GEOMETRIC ALGEBRA
AND
EINSTEIN'S ELECTRON: DETERMINISTIC FIELD THEORIES

A THESIS
SUBMITTED TO
THE COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE AWARD OF THE DEGREE OF
DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE
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JUNE 2010 (REVISED OCTOBER 2012)

Dedicated to the memory of
Sri Ram Dulare Pandey
(Dadaji: My father's elder brother)
and
Smt. Nalini Chakravarti
(Wife of my supervisor)

Acknowledgements

I acknowledge all possible help provided by my supervisor Dr. R. S. Chakravarti, Reader and Head, Department of Mathematics, Cochin University of Science and Technology. I am grateful to him for initiating me into this area of research and for his sincere guidance and non terminating inspiration. I found him not only a good teacher but also a good man as well as a good friend and all this definitely made my life easier in this city of Cochin. Without this the research could not have been accomplished.

I express my sincere thanks to all the teachers, non-teaching staff, the librarian and research scholars of this department for their kind co-operation.

I kindly acknowledge the financial assistance given by University Grants Commission (UGC), New Delhi throughout my research as JRF for the initial two years and SRF for the later years.

I owe very much to my parents, brothers, sisters and especially to my wife and daughter for the moral support they extended for the last five years.

I thank everyone who helped me in preparing this thesis.

I hereby thank the three examiners for studying and assessing this thesis. I have been shown two examiners' reports and I thank these examiners for pointing out omissions and obscurities. Their comments have greatly improved the thesis.

Above all I acknowledge the grace and kindness of my God Lord Shri Ram Ji due to which, I personally feel that I overcame all the difficulties and the realization of the thesis became possible.

Santosh Kumar Pandey

CERTIFICATE

This is to certify that the work reported in this thesis entitled 'Geometric Algebra and Einstein's Electron: Deterministic Field Theories' is a bonafide record of the research work carried out by Mr. Santosh Kumar Pandey under my supervision in the Department of Mathematics, Cochin University of Science & Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Cochin-22
June 2010

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DECLARATION

This thesis entitled *Geometric Algebra and Einstein's Electron: Deterministic Field Theories* contains no material which was accepted for the award of any other degree or diploma in any University and to the best of my knowledge and belief it contains no material previously published by any person except some to which due reference is made in the text of the thesis.

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CHAPTER 1

Introduction

"You know, it would be sufficient to really understand the electron."

Albert Einstein [**2**]

"If I can't picture it, I can't understand it."

Albert Einstein [**41**]

"I think it is quite likely, or at any rate quite possible, that in the long run Einstein will turn out to be correct."

P. A. M. Dirac [**7**]**,** [**38**]

1.1. General Introduction

In this thesis I discuss the work of two physicists, Toyoki Koga (1912-2010) and Mendel Sachs (1927-2012), on the foundations of Quantum Mechanics.

Both of them offer alternatives to the conventional Copenhagen interpretation and each explains, in his opinion, what the Copenhagen interpretation is all about in terms of his theory.

I do not take up these matters here but concentrate on the parts of their work related to the Geometric Algebra of W. K. Clifford and David Hestenes (in the case of Sachs it turned out that the relationship was somewhat weak,).

The title of this thesis refers to Albert Einstein's longstanding interest in the structure of the electron. Einstein is generally considered to be one of the foremost physicists of recent times and one of history's greatest scientists. He was a major early contributor to quantum theory through his work on the photoelectric effect published in 1905. Later, in the 1920's, an axiomatic theory of quantum mechanics came into being, founded on uncertainty and indeterminism. It was accepted by most physicists but Einstein was one of the dissenters.

It is well known that Einstein did not have a clear idea of the structure of the electron. But he had some views on the subject.

The title is only meant to indicate that it studies theories which he might have found acceptable.

Some other founders of quantum theory were also unhappy with quantum mechanics: Planck, Schrödinger and de Broglie. Each had his own point of view but all of them rejected concepts like wave-particle duality, the uncertainty principle and quantum jumps. Einstein believed that there was an underlying deterministic theory waiting to be found.

Most of the dissenters eventually lapsed into silence but Einstein continued searching for a unified field theory for several decades until his death in 1955. Such a theory was to include general relativity and quantum mechanics as special or limiting cases.

Meanwhile, quantum mechanics developed into quantum field theory, a theory which very successfully explains all known physical interactions except gravity.

Attempts to create unified field theories continue. These can be classified into two groups: those that are founded on quantum mechanics (and/or quantum field theory) and attempt to "quantise" general relativity, and other theories that are founded on general relativity and try to make quantum theory consistent with general relativity. The former include string theory, various versions of quantum gravity and noncommutative geometry. Some of the world's best known physicists and mathematicians are interested in these subjects and related areas.

It is noteworthy that P.A.M. Dirac, one of the key figures in the founding of quantum mechanics and quantum field theory, was, by 1970, a dissenter. After that he was not taken seriously by mainstream physicists.

This thesis is concerned with the second group of theories mentioned above. The most famous worker in this area was, of course, Einstein but there have been many others.

Even before Einstein developed the theory of relativity, there were attempts by Lorentz, Poincare, Abraham, Mie and others to describe the electron as an electromagnetic field. These can be considered precursors of deterministic field theories based on general relativity. Still earlier, it had occurred to Faraday that electromagnetism and gravitation may be different aspects of the same phenomenon.

After general relativity was formulated by Einstein and Hilbert, several attempts were made to create unified field theories. Some of the early contributors include Kaluza, Klein, Schrodinger and the great mathematician Hermann Weyl. One worker was H. T. Flint (now virtually unknown) who published in well known journals from the 1920s to the 1960s. R. Penrose advocated a unified field theory based on general relativity recently but was criticised by the “mainstream”.

There has been some more work on deterministic theories of matter: the de Broglie-Bohm theory has gained some acceptance. The paper of U. Enz [**13**] studied a non-relativistic deterministic field model of the electron.

In this thesis we introduce the work of two physicists: Toyoki Koga and Mendel Sachs. Both of them have given deterministic field theories of the electron and other “particles”, along lines that might have been pursued by Einstein.

Our coverage is far from exhaustive, with emphasis on topics related to Geometric Algebra. We discuss only a few basic aspects of their theories.

The original plan suggested by my guide was to study the solutions to Sachs’s equations (which generalise the Dirac equation using Sachs’s ideas about General Relativity) and compare them with Koga’s results. With hindsight, this was quite unrealistic.

In retrospect, the decision to study Sachs was unfortunate (as will eventually become clear to the reader) but we realised it too late.

Koga considers the Schrödinger equation for a single electron and shows that it has a solution which represents a localised field which he calls an elementary field (in his earlier papers, he used the term wavelet for elementary field). A conventional de Broglie wave is, for him, a fictitious representation of an ensemble of elementary fields.

He infers that the usual solutions of the Schrödinger equation refer to ensembles, as do the Uncertainty Principle, Wave-Particle Duality and so on. As mentioned earlier, the latter topics are not the subject of this thesis.

Similarly, Koga obtains a solution of the Dirac equation which describes a localised field. He considers a free electron with positive energy, which is assumed to be a constant of nature. The purpose is not to compete with or replace the work of Dirac and others but rather to lead to a field theory of the electron including gravitation as an essential component, not an add-on.

Along the way, Koga tries, using his solution, to give a geometrical picture of the electron including spin. In this I believe his work was incomplete, as I explain in Section 2.3. I have applied Geometric Algebra to Koga's solution and obtained surprising results.

The term Geometric Algebra refers to the geometrical study of Clifford algebras which was started in the 1960s by David Hestenes. In the 1860s, W. K. Clifford studied a class of associative algebras which unified the earlier work of W. R. Hamilton on quaternions and H. Grassmann on exterior algebras. The subject of applied Clifford algebras, or geometric algebra, which was pioneered by Hestenes (with a view that even if it does not describe new physics, it should give a new insight into the underlying physics and improve our understanding) has exploded in recent years. We translate Toyoki Koga's treatment of the Dirac equation into the language of geometric algebra. This gives a new insight.

In the past, starting with David Hestenes [**15**], Doran and Lasenby [**8, 9, 10**] and many others have studied the conventional theory of the electron using geometric algebra. This work is possibly the first one in which geometric algebra is used to study a deterministic theory of the electron.

When I wrote this thesis I was not aware of the very recent work of Hiley on the application of Geometric Algebra to the de Broglie-Bohm theory of the electron. It seems that Hiley does not consider any specific solution of the Dirac equation.

This thesis can be considered a contribution to Mathematical Physics (chapter 3) and to the History and (possibly) Philosophy of Science (chapters 2, 4, and 5 and part of chapter 3).

The Copenhagen interpretation of Quantum Mechanics is generally held to be an established fact. Koga has contributed to the Philosophy of Science by studying an alternative approach. The work in this thesis clarifies an important part of Koga's theory.

It may be asked why this work has been undertaken by someone who is not trained in these areas. One answer is that so far the Physics community has completely ignored the subjects I have dealt with. Koga is practically unknown, although one of his papers has been plagiarised in a journal from South America. Sachs is relatively well known but nobody seems to have made a serious study of his work. The study by Cyganski and Page [**6**] was apparently discontinued (I suggest a possible reason in Section 4.3). I hope I have contributed to a change in this state of affairs.

Both my supervisor and I had not done any work in Physics earlier. We were also not aware of any work similar to this thesis done by others. I am possibly the first in India to take up such topics.

Coming from Mathematics, we were also unaware that published work in Physics can contain serious errors. It took us years to realise this.

Thus, I made rather slow progress initially. This seems to be a consequence of attempting something totally new.

1.2. Summary of the Thesis

For further information, see the Table of Contents.

Chapter 2 is an introduction to Toyoki Koga's work published in the papers [**24, 25, 23, 26, 21, 22**] and the books [**27, 28**]. Following a path initiated but abandoned by de Broglie in the 1920s, Koga found that the Schrodinger equation for the electron has a solution that represents a localised field (in Galilean, i.e., classical spacetime). He gave a similar solution to the Klein-Gordon equation (in Minkowski space).

Using the latter he obtained a solution to the Dirac equation. Instead of the conventional spinor interpretation of the solution he asserts that the solution consists of four complex scalar functions. The coefficients (Dirac matrices) are assumed to transform as a 4-vector under coordinate (i.e., Lorentz) transformations.

A generalisation of this approach has been used by Fock and others (around 1930) to construct a Dirac equation in curved space. Koga does not appear to have been aware of this old research.

He concludes that the solution represents an anisotropic electron field localised in spacetime. But his arguments are mathematically not complete or satisfactory. He conjectures that the electron field has a form similar to that of a spinning top which has its axis of rotation in a fixed direction. He does not give clear details or mathematical justification for the spin. I take up these issues in chapter 3 and provide the required mathematical argument. I also obtain some significant additional information.

It was my intention to study geometric aspects of the theories of Koga and Sachs. Other topics are not covered in detail; in the very brief accounts given here, it is not possible to be self-contained.

Koga also developed a theory of the electron incorporating its gravitational field, using his substitutes for Einstein's equation. We do not go into this in detail although it was his main goal.

The third chapter deals with the application of geometric algebra to Koga's approach of the Dirac equation. We give a brief history of the concept of spin. Following Koga (but using geometric algebra) we solve the Dirac-Hestenes equation. Our solution suggests that the electron is a localised field which spins and shudders. Most of this material appears in [**33**]. The treatment here is corrected and slightly revised.

