METHOD OF LINES TO SINGULAR PERTURBATION PROBLEMS

THEORY OF DIFFERENTIAL INEQUALITIES WITH APPLICATIONS TO SINGULAR PERTURBATION PROBLEMS AND METHOD OF LINES

Thesis submitted
for the Degree of Doctor of Philosophy

By

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CERTIFICATE

This is to certify that this thesis is a bonafide record of work by Smt.V.M.Sunandakumari, carried out in the Department of Mathematics and Statistics, Cochin University of Science and Technology, Cochin 682 022 under my supervision and guidance and that no part thereof has been submitted for a degree in any other University.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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SYNOPSIS

During recent years, the theory of differential inequalities has been extensively used to discuss singular perturbation problems and method of lines to partial differential equations. The present thesis deals with some differential inequality theorems and their applications to singularly perturbed initial value problems, boundary value problems for ordinary differential equations in Banach space and initial boundary value problems for parabolic differential equations. The method of lines to parabolic and elliptic differential equations are also dealt with.

The thesis is organised into nine chapters. A detailed description of chapters would run as follows:

The first chapter reviews briefly the earlier developments of the theories in the field related to the current work and also discusses the scope of this thesis.

In the second chapter the background material, mainly differential inequality theorems or monotonicity theorems, which shall be needed in the subsequent chapters is presented.

The third chapter is devoted to study the asymptotic behaviour of solutions and/or their derivatives of the following boundary value problems for weakly coupled doubly infinite systems of second order ordinary differential equations with a small parameter multiplying the highest derivative. The boundary value problems are the infinite system of differential equations

(0.1) $\underset{\sim}{P}\,\underset{\sim}{u} : - \varepsilon \underset{\sim}{u}'' + \underset{\sim}{f}(t, [\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon) = \underset{\sim}{0}$,

that is,

(0.2) $P_i\, \underset{\sim}{u}: = -\varepsilon u_i'' + f_i(t, u_i', \underset{\sim}{u}, \varepsilon) = 0, \; t \in D := (a,b), \; i \in Z$

subject to any one of the following boundary conditions

(0.3) $\underset{\sim}{R}\,\underset{\sim}{u} : = (\underset{\sim}{u}\,(a,\varepsilon),\ \underset{\sim}{u}(b,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T$,

(0.4) $\underset{\sim}{R}\,\underset{\sim}{u} : = (\underset{\sim}{u}\,(a,\varepsilon),\ \underset{\sim}{u}'(b,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T$,

(0.5) $\underset{\sim}{R}\,\underset{\sim}{u} : = (\underset{\sim}{u}(a,\varepsilon) - \varepsilon\underset{\sim}{u}'(a,\varepsilon),\ \underset{\sim}{u}'(b,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon), \underset{\sim}{B}(\varepsilon))^T$,

where $\underset{\sim}{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)^T$,

$\underset{\sim}{A}(\varepsilon) = (\ldots, A_{-1}(\varepsilon), A_0(\varepsilon), A_1(\varepsilon), \ldots)^T$,

$\underset{\sim}{B}(\varepsilon) = (\ldots, B_{-1}(\varepsilon), B_0(\varepsilon), B_1(\varepsilon), \ldots)^T$,

ε is a small positive parameter such that $0 < \varepsilon \leq \varepsilon_1$, T stands for the transpose, Z is the set of all integers, and a prime " ' " denotes a differentiation with respect to t. The study is carried out by first obtaining necessary estimates using monotonicity theorems for solutions and/or their derivatives of the above boundary value problems. In turn, these estimates determine the asymptotic behaviour of solutions and/or their derivatives as the small parameter approaches zero.

The results of the third chapter are extended in the fourth chapter for the following differential equations

$$(0.6)\quad \mathbf{P}\,\mathbf{u} := -\varepsilon \mathbf{u}'' + \mathbf{f}(t,[\mathbf{u}]',\mathbf{u},\varepsilon) = \mathbf{0},\ t \in D := (a,\infty),$$

subject to any one of the boundary conditions

$$(0.7)\quad \mathbf{R}\,\mathbf{u} := (\mathbf{u}(a,\varepsilon),\ \mathbf{u}(\infty,\varepsilon))^T = (\mathbf{A}(\varepsilon),\ \mathbf{B}(\varepsilon))^T,$$

$$(0.8)\quad \mathbf{R}\,\mathbf{u} := (-\mathbf{u}'(a,\varepsilon),\ \mathbf{u}(\infty,\varepsilon))^T = (\mathbf{A}(\varepsilon),\ \mathbf{B}(\varepsilon))^T,$$

$$(0.9)\quad \mathbf{R}\,\mathbf{u} := (\mathbf{u}(a,\varepsilon) - \varepsilon \mathbf{u}'(a,\varepsilon),\ \mathbf{u}(\infty,\varepsilon))^T = (\mathbf{A}(\varepsilon),\ \mathbf{B}(\varepsilon))^T.$$

