x'(t^*) = \alpha$ .Corresponding to the variation of α, the tangent changes ,and the maximum intercept becomes a function of the slope α .By its definition ,the upper slope transform is equal to this maximum intercept function. Thus, if the signal $x(t)$ is concave and has an invertible derivative ,then the upper slope transform is equal to its Legendre transform.

The three types of slope transforms are

i) a single valued slope transform for signals processed by erosion systems.

ii) a single valued slope transform for signals processed by dilation systems.

iii) A multi valued transform that results by replacing the suprema and infima of signals with the signal values at stationary points.

5.5 Upper and lower slope transforms

To analyze morphological systems in a transform domain, The following two signal transforms are useful to analyze morphological systems in a transform domain .

Given a signal f ,its upper slope transform [1],[11] is defined as

$$\Im_{\vee}(f)(v) := \bigvee_{x\in R^d} f(x)-\langle x,v\rangle, v\in R^d$$

and its lower slope transform is

$$\Im_{\wedge}(f)(v) := \bigwedge_{x\in R^d} f(x)-\langle x,v\rangle, v\in R^d .$$

These slope transforms provide information about the slope content of signals and a description of morphological systems in a 'slope domain', with functionality similar to the use of Fourier or Laplace transforms in linear systems.

5.5.1(Properties of Upper slope transform).

For f, g $\in Fun(R^d)$, $y \in R^d$, $w \in R^d$,r>0 ,the following are the properties [11] of upper slope transform $X(\alpha)$ and the signal $x(t)$.

Properties of Upper Slope Transform

Signal	Transform
$x(t)$	$X(\alpha) = \bigvee_t x(t) - \alpha t$
$\bigvee_i c_i + x_i(t)$	$\bigvee_i c_i + X_i(\alpha)$
$x(t - t_0)$	$X(\alpha) - \alpha t_0$
$x(t) + \alpha_0 t$	$X(\alpha - \alpha_0)$
$x(rt)$	$X(\alpha/r)$
$x(-t)$	$X(-\alpha)$
$x(t) = x(-t)$	$X(\alpha) = X(-\alpha)$
$rx(t) \;\; , r > 0$	$rX(\alpha/r)$
$x(t) \oplus y(t)$	$X(\alpha) + Y(\alpha)$
$\bigvee_\tau x(\tau) + y(t + \tau)$	$X(-\alpha) + Y(\alpha)$
$x(t) \leq y(t) \;\; \forall t$	$X(\alpha) \leq Y(\alpha) \;\; \forall \alpha$
$\bigvee_t x(t) = X(0)$	$\bigwedge_\alpha X(\alpha) \geq x(0)$
$x(t) \wedge y(t)$	$\leq \; X(\alpha) \wedge Y(\alpha)$
$x(t) + y(t)$	$\leq \; X(\alpha) \square Y(\alpha)$
$x(t) + y(t) \quad , y$ is convex	$X_\vee(\alpha) \oplus Y_\wedge(\alpha)$
$y(t) = \begin{cases} x(t), & \lvert t \rvert \leq T \\ -\infty, & \lvert t \rvert > T \end{cases}$	$Y(\alpha) = X(\alpha) \square T \lvert \alpha \rvert$

The proof of two properties are given below where $\mathcal{A}_\vee$ denotes the upper slope transform

Signal Dilation in Time:

$$
\begin{aligned}
\mathcal{A}_\vee[x(t) \oplus y(t)](\alpha) &= \bigvee_t \left(\bigvee_\tau x(\tau) + y(t - \tau) \right) - \alpha t \\
&= \bigvee_\tau x(\tau) + \left(\bigvee_t y(t - \tau) - \alpha t \right) \\
&= \bigvee_\tau x(\tau) + Y(\alpha) - \alpha \tau \\
&= X(\alpha) + Y(\alpha).
\end{aligned}
$$

Signal Addition in Time: For any signals x, y

$$\mathcal{A}_{\vee}[x(t) + y(t)](\alpha) = \bigvee_t x(t) + y(t) - \alpha t$$
$$\leq \bigvee_t y(t) + \bigwedge_b X(b) + bt - \alpha t$$
$$\leq \bigwedge_b X(b) + \bigvee_t y(t) - (\alpha - b)t$$
$$= \bigwedge_b X(b) + Y(\alpha - b).$$

f is majorized by the affine function $x \mapsto \langle x, v \rangle + b$. Therefore, by computing the infimum of all affine functions $x \mapsto \langle x, v \rangle + f^{\vee}(v)$, a function which majorizes the original function f is obtained. This is known as Adjoint upper slope transform and is given by $\Im_{\vee}^{\leftarrow}(g)(x) = \wedge_{v \in R^d} g(v) + \langle x, v \rangle$, for a function $g : R^d \to R$. The upper slope transform maps the affine function $x \mapsto \langle x, vo \rangle + b$ onto an upper impulse which equals b for v=vo and ∞ elsewhere. By multiplying $\Im_{\vee}^{\leftarrow}$ to this upper impulse,the original input function $x \mapsto \langle x, vo \rangle + b$ is again obtained.

5.5.2 Proposition Let $\Im_{\vee}^{\leftarrow}$ be the adjoint upper slope transform, then $(\Im_{\vee}^{\leftarrow}, \Im_{\vee})$ is an adjunction on Fun(R^d).

Proof. To show that $\Im_{\vee}(f) \leq g \Leftrightarrow f \leq \Im_{\vee}^{\leftarrow}(g)$.

'$\Rightarrow$'; the other implication is proved similarly.

Assume that $\Im_{\vee}(f) \le g$:this means that $f(x) - \langle x, v \rangle \le g(v), x \in R^d, v \in R^d$ .This yields that $f(x) \le \wedge_{v \in R^d} g(v) + \langle x, v \rangle$ *for* $x \in R^d$, i.e , $f \le \Im^{\leftarrow}{}_{v}(g)$.

5.5.3 Lower slope transform

Let $f \in Fun(R^d)$,the lower slope transform of f is $f^{\wedge}(v) = \Im_{\wedge}(f)(v) = \wedge_{x \in R^d} f(x) - \langle x, v \rangle$.

There exists the following relationship with the upper slope transform and the Young – Fenchel conjugate :

$f^{\wedge}(v) = -f^{*}(v) = -(-f)^{\vee}(-v)$.

5.5.4 Proposition (Properties of $\Im_{\wedge}$).

For $f, g \in Fun(R^d), y \in R^d, w \in R^d, r > 0$ and $c \in \overline{R}$:

a) $(f_y)^{\wedge} = (f^{\wedge})_{[-y]}$

b) $(f_{[w]})^{\wedge} = (f^{\wedge})_w$

c) $(f + c)^{\wedge} = f^{\wedge} + c$

d) $(rf)^{\wedge} = rf^{\wedge}(./r)$

e) $f(r.)^{\wedge} = f^{\wedge}(./r)$

f) $f(-.)^{\wedge} = f^{\wedge}(-.)$

g) $(f \square g)^{\wedge} = f^{\wedge} + g^{\wedge}$

Analogous to the result, define $\Im^{\leftarrow}{}_{\wedge}(g)(x) = \bigvee_{v \in R^d} g(v) + \langle x, v \rangle$ as the adjoint lower slope transform.

5.5.5 Proposition $(\Im_{\wedge}, \Im_{\wedge}^{\leftarrow})$ is an adjunction on $Fun(R^d)$.

5.5.6 Morphological transform system

A system (M,X,S)is said to be a morphological transform system if

i) $S[x(t) \oplus y(t)] = X(\alpha) + Y(\alpha)$

ii) $S[x_T(t)] = X(\alpha) * T(\alpha)$

Where M is the collection of Morphological operators defined on set of functionals X and S is a transform.

If $S = A_{\vee}$ upper slope transform then $(M, X, A_{\vee})$ is called morphological slope transform system .Also, if X is a concave class then $A^{*}(x(t)) = xt$ where $A^{*} = A_{\wedge}(A_{\vee})$.

5.5.7 Definition

Let f be an element of $Fun(R^d)$. The function f is u.s.c (upper semi-continuous) if $\forall t, x \in R^d$, $f(x) < t$ implies that $f(y) < t$, for ever in some neighbourhood of x.

The function f is l.s.c (lower semi-continuous) if, $\forall t$ and x in R^d, $f(x) > t$

Implies that $f(y) > t$, for every y in some neighobourhood of x.

The collections of u.s.c and l.s.c functions are denoted by Fun(R^d) and Fun(R^d), respectively.

5.5.8 Proposition

The set Fun(R^d) is a complete lattice under the pointwise partial ordering with the point wise infimum ,and with supremum .

5.5.9 Convex and concave functions.

A function f is concave if its hypograph is convex.

The concave and convex functions are denoted by Fun(Rd) and Fun(Rd) respectively. Note that the subscript characterizes the shape of a concave function.

5.5.10 Proposition

If f is concave ,then dom (f) is a convex set.

If f is convex,then dom(f) is a convex set.

5.5.11 Proposition

a) f is concave iff is a convex set in Rd R

b) f is convex if is a convex set in Rd R.

Concavity and convexity are dual notions in the sense that f is concave iff –f is convex.

As grey scale morphology is usually based on the notion of the hypo graph. Consider concave rather than convex functions. From the duality principle ,it follows that both approaches are equivalent.

The infimum of an arbitrary collection of concave functions is concave .This doesnot hold for the supremum. Define the concave hull of an arbitrary function f as the infimum of all concave functions which lie above f. This is a concave function ,the smallest concave function above f. Dually define the convex hull as the supremum of all convex functions below f.

5.5.12 Proposition

a) The set Fun(Rd) is a complete lattice [1],[2][11] under the pointwise ordering ,with the point wise infimum fi and with supremum .

b) The set Fun(Rd) is a complete lattice under the pointwise ordering ,with the point wise supremum fi and with infimum .