Keeping in mind Koga's experience that standard journals prefer standard viewpoints, the paper [**33**] was submitted to a so-called nonstandard journal established by Louis de Broglie and run (so far) by his students.

It is found that the electron field is anisotropic; it has an axis of symmetry and the electron spins about this axis which has a fixed direction for a free electron. The electron has a constant angular velocity of the order of 10^{21} radians per second. Using two possible Dirac-Hestenes equations obtained from the Klein-Gordon equation, we show that for both positive and negative energy there are two possible orientations (up and down).

In chapter 4 we study some aspects of the work of Mendel Sachs [**35, 36, 37**]. Sachs's stated aim is to show how quantum mechanics is a limiting case of a general relativistic unified field theory. Using 2-spinors and quaternions, Sachs tries to factorise the field equations of general relativity in a manner similar to the process of obtaining the 2-spinor Dirac equations from the Klein-Gordon equation. (Incidentally, we reveal an error in Sachs's equations that may have been overlooked so far, but is quite serious, even fatal).

An examiner has pointed out that Sachs (like lots of physicists) has misunderstood Mach's principle. I assumed that the statement given by Sachs was exactly what came from Mach. This makes most of my comments on this topic in the thesis irrelevant. But the conclusion that Mach's principle can very well be ignored remains valid.

The error by Sachs mentioned above pertains to the limiting values of the metric coefficients as curved space becomes flat. I discuss this in detail in Chapter 4.

According to Koga, the free electron has a definite axis of symmetry and spin. This differs from the conventional (Copenhagen) viewpoint.

Chapter 5 contains a critical study and comparison of the work of Koga and Sachs. In particular, we conclude that the incorporation of Mach's principle is not necessary in Sachs's treatment of the Dirac equation.

Koga and Sachs use similar concepts of electronic mass. In Koga's fundamental equations, constants such as mass m, charge e, and Planck's constant h do not appear. Similarly these constants do not occur in Sachs's general relativistic Dirac equation. Both the theories suggest that the laws of nature are governed by non-linear equations, though the observed phenomena are described by linear equations like the Dirac and Maxwell equations which are good approximations of the non-linear equations in most situations.

Some open problems are also mentioned in this chapter.

According to several authors, writing from the 1920s to the present time, electron spin has not been observed, although it has been used to explain various phenomena and now has many technological applications. Thus, a very important open question is whether spin can actually be experimentally observed and, if so, how.

The work in this thesis suggests that the electron does spin about an axis as Koga states.

Another theme from the 1920s is that in the description of electron spin obtained at that time, the concrete picture of rotation was replaced by an abstract mathematical representation; visualisation or visualisability was entirely lost. The work described here takes a step towards restoring this.

CHAPTER 2

Toyoki Koga's "Foundations of Quantum Physics"

"If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be? What is the structure which can appear under probing with electromagnetic fields as a point charge, but as far as spin and wave properties are concerned exhibits a size of the order of the Compton wave length?"

A. O. Barut [**3**]

"We have perhaps forgotten that there was a time when we wanted to be told what an electron is. The question was never answered. No familiar conceptions can be woven round the electron, it belongs to the waiting list."

A. Eddington [**11**]

2.1. Introduction

In this chapter we give a brief introduction to the work of Toyoki Koga on the foundations of quantum physics. Koga has as his aim the development of a field theory of the electron and other particles that includes gravitation but in which the equations do not contain Planck's constant and the mass and charge of the electron. The "fundamental equations" are covariant in a non-Minkowski sense and their solutions are expected not to have singularities.

Koga accepts the Schrödinger equation for a single electron because of its many successful predictions. But he does not accept the superposition principle. He obtains a new solution with a deterministic interpretation. He shows that a conventional de Broglie wave represents an ensemble of free electrons. Thus, a de Broglie wave does not describe an individual electron but merely an "average" electron.

Koga starts with an analysis of the Schrödinger equation which has a solution (he calls it an elementary field) representing a field that is stable and localised in space. A conventional wave function can be obtained by averaging over an ensemble of elementary fields.

Koga's solution of the Schrödinger equation starts with a modification of old work of de Broglie. It is a field with a singularity. He interprets it as meaning that the electron is a localised field centred around a point. However, he believes that the singularity should not really be there and that it can be removed by considering a suitable nonlinear equation rather than the linear Schrödinger equation.

Koga's book [**27**] contains detailed arguments against the Copenhagen interpretation of Quantum Mechanics and its explanation in terms of his theory. I omit these completely as I have not done any work on this aspect.

It seems to me that Koga's work substantiates Einstein's view that the electron is a localised field and there is a deeper underlying theory. This theory implies that the conventional eigenvalue solutions of Schrödinger's equation stand for ensembles of electrons in static states, rather than individual electrons.

Einstein believed that Quantum Mechanics, namely the Schrödinger equation, was a purely statistical theory which applied only to ensembles. But Koga exhibits a solution (for a single electron) which represents a localised field.

The Schrödinger equation does not describe the electron completely. It is not consistent with Special Relativity. In fact, what Schrödinger first obtained (and discarded) was the Klein-Gordon equation (as it was later called). Later Dirac, by factoring the Klein-Gordon equation, obtained an equation that described the electron relativistically and happened to require electron spin.

Next, Koga makes a similar analysis of the Dirac equation. Here the elementary field consists of four scalar complex valued functions on spacetime. The Dirac matrices transform as a 4-vector. They reflect the anisotropy of the electron. (In chapter 3 we use Geometric Algebra to study the elementary Dirac field in detail.)

Koga uses the formula he obtained earlier (in connection with the Schrödinger equation) to also solve the Klein-Gordon equation which, as is well known, leads to a solution of the Dirac equation. He then interprets it deterministically and reinterprets electron spin as a real phenomenon in physical space rather than just an abstract mathematical property.

The Dirac equation, like the Schrödinger equation, is linear and does not completely reveal the electron structure (its elementary field solution has a singularity, just like the Schrödinger field) Hence Koga develops a set of nonlinear equations including gravitational effects, for a collection of functions that, under certain assumptions of approximation, lead to the concepts of mass and charge. These equations reduce to the Maxwell-Lorentz equations and to the Dirac equation under different limiting assumptions. Hence he calls them the fundamental equations. He uses the same equations, with modified boundary conditions, to study the photon and neutron and the mechanism of strong interaction. We describe in this chapter only the Schrödinger and Dirac elementary fields. For completeness, we give a brief account of the general relativistic theory.

An excerpt from the preface of [**28**] summarises Koga's goals and philosophy. These matters are not discussed in this thesis. They are covered in Chapters VI-IX of [**27**] (Chapter VI is based on the paper [**26**]) along with his ideas on the photon, quantum-electrodynamical phenomena and other elementary particles (which are all fields according to Koga).

"If the principle of general relativity is upheld as fundamental in physics, it appears, according to Einstein, that the unification of theories of matter is to be made in a theory of fields. The governing partial-differential equations are to be non-linear and inhomogeneous, and to contain no symbols representing quantum, mass and charge. These constants are to be interpreted only as almost-invariant integrals of fields, and the concept of action at a distance is an auxiliary device compensating their deficiencies in representing the reality. Thus, the foundations of physics may conceptually be purified and simplified. At the same time, however, it is recognized that those fields cannot be defined in any directly operational sense, and their connections to observed phenomena are to be made only after a process of reorganizing the conceptual structure; that is to introduce again those old and once abandoned concepts such as quantum, mass and charge, on the understanding that the use of them is merely tentative for ad hoc and pragmatic purposes."

2.2. The Schrödinger equation

In 1926 Schrödinger gave the following equation for the electron

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi - U\psi = 0$$

where U is the potential energy and m the mass of the electron (assumed to be a particle for the time being). Substituting

$$\psi = a\exp(iS/\hbar)$$

where a and S are real functions of t and the position vector $\mathbf{r} = (x, y, z)$, one gets, on taking real and imaginary parts,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + U - \frac{\hbar^2\nabla^2 a}{2ma} = 0$$

and

$$\frac{\partial a^2}{\partial t} + \text{div}\,(\frac{a^2\nabla S}{m}) = 0.$$

The fourth term of the real part above was called the quantum potential by David Bohm. If it is absent, that equation becomes the Hamilton-Jacobi equation with trajectories given by

$$\frac{d\mathbf{r}}{dt} = \frac{\mathbf{p}}{m}, \ \frac{d\mathbf{p}}{dt} = -\text{grad}\, U,$$

where $\mathbf{p}$, the momentum, is defined to be ∇S and the energy of the electron is $E = -\frac{\partial S}{\partial t}$ which yields

$$E = \frac{p^2}{2m} + U.$$

In the general case ($\hbar \neq 0$) we get

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{r}} - \left(\nabla U - \frac{\hbar^2}{2m}\nabla(\frac{\nabla^2 a}{a}) \cdot \frac{\partial}{\partial \mathbf{p}}\right)\right) a^2 = 0$$

where the independent variables are $\mathbf{r}$, $\mathbf{p}$ and t. Koga calls this the Liouville equation of a quantum mechanical particle. Trajectories are obtained by taking $\mathbf{p} = m d\mathbf{r}/dt$. The energy is $E = -\partial S/\partial t$ which yields

$$E = \frac{p^2}{2m} + U - \frac{\hbar^2 \nabla^2 a}{2ma}.$$

This is invariant on a trajectory.

Koga points out that there is some resemblance between the present theory of the Schrödinger equation and the theory of de Broglie and Bohm [**19**]. But in Koga's theory, the de Broglie wave, rather than being a real wave guiding the electron, is a fictitious wave constructed by superposing an ensemble of similar and independent elementary fields.

I do not go into these aspects in the thesis.

Now consider a free electron: $U = 0$ up to a constant which we ignore. By a suitable choice of inertial frame, we take $\mathbf{p} = 0$, and assume $\partial a^2/\partial t = 0$. Then the Liouville equation gives $\nabla(\nabla^2 a/a) = 0$ and so

$$\frac{\nabla^2 a}{a} = K \quad \text{(constant)}$$

which has a solution (assuming $0 < K = \kappa^2, \kappa > 0$)

$$a = \frac{\exp(-\kappa r)}{r}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. In the 1920s, de Broglie worked on these lines (but took $K < 0$).

Here $S = -Et$, $E = -\hbar^2\kappa^2/2m$. In general, if the electron has velocity $\mathbf{v}$, we have

$$E = \frac{mv^2}{2} - \frac{\hbar^2\kappa^2}{2m},\ S = -Et + m\mathbf{v}\cdot\mathbf{r}.$$

It should be kept in mind that this solution to the Schrödinger equation is not considered a quantum-mechanical state but is supposed to give a pointwise description of the electron field. There is no superposition principle.

On reading the argument above, it may appear that the energy of a free electon at rest, according to this theory, is negative. But Koga explains that the expression for energy here (in his theory of the Schrödinger equation) is obtained by ignoring the relativistic rest-mass energy in the Dirac equation. In Koga's interpretation, the free electron is a localised field (which he calls an elementary field) rather than a point particle. The function that represents it has a singularity. Presumably, if in reality there is no singularity (the view of Einstein and Koga) then the representation is an approximate one. The reason given by Koga is that a linear equation like the Schrödinger equation cannot perfectly describe reality.