Chapter 5 deals with the following initial value problems

$$(0.10)\quad d\mathbf{x}/dt = \mathbf{u}(t,\mathbf{x},\mathbf{y},\varepsilon),\ \mathbf{x}(a,\varepsilon) = \mathbf{A}(\varepsilon)$$

$$(0.11)\quad \varepsilon\, d\mathbf{y}/dt = \mathbf{v}(t,\mathbf{x},\mathbf{y},\varepsilon),\ \mathbf{y}(a,\varepsilon) = \mathbf{B}(\varepsilon),\ a < t \leq b < \infty$$

where $\varepsilon > 0$ is a small parameter and $\mathbf{x},\mathbf{y},\mathbf{u},\mathbf{v},\mathbf{A}$ and $\mathbf{B}$ are doubly infinite dimensional vector-valued functions. First the asymptotic study, as the parameter ε goes to zero, is carried out for the linear equations of the form

$$(0.12)\quad \varepsilon \underset{\sim}{x}'' + \underset{\sim}{\alpha}\, \underset{\sim}{x}' + \underset{\sim}{\beta}\, \underset{\sim}{x} = \underset{\sim}{\gamma},\ \underset{\sim}{\alpha} = \underset{\sim}{\alpha}(t,\varepsilon),\ \underset{\sim}{\beta} = \underset{\sim}{\beta}(t,\varepsilon)$$
$$\underset{\sim}{\gamma} = \underset{\sim}{\gamma}(t,\varepsilon)$$

and then the results are generalized to the nonlinear systems. Using monotonicity theorems asymptotic estimates for solutions are constructed, under appropriate assumptions, in terms of solutions of the corresponding reduced problems.

Chapter 6 treats linear and nonlinear parabolic differential equations with a small parameter multiplying the time derivative. More precisely, the differential equations

$$(0.13)\quad \varepsilon\, \partial u_i/\partial t + F_i(t,x,\underset{\sim}{u},u_{i,j},u_{i,jk},\varepsilon) = f_i(t,\varepsilon)$$
$$(t,x)\in G_p: = (0,T] \times D,\ D\subset R^m,\ i = 1(1)n,$$
$$D := \{x \mid x = (x_1,\dots,x_m),\ x_i > 0,\ i=1(1)m\}$$

subject to Dirichlet type boundary conditions are considered. Here $\underset{\sim}{u} = (u_1,\dots,u_n)$, $u_{i,jk}$ and $u_{i,j}$ stand respectively for $\partial^2 u_i/\partial x_j\, \partial x_k$ and $\partial u_i/\partial x_j$, $F_i(t,x,\underset{\sim}{u},p,r,\varepsilon)$ is monotone decreasing in r for each i, and $\varepsilon > 0$ is a small parameter. Guided by the experience with ordinary differential equations and using parabolic differential inequality theorems, estimates for solutions of the present problems are obtained. In turn these estimates determine the asymptotic behaviour of solutions as the small parameter ε approaches zero.

In Chapter 7 an attempt is made to study the method of lines (a special case of differential-difference scheme) when applied to nonlinear elliptic differential equations defined on a unit square and semi infinite strip. In fact, a few results are presented on the error estimates and the convergence of the method of lines to the boundary value problems:

(0.14) $$-u_{xx} + f(x,y,u,u_x,u_y,\varepsilon u_{yy}) = 0,$$
$$(x,y) \in G = (0,1) \times (0,1),$$

(0.15) $$u(x,0) = \phi_1(x),\ u(x,1) = \phi_2(x),\ x \in (0,1),$$

(0.16) $$u(0,y) = \psi_1(y),\ u(1,y) = \psi_2(y),\ y \in (0,1),$$

and

(0.17) $$-u_{xx} + f(x,y,u,u_x,u_y,\varepsilon u_{yy}) = 0,$$
$$(x,y) \in G' = (0,1) \times (-\infty, \infty),$$

(0.18) $$u(0,y) = \eta_1(y),\ u(1,y) = \eta_2(y),\ y \in (-\infty,\infty)$$

$$(0.19) \qquad \lim_{|y| \to \infty} u(x,y) = 0, \; x \in (0,1)$$

where ε is a small positive parameter, u_x, u_y, u_{xx} and u_{yy} stand respectively for $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$, and f is assumed to be monotone decreasing in the last argument. Due to the presence of the small parameter ε a modified version of the existing difference scheme is suggested. Using the second order differential inequalities theory, error estimates and hence the convergence of the method of lines are obtained.