Proposition gives that is an opening with invariance domain Funi(Rd) Fun(R d) ,the l.s.c convex functions.

5.5.13 Proposition

a) The operator is an opening on Fun(Rd) with invariance domain the l.s.c convex functions.

b) The operator is a closing on Fun(Rd) with invariance domain the u.s.c concave functions.

5.5.14 Corollary

(a) $\Im^{\leftarrow}{}_{\vee}\Im_{\vee}$ is a closing on $Fun(R^d)$ with invariance domain the u.s.c concave functions, i.e $\Im^{\leftarrow}{}_{\vee}\Im_{\vee} = \beta_u\beta_{\wedge}$.

(b) $\Im_{\vee}\Im_{\vee}{}^{\leftarrow}$ is an opening on $Fun(R^d)$ with invariance domain the l.s.c convex functions, i.e, $\Im_{\vee}\Im_{\vee}{}^{\leftarrow} = \alpha_{\ell}\alpha_{\vee}$

Apply Proposition to the adjunction $(\Im^{\leftarrow}_{\vee}, \Im_{\vee})$ then it implies that $\Im_{\vee}(\underset{j\in J}{\wedge} f_j) = \alpha_{\ell}\alpha_{\vee}(\underset{j\in J}{\wedge} \Im_{\vee}(f_j))$,if f_j is u.s.c and concave for every $j \in J$.

5.5.15 Proposition

For f,g $f^{\wedge}$ and $c \in \bar{R}$:

a) $\Im^{\leftarrow}_{\vee}(f_w) = (\Im^{\leftarrow}_{\vee}(f))_{[w]},$

b) $\Im^{\leftarrow}_{\vee}(f_{[y]}) = (\Im^{\leftarrow}_{\vee}(f))_{-y},$

c) $\Im^{\leftarrow}_{\vee}(f + c) = \Im^{\leftarrow}_{\vee}(f) + c$,

d) $\Im^{\leftarrow}_{\vee}(rf) = r\Im^{\leftarrow}_{\vee}(f)(./r)$,

e) $\Im^{\leftarrow}_{\vee}(f(r.)) = \Im^{\leftarrow}_{\vee}(f)(./r)$,

f) $\Im^{\leftarrow}_{\vee}(f(-.)) = \Im^{\leftarrow}_{\vee}(f)(-.)$,

g) $\Im^{\leftarrow}_{\vee}(f \square g) = \Im^{\leftarrow}_{\vee}(f) + \Im^{\leftarrow}_{\vee}(g)$.

Further, it is easy to verify that $\Im^{\leftarrow}_{\vee}(-f) = -\Im_{\vee}(f)$; (

In other words, $\Im^{\leftarrow}_{\vee}$ is the negative operator of $\Im_{\vee}$. For, the upper slope transform acts on functions of the spatial variable x, whereas the adjoint upper slope transform acts on functions of the slope variable v.

5.5.16 Definition Indicator Function

The upper and lower indicator function [11] corresponding to a set X are defined as $I_{\vee}(X)(x)=\begin{cases}0, x\in X\\ +\infty, x\notin X,\end{cases}$ and

$I_{\wedge}(X)(x)=\begin{cases}0, x\in X\\ -\infty, x\notin X,\end{cases}$ respectively. It is evident that X is closed

$\Leftrightarrow I_{\vee}(X) l.s.c \Leftrightarrow I_{\wedge}(X) u.s.c.$

X convex $\Leftrightarrow I_{\vee}(X) convex \Leftrightarrow I_{\wedge}(X) concave.$

5.5.17 Let $a\in R^d and \ r\in \bar{R}$ & H(a,r) - the hyperplane

H(a,r) $=\{x\in R^d / \langle a,x\rangle = r\}$

Note that H(a,r)=ϕ if r= $\pm\infty$.

$H^-(a.r)=\{x\in R^d / \langle a,x\rangle \le r\}$

$H^+(a.r)=\{x\in R^d / \langle a,x\rangle \ge r\}$.

If r= $-\infty$ then $H^-(a,r)=\phi$ and $H^+(a,r)=R^d$;dually ,if $r=+\infty$ then $H^-(a,r)=R^d$ and $H^+(a,r)=\phi$.The hyper plane H(a,r) supports the set X $\subseteq R^d$ at h if $h\in X\cap H(a,r)$ and $X\subseteq H^-(a,r)$ or $X\subseteq H^+(a,r)$.

5.5.18 Definition Sub linear functions

A function $f : R^d \to \bar{R}$ is said to be positively homogeneous[11] if $-\infty$ for r>0 and $x \in R^d$.It is sub linear if it is both convex and positively homogeneous. For any sub linear function , f(0)=0, $-\infty or + \infty$ .Note that $f \equiv -\infty$ if f(0)= $-\infty$.

The epigraph $U_{\vee}(f)$ is a convex cone .Every sub linear function satisfies the inequality $f(x+y) \le f(x) + f(y)$;

A function with this property is called sub additive.

Example

If $K \subseteq R^d$ is a convex cone ,then the upper indicator function $I_{\vee}(K)$ is sub linear.

5.5.19 Definition

A function $\|.\| : R^d \to \bar{R}_+ = [0, +\infty]$ is called a norm if

i) $\|x\| = 0$ iff x=0;

ii) $\|rx\| = |r|.\|x\|, r \in R, x \in R^d$;

iii) $\|x+y\| \le \|x\| + \|y\|, x, y \in R^d$;

Every norm is a (nonnegative) sub linear function.

5.5.20 Slope transform for sets: the support function

For a set $X \subseteq R^d$ its support function [11] is defined by

$\sigma(X)(v) = \vee_{x \in X} < x, v >, v \in R^d$ and $\sigma(X) = -\infty$ if X=∅ and the operator $\check{\sigma}(f)$ Fun (R^d)→P(R^d) is defined as $\check{\sigma}(f) = \cap_{v \in R^d} \bar{H}(v, f(v))$.

Note that $\sigma(X) = -\infty$ if X=ϕ.

From the observation that the support function is the point wise supremum of the affine functions $v \mapsto \langle x, v \rangle,\ x \in X$.

5.5.21 Proposition The support function $\sigma(X)$ of a set $X \subseteq R^d$ is l.s.c and sub linear.

The operator $\sigma : P(R^d) \to Fun(R^d)$, which maps a set X to the corresponding support function, the slope transform for sets.

There is a simple correspondence between the slope transform for functions and that for sets, namely,

$$\Im_\vee(I_\wedge(X))(v) = \vee_{x \in R^d} I_\wedge(X)(x) - \langle x, v \rangle = \vee_{x \in X} - \langle x, v \rangle$$

$$\sigma_\wedge(X)(v) = \wedge_{x \in X} \langle x, v \rangle.$$

Remark. σ can be considered as the 'upper' slope transform. The lower slope transform should then be defined as $\sigma_\wedge(X)(v) = \wedge_{x \in X} \langle x, v \rangle$.

5.5.22 Distance transform

The distance transform is an operator normally only applied to binary images.

The result of the transform is a grey level image showing the distance to the closest boundary from each point.

Let $\|.\|_p$ denote the norm on R^d given by

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots.. + |x_d|^p)^{1/p}.$$

Given a set $X \subseteq R^d$,its distance transform (also known as its distance function) with respect to p-norm is given by $D_p(X)(x) = \wedge_{y \in X} \|x - y\|_p$.

The distance transform [11] has various applications in image analysis and computer vision. For example, its thresholds at levels r>0 yield the multi scale dilations of X by the balls rB_p ,where B_p is the unit ball with respect to the p- norm. Further (for p=2) , its local maxima provide the points of the skeleton(medial) axis of X^c.

Consider the upper indicator function $l_v(X)$,and the convex conical structuring function $g(x) = \|x\|_p$,

$$D_p(X)(x) = \wedge_{y \in R^d} (l_v(X)(y) + \|x - y\|_p) = (l_v(X)\square g)(x);$$

The distance transform of X can be obtained as the infimal convolution of the upper indicator function of X with the conical norm function. This infimal convolution is equivalent to passing the input signal , i.e ,the set's upper indicator function $I_{\vee}(X)$,through an ETI system with slope response

$$g^{\wedge}(v) = \bigwedge_{x \in R^d} \|x\|_p - \langle x, v \rangle . g^{\wedge}(v) \leq 0 \ .$$

By using Holder's inequality ,$|\langle x, v \rangle| \leq \|x\|_p . \|v\|_q$ where the exponent q is determined by $\frac{1}{p} + \frac{1}{q} = 1$

That is, the distance transform is the output of an ideal cutoff slope – selective filter that rejects all input planes whose vector falls outside the unit ball with respect to the norm , and keeps all the others unaltered.

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Chapter 6

Generalized Structure for Mathematical Morphology *

CONTENTS

6.1 Introduction

When the ETI and DTI systems are related via an adjunction, then there is also a close relationship between their impulse responses. Namely ,let ε be an ETI system, and let Δ be its adjoint dilation. It is easy to show that Δ is a DTI system[11], and therefore $\Delta(f) = f \oplus g$,where g is the lower impulse response. Since it is the generalization,no seperate proof is required for most of the results. However, proofs are given for new propositions. Examples are given for some generalizations. This generalization is helpful for developing the theory of Mathematical Morphology. This chapter give the relation between combinatorial convexity, mathematical morphology and image processing.

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6.2 Different Structures for Morphological Operators

Moore family is defined by using a partially ordered set L .It also satisfies certain properties on L.

6.2.1 Definition Moore family

Let L be a poset.

A subset M of L is a Moore family if every element of L has a least upper bound in M.

$\forall xx \in LL, \exists yy \in M, yM\ y \geq x and x\ and\ \forall zz \in MM, z\ z \geq xx \Rightarrow zz \geq yy]$

A closure operator on L is an increasing, extensive and idempotent operator from L →L.

6.2.2 Proposition

Let L be a poset. There is a one to one correspondence between Moore families in L and closings on L, given as follows.