Koga explains how an ensemble of free elementary fields is represented by a de Broglie wave. He considers the case when all have the same velocity, but his result can be extended to the case of several velocities.

According to Koga, a de Broglie wave only represents an ensemble of electrons, not a single particle. For a system of several electrons, one must consider a separate Schrödinger equation for each of them. There is no such thing as the wave function of a system.

Wave-Partice Duality has the following meaning in Koga's theory: an electron looks like a particle in some experiments and like a wave in others. But it is neither; it is a localised field. Koga explains these things in [**21, 22**] and Chapter IV of [**27**]. See also [**28**].

According to Koga's theory of the Schrödinger equation, a free electron has a spherically symmetric amplitude and has a singularity at its centre. The size is not given by the theory; Koga takes it to be about $1/\kappa$, and assumes $1/\kappa = \hbar/mc$. In the presence of an external potential, the elementary field deforms and there is a "tunnel effect" arising from the interaction between the elementary field, the external electromagnetic field and the internal gravitational field of the electron. This is related to the stability of atoms. (The latter topic is taken up by him later.)

It should be noted that the energy of the (free) electron and κ are supposed to be constants of nature. Koga assumes that κ is closely related to the size of the electron field. As r increases, a decreases to 0. From some point onwards, a can be considered "negligible"; this can be taken to be a bound on the electron's radius. The larger the value of κ, the smaller the radius.

Koga argues that from these considerations, it follows that an electron in an atom is at rest relative to the nucleus, and this "tunnel effect" is complete or maximal when the electron is in an energy eigenstate of the atom. We do not go into details in this brief introduction.

Although, for the purpose of determining the electron radius, we consider a to be negligible beyond some distance from the centre, it should be noted that at any point, the Maxwell field (which Koga obtains from the Dirac field) is the sum of the fields due to individual particles. For this purpose the value of a (of each particle) is not negligible.

2.3. The Dirac equation

In this section we describe Koga's treatment of the Dirac equation. In Chapter 3 we first review the history of the spin concept and then study Koga's theory using Geometric Algebra, which reveals some new information.

Around 1930 several researchers (e.g., Schrödinger, Fock, Ivanenko) tried to modify the Dirac equation to satisfy General Relativity. They realised that a change was needed in the concept of spinor. According to their theory, the components of a spinor field at each point depend only on the point and are invariant under coordinate transformations. But at each point a collection of four independent vectors, called a tetrad, is defined and is a continuous function of spacetime. The spinor transformation law at each point is valid with respect to tetrad rotations rather than coordinate transformations. (See the books by Anderson [**1**] or Lord [**29**].) Remarkably, Koga's ideas on the Dirac equation can be considered a special case of this approach for flat space and tetrads consisting of orthogonal unit vectors related by Lorentz transformations. Koga gives no indication that he was aware of the old work mentioned above. It seems he was not.

As in the Schrödinger case, in Koga's theory of the Dirac equation the solution is not a quantum-mechanical state but gives the properties of the electron field at each point of spacetime.

Koga shows that at points far from the electron the (Dirac) ψ field reduces to an electromagnetic field, satisfying Maxwell's equations, if properly interpreted (see [**25**] or [**27**], Chapter V).

For an electron in an (external) electromagnetic field, the Dirac equation is, in modern notation,

$$i\hat{\gamma}^{\mu}(\partial_{\mu} - ieA_{\mu})|\psi\rangle = m|\psi\rangle$$

where, in the Copenhagen interpretation, $|\psi\rangle$ is considered a 4-spinor and the matrices $\hat{\gamma}^{\mu}$ are invariant under Lorentz transformations. Koga argues that $|\psi\rangle$ must be a collection of four scalar functions of spacetime and the Dirac matrices must transform as the components of a 4-vector (a generalisation of this was used by Fock and others in the 1920s in an attempt to make the Dirac equation general relativistic). This will ensure that the anisotropy embodied in the equation (his words) does not rotate together with the coordinate system when we make a coordinate transformation.

Koga gives a solution to the Dirac equation for a free electron ($A_\mu = 0$) starting with a solution to the Klein-Gordon equation

$$\left(\hbar^2 \frac{\partial^2}{\partial t^2} - \hbar^2 c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + m^2 c^4\right) \varphi = 0$$

where we work in Minkowski space: a point is given by $(ct, x, y, z) = (x^0, x^1, x^2, x^3)$ and the metric is $\eta_{ij} dx^i dx^j = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

Koga studies the Klein-Gordon equation like he did the Schrödinger equation. He gives the following solution:

$$\varphi = a \exp(iS/\hbar)$$

$$\text{where} \quad S = -Ect + \mathbf{p} \cdot \mathbf{r}$$

$$\text{and} \quad a = \frac{\exp(-\kappa |\mathbf{r}'|)}{|\mathbf{r}'|}.$$

Here $\mathbf{r}' = \dfrac{\mathbf{r} - \mathbf{u}t}{\sqrt{1 - u^2/c^2}}$ ($\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} =$ position vector),

$\mathbf{p} = \dfrac{\mathbf{u}E}{c}$ ($\mathbf{u} =$ 3-velocity of the electron),

$E^2 = \dfrac{m^2 c^2 - \hbar^2 \kappa^2}{1 - u^2/c^2}$ ($cE =$ energy).

As in the case of the Schrödinger equation, here also E and κ are closely related. But the energy of the electron is now cE.

The energy cE is defined by $E = -\partial S/\partial(ct)$. The momentum $\mathbf{p}$ is ∇S (here ∇ is the 3-dimensional gradient).

Koga gives a justification for considering E to be independent of t and $\mathbf{p}$ independent of $\mathbf{r}$. I omit it here.

For a free electron at rest, we have $E^2 = m^2c^2 - \hbar^2\kappa^2$. Since κ is a constant of nature, so is E^2.

The Klein-Gordon equation yields the Dirac equation as follows: it can be written as

$$D_0 D_1 \varphi = 0 = D_1 D_0 \varphi.$$

In Koga's notation, we take

$$D_0 = \beta(\hbar\frac{\partial}{\partial t}) + \beta\boldsymbol{\alpha}\cdot i\hbar c\frac{\partial}{\partial \mathbf{r}} - mc^2,$$
$$D_1 = \beta(\hbar\frac{\partial}{\partial t}) + \beta\boldsymbol{\alpha}\cdot i\hbar c\frac{\partial}{\partial \mathbf{r}} + mc^2.$$

Here β is a 4×4 matrix and $\boldsymbol{\alpha}$ is a triple of 4×4 matrices satisfying well-known commutation relations.

Then, if φ is a solution of the Klein-Gordon equation (more precisely, a 4-tuple of scalar solutions) and $|\psi\rangle = D_1\varphi$ then $|\psi\rangle$ satisfies $D_0|\psi\rangle = 0$, which is the Dirac equation.

It should be noted that although Koga uses D_0 to get the Dirac equation it would be equally justified to use D_1 to write an equation for the electron. For each of these "Dirac equations", there are two possible energies: positive and negative.

Koga takes $\varphi_j = a\exp(\frac{iS}{\hbar})A_j\exp(i\theta_j)$ for $j = 1, 2, 3, 4$ as four solutions to the Klein-Gordon equation, and

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = D_1 \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix}.$$

Koga concludes that the field representing a free electron (as above) is circularly symmetric only about the axis which passes the point $\mathbf{r} = \mathbf{u}t$ and is parallel to the z-axis. We omit the details. His arguments are mathematically not complete or satisfactory [**32**] as will be explained shortly.

Koga's solution to the Dirac equation, as I understand it, is not meant to be a replacement of the work done by Dirac in the 1920s, which culminated in the prediction of the positron.

Rather, it is an improvement of Koga's solution of the Schrödinger equation and is a step towards his goal of a theory including gravitation.

It gives information about the electron field, including the field in the very small region which many consider a point. He wanted, among other things, to prove that a free electron has an axis of symmetry and spins around it like a top.

For this purpose, Koga considers a rotation of the coordinate system about the z axis by an angle φ. He looks at what happens to the four complex components of the solution: ψ_1, ψ_2, ψ_3, ψ_4.

For instance, the expression for ψ_1 has four terms, each containing only one of the angles θ_j in its argument.

After the rotation, he shows that in the case of ψ_1, θ_4 gets replaced by $\theta_4 - \varphi$.

But at this point, he asserts that since the angles θ_j are arbitrary, we can simply ignore the effect of φ!

I consider this unsatisfactory.

In chapter 3 we analyse the Dirac equation and Koga's solution using geometric algebra. Our analysis suggests that the electron spins and shudders. However, there is no interference between states; in fact there are no states in Koga's theory, only pointwise descriptions of the electron field.

The theory discussed above is, of course, only approximate. It suggests a single spin frequency although, in reality, there may be terms with several frequencies.

Koga shows that the Dirac equation implies the Maxwell-Lorentz equation of the electromagnetic field, provided the comparison of the two is restricted to their time-independent (or slowly varying) solutions. The reason for this restriction is that the Dirac field may contain some high frequency terms which are averaged out in the Maxwell fields (he uses the term "coarse-grained") . He derives the correct value of the magnetic moment of the electron (which, as is well known, equals the Bohr magneton). The only assumption he makes here is that the Dirac field is localised, and no specific solution is used.

2.4. Koga's general relativistic theory of the electron

Koga considers the Schrödinger equation, and also the Dirac equation, inadequate to describe the structure of the electron. For him these theories are merely pointers to a theory that includes gravitation: specifically, the gravitational field of the electron itself. Here I am only trying to give a very brief glimpse of Koga's work. Hence there is no complete explanation of the notation, etc. A reader unfamiliar with general relativity can omit this section.

Koga assumes a geometry that is more restricted than general Riemannian geometry but more general than the geometry of Minkowski spacetime (which he calls Euclidean geometry).

For the matter field he gives the following two sets of non-linear equations:

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{ij})}{\partial x^j} - g^{ij}\frac{\partial\eta}{\partial x^j} = 0 \qquad (1)$$

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{*ij})}{\partial x^j} - g^{ij}\frac{\partial\xi}{\partial x^j} = 0 \qquad (2)$$

Here g is the determinant of the metric tensor g^{ij}; F^{ij} is an anti symmetric tensor and F^{*ij} is conjugate of the F^{ij}; ξ and η are scalars.

Koga further considers that Einstein's equations:

$$R_{ij} - \frac{1}{2}g_{ij}R = -KT_{ij}$$

are not useful to study the internal field of the electron because they are quite complicated and have not been verified on a microscopic scale. He replaces them by a set of four equations containing only first order derivatives of the metric tensor,

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^i} = ag_{ij}(F^{*jk} - g^{jk}\xi)\frac{\partial\eta}{\partial x^k} + bg_{ij}(F^{jk} - g^{jk}\eta)A_k \qquad (3)$$

(Here a and b are constants and A_k represents an external electromagnetic field of macroscopic scale.)

Koga calls all these the fundamental equations. The fundamental equations given by (1) and (2) reduce to the Dirac equation (as an approximation) and to the Maxwell-Lorentz equation (in another approximation).

In order to get the Dirac equation or the Maxwell-Lorentz equation from (1) and (2), the equation for the metric tensor field given by (3) is used to linearise the nonlinear terms occurring in (1) and (2).