The method of lines, more precisely longitudinal line method, for the following problems is discussed in Chapter 8.

$$(0.20) \qquad u_t - f(t,x,u,u_x,\varepsilon u_{xx}) = 0, \; (t,x) \in (0,T] \times (0,a), \; a < \infty$$

$$(0.21) \qquad u(0,x) = \eta(x), \; x \in [0,a]$$

$$(0.22) \qquad u(t,0) = \eta_o(t), \; u(t,a) = \eta_1(t), \; t \in (0,T]$$

and

$$(0.23) \qquad u_t - f(t,x,u,u_x, \varepsilon u_{xx}) = 0, \; (t,x) \in G'_p,$$

$$G'_p := (0,T] \times R, \; R = (-\infty, \infty)$$

$$(0.24) \qquad u(0,x) = \eta_2(x), \; x \in R$$

The last chapter continues with the study of the method of lines begun in Chapter 7. In this chapter a weakly coupled system of two parabolic differential equations with a small parameter multiplying the highest spatial derivatives is considered and results similar to that of given in Chapter 8 are given.

LIST OF ABBREVIATIONS

DE	Differential Equation(s)
ODE	Ordinary Differential Equation(s)
PDE	Partial Differential Equation(s)
IC	Initial Condition(s)
BC	Boundary Condition(s)
IVP	Initial Value Problem(s)
BVP	Boundary Value Problem(s)
IBVP	Initial Boundary Value Problem(s)
LBVP	Linear Boundary Value Problem(s)
NBVP	Nonlinear Boundary Value Problem(s)
SPP	Singular Perturbation Problem(s)
MOL	Method of Lines

CONTENTS

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Chapter 1

INTRODUCTION

The theory of differential inequalities plays an important role in the study of initial value problems (IVPs), boundary value problems (BVPs) for the differential equations (DEs). These problems pose the mathematician with a class of problems, most of which fall into one of the following categories (the dividing lines need not be sharp): (i) the existence problem; (ii) the uniqueness problem and the related problems of continuous dependence on the various given data; (iii) qualitative and quantitative properties of solutions, namely the asymptotic behaviour, the oscillatory behaviour of solutions, validity of the maximum principle, numerical determination of the solutions and so on.

A great variety of methods is available in the literature for solving the above mentioned problems. One of the methods is the 'method of differential inequalities' which has become more popular only in the past two decades. Though the phrase 'method of inequalities' will be made more clear and precise by the contents of the thesis, the basic idea underlying in this method and its application shall be given now.

Consider the problem: if $f(t)$ is a real-valued function defined on [0,T], then the integral $u(t) = \int_0^t f(\tau)d\tau$ is to be defined appropriately. The method given by Perron [1] to solve this problem proceeds as follows. First this problem is rewritten as an IVP consisting of the DE $u'=f(t)$ and the initial condition (IC) $u(0)=0$. Then one considers the functions $v(t)$ and $w(t)$ satisfying the inequalities

$$v' \leq f(t), \quad v(0) = 0$$
$$w' \geq f(t), \quad w(0) = 0.$$

It can be proved that $v \leq w$ and the solution of this problem is obtained from $u = \inf w$ or $\bar{u} = \sup v$ (integrability is defined by $u = \bar{u}$ and is thus identical with uniqueness). Soon after Perron's works, a sizable number of problems related to DEs and integral equations were attempted using Perron's method. Most of these works dealt with the problems of existence of solutions. But only recently the systematic use of differential inequalities in problems (ii) and (iii) above is found in the literature. A comprehensive account of the theory and applications of differential inequalities to DEs is well presented in [2-4].

Some basic concepts, definitions and methods which shall be used frequently in the rest of the chapters of this thesis are presented now.

Many of the theorems presented in Chapter 2 of this thesis have similar structures and this structural similarity is defined more appropriately as 'problems of monotonic type' coined by Collatz [5]. Such a problem exists if it can be written in the form Au=0 with the help of an operator A, and if either

$$(1.1) \qquad Av < Aw \Longrightarrow v < w$$

or

$$(1.2) \qquad Av \leq Aw \Longrightarrow v \leq w$$

holds true.

The theorems having structure either (1.1) or (1.2) are in general called monotonicity theorems. To illustrate this definition consider the IVP

$$(1.3) \qquad \begin{cases} u' &= f(t,u), \ t \in J_0 := (0,T] \\ u(0) &= \eta, \ u \in U := C^1(J_0) \cap C(J), \ J := [0,T] \end{cases}$$

which can be written as

$$(1.4) \qquad Au := F \Longleftrightarrow \begin{cases} Pu := u' - f(t,u) = 0, \ t \in J_0, \\ Ru := u(0) = \eta. \end{cases}$$

Here C: = D means 'C is defined by D', a prime "'" denotes a differentiation with respect to t. Then for this problem one has a monotonicity theorem of the type (1.1), that is,

$$\left.\begin{array}{ll} v'-f(t,v) & < w'-f(t,w) \\ v(0) & < w(0) \end{array}\right\} \Longrightarrow v(t) < w(t).$$