To a Moore family M, associate the closing defined by setting for every x∈L; (x) is equal to the least y∈M such that y≥x.

To a closing, one associates the Moore family M which is the invariance domain of M = Inv (i.e. M=$\{\varphi(x)/x \in L\}$.

6.2.3 Convex geometry

Let S be a set ,consider the family T of subsets of S with the following properties:

$\varphi \epsilon T, S \epsilon T,$

$A, B \epsilon T$ *implies* $A \cap B \epsilon T$

This family defines a closure operator $\phi(X) = \cap\{A \epsilon T, X \subseteq A\}$.

Every closure operator defines a family T' with the above properties. Elements of T or elements defined by ϕ will be called convex. We call the pair (S, ϕ) is a Convex geometry if ϕ verifies the anti-exchange axiom[6]

$\forall x, y \notin \phi(X), x \neq y, x \in \phi(X \cup y)$ *implies* $y \notin \phi(X \cup x)$

In the same way, If $\phi(X) \neq S, \exists p \in S \backslash \phi(X), \phi(X \cup p) = \phi(X) \cup p$

Corresponding to a partially ordered set ,we have a graphical representation, known as Hasse Diagrams. So we can infer that Poset give some geometrical representation. In view of this we can define Poset Geometry.

6.2.4 Poset geometry

Let P be a partially ordered set and X be a subset of P, Define $D_p(X) = \{y \epsilon P, y \leq x for\ some\ x \epsilon X\}$, (P, D_p) is a convex geometry called Poset geometry which are characterized by the following:

The convex geometry (S,ϕ)arises from the poset geometry on a Poset P if and only if $\phi(A \cup B) = \phi(A) \cup \phi(B) \forall A, B \subseteq S$.

Definitions of Dilation and Erosion is given below. We also give the definition of Alexandroff space ,in order to link it with Morpholgy.

6.2.5 Dilation

A dilation is defined by an operator $\delta: P(S) \rightarrow P(S)$ with the following properties:$\delta(\phi) = \phi, A, B\ \epsilon P(S), \delta(A \cup B) = \delta(A) \cup \delta(B)$

6.2.6 Erosion

An erosion is defined by an operator $\varepsilon: P(S) \rightarrow P(S)$ with the following properties: $\varepsilon(S) = S, A, B \epsilon P(S), \varepsilon(A \cap B) = \varepsilon(A) \cap \varepsilon(B)$

6.2.7 Alexandroff space

A topological space is an Alexandroff space if the intersection of any family of open sets is open(resp.the union of any family of closed sets is closed)

Let δ be a dilation on S. For any dilation, define a binary relation as follows:

xRy is equivalent to $x \in \delta(y)$, for $x, y \in S$ or xR'y is equivalent to $\delta(x) \subseteq \delta(y), \forall x, y \in S$

6.2.8 Result

Let δ be a dilation on S. R its binary relation canonically associated with it. Then the following are equivalent.

i) R is reflexive and transitive

ii) xRy is equivalent to $\delta(x) \subseteq \delta(y)$

iii) defines a dual Moore family.

Proof. .i) $\Rightarrow$ ii) .Since R is reflexive and transitive, δ(x)□ δ(x) and δ(x)□ δ(y).Therefore $\delta(x) \subseteq \delta(y)$.

Ii) $\Rightarrow$ iii) .Since $\delta(x) \subseteq \delta(y)$,by definition δ defines a dual Moore family.

iii) $\Rightarrow$ i). Since δ defines a dual Moore family,it is both reflexive and transitive.

6.2.9 Proposition

Let S be a set . Let N: S→ P(S) corresponding to ρ by

$\forall x, y \in S, x \in N(y) \Leftrightarrow y \in N(x) \Leftrightarrow x\rho y$ and N(x) = $\{y \in S/x\rho y\}$

Then (i) N separates S in a primary sense

(ii) (S, N) is a Poset geometry.

(iii) (S,N) is a To Alexandroff space.

6.2.10 Proposition

(S,N) is separated in a primary sense if N verifies the following two properties.

For any family (x_i), $_i\in$ I of elements and for any element x$\in S$, verifying

N(x)$\subseteq \cup_{i\in I} N(x_i)$, $\exists j \in I$ such that N(x) $\subseteq$ N(x_j)

N(x) = N(y) is equivalent to x=y for any x, y $\in$S

6.2.11 Definition

$\varphi = \varepsilon o \delta$ is called a morphological closure and

$\varphi(x) = \{y \in S/\delta(y) \subseteq \delta(x)\}$

6.2.12 Result

Let S be an infinite space and let N be defined by R. Then (S,N) is a convex geometry if and only if $\forall X \subseteq S,(S - N(x),\psi)$ is a To-Alexandroff space [1],[6]where $\psi(A) = \cup_{y\in A}[N(N(X) \cup y) \cap S - N(X)]$

6.2.13 Proposition

Let $\delta: P(v) \to P(w) and\ \varepsilon: P(w) \to P(v)$ such that $N: v \to P(w)\ where\ N(v) = \delta(\{v\}), \forall v \in V$ and $\delta = \delta_N$, $\varepsilon = \varepsilon_N$.

Define $\delta_{\bar{N}}(Y) = V - \varepsilon_N(W - Y), \forall Y \epsilon P(W)$ and $\varepsilon_{\bar{N}}(X) = W - \delta_N(V-X)$, $X \in P(V)$.

Then $\delta_{\bar{N}}$ and $\varepsilon_{\bar{N}}$ are dual by complementation of ε_N and δ_N.

Also $\varepsilon_{\bar{N}}$ is a dilation and $\delta_{\bar{N}}$ is an erosion. Also $\delta_{\bar{N}} = \delta_{\rho^{-1}}$ and $\varepsilon_{\bar{N}} = \varepsilon_{\rho^{-1}}$ where ρ^{-1} is defined as $w\ \rho^{-1}\ v \Leftrightarrow v\ \rho\ w$ and $v\ \rho\ w \Rightarrow w \in \delta(v) = N(v)$.

6.2.14 Proposition

Let δ be a dilation and ε an erosion. Let ρ be the relation defined as before .Then

i) ρ is reflexive and transitive.

ii) $v \rho w$ is equivalent to $\delta_N(x) \subseteq \delta_N(y)$.

iii) δ_N defines a Dual Moore family.

iv) ε_N defines a Moore family.

6.3 Generalized Structure for Mathematical Morphology

6.3.1 Morphogenetic field

Let $X \neq \varphi$ and $W \subseteq P(X)$ such that i) ϕ, X $\in$ W , ii) If B $\in$ W then its complement $\overline{B}$ $\in$ W iii) If $B_i \in W$ is a sequence of signals defined in X, then $\bigcup_{n=1}^{\infty} Bi \in$ W.

Let A= $\{\phi : W \to U / \phi(\cup A_i) = \vee\phi(A_i) \,\&\, \phi(\wedge A_i) = \wedge\phi(A_i)\}$. Then W_U is called Morphogenetic field [22]where the family W_u is the set of all image signals defined on the continuous or discrete image Plane X and taking values in a set U .The pair (W_u, A) is called an operator space where A is the collection of operators defined on X.

6.3.2 Morphological space

The triplet (X, W_u, A) consisting of a set X, a morphogenetic field W_u and an operator A(or collection of operators) defined on X is called a Morphological space.

Note: If $X = Z^2$ then it is called Discrete Morphological space

6.3.3 Definition

Let (X, W_u, A) be a morphological space and (W_U, A) be an operator space in (X, W_u, A).

If X is a class of concave functions then (X, W_u, A) is called concave morphological space. If X is a class of convex functions then (X, W_u, A) is called convex morphological space.[22]

6.3.4 Proposition

Every convex morphological space has * property.

6.4 Results in Generalized Structure

6.4.1 Definition * property

Let (X,W_u,A) be a morphological space and (W_U,A) be an operator space in (X,W_u,A).

Let $x(\alpha)\in X$, then $x(\alpha)$ has at least one maxima or minima in X.

6.4.2 Proposition

Every convex morphological space is optimizable.

6.4.3 Definition

Let (X,W_u,A) be a morphological space and (W_U,A) be an operator space in (X,W_u,A).

If ϕ is an operator in A and in particular if ϕ satisfies (or defines a rule) in W_U then the operator space (W_U,ϕ) is called a geometrical space and ϕ defines a morphological geometry in W_U.

6.4.4 Proposition

Let (X,W_u,A) be a morphological space and (W_U,A) be an operator space in (X,W_u,A)

Suppose that $\varphi\in A, S\in W_U . X_1, X_2\in W_U \Rightarrow X_1\cap X_2\in W_U$ and

$\phi(S)=\cap\{X_1 \in W_U / S \subseteq X_1\}$ then ϕ defines a morphological geometry, known as convex geometry if

$\forall x, y \notin \phi(S), x \neq y, x \in \phi(S \cup Y) \Rightarrow y \notin \phi(S \cup x)$. (W_U, ϕ) is called a convex geometrical space.[22]

Also if $\phi(S) \neq X$ then $\exists z \in X - \phi(S)$ and $\phi(S \cup z) = \phi(S) \cup z$.

6.4.5 Definition Poset Geometry

Let $(P, \leq)$ be a poset and X be a subset of P. Define $T_p(X) = \{y \in P / y \leq x, \text{for some } x \in A\}$. Let (X, W_u, A) be a morphological space and $W_u = P, A = T_p$ .Then the operator space is (P, T_P) defines a geometry known as poset geometry and (P, T_P) is called a poset geometrical space.

6.4.6 Proposition

Let (X, W_u, A) be a morphological space and let (W_U, ϕ) be a poset geometrical space [22]in (X, W_u, A) .Then (W_U, ϕ) is called a convex geometrical space iff $\phi(X_1 \cup X_2) = \phi(X_1) \cup \phi(X_2), \forall X_1, X_2 \in X$.