Koga further notes that none of the above fundamental equations (1), (2) and (3) contain the constants such as mass m, charge e and the Planck's constant h. By substituting the symbol of mass for a certain function which is assumed to be almost invariant, and making other approximations he gets the Dirac equation for the electron. By changing the boundary conditions, similar equations are obtained for nucleons. Similarly, by substituting the symbol of electric charge for another function and applying an averaging process the Maxwell-Lorentz equations are obtained.

Koga rejects Mach's principle (which, according to many physicists, states that the mass of a body is entirely due to its interaction with the rest of the universe, but mainly nearby matter) on the ground that the concepts of mass, force and action at a distance have no place in the fundamental equations and the Mach principle is only conceivable in terms of such classical mechanical concepts and is hence not compatible with relativity. We will see more of Mach's principle in Chapter 4.

An examiner has pointed out that the statement above is not Mach's principle, but only a misunderstanding of it that is widespread among physicists including Sachs (who I got it from). A correct version (given by the examiner) is that inertial forces (and hence inertia, not mass) arise due to interaction with other matter.

Actually, Koga states (and rejects) the Mach principle as "the inner structure of the electron is a reflection of the external universe". He does not give any reference for this statement. This seems just an example of the misunderstanding mentioned above; it is even possible that Koga got it from Sachs.

Koga gives arguments for rejecting his version of Mach's principle. I don't go into this in detail.

All this is the subject of the paper [**26**] and Chapter 6 of the book [**27**]. In the book he also studies the photon, quantum electrodynamical phenomenon and other particles like the proton, neutron and pions using the same ideas.

CHAPTER 3

Geometric Algebra and its application to Koga's approach to the Dirac equation

"The lack of a concrete picture of the spin leaves a grievous gap in our understanding of quantum mechanics."

H. C. Ohanian [**31**]

"A consistent interpretation of fermion spin seems to be that it has a definite direction, much like the polarization direction of a classical electromagnetic field."

W. E. Baylis [**4**]

"Experimental procedure involving static and slowly rotating magnetic field can be devised to measure the spin direction."

W. E. Baylis [**4**]

3.1. Introduction

In this chapter we first introduce Clifford algebras and their geometrical aspects to the extent we need. We then take up the Dirac equation in the form introduced by Hestenes. We adapt Koga's method of solution of the Dirac equation (given in the last chapter) to this context. Our solution reveals some aspects of the electron: we get a spinning and shuddering field which is localised in spacetime.

The purpose of Koga's solution to the Dirac equation is to describe the electron as a localised field. It is not meant to replace the work of Dirac but instead looks at aspects which Dirac did not touch.

I use geometric algebra which was developed by Hestenes in the 1960s using ideas of Clifford from the 19th century. The notation here is mostly that of Doran and Lasenby [**10**] except that the specific solution to the Klein-Gordon equation comes from Koga; its origin lies in de Broglie's work on the Schrödinger equation.

The solution to the Dirac equation using Geometric Algebra and its interpretation (Sections 3.5 and 3.6) was not what I expected.

I thought I would just get a spinning field as Koga stated.

The first surprise was the Klein-Gordon term (which, with hindsight, I should have expected).

Then the rest of the solution also did not represent a spinning field.

Almost miraculously, it turned out that it could be broken up into a spinning term and another term showing a one dimensional oscillation modulated by a scalar factor, which makes it a localised field.

The last two terms (as mentioned above) can be considered corrections to the Klein-Gordon term, which stands for a particle (or field) without spin.

This seems roughly parallel to the history of the subject: Schrödinger first considered (and rejected) the Klein-Gordon equation, and spin was discovered a bit later. Zitterbewegung or shudder came still later.

In the second section we give the definition and some basic properties of Clifford algebras. In the third section we introduce the examples of geometric algebras that we need. After that we describe the Dirac-Hestenes equation and our solution which is based on Koga's work in [**24**],[**23**], [**27**] (Chapter V) and [**28**] (Chapter V). Finally we estimate the angular velocity of the electron, show that the theory gives two opposite values of spin, and estimate the size of the electron.

In order to put the results of this chapter in perspective we now give a brief (and very incomplete and greatly oversimplified, but adequate for the present purpose) summary of the history of the spin concept. Much of this material is taken from the paper by Morrison [**30**].

To explain the anomalous Zeeman effect Pauli found it necessary to assume that the electron has, in addition to the magnetic moment of orbital motion, another magnetic moment independent of orbital motion. Uhlenbeck and Goudsmit asserted that this arises from spin, i.e., the angular momentum of a spinning electron. The magnetic moment in any direction, when measured, can only take two possible values; the mechanical spin momentum is always $\pm\hbar/2$. In the absence of an external magnetic field the energy effect of spin is small compared to that of charge and mass.

So the electron (considered a point particle) has three space coordinates and at least one more degree of freedom. Its wavefunction can be written as $\psi(x, y, z, \sigma)$ where $\sigma = \pm 1$ or

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where $\psi_1 = \psi(x, y, z, +1)$ and $\psi_2 = \psi(x, y, z, -1)$.

This was the starting point of Pauli's approach to spin: he showed that it requires the splitting of the wavefunction into two components. Presumably, at this stage nobody suspected that relativity was involved.

Very soon afterwards (from our perspective) Dirac obtained his relativistic equation which requires a further splitting of each of the two components of Pauli's wavefunction. The mathematical formalism of the Dirac equation and group representation theory require the existence of spin to guarantee conservation of angular momentum and to construct the generators of the rotation group. Now spin was far more than an ad hoc hypothesis required to account for specific effects. However, a physical understanding of spin was still lacking.

The above account does not reveal a large part of the history of spin. Specifically, it should be noted that the pioneers of quantum mechanics like Pauli, Heisenberg and Bohr each first rejected and then accepted the spin hypothesis, each for his own reasons (we need not go into them) without any theoretical consensus. Neither relativity nor classical mechanics nor quantum mechanics gave a full understanding of what spin is.

This state of affairs prevails even now, although spin has become a fundamental feature of scientific and medical technology: consider, for instance, electron spin resonance (ESR) or magnetic resonance imaging (MRI).

Various authors, from the 1920s to the present (for example, [**5, 16, 35, 31, 30**]) have mentioned that spin has neither been completely understood nor observed explicitly, although there are now extensive applications of spin.

There seems to be only one exception (other than Koga). According to Ohanian [**31**], Belinfante in the 1930s proved, in the context of conventional quantum mechanics, that spin may be regarded as a circulating flow of energy in the wave field of the electron. For some reason, this has been ignored.

In this situation, Koga's solution to the Dirac equation and its refinement using geometric algebra possibly hold the key to understanding what spin really is: maybe the mysterious phenomenon that various observations hint at is really a spinning and shuddering electron field.

3.2. Basic concepts of Geometric Algebra

Except for the definition in terms of the tensor algebra, Sections 3.2, 3.3 and 3.4 are the work of Hestenes as described by Doran and Lasenby.

The notation I for an ideal of the tensor algebra should not be confused with its use later in this chapter for the unit pseudoscalar.

Let V be a real, finite dimensional vector space with a quadratic form Q or equivalently a symmetric bilinear form B. The Clifford algebra $Cl(V)$ of V (with the bilinear form B) is the quotient of the tensor algebra $T(V)$ of V by the ideal I generated by all elements $v \otimes v - B(v,v)$ where v is a vector in V. In $Cl(V)$, any two elements a and b have a product called the geometric product, denoted ab. This is the image of the tensor product $a \otimes b$ (which is in $T(V)$). The members of $Cl(V)$ are called multivectors. We have $v^2 = B(v,v)$ for any vector v. If u and v are vectors then

$$(u+v)^2 = u^2 + uv + vu + v^2.$$

is a scalar. We define the inner product $u \cdot v$ by

$$u \cdot v = \frac{1}{2}(uv + vu) = \frac{1}{2}((u+v)^2 - u^2 - v^2).$$

Then $uv = u \cdot v + u \wedge v$, where $u \wedge v = \frac{1}{2}(uv - vu)$ is called the outer product, which is antisymmetric.

It turns out that the outer product generalises to the whole of $Cl(V)$, and gives an associative algebra. It is antisymmetric on vectors.

The outer product records the dimensionality of the object formed from a set of vectors, for instance, two vectors determine a plane (though the origin).

Any multivector which can be written purely as an outer product of a set of vectors is called a blade. If the number of vectors is r, the blade is said to have grade r. It is a fact that every blade can be written as a geometric product of orthogonal unit vectors with a scalar coefficient. (u and v are orthogonal if $u \cdot v = 0$.)

Let $(e_1, \ldots, e_n)$ be an orthogonal basis for V. Then a basis for $Cl(V)$ can be built up as

$$\{1\} \cup \{e_i\} \cup \{e_i e_j | i < j\} \cup \{e_i e_j e_k | i < j < k\} \cup \cdots \cup \{e_1 e_2 \ldots e_n\}$$

The rth set here is a basis for the subspace of grade r multivectors and has $\binom{n}{r}$ numbers. Thus $Cl(V)$ has dimension 2^n.

3.3. Some important geometric algebras

Here we look at the specific geometric (Clifford) algebras that we will use to study the Dirac equation.

The Dirac algebra or spacetime algebra (STA) [**15**] is the Clifford algebra of Minkowski spacetime. It is generated by four orthogonal unit vectors (which are taken parallel to the coordinate axes). These are denoted by γ_μ ($\mu = 0, 1, 2, 3$). We take γ_0 to be timelike with $\gamma_0^2 = 1$ and γ_i spacelike, with square -1, for $i = 1, 2, 3$. The bilinear form is the Minkowski metric

$\eta_{\mu\nu}$. Here we take $\eta_{00} = \eta^{00} = 1$, $\eta_{11} = \eta^{11} = -1$ and so on. Hence $\gamma^0 = \gamma_0$, $\gamma^1 = -\gamma_1$ etc..

We have

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}$$

We note that Minkowski spacetime has a reciprocal basis $\{\gamma^\mu\}$, where $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$ and $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$ (the Einstein summation convention applies). A vector space basis for STA is

$$\{1\} \cup \{\gamma_\mu\} \cup \{\gamma_\mu\gamma_\nu | \mu < \nu\} \cup \{I\gamma_\mu\} \cup \{I\}$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3$ is called the pseudoscalar. The elements $\gamma_\mu\gamma_\nu$ are bivectors and $I\gamma_\mu$ are trivectors. In the given basis, the vectors and trivectors are odd (they have odd grade) and the rest are even.

The even multivectors form a subalgebra of STA, called its even subalgebra. It has dimension 8 whereas STA in 16 dimensional. If we write $\sigma_i = \gamma_i\gamma_0$ $(i = 1, 2, 3)$ then the even subalgebra of STA is generated by σ_1, σ_2, σ_3. They are orthogonal unit vectors of the subalgebra and bivectors of STA. We have

$$\sigma_1\sigma_2 = I\sigma_3,\ \sigma_2\sigma_3 = I\sigma_1,\ \sigma_3\sigma_1 = I\sigma_2$$

and $I = \sigma_1\sigma_2\sigma_3$. This algebra can be considered the geometric algebra of 3-dimensional Euclidean space with σ_i being the unit vectors along the coordinate axes. It is frequently called the Pauli algebra.