It is to be noted that

$$\phi < \psi \text{ in } J \Longleftrightarrow \phi(t) < \psi(t),\ t \in J.$$

There are many interesting consequences for a problem which is of monotonic type. Infact, consider a problem written as $Au = 0$ for which a monotonicity theorem of type (1.2) is valid. Then this problem has a unique solution if it exists. For, let u_1, u_2 be two solutions of $Au = 0$. Now $Au_1 = 0 = Au_2 \Longleftrightarrow Au_1 \leq Au_2$ and $Au_2 \leq Au_1$ which yield $u_1 \leq u_2$ and $u_2 \leq u_1$, that is, $u_1 \equiv u_2$. Also, in this case the construction of lower and upper bounds becomes simpler. For, it is enough to look for two functions v and w satisfying the inequalities $Av \leq 0 \leq Aw$ which in turn implies $v \leq u \leq w$ where u is a solution of $Au = 0$.

A great variety of monotonicity theorems and their applications to DEs is well presented in [3] and [6]. Some authors prefer the name comparison theorems instead of monotonicity theorems. But this name is mainly used only for the BVPs concerned with second order ordinary differential equations (ODEs). Some authors prefer to call the operator A which satisfies the implication (1.2) (or (1.1)) as an inverse-isotone operator with (or without) the admission of equality sign [6].

Closely related to monotonicity theorems is the concept of quasi-monotonicity. As has already been observed, the IVP(1.4) is always of monotonic type in the sense of (1.1). This is not true, in general, for an IVP consisting of systems of DEs. Infact, the IVP

$$(1.5) \qquad \underset{\sim}{A}\,\underset{\sim}{u}: = \underset{\sim}{F} \Longleftrightarrow \begin{cases} \underset{\sim}{P}\,\underset{\sim}{u}: = \underset{\sim}{u}'(t)-\underset{\sim}{f}(t,\underset{\sim}{u}) = \underset{\sim}{0}, & t \in J_o, \\ \underset{\sim}{R}\,\underset{\sim}{u}: = \underset{\sim}{u}(0) = \underset{\sim}{\eta}, \end{cases}$$

where $\underset{\sim}{u} = (u_1, \ldots, u_n)$, $\underset{\sim}{f} = (f_1, \ldots, f_n)$ and $\underset{\sim}{\eta} = (\eta_1, \ldots, \eta_n)$, need not be a problem of monotonic type. But when the vector-valued function $\underset{\sim}{f}$ has the following monotonicity property the IVP (1.5) becomes a problem of monotonic type (1.1): $f_i(t,\underset{\sim}{u})$ is increasing with u_j, $j \neq i$, $i,j = 1(1)n$. In this case $\underset{\sim}{f}$ is said to be quasimonotone increasing with $\underset{\sim}{u}$.

Next comes the concept of subfunctions and superfunctions. A function v (respectively w) is called a subfunction (respectively superfunction) with respect to a given problem if it is defined in the appropriate domain and if $v \leq u$ (respectively $u \leq w$) for every solution u of the problem. It is to be noted that every upper bound to the solution is a superfunction.

An essential part of this thesis is concerned with theorems on (error) estimations. They are all formulated according to some uniform procedures. Very often an inequality of the form $|v-u| \leq \rho$ is established. Here u is a solution of the given problem, say Au=0, and v is an approximate solution for whose 'defect' Av one knows a 'defect bound' $|Av| \leq \delta$. The bound ρ is either determined from a problem with a similar structure $\Omega\rho = 0$ or from the earlier experience of dealing problems similar to Au = 0. To illustrate this consider the IVP (1.3) and let u be its solution. For a given approximate solution v(t), the quantities $v'-f(t,v)$ and $v(0)-\eta$ can be determined; that is, more precisely, let v(t) satisfy the following inequalities

(1.6) $\quad | v'-f(t,v) | \leq \delta , \; | v(0)-\eta | \leq \varepsilon ,$

where δ and ε are constants. Further, let f(t,z) satisfy the Lipschitz condition

(1.7) $\quad | f(t,z)-f(t,\bar{z}) | \leq L(z-\bar{z}), \; z \geq \bar{z} ,$

where L is a positive constant. Let ρ be the solution of the linear DE

$$\rho' = L\rho + \delta , \; \rho(0) = \varepsilon .$$

Then one can establish the error estimate

$$| v-u | \leq \rho .$$

The inequality $|v-u| \leq \rho$ can also be proved in another way by using an appropriate criteria to show that $v-\rho$ and $v+\rho$ are a subfunction and a superfunction respectively for the original problem. This way of getting error estimates are possible mainly because the problems considered in this thesis are mostly of monotonic type.

Next comes the definition of singular perturbation problems (SPPs). Consider, for example, a family of BVPs (IVPs or IBVPs) P_ε depending on a small parameter ε. Under certain conditions, a 'solution' $y_\varepsilon(x)$ of P_ε can be constructed by the well known 'method of perturbation'- that is, as a power series in ε with the first term y_0 being the solution of the problem P_0(the reduced problem of P_ε obtained by setting $\varepsilon = 0$). When such an expansion converges as $\varepsilon \to 0$ uniformly in x, one has a regular perturbation problem. When $y_\varepsilon(x)$ does not have a uniform limit in x as $\varepsilon \to 0$, this regular perturbation method will fail and one gets a SPP.