6.4.7 Proposition

Let (X,W_u,A) be a morphological space. Then for $\gamma(x)\in W_U$, $(X,\gamma(x))$ is called an anti matroid if $(X,\gamma(x))$ satisfies the following.

i) $\varphi\in\gamma(x), \gamma(x)$ is closed under union.

ii) For $S\in\gamma(x), S\neq\phi, \exists x\in S$ such that $S-x\in\gamma(x)$.

6.4.8 Proposition

Let (X,W_u,A) & $(Y,W_u,\bar{A})$ be a morphological spaces. The pair $(A,\bar{A})$ is called an adjunction iff $A(X)\leq Y \Leftrightarrow X\leq \bar{A}(Y)$ where $\bar{A}$ is an inverse operator of A.

6.4.9 Proposition

Let (X,W_u,δ) & (Y,W_u,ε) be a morphological spaces with operators dilation and erosion on A. Then $\delta(X)\leq Y \Leftrightarrow X\leq \varepsilon(Y)$.

6.4.10 Proposition (For lattice)

Let $(X,W_u,A)\&(Y,W_u,\bar{A})$ be a morphological spaces. The pair $(A,\bar{A})$ is called an adjunction iff $\forall u,v\in X,\exists$ an adjunction $(l_{u,v},m_{v,u})$ on U such that $\bar{A}(x(u))=\underset{v\in X}{\vee} m_{v,u}(x(v))$

and $A(y(v))=\underset{u\in X}{\wedge} l_{u,v}(y(u))$, $\forall\ u,v\in X,x,y\in W_U$.

6.4.11 Definition

The operator $\phi=\varepsilon\circ\delta$ defines a closure called morphological closure and $\phi^*=\delta\circ\varepsilon$ defines a kernel, called morphological kernel.

6.4.12 Lemma

Let (X,W_u,A) be a morphological space.

$\phi^*(S)=\cup\{X_1\in W_U / X_1\subseteq S\}$ defines a kernel operator in A. The pair $(X,\phi^*(S))$ is an anti matroid if ϕ^* satisfies the axiom:

For $\phi^*(S)\neq\varphi,\exists z\in\phi^*(S),\phi^*(S-z)=\phi^*(S)-z$.

Proof:

Since $\phi^*(S)\in W_U$ where W_U is a morphogenetic field in a morphological space (X,W_u,A) $\phi^*(S)$ is an anti matroid.

Direct proof.

Since $\phi^*(S-z) \subseteq S-z$,so $z \notin \phi^*(S-z)$ .From

monotonicity, $S-z \subseteq S \Rightarrow$ $\phi^*(S-z) \subset \phi^*(S)$ .There fore

$\phi^*(S-z) \subset \phi^*(S)$

Conversely, $\dot{\phi}^*(S)-z \subseteq S-z \Rightarrow \phi^*(\dot{\phi}^*(S)-z) \subseteq \phi^*(S-z)$.

There fore $\phi^*(S)-z) \subset \phi^*(S-z) \Rightarrow \phi^*(S)-z) = \phi^*(S-z)$.

6.4.13 Theorem

Let (X,W_u,A) be a morphological space. (X,ϕ) defines a convex geometry iff (X,ϕ^*) is an antimatroid.

6.4.14 Definition Seperation

Let (X,W_u,A) , $(X,W_u,\bar{A})$ be morphological spaces. Let $(A,\bar{A})$ be adjunctions. (X,A) is separated in a primary sense if A verifies the following two properties.

Let $x \in X$, $A(x) \subseteq \bigcup_{\forall x_i \in W_U} A(x_i) \Rightarrow \exists j \in I$ such that $A(x) \subseteq A(x_j)$

$A(x) = A(y) \Rightarrow x = y \forall\ x,y \in X$ and $\phi = A \circ \bar{A}$ defines a morphological closure.

6.4.15 Theorem

Let (X, W_u, A) be a morphological space and $\phi = A \circ \bar{A}$ be the morphological closure. Then the following statements are equivalent.

A separates X in a primary sense.

(W_U, ϕ) is a morphological geometrical space.

(W_U, ϕ) is a poset geometrical space.

6.4.16 Theorem

Let (X, W_u, A) be a morphological space and let X be an infinite set and $\phi = A \circ \bar{A}$ be the morphological closure. Then the following statements are equivalent.

1) A separates X in a primary sense.

2) (W_U, ϕ) is a morphological geometrical space.

3) (W_U, ϕ) is a poset geometrical space

(W_U, ϕ) is a T_0 Alexandroff space.

Proof:

3)$\Rightarrow$ 4)

Let (W_U,ϕ) be a poset geometrical space. $\forall y \in \phi(y), \exists B \subseteq Y$,B being a finite set such that $y \in \phi(B)$.$\therefore$ $y \in \phi(y) \Rightarrow y \in \phi(z)$ for some z$\in$ Y.$\therefore$ (W_U,ϕ) is an Alexandroff space

$\because \forall\ x\ ,\ y\ \in\ \phi\ (\ y\ )\ , x \neq y, x \in \phi(Y \cup y) \Rightarrow y \notin \phi(Y \cup x)$,$\therefore$ is a T_0 space.

4) $\Rightarrow$ 1)

Let (W_U,ϕ) is a T_0 Alexandroff space.

$\because \quad \phi = A \circ \bar{A},\ \phi(Y) = \left\{ y \in X / \bar{A}(y) \subseteq \bar{A}(Y) \right\}$.$\therefore$ $\forall x \neq y, \bar{A}$ *seperates* X in a primary sense.

1)$\Rightarrow$ 2)

Let $\bar{A}$ separates X in a primary sense .Since $\bar{A}(x) \subseteq \bar{A}(y)$

$\phi(y) = A \circ \bar{A}(y)$ and

$\phi(y) = \left\{ y \in X / \bar{A}(Y) \subseteq \bar{A}(y) \right\}$.$\phi(Y) = \cap \{ Y_1 \in W_U / Y \subseteq Y_1 \}$

$\Rightarrow \{ y \in X / yRx \forall x \in Y \}$

$\therefore (W_U, \phi)$ is a morphological geometrical space.

2) $\Rightarrow$ 3)

Let (W_U, ϕ) is a morphological geometrical space. $\Rightarrow \phi(Y)$ is an ideal of X. $\Rightarrow \phi(Y)$ is closed and R is an order relation. $\therefore (W_U, \phi)$ is a poset geometrical space.

Hence the result.

6.4.17 Definition Self Conjugate Operator Space

An operator space (W_u, A) is called self conjugate if it has a negation.

Example A clodum V has conjugate a * for every ‘a’ such that (avb)* = a* $\wedge$ b* and $(a*b)^{*'} = (a^{*} *' b^{*})$ [23]

Example If V is a blog [4] then it becomes self conjugate by setting

$$a^* = \begin{cases} a^{-1}, \text{ when } V_{\inf} < a < V_{\sup} \\ V_{\sup}, \text{ when } V_{\inf} = a \\ V_{\inf}, \text{ when } V_{\sup} = a \end{cases} \quad [23]$$

Example If X is a concave class then A* x (t)= x(-t) where A* = A $\wedge$ (A$\vee$).

6.4.18 Definition Self Conjugate Morphological Space

If the operator space (W_u, A) is self conjugate then the morphological space (X, W_u, A) is called a self conjugate morphological space.

6.4.19 Definition Operatable Functions

Let (X, W_u, A) be a morphological space. The collection K (X, W, A) of operatable functions consists of all real valued morphologically operatable functions x(t) defined on X such that x(t) has finite operatability with respect to A. A morphologically operatable function $x \in K$ iff $|x| \in K$ .ie. iff $|A(x(\alpha))| \leq A|x(\alpha)|$

6.4 .20 Definition Morphological Transform Systems

Let (X, W_u, A) be a perfect morphological space and K= K (X,W_u, A) be an operatable space. K is called a morphological transform system if

$A[x_T(t)] = X(\alpha) \circ T(\alpha)$

Remark Since K is an operatable space,

1) $A[x(t) + y(t)] = X(\alpha) + Y(\alpha)$

2) $A[x_T(t)] = X(\alpha) \circ T(\alpha)$

6.4.21 Definition Morphological Slope Transform System

If A= Av in the previous definition, then K is called a Morphological slope transform system where Av is the upper slope transform.

Let (X, W_u, A) be a self conjugate morphological space. If X is a concave class then A * (x(t))= x(-t) where A*= A$\wedge$(A v) and A$\wedge$ is the lower slope transform. Also Av ($\vee x_c$) = $\sum_{\forall c} A_\vee(x_c)$

6.4.22 Proposition (Characterization of Slope Transforms).*

A Slope transform is an extended real valued function Av (or A$\wedge$) defined on a Morphogenetic field W_u such that

1. Av (ϕ)= 0
2. Av (x_c) ≥ 0 $\forall$ $x_c \in W_u$
3. Av is countably additive in the sense that if (x_c)is any disjoint sequence[or sampling Signal]then Av ($\vee x_c$) = $\sum_{\forall c} A_\vee(x_c)$

Remark Av takes $+\infty$ i.e Av (x_c) $=\infty$ if x(t) = ∞

Av (α) $> -\infty, \forall\alpha$ unless x(t) $= -\infty, \forall t$

If x=∞ - then Av = $-\infty$

6.4.23 Proposition Let K be a morphological transform system. Let X be a class of concave functions. Let $x(\alpha) \in$ X with each $x(\alpha)$ has an invertible derivative. Then $A_\vee(x(\alpha)) = L(x(\alpha))$ where L is the Legendre transform and Av is the upper slope transform.

Algebraic structures are important for defining Morphological operators. Many properties of the algebraic structure may applicable to these operators as well.