Let ψ be a function defined on Minkowski space, taking values in STA. Then the (4 dimensional) gradient of ψ is defined as

$$\nabla\psi = \gamma^\mu \frac{\partial\psi}{\partial x^\mu}.$$

The Laplacian of ψ is $\nabla^2\psi = \nabla \cdot \nabla\psi$ which gives

$$\nabla^2\psi = \frac{\partial\psi}{(\partial x^0)^2} - \frac{\partial\psi}{(\partial x^1)^2} - \frac{\partial\psi}{(\partial x^2)^2} - \frac{\partial\psi}{(\partial x^3)^2}.$$

The contravariant and covariant coordinates of a point satisfy $x^0 = x_0 = ct$, $x^1 = -x_1 = x$, etc..

We have mentioned earlier that γ_0 is timelike and γ_1, γ_2, γ_3 are spacelike. What we mean by this is that there is an inertial observer following a timelike path with unit speed, where the velocity vector is taken to be γ_0. Then γ_1, γ_2, γ_3 are chosen so that they form a right-handed set of orthogonal spacelike vectors perpendicular to γ_0. Thus we split any event (which means a point of spacetime) into time and space components. Another observer with a different velocity would perform a different split.

3.4. The Dirac-Hestenes equation

We now consider the Dirac equation in the form given by Hestenes (sometimes called the Dirac-Hestenes equation [**8, 9, 10**]) and its relation to the conventional Dirac equation which is written in terms of matrices and column vectors of complex functions.

We use the term spinor below but it may be a bit inappropriate. We are just concerned with complex vector spaces, not with any transformation law.

Firstly, there is an isomorphism (of real vector spaces) between 2-dimensional complex space ("spinor space") and the subspace of STA spanned by 1, $I\sigma_1$, $I\sigma_2$ and $I\sigma_3$ given by

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3.$$

This extends to an isomorphism between 4-dimensional complex space ("4-spinors") and the even subalgebra of STA:

$$|\psi\rangle = \begin{pmatrix} |\varphi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \varphi + \eta\sigma_3$$

where $|\varphi\rangle \leftrightarrow \phi$ and $|\eta\rangle \leftrightarrow \eta$ as stated earlier. Fix $k \in \{1, 2, 3\}$. Consider the linear operator $\psi \mapsto \sigma_k \psi \sigma_3$, where $\psi \in \text{span}(1, I\sigma_1, I\sigma_2, I\sigma_3)$. It turns out that this corresponds to the operator $|\psi\rangle \mapsto \hat{\sigma}_k|\psi\rangle$ on complex 2-space where $\hat{\sigma}_k$ is a

Pauli matrix:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Similarly, for $\mu = 0, 1, 2, 3$ consider the following operator:

$$\psi \mapsto \gamma_\mu \psi \gamma_0$$

on the even subalgebra of STA. In complex 4-space this corresponds to

$$|\psi\rangle \mapsto \hat{\gamma}_\mu |\psi\rangle$$

where $\hat{\gamma}_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ ($I_2 = 2 \times 2$ identity matrix)

$$\hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}$$

for $k = 1, 2, 3$. Similarly the operator

$$\psi \mapsto \psi I \sigma_3$$

corresponds to

$$|\psi\rangle \mapsto i|\psi\rangle.$$

Now we come to the Dirac equation in its two forms. We note that $|\psi\rangle$ satisfies the Dirac equation

$$i\hat{\gamma}^{\mu}\frac{\partial|\psi\rangle}{\partial x^{\mu}} = m|\psi\rangle$$

if and only if ψ satisfies the equation

$$\nabla\psi I\sigma_3 = m\psi\gamma_0.$$

The latter is the Dirac-Hestenes equation. We leave the simple proof to the reader (note that $\gamma_0^2 = 1$ is used).

The explicit map given is for column spinors written in the so-called Dirac-Pauli representation. Some other representations also occur in the literature. Similar expressions can be given for other representations since they are equivalent via unitary transformations.

The gradient operator ∇ here is 4-dimensional and is defined in Section 3.3. It should not be confused with the 3-dimensional gradient used earlier.

The Dirac-Hestenes equation above comes from the operator that Koga calls D_0. A similar equation is obtained from Koga's D_1:

$$\nabla\psi I\sigma_3 = -m\psi\gamma_0.$$

This is an equally legitimate equation. Considering both equations yields four combinations of spin and energy.

3.5. A solution to the Dirac-Hestenes equation

We now look at a solution of the Dirac equation using geometric algebra, i.e., the Dirac-Hestenes equation. For convenience we take $c = 1$ and $\hbar = 1$.

Let φ be a multivector field, namely a function on spacetime taking values in STA. The Klein-Gordon equation for a free electron is

$$\nabla^2\varphi + m^2\varphi = 0$$

where $m > 0$ is the electron's rest mass.

As stated in Chapter 2, de Broglie and Koga took as a solution to the original Klein-Gordon equation (which happens to have the same form as the Geometric Algebra version)

$$\varphi = ae^{iS}$$

where a and S are real scalar fields in spacetime. Expressions can be written out for a and S:

$$S = -Et + \mathbf{p} \cdot \mathbf{r},$$

$$a = \exp(-\kappa r')/r'$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3$, $\mathbf{r}' = (\mathbf{r} - \mathbf{u}t)/\sqrt{1 - u^2}$ where $\mathbf{u}$ is the electron's velocity, κ

is a positive constant, $\mathbf{p} = \mathbf{u}E$ the momentum and $E^2 = (m^2 - \kappa^2)/(1 - u^2)$ where E is the energy of the electron.

We now consider an inertial frame in which the electron is at rest: $\mathbf{u} = 0$, $\mathbf{p} = 0$, $\mathbf{r}' = \mathbf{r}$. We assume that the origin is, in some sense, the 'centre' of the electron.

We write $\varphi = ae^{SI\sigma_3}$ for a solution to the Klein-Gordon equation. We can do this because $(I\sigma_3)^2 = -1$ in STA. By replacing i by $I\sigma_3$, we are giving a special role to the x^3-axis as we shall see soon.

It was Hestenes's idea to replace $i = \sqrt{-1}$ with $I\sigma_3$.

The function φ given here is $a(\cos S + (\sin S)I\sigma_3)$. It is even-valued since $a \cos S$ and $a \sin S$ are scalars while $I\sigma_3 = \gamma_2\gamma_1$ is a bivector.

To solve the Dirac-Hestenes equation, we use the following fact. If φ is a solution to the Klein-Gordon equation, and $\psi = \nabla\varphi I\sigma_3 + m\varphi\gamma_0$, then ψ satisfies the Dirac-Hestenes equation. If φ is odd then ψ is even and vice versa.

This can easily be verified. We omit the proof.

We want ψ to be even because of the correspondence mentioned earlier between complex 4-space and the even subalgebra of STA.

We now ensure that φ is an odd solution of the Klein-Gordon equation by taking

$$\varphi = ae^{SI\sigma_3}\gamma_0.$$

Obtaining the expression for ψ involves the application of the usual rules of calculus to the 4-dimensional gradient operator ∇: the product rule, the chain rule and so on. It should be borne in mind that some of the products are not commutative. In our context, $\mathbf{p} = 0$ and so $S = -Et$ is a function of t alone. Similarly, from $\mathbf{u} = 0$ it follows that $\mathbf{r}' = \mathbf{r}$ is independent of t. Thus a is a function of x, y and z. The relations between covariant and contravariant vectors and components were stated in Section 3.3.

We have

$$\nabla\varphi = (\nabla a)e^{SI\sigma_3}\gamma_0 + a\nabla(e^{SI\sigma_3})\gamma_0$$

where $a = e^{-\kappa r}/r$, $r = \sqrt{x^2 + y^2 + z^2}$ and $S = -Et$.

The computation of ∇a has to be done carefully observing the covariant-contravariant relationships mentioned above.

Let $\mathbf{R} = \mathbf{r}(\frac{1}{r^2} + \frac{\kappa}{r})$.

We get $\nabla a = a\mathbf{R}$ and $\nabla(e^{SI\sigma_3}) = (\nabla S)e^{SI\sigma_3}I\sigma_3$ with $\nabla S = -E\gamma^0 = -E\gamma_0$.

All these put together yield the solution

$$\psi = \mathbf{R}\varphi I\sigma_3 + (E+m)\varphi\gamma_0.$$

The quantities a and $\mathbf{R}$ were defined by Koga. The rest of the notation comes from Doran and Lasenby [**10**].

With hindsight, all this work seems simple and obvious but it was not so easy when we worked on it!

We will interpret this equation in the next section.

3.6. Interpretation of the solution

We now rewrite the expression for ψ obtained in the last section,

$$\psi = \mathbf{R}\varphi I\sigma_3 + (E+m)\varphi\gamma_0,$$

as a sum of terms, each with some physical (or geometrical) significance.

The solution complements the information about the electron given by Dirac rather than replacing it. It is not similar to any well known solution. For example, Doran and Lasenby only talk about plane waves. So does Baylis ([**4**], Section 19.6). Hestenes considers a point electron moving along a helix in Minkowski space, corresponding to uniform circular motion in 3-space. He calls it the zitterbewegung interpretation of quantum mechanics.

All these authors accept at least part of the Copenhagen interpretation of quantum mechanics, e.g., the uncertainty principle. It should be remembered that in Koga's theory, there is no superposition (or, rather, superposed states represent ensembles, not individual electrons).

The solution given in this thesis is, of course, only a linear approximation to a nonlinear phenomenon. It has a singularity at the centre which must be presumed to be un-physical. Therefore it does not accurately describe the electron field at points very close to the centre. Also, it assumes a single frequency which must be approximately true at best. Hopefully, it is fairly accurate at other points (not too far from the centre) and can someday get some experimental support.

Hestenes has done a great deal of work on the Dirac theory over several decades, including zitterbewegung, using Geometric Algebra (his work on zitterbewegung has not been accepted by physicists, probably because it significantly modifies the original theory of Schrödinger). A lot of his work is related to this thesis, although he does not give any similar solution. Hestenes only considers plane wave solutions.

The second term-above, $(E + m)\varphi\gamma_0$, is just an even multivector solution of the Klein-Gordon equation. It stands for a 'particle' (actually a localised field) without spin. The other term is

$$a(\frac{1}{r^2} + \frac{\kappa}{r})\mathbf{r}e^{SI\sigma_3}\gamma_0 I\sigma_3.$$

Here $a(\frac{1}{r^2} + \frac{\kappa}{r})$ is a spherically symmetric scalar field. We now concentrate our attention on the rest of the term:

$$\mathbf{r}e^{SI\sigma_3}\gamma_0 I\sigma_3.$$

We see that γ_0 commutes with $I\sigma_3$ and with $e^{SI\sigma_3}$ and that

$$\begin{aligned}\mathbf{r}\gamma_0 &= (x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)\gamma_0 \\ &= x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3.\end{aligned}$$

This is a vector in 3-dimensional Euclidean space. Thus we get

$$\begin{aligned}&\mathbf{r}\gamma_0 e^{SI\sigma_3} I\sigma_3 \\ &\quad = \mathbf{r}\gamma_0 e^{SI\sigma_3} e^{(\pi/2)I\sigma_3} \\ &\quad = (x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{(S+\pi/2)I\sigma_3} \\ &\quad = (x^1\sigma_1 + x^2\sigma_2)e^{(S+\pi/2)I\sigma_3} + x^3\sigma_3 \\ &\qquad + x^3\sigma_3(e^{(S+\pi/2)I\sigma_3} - 1)\end{aligned}$$

In the last expression, the first line stands for the vector obtained by rotating $x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3$ in the $\sigma_1\sigma_2$ plane through the angle $S + \pi/2$.