In this thesis two aspects of SPPs are considered. One aspect is to study the asymptotic behaviour of the solutions and/ or their derivatives as the small parameter goes to zero by constructing the necessary estimates. This helps in recognizing the boundary layer nature of the solutions and/ or their derivatives. Such problems are considered in Chapters 3-6. The other aspect is to apply the method of lines (MOL) (explained in the next paragraph) to SPPs. Problems of this type are discussed in Chapters 7-9.

The last three chapters of this thesis are devoted to the study of MOL. To understand the basic ideas underlying in this method an example is provided here. Consider the one-dimensional heat equation

$$(1.8) \qquad \partial u/\partial t = \partial^2 u/\partial x^2, \ (t,x) \in G_p := (0,T] \times (0,a)$$

subject to the initial and boundary conditions (BCs)

$$(1.9) \qquad \begin{cases} u(0,x) = \eta(x), \ x \in [0,a] \\ u(t,0) = \eta_0(t), \\ u(t,a) = \eta_1(t), \ t \in (0,T]. \end{cases}$$

By using discretization in the spatial variable only, that is, by putting $x_i = ih$ for $i=0(1)n$, where $h=a/n$ (n fixed), and replacing $u(t,x_i)$ by $v_i(t)$, $\frac{\partial^2 u(t,x_i)}{\partial x^2}$ by $\delta^2 v_i = (v_{i+1} - 2v_i + v_{i-1}) / h^2$, the heat equation is replaced by

$$(1.10) \qquad v_i'(t) = \delta^2 v_i(t) = (v_{i+1} - 2v_i + v_{i-1})/h^2, \ i=1(1)n-1.$$

The corresponding conditions are

$$(1.11) \quad \begin{cases} v_i(0) = \eta(x_i), & i = 1(1)n-1, \\ v_o(t) = \eta_o(t), & v_n(t) = \eta_1(t). \end{cases}$$

The new problem, that is, (1.10)-(1.11) is an IVP for a system of n-1 ODEs. This approximation scheme which is a special case of differential-difference method is called the line method or straight-line method or method of lines. There is another method in which discretization is performed in the time variable t only. In this method the original problem gets transformed into a BVP for systems of second order ODEs and is called the 'method of Rothe'. This method is also called 'transversal' line method whereas the method discussed above is called the 'longitudinal' line method. In this thesis only the longitudinal line method is considered for parabolic equations. Similar line method is also discussed for the elliptic equations. There is ample literature on the line method but most of the works are confined only to the transversal line method. Also the present approach for the MOL in this thesis differs from that of many authors in so far as it makes systematic use of the theory of ordinary differential inequalities. The survey article by Liskovets [7] contains a large bibliography on MOL.

A brief outline of the works presented in different chapters of this thesis together with a short review of the works closely related to them is given now.

Related to the IVP (1.4) there is a classical result on differential inequalities which is given in the following theorem.

Theorem 1.1. [3]

Consider the IVP (1.4). For the functions $v, w \in U$ the following implication is true.

$$(1.12) \qquad Av < Aw$$

that is,

$$\begin{cases} Pv < Pw, \ t \in J_0 \\ Rv < Rw \end{cases}$$

$$\Longrightarrow$$

$$(1.13) \qquad v(t) < w(t), \ t \in J_0.$$

Various generalizations of the above theorem are available. The following theorem is one among them.

Theorem 1.2. [3]

Consider the IVP (1.4). Suppose that f satisfies the Lipschitz condition

$$| f(t,x)-f(t,y) | \le L |x-y|, L > 0.$$

Then for every $v,w \in U$ the following implication is true.

$$(1.14) \qquad Av \le Aw \Longrightarrow v \le w.$$

These monotonicity theorems are extended in many directions. Infact monotonicity theorems are available for systems of first order, second order ordinary and partial DEs. For details one can refer to [3] and [4]. Adams and Spreuer [6,8] also gave monotonicity theorems for more general case. But their statements and proofs are slightly different from that of given by Walter [3]. Monotonicity theorems presented in this thesis follow the ideas of Adams and Spreuer. The following is a monotonicity theorem for more general integro-differential equations.