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Chapter 7

Partial Self Similarity ,Mathematical Morphology and Fractals

CONTENTS

7.1 Introduction

Scale-space is an accepted and often used formalism in image processing and computer vision. Today, this formalism is so important because it makes the choice at what scale visual observations are to be made explicit. A Scale space can be described as a family of filters which transform a given signal (Image) into a simplified signal (image).Morphological Dilation and Erosion ,with structuring functions of increasing size in the scaling parameter ,define a class of scale spaces. Fractals are mathematical sets with a high degree of geometrical complexity. It can model many natural phenomena .

Examples include physical objects such as clouds ,mountains, trees and coastlines ,image intensity signals eminating from certain fractal surfaces etc.

7.2 Basic Concepts.

7.2.1 Definition Sensitive operator

Let (X, W_u, A) be a Morphological space[16]. Let B_1 be the neighbourhood of $x \in X$ i.e., N(x)=$B_1 \subseteq X$ .Then $\forall x \in X$, $x \in B_1, y \in B_1$ $\exists B_2$ such that $B_1 \subseteq B_2 \subseteq X$ and $\alpha^n(x) \in B_2$ and $\alpha^n(y) \notin \overline{B}_2$ $n \in Z^+$. Then $\alpha \in A$ is called a sensitive operator and the operator space [16] (W_u, A) is called a sensitive space.

Example: Dilation is sensitive. Constant signals f(x)= c are not sensitive.

7.2.2 Proposition

Let N: X→P(X) be defined such that $N(x) = \{y \in X / x\rho y\}$ where ρ is the relation, dilation defined between x and y $\forall x, y \in X$. i.e., $x\rho y \Rightarrow y = \delta(x)$ where δ is the dilation and for $\alpha \in A$, $\alpha = \delta \Rightarrow$ δ^n (x) $\in B_2$ =N(x), $\delta^n(y) \notin \overline{B}_2$ $n \in Z^+, x, y \in B_1$ 1. Thus δ is sensitive.

7.2.3 Definition Perfect Set

Let F$\subseteq X$. Define $S(F)=\{\alpha/\alpha\in(W_u,A) besensitive[6] by \alpha\in F\}$. If S(F) $\neq\phi$ and (X, W_u, A) is a convex morphological space [16] then F is called Perfect.

7.2.4 Definition Stirring Operator

Let (X, W_u, A) be a Morphological space and let U, V$\subseteq X$ be two sets. Let α $\in$A. Then α is called stirring [6] if given any neighbourhoods N_1 and N_2,of U and V, $\forall x\in U$, y$\in$V in X, $\exists k\in Z^+$ such that

$$\alpha^k(N_1)\cap\alpha^k(N_2)\neq\phi.$$

α is strongly stirring if $\exists k\in Z^+$ and a set G in X such that $G\subseteq\alpha^k(N_1)\cap\alpha^k(N_2)$.

7.2.5 Definition Partial Similarity

Let (X, Wu, A) be a Morphological space.

Let K⊆X. K is called Partial self similar or α *similar* if $\exists K_1,K_2,....K_t$ such that $K=\bigcup_{i=1}^{t}K_i$ and for each K_i, $\exists$contraction maps $\varphi_{(i,j,k)}$, for i=1..... t,r=1 t ,j=1,....t and k=1. $w(i,j)$ with $w(i,j)$>0 such that $K_i=\bigcup_{j,k}\varphi_{(i,j,k)}(K_j)$.

7.2.6 Definition Scale space

Let S a scaling on an image space L. The family {T(t)},t > 0 of operators on L is called an (S,+) scale – space if T(t).T(s)=T(t+ s) ,s ,t > 0 and T(t).S(t)=S(t).T(1) ,t > 0

7.2.7 Proposition

The erosion $\varepsilon(f) = f \ominus b$ with a convex structuring element b induces an ($S^{1/2}$, + ½) scale space and f is ½ similar.

7.2.8 Definition Anamorphic Scaling

A family $S = \{S(t)/t > 0\}$ of operators on L is called a scaling if S(1) = identity element.

S(t)S(s)=S(ts) for s,t > 0.Two scalings S and $\dot{S}$ are said to be anamorphic if $\exists$ an increasing bijection γ on T such that $S(\gamma(t)) = S'(t)\ \forall t \in T$ Also $\gamma(1) = 1$, $\gamma(st) = \gamma(s)\gamma(t) fors,t \in T$.

7.2.9 Proposition

Anamorphic scaling are $\alpha -$ similar

7.2.10 Proposition

The erosion $\varepsilon(f) = f \ominus b$ with b∈SP(k) for K>1 induces a $(S^{\alpha},+v)$ scale space if $v = 1-\alpha + K^{*}(2\alpha - 1)$ which implies that f is α - similar . b is called the structuring function.

7.2.11 Proposition

Let (X, Wu, A) be a Morphological space. Let f be α similar. Then $\exists \psi \in A$ such that $\psi^{\alpha}(f) = \alpha\psi(f)$.

7.2.12 Definition

The cross – section $X_t(f)$ [1],[2],[5] of f at level t is the set obtained by thresholding f at level t.

$X_t(f) = \{x / f(x) \geq t\}$, where $-\infty < t < \infty$

7.2.13 Proposition

If f is a fractal then $\exists i \in I$ such that ∀i, $X_{t_i}(f)$ are self similar and $X = \bigcup_{\forall i} X_t(f)$.

7.3 Morphological Fractals

7.3.1 Surface area of a compact set

Morphological operators extracts the impact of a particular shape on images using structuring elements. It encodes the primitive shape information. The transformed image is obtained by using a structuring element .Therefore it can be treated as a function of the structuring element.

Dilation of a set X with a structuring element Y is given by the expression $X \oplus Y = \{x / Y^x \cup X \neq \phi\}, Y^x$ denotes the translation of a set Y with x.

Dilation operation can be used to define the surface area of a compact set.

Surface area [19]of a compact set X with respect to a compact convex structuring element Y which is symmetrical with respect to the origin is given by

$$S(X,Y) = \lim_{\rho \to 0} \frac{V(\partial X \oplus \rho Y)}{2\rho}$$

Where ∂X is the boundary of set X and $\oplus$ denotes the dilation of the boundary of X by the structuring element Y and ρ is a scaling factor. Volume of a set X is denoted by $V(X)$.

7.3.2 Particular Case – Fractals :

If the object is regular, the surface area will not change with ρ_i.For a fractal object, S is increases exponentially with decreasing ρ.

7.3.3 Definition Fractal Identification:

An image is segmented into the regions $R_1, R_2, ...R_n$ if $\exists$ a relation ρ on Regions such that $R_i \rho R_j$ if $R_i \cap R_j = \phi$ and $\bigcup R_i = X$.

Also Image Property of $R_i \cap R_j = \phi$,if $i \neq j$ .If *Image Property of* R_i = *Property of* R_j then each R_i is a fractal.

Note: Converse is not always true. For every Fractal, it is not necessary that Image *Property of* R_i = *Property of* R_j

7.4 Class of Fractal Graphs- G(k ,t)

7.4.1 Definition

Let $F(p) = \prod_{i=1}^{m} f_i(p_i)$ ....(1) where ($p = (p_1, p_2, ...p_m)$ and f_i, $i = 1,2,....m$ is a set of completely defined functions and F is uniquely defined on R.

Define G(F) as $p \in G(F)$ iff F(p)=1.i.e F is a characteristic function of $G(F)$.The set of graphs which can be generated from (1) by allowing each f_i to vary over all possible logic functions is defined as Class of Fractal Graphs, [18] denoted by $G(k,t)$ where the vectors ($k=(k_1,k_2,...k_m)$) and t=($t=(t_1,t_2,...t_m)$).

7.4.2 Definition Compression

Let R be a rectangular plane and is divided into $2^{n_1}\times 2^{n_2}$ grids represented by R($2^{n_1}\times 2^{n_2}$).G is a graph on R and $F:R\to\{0.1\}$ is its characteristic function.

Given two integers $s_1 \& s_2, 0<s_1<n_1, 0<s_2<n_2$ construct a rectangular plane $R'(2^{n_1-s_1}\times 2^{n_2-s_2})$ regarding its left lower corner as an origin. A function $F':(R'\to\{0,1\})$ is defined as follows.

$\forall p'=(x',y')\in R'(2^{n_1-s_1}\times 2^{n_2-s_2})$,if$\exists$ integers $\alpha_1 \& \alpha_2$ where $0\le\alpha_1<2^{s_1}$, $0\le\alpha_2<2^{s_2}$ such that $F(p)=1$ where $p=(x,y)\in R$, $(x=x'.2^{s_1}+\alpha_1)$

And $(y=y'.2^{s_2}+\alpha_2)$ then $F'(p')=1$,otherwise $F'(p')=0$

Graph G' with F' as its characteristic function is called a compressed graph of G based on (s_1,s_2) and a compressed graph of G is denoted by $G'=C_{(s_1,s_2)}(G)$ and $F'(p'), p'\in R'$ is given below.

$$F'(p')=\begin{cases}1, if \exists a\in R'(2^{s_1}\times 2^{s_2}) such that F(p)=1 where p=(p',a)\\ 0, otherwise\end{cases}$$

7.4.3 Definition Similarity

Assume that $G^1 \& G^2$ are two graphs on a plane. If $\exists$ a compression transformation $C(s_1,s_2)$ such that G^1 is compressed into graph $G^3 = C(s_1,s_2)(G^1)$ and G^2 can coincide with G^3 through translating G^2 ,then $G^1 \& G^2$ are similar, denoted by $G^1 \to G^2$.

Note: Compression is nonreversible. Therefore Similarity is an asymmetric relation.

7.4.4 Definition Self Similarity:

Let G be a graph. If $\exists$ a partition $G_1, G_2, ... G_t$ of G and G_i is a proper sub graph of G such that G and each non empty sub graph G_i of G are similar.

For any two non empty sub graphs $G_i and G_j$, $G_i and G_j$ are similar or $G_j and G_i$ are similar, [1], [4],[11].Then G is said to be a self similar graph.