This can be written using a rotor (this is a general method of describing a rotation of a multivector in an arbitrary plane; see [**10**], Section 2.7.1):

$$e^{-\frac{(S+\pi/2)}{2}I\sigma_3}(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{\frac{(S+\pi/2)}{2}I\sigma_3}.$$

Since $S = -Et$ and E is constant, this gives a field rotating with uniform angular velocity in the $\sigma_1\sigma_2$ plane, i.e. about the σ_3 axis. (It must be noted that a rotation is always in a plane, in a space of any dimension. It is only in a 3 dimensional space that there is an axis of rotation.)

The last term

$$x^3\sigma_3(e^{(S+\pi/2)I\sigma_3} - 1)$$

is a field whose value at each point $x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$ is independent of x^1 and x^2 and proportional to x^3. It represents an oscillatory motion that gets larger with $|x^3|$. That is not strictly true because it is multiplied by the scalar factor $a(\frac{1}{r^2} + \frac{\kappa}{r})$. It is thus a localised field which vanishes at infinity. This motion seems vaguely similar (or at least analogous) to Schrödinger's Zitterbewegung.

In 1930 Schrödinger analysed the wave packet solutions of the Dirac equation for a free electron. An interference between positive and negative energy states produces what appears to be a fluctuation (at the speed of light) of the position of the electron (assumed to be a point particle).

Hestenes introduced what he called the zitterbewegung interpretation of quantum mechanics. He also assumed that the electron is a sizeless point but moving along a helix.

Neither of these bears much resemblance to the shudder described here (which has nothing to do with negative energy).

It seems unlikely to me that there is anything similar to this work in the literature.

One difference between this theory and Koga's should be pointed out: here, the fields are all in a real Clifford algebra. Complex numbers have been eliminated. Like Koga, we interpret the solution as suggesting that the electron is a localised field, with the value of ψ at each point giving the properties of the electron at that point, at least in principle.

Koga shows how Maxwell's equations in differential form can be obtained from the Dirac equation at points far from the centre. He does this by identifying various components of the solution with appropriate components of Maxwell fields. He does not use his solution or any other, but only the assumption that the solution is a deterministic localised field. Using this, Koga derives the correct value for the magnetic moment of the electron.

3.7. Some information obtained from the solution

If we include $\hbar$ and c, our solution to the Dirac-Hestenes equation can be written as

$$\psi = \hbar c \mathbf{R} \varphi I \sigma_3 + (Ec + mc^2)\varphi\gamma_0,$$

and the term expressing the spinning field is given by

$$e^{-\frac{1}{2}(\frac{S}{\hbar}+\frac{\pi}{2})I\sigma_3}(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{\frac{1}{2}(\frac{S}{\hbar}+\frac{\pi}{2})I\sigma_3}.$$

Since $S = -Ect$, the angular velocity is

$$\omega = -\frac{Ec}{\hbar}.$$

It should be noted that the negative sign in the expression for ω is not a typographical error. But I omit it later because I am only interested in the absolute value.

We have

$$E^2c^2 = m^2c^4 - \hbar^2c^2\kappa^2 > 0.$$

Here Ec stands for the energy of the electron. Thus we obtain

$$0 < \kappa < \frac{mc}{\hbar}.$$

Taking

$$m = 9.1094 \times 10^{-31}\text{Kg},$$

$$c = 2.9979 \times 10^{8}\text{m/S}$$

$$\text{and } \hbar = 1.0546 \times 10^{-34}\text{J.S.}$$

We find that $\kappa < 2.5896 \times 10^{12}$ per metre.

The theory assumes that κ is constant but the value of κ is not given by the theory. Now we can roughly calculate the spin angular velocity. With the above values of mass m and speed of light c, the value of m^2c^4 is of the order of 10^{-27}. Similarly the value of $\hbar^2c^2$ is of the order of 10^{-52}. Taking the value of κ in the above range, the spin angular velocity is roughly of the order of 10^{21} radians/second, by using the relation

$$\omega = \frac{Ec}{\hbar} = (\frac{m^2c^4 - \hbar^2c^2\kappa^2}{\hbar^2})^{1/2}.$$

Now we come to the well known concept of spin up and down states.

Let $c = 1$ and $\hbar = 1$.

The equation $\nabla\psi I\sigma_3 = m\psi\gamma_0$ has solutions

$$\psi = \mathbf{R}\varphi I\sigma_3 + (E + m)\varphi\gamma_0$$

with $\varphi = ae^{SI\sigma_3}\gamma_0$ and

$$\psi = \mathbf{R}\varphi I\sigma_3 + (-E + m)\varphi\gamma_0$$

with $\varphi = ae^{-SI\sigma_3}\gamma_0$.

Similarly, the equation $\nabla\psi I\sigma_3 = -m\psi\gamma_0$ has the solutions (for the same values of φ as above)

$$\psi = -\mathbf{R}\varphi I\sigma_3 + (\mp E + m)\varphi\gamma_0$$

(using the same method and multiplying the right hand side by -1).

These give us all four combinations of energy and spin as follows. Because of the relation between energy and phase, and between phase and angular velocity in Koga's theory, we are forced to consider both Dirac equations. That is not sufficient to get all possibilities. But multiplying by -1 changes the sign of the energy while leaving the spin direction unchanged. For positive energy, the solutions of the two equations have conjugate Klein-Gordon terms and opposite spins. Similarly for negative energy. (We are considering an analogue of complex conjugation, treating $I\sigma_3$ as i.)

In Koga's view, the motion of a single electron is always causal and continuous. There are no acausal jumps from positive to negative energy states. One should abandon the idea of a vacuum filled with negative energy electrons and pairs of virtual electrons and photons ([**27**], Chapter III, Sections 3.2 and 3.3).

Sachs ([**38**]) also mentions that energy is defined in his field theory in terms of continuous change only, which rules out jumps from positive to negative energy.

Finally, the present theory enables us to put a bound on the size of the electron field.

Assuming that speeds greater than c do not occur, we must have $\omega r < c$ for any point in the electron. Here r is the distance from the point to the axis of rotation. Thus

$$r < c/\omega.$$

As an examiner pointed out, the classical electron model of the early 20th century implied a bound on the electron size from that on the surface velocity. I was unaware of this. But the spin angular velocity is obtained here from the present theory. This is not a classical electron.

CHAPTER 4

Mendel Sachs: General relativity leads to quantum mechanics

"As Einstein originally anticipated, quantum mechanics is a derivative feature of matter in the local domain, where the curved spacetime approaches the flat spacetime representation, as an approximation."

M. Sachs [**37**]

"Of course, empirical verification is a necessary requirement of any scientific theory. But it is not sufficient. For a true scientific theory must also be both logically and mathematically consistent."

M. Sachs [**36**]

4.1. Introduction

In this chapter we try to give an idea of the core of Sachs's work [**35, 36, 37**], namely his argument that "General Relativity implies Quantum Mechanics." (Actually, what he tries to prove is that general relativity and the Dirac equation imply quantum mechanics.) We only give an outline and do not go into details.

In Sachs's work there is some ambiguity in the mathematics (for instance, in the choice of conjugate or in the correspondence between Pauli matrices and unit quaternions) but, hopefully, not in the physical implications. This ambiguity is visible at some places in this chapter. Anyone who finds it worthwhile to do further work following Sachs needs to remove the ambiguity.

In the next section we give a brief introduction to the quaternions.

Then we describe how he tries to use quaternions to factorise the metric of general relativity and remove unnecessary symmetries (symmetries which are not required by the postulates of general relativity).

After this we come to Sachs's field equations, which are a factorisation of Einstein's equations. These involve his quaternion metric coefficients which are introduced in the previous section.

Next we see Sachs's derivation of generally covariant two component spinor equations corresponding to the Dirac equations in two component form, namely the Majorana equations.

Finally we come to the derivation of quantum mechanics as a limiting case.

An examiner has revealed that Sachs's interpretation of the Mach principle (which I took for granted) is wrong, and is, in fact, a widespread misconception among physicists. Mach asserted that inertial forces (i.e., inertia, and not mass, as Sachs assumes) arise due to interactions with other matter. This removes the basis for Sachs's interpretation of his derivations. It does not affect the validity of his predictions; they may be correct or wrong. Unfortunately, it also makes parts of this thesis irrelevant!

I believe that Sachs's derivation is fatally flawed because his generalised Dirac equations do not reduce to the ordinary ones in the flat space limit. Therefore the discussion in this chapter is only hypothetical: it is based on what would be true if the argument was correct.

4.2. The Quaternions

We begin with a convenient approach to the quaternions. Our notation resembles that of previous chapters but is not necessarily the same.

The quaternions form a 4-dimensional real algebra. A vector space basis for the quaternions is $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$. We have $\mathbf{i}^2 = -\mathbf{1}$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$ and so on.

The conjugate of a quaternion $a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is defined by analogy with the complex conjugate: it is $a_0\mathbf{1} - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. This is the usual conjugate and corresponds to space reflection. Another conjugate, defined by Sachs as the negative of the above, corresponds to time reversal.

The quaternions form a division ring which is not commutative. It can be represented by 2×2 complex matrices as we now describe.

Let $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ where $i = \sqrt{-1}$. These are called the Pauli spin matrices. They satisfy $\sigma^1\sigma^2 = i\sigma^3 = -\sigma^2\sigma^1$ and so on.

The real algebra generated by these matrices is 8 dimensional. It is isomorphic to the geometric algebra of 3 dimensional space. A vector space basis for it is $\{\sigma^0, i\sigma^0, \sigma^1, i\sigma^1, \sigma^2,$

$i\sigma^2, \sigma^3, i\sigma^3, \}$. In this basis the subset $\{\sigma^0, i\sigma^1, i\sigma^2, i\sigma^3, \}$ generates the even subalgebra which is isomorphic to the algebra of quaternions. We have $(\sigma^1)^2 = \sigma^0$ and hence $(i\sigma^1)^2 = -\sigma^0$. An isomorphism between this algebra and the quaternions is obtained, for example, by taking $\sigma^0 \leftrightarrow \mathbf{1}$, $i\sigma^1 \leftrightarrow \mathbf{i}$, $i\sigma^2 \leftrightarrow \mathbf{j}$, $i\sigma^3 \leftrightarrow -\mathbf{k}$. Alternatively, we can take $\sigma^0 \leftrightarrow \mathbf{1}$, $i\sigma^1 \leftrightarrow -\mathbf{i}$, $i\sigma^2 \leftrightarrow -\mathbf{j}$, $i\sigma^3 \leftrightarrow -\mathbf{k}$.

Incidentally, the algebra generated by $\sigma^0, \sigma^1, \sigma^2, \sigma^3$ is isomorphic as a real algebra to the algebra of all complex 2×2 matrices, sometimes described as the algebra of "complex quaternions". This is the structure we get if, in the definition of quaternions above, we allow complex coefficients. It is not a division algebra.