Now consider the weakly coupled system of second order integro-differential equations which includes the BCs [6,8,9]:

$$(1.15) \qquad A_i \underset{\sim}{u}:=F_i(x,\underset{\sim}{u},u_{i,j},u_{i,jk}, \int_{\overline{G}} K_i(x,t,\underset{\sim}{u}(t))dt) = 0,$$

on $\overline{G}: = G \cup \partial G$, $j,k = 1(1)m$, $i = 1(1)n$, $G \subset R^m$ being a bounded domain with ∂G as its boundary, where

(i) $\underset{\sim}{u} = (u_1, \ldots, u_n)$, $\underset{\sim}{x} = (x_1, \ldots, x_m)$; $u_{i,j}$ and $u_{i,jk}$ stand for $\partial u_i/\partial x_j$ and $\partial^2 u_i/\partial x_j\, \partial x_k$ respectively;

(ii) F_i, for fixed i, is defined on $\bar{G} \times R^n \times R^m \times R^{m^2} \times R$, K_i, for fixed i, is defined on $\bar{G} \times \bar{G} \times R^n$;

(iii) the second derivatives are admitted in F_i for $x \in \partial G$ provided the matrix $(u_{i,jk})$ then possesses only derivatives tangential to ∂G;

(iv) $F_i(x,\underset{\sim}{u},p,q,r)$, for fixed i, is weakly monotone decreasing with respect to the $m \times m$ matrix 'q' on $\bar{G}$ [3, p.304];

(v) for $x \in \partial G$: (a) an outer normal derivative $\partial(\ )/\partial n_a$ [3, p.245] exists everywhere on ∂G; (b) F_i is weakly monotone increasing with respect to 'p' where p denotes the derivative $\partial(\ )/\partial n_a$;

(vi) the integral $\int_{\bar{G}}$ may be of the Stieltjes type and thus may include $\underset{\sim}{u}(t)$ at points of $\bar{G}$ with $t \neq x$;

(vii) $u_i \in U := C^2(\bar{G})$, $i = 1(1)n$, provided that the second order derivatives appear in F_i for some i and $x \in \bar{G}$; otherwise smoothness requirements are correspondingly weaker.

The above problem (1.15) represents many types namely

(i) IVPs for ODEs (single or system)

(ii) BVPs for ODEs of second order (single or system)

(iii) BVPs for elliptic DEs (single or system)

(iv) IBVPs for parabolic DEs (single or system).

The differential inequality theorem given by Adams and Spreuer [6,8] is as follows.

Theorem 1.3.

Consider the problem (1.15) and assume:

(i) F_i, for fixed i, is weakly monotone decreasing with respect to (a) $u_j(x)$ for $j \neq i$, $i,j=1(1)n$, and (b) $u_k(t)$ for $t \neq x$ and $k = 1(1)n$;

(ii) there exists a vector function $\underset{\sim}{s} = (s_1, \ldots, s_n)$, named as a 'test function', such that

$$s_i = s_i(x,\alpha) > 0 \text{ on } \bar{G},\ s_i \in U,\quad \alpha > 0,$$

$$s_i(x,0) = 0,\ \liminf_{\substack{x \in \bar{G} \\ \alpha \to \infty}} s_i(x,\alpha) = \infty,$$

$$A_i(\underset{\sim}{u}+\underset{\sim}{s}) - A_i\underset{\sim}{u} > 0 \text{ on } \bar{G} \text{ for every } \underset{\sim}{u},\ u_i \in U,\ i=1(1)n.$$

Then the operator $\underset{\sim}{A} = (A_1, \ldots, A_n)$ is inverse-isotone with admission of equality sign, that is, the following implication is true for all $\underset{\sim}{y} = (y_1, \dots , y_n)$ and $\underset{\sim}{z} = (z_1, \dots , z_n)$, $y_i, z_i \in U$, $i=1(1)n$:

$$(1.16) \quad \begin{cases} A_i \underset{\sim}{z} \leq A_i \underset{\sim}{u} \leq A_i \underset{\sim}{y},\ i = 1(1)n \\ \Longrightarrow \\ z_i(x) \leq u_i(x) \leq y_i(x) \text{ on } \overline{G},\ i = 1(1)n. \end{cases}$$

A more rigorous proof for the above theorem (in the case of a parabolic system) than the one given in [6,8] is presented in [10]. The results given in [6,8,10] are confined only with finite systems of DEs and also the equations are defined only on bounded regions. The second chapter of this thesis gives monotonicity theorems for doubly infinite weakly coupled systems of first, second order ODEs defined on bounded, unbounded real intervals and finite systems of parabolic DEs defined on unbounded regions. Also slightly modified versions of existing monotonicity theorems for finite systems of first and second order ODEs are presented.

Several authors [11-13] used maximum principle in DEs to study some SPPs for partial differential equations (PDEs) as well as for ODEs. The work of W. Eckhaus and E.M.DeJager [14] is a direct application of the maximum principle to such problems. This basic and useful tool was further exploited by Dorr, Parter and Shampine [15] in the study of SPPs for second order ODEs. The DEs considered by them are of the form

(1.17) $\varepsilon y'' = f(t,y,y',\varepsilon)$,

where $\varepsilon > 0$ is a small parameter.