7.4.5 The Order of Self Similarity

G is a self similar graph, if any proper sub graph among all partitions of G which satisfy the definition of self similarity is not a self similar graph, then the order of similarity of G is 1.

If the maximal order of sub graphs among all partitions of G which satisfy the definition of self similarity is r, then the order of self – similarity of G is r+1

7.4.6 Definition Mutual Similarity

Let G be a graph. If $\exists$ a partition $G_1, G_2, ...G_t$ of G and G_i is a proper sub graph of G such that for any two non empty sub graphs $G_i and G_j$, $G_i and G_j$ are similar or $G_j and G_i$ are similar, then G is a mutually similar graph.

7.4.7 Definition Fractal Graphs $G(k,t)$

If a graph G can be partitioned into several sub graphs and the sub graphs are mutually similar and each sub graph can further be partitioned into mutually similar sub graphs etc, then G is said to be a mutual- similar graph. $G(k,t)$ is a mutual similar graph.[18]

7.4.8 Generation of $G(k,t)$ using Mathematical Morphological operators.

For $G \in G(k,t), \exists$ a unique compressed coding $M(k,t)$ denoted by $M(G)$, where $M(G)$ = $m(k_1,t_1), m(k_2,t_2), ...m(k_m,t_m)$ and $m(k_i,t_i) \in M(k_i,t_i)$.

$\forall i$ a point set $B_G(t)$ is given by

$B_G(t) = \{0,0,..p_i,0,.....0 / M(k_i,t_i)(p_i) = 1, p_i \in R_i\}$ where $M(k_i,t_i)(p_i)$ denotes the p_i th element in matrix $M(k_i,t_i)$.

7.4.9 Matrix representation of $G(k,t)$

$M(k_i,t_i)$ denotes [18]the whole set of $2^{k_i} \times 2^{t_i}$ non zero matrices with 0 or 1 as their elements and $k = (k_1,k_2...k_m)$, $t = (t_1,t_2..t_m)$, and $M(k,t)$ denotes the cartesian product space of $m(k_1,t_1), m(k_2,t_2),.......m(k_m,t_m)$. $\forall G \in G(k,t)$, G can be generated by the dilation $G = B_G(1) \oplus B_G(2) \oplus ... \oplus B_G(m)$.

7.4.10 Procedure for generating $G(k,t)$

Step 1

Let $G \in G(k,t)$.

Step 2

For any i, construct $B_G(t)$ where $B_G(t) = \{0,0,..p_i,0,.....0 / M(k_i,t_i)(p_i) = 1, p_i \in R_i\}$ and

$M(G)$ = $m(k_1,t_1), m(k_2,t_2),.......m(k_m,t_m)$ and $m(k_i,t_i) \in M(k_i,t_i)$.

Step 3

Vary i from m to 1.

Step 4

Let $C = D \oplus B_G(t)$ -----(*) where $D = (0,0,..,0)$. Step 5

If $i > 1$ replace D by C and let $i = i - 1$ and again find C using * .

Step 5

If $i = 1$ then C in * is the required graph.

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Chapter 8

Morphological operators as adjunctions

CONTENTS

8.1 Introduction

Adjunctions are pairs of operators which satisfy, some mathematical property. In mathematical Morphology Dilation and erosion are fundamental operators. These operators form an adjunction between two spaces. These operators are dual operators. In this chapter, a survey of adjunctions is materialized.

8.2 Various Adjunctions in Mathematical Morphology

8.2.1 Poset Morphological Adjunction

Let A and B two posets with two operators $\delta: A \rightarrow B$ and $\varepsilon : B \rightarrow A$. $(\varepsilon,8)$ is an adjunction if $\forall a \in A$, $\forall a \in B$, $\delta(a) \leq b \Leftrightarrow a \leq \varepsilon(b)$

δ Is lower adjoint of ε and ε is upper adjoint of δ.[5],[22]

8.2.2 Definition Lattice Morphological Adjunction

Let L, M be complete lattices, let ε: L⟶M and δ: M⟶L. The pair (ε,δ) is called an adjunction between L &M if $\delta(y) \leq X \Leftrightarrow Y \leq \varepsilon(X)$ $\forall X \in L$, $Y \in M$.[2],[4],[22]

8.2.3 Definition Gray scale adjunction

The set L = Fun(R^2, $\bar{R}$) is a complete lattice under the point wise ordering. Every translation invariant adjunction on L is of the form (ε_b, δ_b) with

$$\varepsilon_b (f) (x) = \wedge_{h \in R^2} (f(x-h) + b(h))$$

$$\varepsilon_b (f) (x) = \vee_{h \in R^2} (f(x+h) - b(h))$$

The function b is a structuring element.

8.2.4 Proposition

Let L, M be complete lattices. A pair of operators (ε, δ) is an adjunction if $\varepsilon(\wedge_{i \in I} Xi) = \wedge_{i \in I} \varepsilon(Xi)$ and $\delta(\vee_{j \in J} Y_j) = \vee_{j \in J} \delta(Y_j)$ for arbitrary collections $\{x_i / i \in I\} \subseteq L$ and $\{y_j / j \in J\} \subseteq M$.

8.2.5 Definition Adjoint upper slope transform: Given a signal f, its upper slope transform is defined as $S_V(f)(v) = \bigvee_{x \in R^d} f(x) - < x, v >$ $v \in R^d$ and its lower slope transform is $S_\wedge(f)(v) = \bigwedge_{x \in R^d} f(x) - < x.v >$, $v \in R^d$ [21],[22]

The adjoint upper slope transform $S^{\leftarrow}{}_V$ is defined as $S^{\leftarrow}{}_V g(x) = \bigwedge_{v \in R^d} g(v) + < x, v >$ is an adjunction on Fun(R^d).

i.e $S_V(f) \leq g \Leftrightarrow f \leq S^{\leftarrow}{}_V(g)$.

8.2.6 Definition Adjoint lower slope transform

Adjoint lower slope transform is defined as

$S^{\leftarrow}{}_\wedge(g)(x) = \bigvee_{v \in R^d} g(v) + < x, v >$ and $(S_\wedge, S^{\leftarrow}{}_\wedge)$ is an adjunction on Fun(R^d).

8.2.7 Definition

For a set $X \subseteq R^d$, its support function $\sigma(x)$ [4],[21] is defined by $\sigma(X)(v) = \bigvee_{x \in X} < x, v >$, $v \in R^d$ and $\sigma(X) = -\infty$ if $X = \emptyset$ and the operator $\overset{\leftarrow}{\sigma}(f)$ Fun (R^d)$\rightarrow$P(R^d) is defined as $\overset{\leftarrow}{\sigma}(f) = \bigcap_{v \in R^d} \bar{H}(v, f(v))$.

The pair ($\overset{\leftarrow}{\sigma}, \sigma$) constitutes an adjunction between Fun(R^d) and P (R^d).

i.e. $\sigma(X) \leq f \Leftrightarrow X \subseteq \overset{\leftarrow}{\sigma}(f)$

8.2.8 Definition The polar[4],[21],[22] X^o of a set $X \subseteq R^d$ is defined by $X^o = \{y \in R^d / \langle x,y \rangle \leq 1 \ \forall x \in X\}$ and the operator π is defined as $\pi(X) = X^o$. The pair (π, π) is an adjunction between $P'(R^d)$ and $P(R^d)$ and

$(\cup_{i \in I} X_i)^o = \cap_{i \in I} X_i^o, \forall X_i \subseteq R^d, i \in I$

8.2.9 Definition

Let $\gamma(X)(x) = \inf\{r>o / x \in \gamma x\}$ and

$\vec{\gamma}(t) = \{x \in R^d / \forall r>o / f(rx) \leq r\}$ then $(\gamma, \vec{\gamma})$ is an adjunction between $P^1(R^d)$ and $Fun(R^d)$.

8.3 Complete Lattice Adjunctions

8.3 .1 Definition Complete power lattice adjunction

Let L be a complete lattice and P an arbitrary non empty set. Denote the elements of the power set L^P by X. For a given $p \in P$, the value of X at p is denoted by X_p. L^P is a complete lattice, known as a complete power lattice with the point wise ordering $X \leq Y$ iff $Xp \leq Yp$, $p \in P$ wherever X, $Y \in L^P$. The pair (ε, δ) is an adjunction [1],[9],[10],[16] between the complete power lattices L^P and M^k iff $\exists$ adjunctions such that

$(\varepsilon(X))_k = \wedge_{p \in P} \varepsilon_{p,k}(X_p)$ and $(\delta(Y))_p = \vee_{k \in K} \delta_{k,p}(Y_k)$ for $X \in L^P, Y \in M^k$

8.3.2 Definition h-adjunction

Let R be a complete lattice and T be a non-empty set. Let h: T⟶R be a surjective mapping. Define an equivalence relation $=_h$ on T as follows.

$t =_h t^1 \Leftrightarrow h(t) = h(t^1)$, $t, t^1 \in T$

and $t \leq_h t^1 \Leftrightarrow h(t) \leq h(t^1)$, $t, t^1 \in T$

$\leq h$ is an h ordering[1],[9],[16]. $\overleftarrow{h}$: R⟶T is called semi inverse of h and $h\overleftarrow{h}(r)=r \;\; \forall r \in R$.

Let ε, δ be two mappings with the property that for s, t ∈T, $\delta(s) \leq_h t \Leftrightarrow \leq_h \varepsilon(t)$ then (ε,8) is called an h-adjunction.[1],[5]

8.3.3Definition

A mapping : T⟶T which is h-increasing, bijective and has an h-increasing inverse ψ^{-1} is called an h-isomorphism [1] on T, and also (ψ, ψ^{-1}) and (ψ^{-1}, ψ) are h-adjunctions on T.