We will identify the quaternions with the 4-dimensional algebra described above. We identify $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with σ^0, $-i\sigma^1$, $-i\sigma^2$, $-i\sigma^3$ respectively as Sachs does in his introduction to quaternions in [**35**].

It should be noted that the i factor is essential in the correspondence between quaternions and matrices given here. Sachs ignores this when taking the limit of his metric as the curvature of space vanishes. This is totally unjustified.

4.3. Factorisation of the metric

Sachs tries to factorise the metric of General Relativity and Einstein's equations. This is the only factorisation that I am aware of. Sachs states that it is motivated by some work of Einstein and Mayer in the 1930s [**12**].

The approach of Sachs to the generalisation of general relativity starts with the observation that the structure of the 10 parameter Poincaré group is represented by the metric tensor $g^{\mu\nu}$. This includes reflection symmetry, which is not required by the postulates of general relativity. By removing these symmetries from the underlying group he gets a 16 parameter group which he calls the Einstein group. Sachs tries to achieve this by factorising the expression for ds^2. Write

$$ds = q^{\mu}(x)dx_{\mu},$$

$$ds^* = q^{*\mu}(x)dx_{\mu}$$

where the asterisk stands for the quaternion conjugate. Then we have

$$ds^2 = g^{\mu\nu}dx_{\mu}dx_{\nu} = ds ds^*$$

which gives (with a normalisation factor)

$$g^{\mu\nu} = -\frac{1}{2}(q^{\mu}q^{*\nu} + q^{\nu}q^{*\mu}).$$

In the limit of flat spacetime (special relativity) the metric $g^{\mu\nu}$ must approach the Lorentz metric $\eta^{\mu\nu}$. Sachs assumes that this corresponds to the following limits on the fields $q^\mu(x)$:

$$q^0(x) \to \sigma^0 = 2 \times 2 \text{ identify matrix}$$

$$q^1(x) \to \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$q^2(x) \to \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$q^3(x) \to \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The latter are the Pauli spin matrices. One reason why this is not justified is that the quaternion algebra is 4 dimensional whereas the Pauli algebra is 8 dimensional.

The assumption that the quaternion metric coefficients approach the Pauli spin matrices in the flat space limit is crucial to Sach's arguments.

But it is not true; the Pauli spin matrices are not quaternions, as I have explained. This is a serious, even fatal, flaw in his derivation. Anyone who wishes to work on Sachs's theory needs to correct this, possibly by adding some i factors at various places.

The resulting expressions may be quite different from what Sachs has obtained. It is surprising that this has not been commented on so far. I suspect that it has been noticed by others. For example, why did Cyganski and Page abandon their study [**6**]?

It may be appropriate to mention that I spent a long time (months) trying to understand (i.e., derive) Sachs's limit before I realised (near the end of my thesis period) that it was irreparably wrong.

4.4. Factorisation of Einstein's equations

We are concerned with spinor variables in a curved spacetime. A spinor field has a covariant derivative

$$\eta_{;\mu} = \partial_\mu \eta + \Omega_\mu \eta$$

where the spin affine connection is

$$\begin{aligned}\Omega_\mu &= \frac{1}{4}(\partial_\mu q^\rho + \Gamma^\rho_{\tau\mu} q^\tau) q^*_\rho \\ &= -\frac{1}{4} q_\rho (\partial_\mu q^{*\rho} + \Gamma^\rho_{\tau\mu} q^{*\tau}).\end{aligned}$$

It transforms as follows when $x_\mu \to x'_\mu$:

$$\Omega_\mu \to \Omega'_\mu = S\Omega_\mu S^{-1} - (\partial_\mu S) S^{-1}$$

where $\{S(x)\}$ are the spinor transformations of the Einstein group:

$$S(\theta^\mu_\nu(x)) = \exp\frac{q_\mu q_\nu \theta^\mu_\nu(x)}{2}$$

where $\theta^\mu_\nu(x)$ define the sixteen continuous transformations of the Einstein group.

Sachs obtains the following field equations. A reader unfamiliar with General Relativity need not bother about the presence of Einstein's equations here since I do not deal with them in this thesis.

The equations below constitute a factorisation of Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = KT_{\mu\nu},$$

just as the Klein-Gordon equation in special relativity

$$\Box\varphi = m^2\varphi$$

(where $\Box$ is the 4-dimensional Laplacian in Minkowski space defined in Section 3.3) factors into 2-component spinor equations:

$$\sigma_\mu \partial^\mu \eta = -m\psi,$$

$$\sigma^*_\mu \partial^\mu \psi = -m\eta.$$

Sachs's field equations are

$$\frac{1}{4}(K_{\rho\lambda}q^{\lambda} + q^{\lambda}K^{\dagger}_{\rho\lambda}) + \frac{1}{8}Rq_{\rho} = kT_{\rho},$$
$$-\frac{1}{4}(K^{\dagger}_{\rho\lambda}q^{*\lambda} + q^{*\lambda}K_{\rho\lambda}) + \frac{1}{8}Rq^{*}_{\rho} = kT^{*}_{\rho}$$

where the asterisk denotes the quaternion conjugate and the dagger stands for the Hermitian adjoint. Here $K_{\rho\lambda}$ is the spin curvature tensor defined by

$$K_{\rho\lambda} = \Omega_{\lambda;\rho} - \Omega_{\rho;\lambda},$$

R is the scalar curvature field and the right hand sides of the two equations are the source terms.

4.5. A general relativistic Dirac equation

In the late 19th century, the physicist and philosopher Ernst Mach made the assertion (according to Sachs) that a body's mass is a measure of its coupling to all of the other matter in a closed system. Strictly speaking, the only closed system is the universe. Einstein called this view the Mach principle [**40**]. Sachs argues that the Mach principle (as stated by him) holds in a generalised form: all the intrinsic qualities of a body (or what are usually considered intrinsic) such as mass and electric charge are only measures of the coupling between the observed matter and the rest of the closed system.

As clarified by an examiner, the principle that Mach stated was that inertial forces arise due to interaction with other matter. Thus it is the inertia of a body, not its mass, that arises like this. Consequently, there does not seem to be any nontrivial "generalised Mach principle" in the sense of Sachs: that is just a generalisation of a misunderstanding of Mach's principle.

We now consider Sachs's "derivation of quantum mechanics", which is the core of his study of the properties of matter and spacetime.

This section is only a brief review of Sachs's work, not a full-fledged derivation. As stated earlier, in order to pass from the Majorana equations to the generalisation attempted by Sachs, one needs to add i factors at various places which would totally change the expressions.

The relativistic equation

$$E^2 = p^2c^2 + m^2c^4$$

leads to the Klein-Gordon equation which factors into the following pair of 2-component spinor equation called the Majorana equations:

$$(\sigma^{\mu}\partial_{\mu} + I)\eta = -m\chi$$

$$(\sigma^{*\mu}\partial_{\mu} + I^{*})\chi = -m\eta$$

where $\chi = \epsilon\eta^*$, η^* being the complex conjugate of η. Here χ and η are time reversed spinor variables and ϵ is the Levi-Civita matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We follow the convention that $\hbar = 1, c = 1$. The inertial mass of the particle is m.

Sachs does not precisely state what he means by a spinor. Are its components fixed at each point of spacetime or do they change with coordinate transformations?

What do the Majorara equations (in Sach's work) mean? What is the "particle"? Is it a point mass? What is electron spin, in terms of his equations? Sachs does not consider such questions at all.

In Sachs's view the function I is the interaction term due to all the other matter of the closed system. (More conventionally, I is considered to be the effect of, for example, a background electromagnetic field.) Sachs points out that Dirac brought back reflection symmetry by combining the above 2-spinor equations into a 'bispinor' wave equation (the usual Dirac equation)

$$(\gamma^\mu \partial_\mu + I)\psi = -m\psi.$$

Now we consider what happens to the Majorana equations when we pass from Minkowski spacetime to Riemannian (curved) spacetime (assuming the correctness of Sachs's argument). We need to replace $\sigma^\mu \partial_\mu \eta$ by

$$q^\mu \eta_{;\mu} = q^\mu(\partial_\mu + \Omega_\mu)\eta.$$

Then we get the equation

$$q^\mu(\partial_\mu + \Omega_\mu)\eta + I\eta = 0,$$

$$q^{*\mu}(\partial_\mu + \Omega_\mu^\dagger)\chi + I^*\chi = 0.$$

Define matrix fields Λ_+ and Λ_- by

$$\Lambda_+ = q^\mu \Omega_\mu + \text{ its hermitian conjugate},$$

$$\Lambda_- = q^\mu \Omega_\mu - \text{its hermitian conjugate}.$$

Let τ denote the time-reversal operation. Using $\tau q^\mu = q^{*\mu}$ we get

$$\tau \Omega_\mu = -\Omega_\mu^\dagger$$

and then

$$\tau \Lambda_\pm = \pm \epsilon \bar{\Lambda}_\pm \epsilon$$

which gives

$$(\tau \Lambda_\pm)\Lambda_\pm = \pm |\text{det}\Lambda_\pm| \exp(i\delta)\sigma_0$$

where $\delta = 0$ if det $\Lambda_\pm < 0$, $\delta = \pi$ if det $\Lambda_\pm > 0$. We write these as a matrix equation

$$|(\tau\Lambda_\pm)\Lambda_\pm|\eta = \pm(2\alpha_\pm)^2\eta\exp(i\delta)$$

where $(2\alpha_\pm)^2 = |\det\Lambda_\pm|$. After factorisation (of which we omit the details)

$$q^\mu\Omega_\mu\eta = (\alpha_+ + i\alpha_-)\chi = \lambda\exp(i\gamma)\chi$$

where

$$\lambda = |\alpha_+ + i\alpha_-| = \frac{1}{2}(|\det\Lambda_+| + |\det\Lambda_-|)^{1/2}.$$

This plays the role of the inertial mass in the quantum mechanical equations.

Thus we have a pair of equations of which the second is the time-reversal of the first:

$$\begin{aligned} q^\mu\Omega_\mu\eta &= \lambda\exp(i\gamma)\chi \\ -q^{*\mu}\Omega^\dagger_\mu\chi &= \lambda\exp(-i\gamma)\eta. \end{aligned}$$

Here $\gamma = \tan^{-1}(\alpha_-/\alpha_+)$.

The generally covariant extensions of these are

$$q^\mu\eta_{;\mu} = q^\mu\partial_\mu\eta + \lambda\exp(i\gamma)\chi = -I\eta,$$

and its time-reversal

$$q^{*\mu}\chi_{;\mu} = q^{*\mu}\partial_\mu\chi + \lambda\exp(-i\gamma)\eta = -I^*\chi.$$

Sachs now shows that the phase factor $\exp(\pm i\gamma)$ in the above equation can be transformed away. The equation of continuity in special relativity is

$$\partial_\mu(\eta^\dagger\sigma^\mu\eta) = 0.$$

Using the product rule for covariant differentiation and for ordinary differentiation, the following equation is obtained:

$$q^\mu\eta'_{;\mu} = q^\mu\partial_\mu\eta' + \lambda\chi' = -I'\eta'$$

where

$$\eta' = \eta\exp(-i\gamma/2),$$

$$\chi' = \chi\exp(i\gamma/2)$$

$$, I' = I + \frac{1}{2}iq^\mu\partial_\mu\gamma.$$

The corresponding time-reversed equation is

$$q^{*\mu}\chi'_{;\mu} = q^{*\mu}\partial_\mu\chi' + \lambda\eta' = -I'^*\chi'.$$

Sachs now considers what happens to the above equations (his generalisation of the Dirac equations) in various limits. We see this in the next section.