They obtained estimates for solutions of the equation (1.17) and then these estimates are used to study the asymptotic behaviour of solutions as the small parameter ε goes to zero. Following them, a good number of research articles have started appearing in the literature which made use of the theory of differential inequalities to study such SPPs [16-24]. Some of the results of [15-20,25] are direct extensions from single DE to systems of DEs by N.Ramanujam and U.N.Srivastava [26,27]. Also a few more new results are provided by them.

In Chapter 3 of this thesis two point BVPs, described by doubly infinite weakly coupled systems of second order ODEs defined on bounded intervals with a small parameter multiplying the highest derivative, are studied using monotonicity theorems of second order differential inequalities in a Banach space. Estimates for solutions are obtained which are then utilized to derive results on the limiting behaviour of solutions as the small parameter goes to zero.

The results so far obtained by many authors and in Chapter 3 of this thesis for SPPs are confined with the DEs of the form (1.17) which are defined only on bounded intervals. The study of asymptotic behaviour of solutions of such equations defined on unbounded intervals is carried out in Chapter 4.

In Chapter 5 singularly perturbed IVPs are considered. The scalar IVP

$$(1.18)\begin{cases}\varepsilon u'' + f(t,u,u',\varepsilon) = 0, \ 0 \leq t \leq b < \infty ,\\ u(0,\varepsilon) = A(\varepsilon), \ u'(0,\varepsilon) = B(\varepsilon),\end{cases}$$

where $\varepsilon > 0$ is a small parameter, has been considered by Baxley [25]. Under suitable conditions on 'f' which are quite different from the standard uniform Lipschitz condition, he obtained bounds for $u(t,\varepsilon)$ and $u'(t,\varepsilon)$ as $\varepsilon \to 0+$ by the use of the maximum principle. These bounds do not appear to contain the boundary layer term which is important for the analysis of the nonuniformity occuring in the derivative of the solution as $\varepsilon \to 0+$. From standard differential inequality theorems for first order ODEs [28], Howes [19] deduced the existence and asymptotic behaviour of solutions of the following scalar IVP:

$$(1.19)\begin{cases}a(t,\varepsilon)u' = f(t,u,\varepsilon), \ t \in (0,T],\\ u(0,\varepsilon) \quad = A,\end{cases}$$

where $a(t,\varepsilon) = a(t,0) + O(\varepsilon)$, $a(t,\varepsilon) > 0$.

Here and throughout this thesis, O denotes the Landau order symbol.

Weinstein and Smith [29], using elementary comparison techniques, studied the behaviour of solution of the DE

$$(1.20) \qquad \varepsilon u'' + b(t,\varepsilon)u' + c(t,\varepsilon)u = f(t,\varepsilon),$$

for small values of the positive parameter ε and for values of t on a given interval $[0,t^*]$. They obtained estimates for solution of the equation (1.20) which are valid only when the coefficients $c(t,\varepsilon)$ and $b(t,\varepsilon)$ satisfy the over-damping condition

$$(1.21) \qquad 4\varepsilon c_1 < b_o^2,$$

where

$$(1.22) \qquad c(t,\varepsilon) \leq c_1 \text{ and } b(t,\varepsilon) \geq b_o > 0.$$

Of course, it is mentioned in [29] that the condition (1.21) can be removed by using either a theorem available in it or the generalized maximum principle [30].

In [27] authors considered the same equation (1.20) and obtained estimates for solutions and its derivatives. The results are obtained using the theory of first order differential inequalities. They then generalized the results to finite systems of first order equation and higher order equations. Chapter 5 of the present thesis extends the results of [27] to the IVP

$$(1.23)\quad \begin{cases} d\underset{\sim}{x}/dt = \underset{\sim}{u}(t,\underset{\sim}{x},\underset{\sim}{y},\varepsilon),\ \underset{\sim}{x}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \\ \varepsilon\, d\underset{\sim}{y}/dt = \underset{\sim}{v}(t,\underset{\sim}{x},\underset{\sim}{y},\varepsilon),\ \underset{\sim}{y}(a,\varepsilon) = \underset{\sim}{B}(\varepsilon),\ a < t \le b < \infty, \end{cases}$$

where $\varepsilon > 0$ is a small parameter and $\underset{\sim}{x}$, $\underset{\sim}{y}$, $\underset{\sim}{u}$, $\underset{\sim}{v}$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are doubly infinite-dimensional vector functions. Under appropriate assumptions and using the theory of first order differential inequalities in a Banach space, asymptotic estimates for solutions of the above IVP are constructed in terms of solutions of the reduced problem

$$(1.24)\quad \begin{cases} d\underset{\sim}{X}/dt = \underset{\sim}{u}\,(t,\underset{\sim}{X},\underset{\sim}{Y},0),\ \underset{\sim}{X}(a) = \underset{\sim}{A}(0), \\ \underset{\sim}{v}(t,\ \underset{\sim}{X},\ \underset{\sim}{Y},\ 0) = \underset{\sim}{0}. \end{cases}$$

The estimates so obtained contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of solutions as functions of t and ε

In [31], the authors obtained estimates for the solutions of systems of PDEs of parabolic type with a small parameter multiplying the time derivative. These estimates determine the asymptotic behaviour of solutions. They considered the DEs which are defined only in the bounded regions. These results are extended in Chapter 6 of this thesis to parabolic DEs defined on unbounded regions.