8.3.4 Definition

(ε, Δ) is an adjunction on $Fun(E,T)^P$ iff ∃adjunctions $(\tilde{e}_{p,q,x,y}, \tilde{d}_{qp,y,x})$ on T,$\forall x, y \in E$ p ,q ∈P such that $(\varepsilon(F))_p(x) = \bigwedge_{p \in P} \bigwedge_{x \in E} \tilde{e}_{p,q,x,y} (F_p(x))$

$(\Delta(F))_p (x) = \bigvee_{q \in P} \bigvee_{y \in E} \tilde{d}_{qp,y,x}(F_q(y))$for F ∈ Fun $(E,T)^P$

8.3.5 Definition

An operator ψ is called an H-operator[1],[9],[10],[16] if $\psi(F_z) = [\psi(F)]_z$, $F \epsilon$ Fun(E,T)^P, $z \epsilon E$.For T=$\bar{R}$, an operator ψ : Fun (E, $\bar{R}$)$^P \longrightarrow$ **Fun (E,$\bar{R}$)P** is called a T-operator if it is an H-operator and if $\psi(F + t) = \psi(F) + t$ for $F \epsilon$ Fun (E$\bar{R}$)^P and $t \epsilon R^P$. ∴ Every T adjunction on Fun (E, $\bar{R}$)^P is given by

$$(\varepsilon(F))_P (x) = \Lambda_{q\epsilon P} \Lambda_{z\epsilon E}[F_q (x + z) - B_{p,q}(z)]$$

$$(\Delta(F)_p)(x) = V_{q\epsilon P} V_{z\epsilon E}[(F_q (x - z) + B_{q,p}(z)]$$

8.3.6 Proposition

The pair (e ,d) on R^P given by e(t) = t–b, d(t) = t +b where $b \epsilon R^P$ defines an h- adjunction iff $h(s) \leq h(t) \Leftrightarrow h(s+b) \leq h(t +b)$ for s, $t \epsilon R^P$.

8.3.7 Proposition

Let B: E$\longrightarrow$ R^P be a function such that h is an h adjunction [1],[16] for every $b \epsilon \{B(z)/z \epsilon E\}$. Then the pair (ε, Δ) of operators on Fun(E, $\bar{R}^P$) determined by the expressions $h(\varepsilon(F)(x)) = \Lambda_{z\epsilon E} h(F(x + z) - B(z))$ and $h(\Delta(F)(x) = V_{z\epsilon E} h(F(x - z) + B(z))$ defines and H-invariant h – adjunction on Fun(E, $\bar{R}^P$) .

8.4 Generalized Convolution Adjunctions

A T – invariant dilation Δ_H (F) $= F \circledast H$ can be represented via scalar dilation[9],[10],[22] as Δ_H (F)(x)=$\bigvee_{y\epsilon E} F(y) * H(x-y) = \bigvee_{y\epsilon E} \lambda_{H(x-y)}(F(y))$ where the V-translation $\lambda_a(v) = a * v$ is a scalar dilation and let $\lambda^{\leftarrow}{}_a$ be the scalar adjoint erosion of λ_a.

Therefore,

$\lambda_{H(x-y)}(v) = d_{y,x}(v), \lambda^{\leftarrow}{}_{H(x-y)}(w) = e_{x,y}(w) \Rightarrow (\lambda^{\leftarrow}{}_{H(x-y)}, \lambda_{H(x-y)})$ is the scalar adjunction of V translations with the scalar adjunction$(e_{x,y}, d_{y,x})$ and the adjoint signal erosionΔ_H is $\varepsilon_H(G)(y) = \bigwedge_{x\epsilon E} \lambda^{\leftarrow}{}_{H(x-y)}(G(x))$. If V is a blog, i.e. V_G = V– {V_{inf}, V_{sup}}is a group under '*'multiplication: let V* denote the conjugate of each scalar vϵV. ∴ The scalar adjoint erosion can be written as $\lambda^{\leftarrow}{}_a$(w)=a*w. ∴ The adjoint signal erosion becomes

$\varepsilon_H(G)(y) = \bigwedge_{x\epsilon E} G(x) * [\ H(x-y)]$

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CONTENTS

9.1 Signal Processing

Mathematical Morphology is developed from set theory and Integral geometry. It is concerned with the shape of a signal wave form in the complete time domain. Morphological filters are useful for suppressing noise [9],[10]. In signal processing ,a structuring element is using for collecting information from the signals. By moving the structuring element constantly on the signal, interrelationship among every part of the signal is obtained.

Using Morphological operators, the radical shape of the distributed signal can be recognized, reconstructed and enhanced, even if the original signal is mixed with strong noise or serious distortion.[9],[10]

Morhological transform can decompose a complicated signal into several parts that have a different physical significance .It can pick up the signal from the back ground and keep its main shape trait at the same time .[10]

Erosion of $f(n)$ by $g(m)$ is a kind of shrinking transform ,which can make the target signal contract while holes enlarging. Dually ,Dilation is an expanding process, which realizes the target signal enlarging together with holes contracting. Generally ,Erosion and Dilation are not reversible to each other .So conjugation of them can form new Morphological operators.[11]

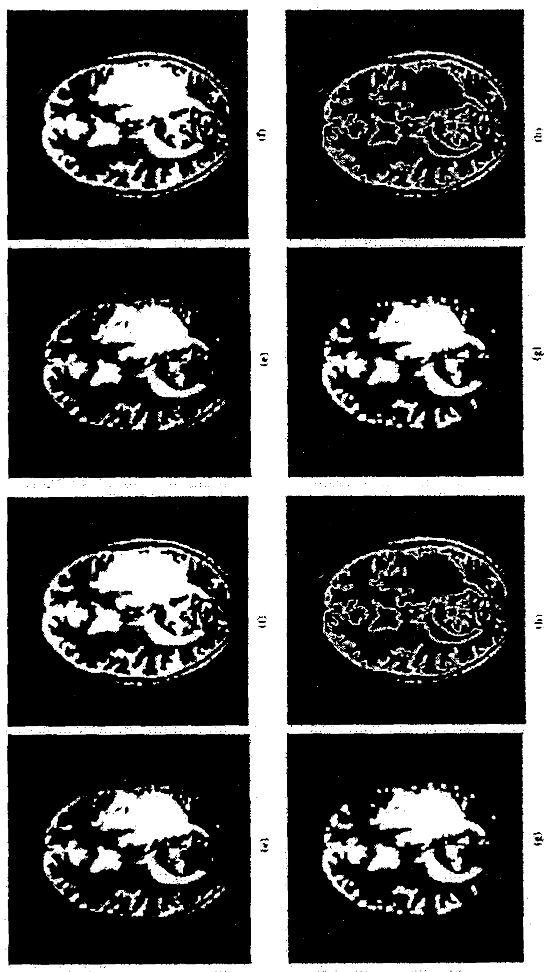
9.2 Robotics

Mathematical Morphology is a useful tool in image analysis, commonly used to extract components of the image like contours, skeletons and convex forms. In Robotics ,path planning is strongly influenced by the precision of the acquisition process .Thus it can be modified by the quality of the information obtained from the environment , and the attributes of the system and the environment in which it works. [19]Image segmentation is an essential part of any intelligent system. Since it is used in further processing .It is used in feature extraction, object or face recognition ,among others.[22]

9.3 Medical Imaging using Mathematical Morphology

Many medical imaging techniques use mathematical morphology (MM), with discs and spheres being the structuring elements (SE) of choice. Given the non-linear nature of the underlying comparison operations (min, max, AND, OR), MM optimization can be challenging.

Examples in Medical Applications :



Example of Morphological operations on MR brain image using a structuring element of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ a) The original MR brain image b) The threshold MR brain image for morphological operations c) Dilation of the threshold MR brain image d) resultant image after 5 successive dilations e) Erosion of the threshold MR brain image f) closing of the image g) opening of the image h) morphological boundary detection on the threshold MR brain image .

9.4 Oil Spills Detection

An important cause of marine pollution is oil spills. [25] Oil spills are constantly present in the main ship traffic routes. SAR images are being widely used for monitoring such a kind of pollution. However, they present some drawbacks which make difficult to develop a fully automatic oil spills detection system. A polluted area will appear in the SAR image as a zone darker than its surrounding. Therefore, we have to process the image to detect and segment dark spots. This procedure will only possible if there is contrast between the dark spots and their background. Non linear filters are more adequate for obtaining a correct estimation of the background. Morphological filters are useful for tracking the slow variations of the background while preserving the contours of the dark spots. When a pixel belongs to a dark spot, there should be a directional Structuring Element totally included in it. Then, the output of this directional closing will keep the level of the dark spot. Accurate image segmentation is implemented to extract the candidates to be oil sticks. An accurate extraction of possible oil sticks is performed .Top – hat filter is commonly using for this.

9.5 Dynamic Mathematical Morphology

Object boundaries contain important shape information in an image. Mathematical morphology is shape sensitive and can be used in boundary detection. Dynamic mathematical morphology only operates on the parts of interest in an image and reacts to certain characteristics of the region. The next position of the structuring element is dynamically selected at each step of the operation. The technique is used to detect object boundaries.

9.6 Conclusion

As a discipline mathematical morphology has its roots in the pioneering work of G. Matheron (1975) and J. Serra (1982). It is a powerful tool for solving problems ranging over the entire imaging spectrum, including character recognition, medical imaging, microscopy, inspection, metallurgy and robot vision (Matheron, 1975, Serra, 1982, Dougherty and Astola, 1994, Gonzalez and Woods, 1992, Haralick and Shapiro, 1992, Pitas and Venetsanopoulos, 1990, Serra, 1989, Serra and Soille, 1994, Maragos, et al., 1996, Heijmans and Roerdink, 1998). Morphology is now a necessary tool for engineers involved with imaging applications. Morphological operations have been viewed as filters the properties of which have been well studied (Heijmans, 1994). Another well-known class of non-linear filters is the class of rank order filters (Pitas and Venetsanopoulos, 1990). Soft morphological filters are a combination of morphological and weighted rank order filters (Koskinen, et al., 1991, Kuosmanen and Astola, 1995). They have been introduced to improve the behaviour of traditional morphological filters in noisy environments. The idea was to slightly relax the typical morphological definitions in such a way that a degree of robustness is achieved, while most of the desirable properties of typical morphological operations are maintained. Soft morphological filters are less sensitive to additive noise and to small variations in object shape than typical morphological filters. They can remove positive and negative impulse noise, preserving at the same time small details in images.