4.6. Quantum mechanics from general relativity: the limiting cases of the generally covariant equations

Sachs actually tries to take the reader in a circle from the Dirac equation to the Dirac equation, although he first suggests that he is going to derive quantum mechanics from general relativity.

The fundamental equations obtained by Sachs are (without the primes)

$$(q^{\mu}\partial_{\mu} + I)\eta = -\lambda\chi,$$
$$(q^{*\mu}\partial_{\mu} + I^{*})\chi = -\lambda\eta.$$

In the limit as $g^{\mu\nu} \to \eta^{\mu\nu}$ (the special relativistic limit as gravity is reduced to zero) Sachs assumes that $q^{\mu} \to \sigma^{\mu}$, and thus gets the Majorana 2-component spinor equations. These lead to the Dirac equation. In the limit of low velocities he gets the Schrödinger equation.

If η and χ represent 'one particle' (in the covariant equation) then I represents the effect of all the 'other matter'. This makes the equations nonlinear.

One possible limiting assumption is that there is 'weak coupling' or 'low recoil': the matter we are interested in has a negligible impact on the 'other matter'. For example, a particle in an accelerator has essentially no effect on the accelerator. Then the interaction term I can be taken to be a function of space-time, which makes the equations linear. In this limit he assumes that the spin affine connection $\Omega_\mu \to 0$ and $\lambda \to mc/\hbar$. This is the usual situation in all applications of quantum mechanics.

According to Sachs, another limit is possible. This was originally a speculation of Einstein. If some matter is isolated from the rest of the 'closed system', then, as the remaining matter is removed, I approaches 0. In this limit he assumes that the spin affine connection $\Omega_\mu \to 0$ and $\lambda \to 0$. In this case, rather than the usual description of a free particle, Sachs believes that not only the mass but the charge of the particle also goes to 0. This would be in full accord with the generalised Mach principle.

As mentioned earlier, an examiner has explained that Sachs misunderstands Mach's principle. His generalised Mach principle is the result of this misunderstanding.

CHAPTER 5

Comparison and conclusions

"There is no quantum world, there is only an abstract quantum physical description. It is wrong to think the task of physics is to find out how nature is physics concerns what we can say about nature."

N. Bohr [20]

"Do you really belive that the moon exists only when you look at it?"

A. Einstein

"Physicists do want to find out 'how nature is' and feel they are doing this with quantum mechanics, yet the official view which most workers claim to follow rules out the attempt as meaningless!"

P. R. Holland [19]

5.1. Introduction

In this chapter we first make some comments about the work of Toyoki Koga (in Section 5.2) and Mendel Sachs (in Section 5.3). We then compare the two in Section 5.4 and consider what information the two put together give us. We also say a few words about the aspects of their theories that we do not discuss. Finally we consider some unsolved problems.

5.2. The Field Theory of Toyoki Koga

Koga states that his study of the Schrodinger equation was done around 1952. But he then heard about David Bohm's work and gave up the idea of getting his research in this area published. About 20 years later he happened to see Bohm's paper for the first time. He felt that Bohm had not done very much and sent his old paper for publication [**21**]. This led to his study of the Dirac equation and his "general-relativistic" theory in which he uses his substitutes for Einstein's equation. In Koga's view, his work on the Schrodinger and Dirac equations was of value mainly as clues to the latter theory. In fact, he does not give much importance to his solutions to the Schrodinger and Dirac equations. It appears that for him their significance is merely as a proof that deterministic solutions exist. However, in chapter 3 we have been able to extract some significant information about the electron using both his approach to the Dirac equation and geometric algebra. This is a branch of mathematics of which he seems to have been unaware, as I explain below.

Koga commented on various topics about which he had strong views, especially the use of mathematics without a clear physical understanding.

He also commented on several physicists. For instance, he described David Bohm (referring to his later work with Hiley) as "an eloquent purveyor of bizarre fantasy". Another example is his comment on the Generalised Mach Principle given later in this chapter.

Koga condemned the use of spinors in physics but had nothing to say about Clifford algebras. It seems that he was not aware of their existence. In particular, he was surely not aware of the geometric interpretation of Clifford algebras due to Hestenes.

It is a remarkable fact that Koga's substitute for spinor fields in the Dirac equation is consistent with the work of Fock which was done almost 50 years earlier.

In the last chapter of his book Foundations of Quantum Physics [**27**] Koga mentions some unresolved problems in his scheme. One more point, also mentioned by Koga, is that the

Dirac equation does not give any information about the precession of the electron's spin axis; nor does his more general theory.

One aspect we have ignored is the role of probability theory. According to Koga, the principles like uncertainty, spontaneous tranitions and wave-particle duality are the consequence of mistaking an ensemble representation of a physical system for a single-sytem representation. He makes a similar criticism of Bell's arguments for his inequalities. In his opinion, in view of the successes of quantum mechanics, these ideas can only be treated as working rules and not as fundamental principles. Heisenberg's equations, according to Koga, are only equivalent to Schrödinger's theory for steady or static states.

5.3. Mendel Sachs's Derivation of Quantum Mechanics

Sachs starts with the Dirac equation in the Majorana form and tries to change it (unsuccessfully as I have explained) so as to make it general-relativistic. He then takes this modified Dirac equation and considers the limit as the curvature of spacetime vanishes, i.e. the flat space limit. The interaction term in his generally covariant equations makes them nonlinear since it partly depends on the solutions to the equations.

He considers two limits: firstly the case in which the interaction term has a negligible dependence on the solution which makes it a function of spacetime alone, thus making the equation linear, and secondly the limit as the interaction term approaches 0, corresponding to a free particle. (He says that the latter limit is unachievable in principle).

Sachs gives an argument to show that the continuous distribution of mass values approaches a discrete mass spectrum in the limit of uncoupling. We omit the details of all this.

To conclude, we would like to mention that Sachs's derivation do not appear to reqire the use of the Mach principle at all.It must also be noted that Sachs does not consider the meaning of his solutions. The question of the shape and size of the electron and whether it has an axis of symmetry or rotation are completely ignored by Sachs. Thus, although he makes a large number of predictions, he does not give any insight into the basic nature of the electron.

5.4. Conclusions from a comparison

What can we conclude from reading the work of both Toyoki Koga and Mendel Sachs?

As an examiner mentions, the only connection between the works of Sachs and Koga is through the Dirac equation. Koga does not use Geometric Algebra but I used it to study his solution. I found, in addition to a term that agrees with his view (the electron is like a spinning top), a sort of shudder, the existence of which he did not suspect.

Sachs uses the quaternions, an example of a geometric algebra, but he doesn't treat them as geometric objects. In order to make a comparison, one would have to re-do Sachs's work correctly; the mathematical derivation needs to be corrected and the physical meaning clarified.

Both believe that Einstein was on the right track in advocating a unified field theory based on General Relativity.

Koga states that Einstein made a serious error in assuming that the Schrödinger equation has only ensemble solutions. Einstein therefore discarded quantum mechanics at the outset of his search and apparently intended to express the existence of matter immediately in terms of the metric tensor. He was unsuccessful.

Koga feels that considering a field of matter may be an essential "scaffold" for reaching the ultimate representation. Koga shows that the Schrödinger and Dirac equations do have deterministic solutions. This suggests necessary properties of the general nonlinear equations. We do not deal with this matter in detail here.

In contrast, the main goal of Sachs appears to be to prove that by starting from a generalisation of General Relativity (and the Dirac equation, although he prefers not to mention it) one can obtain Quantum Mechanics as a limiting case.

Sachs does not bother much about the meaning of his equations or of their solutions. For instance, he does not state whether the solutions are deterministic or not!

He also does not consider the geometric meaning of a spinor field. Thus he never mentions what electron spin is or might be, except that in one place he states that the electron is not a hard little spinning object.

Unlike Koga, who believes that matter has intrinsic properties and an electron is like a small universe, Sachs emphasises the Generalised Mach Principle that matter has no intrinsic properties and all properties are due to interactions, mainly with nearby matter. (Koga commented that if this were applied to people it would imply that an individual is just a manifestation of the surrounding society!) It seems impossible to prove or disprove Mach's principle. There does not seem to be much justification for accepting it. Although Sachs states that his results are in full accord with this principle, it seems that they don't imply it.

Sachs appears to achieve his goal of formally explaining quantum mechanics (if his derivation can be corrected, of course) but without giving any geometrical insight. He does make numerous predictions about phenomena involving various scales from the microscopic to the astronomical but does not give any real understanding of what is going on. In this respect he seems to be no better than the physicists who believe in the Copenhagen interpretation which he criticises.

Both Koga and Sachs believe that matter consists of fields. Koga thinks that an electron field (for instance) has a boundary, although it may not be sharply defined. Thus one can talk about the size of an electron (it must be very small since the electron is conventionally considered a point particle). Sachs, on the other hand, thinks that electrons are like waves, which have no boundary. It should be mentioned that he explains away entanglement on this basis.

Koga seems to be a master of physical approximation. He states that every equation used in physics is an approximation (which may be very accurate in some situations but not in others). Sachs does not seem to be aware of this. He puts great stress on the importance of finding exact solutions.

Putting the two theories together, it seems that electrons and other "particles" are actually fields. Whether they are actually bounded or are like waves, it seems obvious that each electron is located almost entirely in the neighbourhood of a point. Spin and shudder appear to be real physical phenomena. A theory of matter must involve gravitation.

Electron spin, although it has not been directly observed, is a reality and is extensively applied. The electron apparently has some other intrinsic motion. This, according to some, is the cause of spin. The analysis given in this thesis, based on ideas of de Broglie and Koga, suggests that there is a one-dimensional shuddering motion parallel to the axis of spin, and it has no connection with negative energy.

We close this section by reminding the reader that both Koga and Sachs did a great deal more than what it described in this thesis. Koga's work was published in the 1970s but it was preceded by decades of thought about these questions. Sachs has been publishing papers for more than half a century.

5.5. Open Problems

On reading Koga and Sachs, some questions occur almost inevitably.

- Is there any evidence for deterministic electrons?
- How does one explain the stability of atoms and molecules? What role does gravity play?

• What is the relation between eigenvalue solutions and stable states? Koga's explanation is not very clear while Sachs's is worse.

• A theory which explains transitions between stable states is needed.

• An experimental verification of spin would be very desirable.

• How to explain the precession of the axis of spin?

• Why does the electron spin?

According to many physicists, in papers and books written from the 1920s onwards to the 21st century, the consequences of electron spin have been observed but not spin itself.

It is widely asserted that nobody knows what spin is.

The work of Belinfante on spin and circulation is, according to Ohanian [**31**], unknown to the world of physicists.

For instance, Mendel Sachs says that an electron is not a hard little spinning object but stops there and does not say what it is according to him!

A significant exception is Toyoki Koga. According to Koga's theory, as clarified here, an electron is a soft little spinning and shuddering object.

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