Currently, Mathematical Morphology allows processing images to enhance fuzzy areas, segment objects, detect edges and analyze structures. The techniques developed for binary images are a major step forward in the application of this theory to gray level images. One of these techniques is based on fuzzy logic and on the theory of fuzzy sets.

Fuzzy sets have proved to be strongly advantageous when representing in accuracies, not only regarding the spatial localization of objects in an image but also the membership of a certain pixel to a given class. Such inaccuracies are inherent to real images either because of the presence of indefinite limits between the structures or objects to be segmented within the image due to noisy acquisitions or directly because they are inherent to the image formation methods.

Morphological Segmentation for Character Extraction from Scene Image [24]

Source Image: Kentucky

Result Image: Kentucky

9.7 Future Prospects

Uses of Mathematical Morphology are mainly in the following areas:

1) Image enhancement 2)Image segmentation 3)Image restoration 4)Edge detection 5)Texture analysis 6)Particle analysis 7)Feature generation 8) Skeletonization 9)Shape analysis 10)Image compression11)Component analysis 12)Curve filling 13)General thinning 14) Feature detection 15)Noise reduction 16) Space-time filtering

Despite modern technologies (immunophenotyping, molecular probing, etc.) cytomorphologic examination of stained peripheral blood smears by microscopy remains the main way of diagnosis in a large variety of diseases (e.g. leukaemic disorders). Using tools from mathematical morphology for processing peripheral blood colour images, there exist an image-based approach, to provide an objective and understandable description of lymphocyte populations according to a specifically designed ontology. This ontology-based framework needs a conceptualization of the problem from a morphological viewpoint, the introduction of an adapted language, the generation of representative image databases, the development of image processing and data classification algorithms to automate the procedure and the validation of the system by human expertise.

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List Of Publications

The following papers were published/ presented as a part of the subject

1) Convex Geometry and Mathematical Morphology in a Generalized Structure, International Journal of Computer Applications (0975 – 8887), Volume 6– No.3, September 2010.
2) Morphological Systems, Proceedings of the National Seminar on Digital Topology and Morphology, Mar Thoma College for Women, Perumbavoor,2009.
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5) An Over View of Morphological Operators as adjunctions, Journal of Mathematical Imaging and Vision (Communicated).
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Appendix A

Distance Transforms

Distance measures are positive definite, symmetrical and satisfy the triangle inequality. Distances in the image can be approximated by the distances between neighbouring pixels .The well –known distance measures are City block and Chess board distance. The city block distance between two points $P=(x,y)$ and $Q=(u,v)$ is defined as : $d_4(P,Q)=|x-u|+|y-v|$.

The chess board distance between P,Q is $d_4(P,Q)=\max(|x-u|,|y-v|)$.

The Euclidean distance between two points $P=(x,y)$ and $Q=(u,v)$ is defined as: $d_e(P,Q)=\sqrt{(x-u)^2+(y-v)^2}$ .The subscripts ''4'' and ''8'' indicate the 4- neighbour and 8- neighbour and ''e'' denotes Euclidean distance.

The chess board and city block methods are useful for distance transformation. They are very sensitive to the orientation of the object. Mathematical Morphological approach to distance transform is useful for the decomposition of global operations to local operations.

Properties of Lattices

I In a lattice, the following properties hold:

Commutativity

$x \wedge y = \inf(x, y) = \inf(y, x) = y \wedge x.$

$x \vee y = \sup(x, y) = \sup(y, x) = y \vee x.$

Associativity

$x \wedge (y \wedge z) = \inf(x, (y \wedge z))$

$= \inf(x, \inf(y,z))$

$= \inf(x,y,z)$

$= \inf(\inf(x, y), z))$

$= \inf((x \wedge y), z)$

$= (x \wedge y) \wedge z.$

Now, $x \vee (y \vee z) = \sup(x, (y \vee z))$

$= \sup(x, \sup(y,z))$

$= \sup(x, y,z)$

$= \sup(\sup(x,y), z)$

$= \sup((x \vee y), z)$

$= (x \vee y) \vee z.$

Absorptivity

Now, $x \wedge (x \vee y) = \inf(x, x \vee y)$

$= \inf (x, \sup (x, y))$

$= x$ [since $x \varepsilon \sup (x, y)$]

and

$x \vee (x \wedge y) = \sup (x, x \wedge y)$

$= \sup (x, \inf (x, y))$

$= x$ [since $\inf (x, y) \varepsilon x$].

Idempotency

$x \wedge x = \inf (x, x) = x$ and $x \vee x = \sup (x, x) = x$.

If a lattice L under the operations $\wedge$, $\vee$ satisfies all the above properties then

($L, \wedge , \vee$) is an algebraic lattice.

Granulometry

Convex structuring elements are scalable: Dilation / erosion with structuring

Element nB is equivalent to n dilations / erosions with structuring element B.

Scalability is important for implementations in hardware .Rectangular structuring elements are separable. Multiple openings or closings with the same structuring element do not alter the image any more (idempotency). Opening and closing are so-called sieve operations .

Using structuring elements of increasing size, one can remove small-, middle- and coarse-scale structures step by step. Such a morphological image decomposition into structures of different size is called granulometry.

Pattern Spectrum

Pattern Spectrum is known as granulometric size density. It is employed to measure the size distribution of an object.

Pattern spectrum $PS_{r_i,k}(F)$ of a set F in terms of SE $r_i k$ is defined as: $PS_{r_i,k}(F) = \begin{cases} Card((F \circ r_i k) - (F \circ r_{i+1} K)), i \geq 0 \\ Card((F \bullet r_{-i} k) - (F \bullet r_{-i-1} K)), i < 0 \end{cases}$ where ***Card(F)*** denotes the cardinality of set F

Recursive Dilation

Recursive Dilation is defined as: $F \overset{i}{\oplus} K = \begin{cases} F, i = 0 \\ (F \overset{i-1}{\oplus} K) \oplus K, i \geq 1 \end{cases}$ where i is defined as scalar factor and K as its base.

Recursive Dilation is employed to compose SE series in the same shape but different sizes.

Recursive Erosion

Recursive Erosion is also called successive erosion which is defined as:

$$F \overset{i}{\Theta} K = \begin{cases} F, i = 0 \\ (F \overset{i-1}{\Theta} K) \Theta K, i \geq 1 \end{cases}$$

When performing recursive erosions of an object, its components are progressively shrunk until completely disappeared. It is Useful for distance transform and segmentation.

Fourier Transforms and Morphological Slope Transforms – A comparison

There exists a morphological system theory that resembles linear system theory.

The slope transform is the morphological analogue to the Fourier transform.

It transforms (tangential) dilation into addition. Parabolas / paraboloids as structuring functions are the morphological analogues to Gaussians in linear system theory. In linear system theory, Gaussian convolution plays a fundamental role: Gaussians remain Gaussians under the Fourier transform. Gaussians are the only separable and rotationally invariant convolution kernels.

Analog results for morphology:

Paraboloids remain paraboloids under the slope transform. Paraboloids are the only structuring functions that are separable and rotationally invariant. Morphological _filters are invariant under monotonously increasing grey scale transformations. By replacing a grey value by its maximum or minimum within a neighbourhood, dilation and erosion are obtained. Dilation and erosion are used for shape analysis. Sequential combinations of erosion and dilation create openings and closings. They act as morphological low pass filters. Granulometries are examples for morphological band pass filters. Top hats result from computing differences between closing, original image, and opening. They act as morphological high pass fillers.

APPENDIX B

Operators In Software

Creating a structuring element

The following function creates structuring elements. It creates standard structuring element.

```
IplConvKernel* cvCreateStructuringElementEx(int cols,
                                            int rows,
                                            int anchor_x,
                                            int anchor_y,
                                            int shape,
                                            int* values=NULL)
```

where cols and *rows* is the number of columns and rows in the structuring element

anchor_x and *anchor_y* point to the anchor pixel. The pixel that is checked for when the transformation should be made or not.

shape - choose from three standard structuring elements.

CV_SHAPE_RECT

CV_SHAPE_CROSS

CV_SHAPE_ELLIPSE

CV_SHAPE_CUSTOM

Set *shape* to *CV_SHAPE_CUSTOM*, also supply the custom element. This is done using values. This parameter is used only if *shape* is set to custom.

values should be a 2D matrix, corresponding to the structuring element itself.

If values is NULL (and *shape* is custom), then all points in the structuring element will be considered nonzero (a rows*cols sized rectangle).

Dilation

This operation is the basic building block of morphology. The function is.

```
void cvDilate(const CvArr* src,
        CvArr* dst,
        IplConvKernel* element=NULL,
        int iterations=1);
```

The function takes four parameters:

src: The image to dilate

dst: This is where the dilated image is stored *element*: (optional) The structuring element (use cvCreateStructuringElementEx to create one). If not specified, a 3×3 square is used.

iterations: (optional) Number of times to dilate *src*. If not specified, this is set to 1.

Use the same image as *src* and *dst*.

Erosion

Erosion is also a basic function of morphology. The function is:

```
void cvErode(const CvArr* src,
          CvArr* dst,
          IplConvKernel* element=NULL,
          int iterations=1);
```

The parameters are the same as dilation. Perform erosion instead of dilation.

src: The image to erode

dst: This is where the eroded image is stored

element: (optional) The structuring element (use cvCreateStructuringElementEx to create one). If not specified, a 3×3 square is used.

iterations: (optional) Number of times to erode *src*. If not specified, this is set to 1.

This is also an in-place operation. *src* and *dst* can point to the same image.