The MOL is widely used in discussing the existence, uniqueness and finding numerical solutions of PDEs. In Chapter 7 an attempt is made to apply the MOL to elliptic DEs with a small parameter multiplying one of the highest derivatives. Due to the presence of the small parameter a modified version of the existing difference scheme is suggested. The theory of second order differential inequalities is used to obtain error estimates for this method which, in turn, determine the convergence of this method.

In Chapters 8 and 9 the MOL is further applied respectively to single parabolic DE and to systems of parabolic DEs with a small parameter multiplying the highest spatial derivatives.

The possible extensions of the work presented in this thesis are given at appropriate places in the respective chapters.

Chapter 2

MONOTONICITY THEOREMS*

Monotonicity theorems for doubly infinite weakly coupled systems of ODEs defined on bounded, unbounded real intervals and for parabolic DEs defined on unbounded regions are developed in this chapter. Further necessary definitions and notations which shall be used in the rest of this thesis are given. Also if a given system of DEs does not satisfy the quasi-monotonicity condition then a method of constructing an adjoint system for which the quasi-monotonicity condition is satisfied is presented.

1. MONOTONICITY THEOREMS FOR WEAKLY COUPLED FINITE SYSTEMS OF ODEs DEFINED ON BOUNDED INTERVALS.

Definition 2.1.

A vector-valued function $\underset{\sim}{g} : R^m \longrightarrow R^n$ is quasi-monotone increasing (decreasing) in $\underset{\sim}{u} \in R^m$ if $g_i(\underset{\sim}{u})$ is increasing (decreasing) with u_j, $j \neq i$, $i = 1(1)n$, $j = 1(1)m$. Here R^n is n-dimensional Euclidian space.

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* Some portion of this chapter is to appear in SIAM J. NUMER. ANAL. Vol.25, No.2, April 1988 and Bullettino U.M.I (7) I.B(1987).

Theorem 2.2.

Consider the BVP

$$(2.1)\quad \begin{cases} P_i\,\underset{\sim}{v} := -v_i'' + f_i(x, v_i', \underset{\sim}{v}) = 0, \quad x \in D := (a,b), \\ R_i\,\underset{\sim}{v} := \begin{cases} v_i(a) = A_i \\ v_i(b) = B_i, \quad i = 1(1)m-1, \end{cases} \end{cases}$$

where $\underset{\sim}{v} = (v_o, \ldots, v_m)$, $v_i \in U := C^2(D) \cap C(\bar{D})$, $\bar{D} := [a,b]$.

Assume

(i) $\underset{\sim}{f} = \underset{\sim}{f}(x, p, \underset{\sim}{z}) = (f_1, \ldots, f_m)$ is quasimonotone decreasing in $\underset{\sim}{z}$;

(ii) $\underset{\sim}{v}$ is a solution of the BVP (2.1);

(iii) there exists a 'test function' $\underset{\sim}{s} : \bar{D} \longrightarrow R^{m+1}$ with the properties: $\underset{\sim}{s} > 0$ on $\bar{D}$, that is, $s_i(x) > 0$, $s_i \in U$, $i = 0(1)m$,

$$(2.2)\qquad P_i(\underset{\sim}{z} + \alpha\underset{\sim}{s}) - P_i\underset{\sim}{z} > 0, \quad i = 1(1)m-1,$$

for every positive real number α and $\underset{\sim}{z}$, $z_i \in U$, $i = 0(1)m$.

Then for every $\underset{\sim}{u} = (u_o, \ldots, u_m)$, $\underset{\sim}{w} = (w_o, \ldots, w_m)$, w_i, $u_i \in U$, $i=0(1)m$, the following implication is true:

$$(2.3)\quad \begin{cases} u_o(x) \le v_o(x) \le w_o(x) \\ u_m(x) \le v_m(x) \le w_m(x), \ x \in \bar{D} \\ R_i \underset{\sim}{u} \le R_i \underset{\sim}{v} \le R_i \underset{\sim}{w} \\ P_i \underset{\sim}{u} \le P_i \underset{\sim}{v} \le P_i \underset{\sim}{w}, \ i = 1(1)m-1 \end{cases}$$

$$\Longrightarrow$$

$$(2.4)\quad u_i(x) \le v_i(x) \le w_i(x), \ x \in D, \ i = 1(1)m-1.$$

Proof

In the following a proof is given to establish the right inequality of (2.4). Define

$$\alpha = \max_{i=0(1)m} \left[\max_{x \in \bar{D}} \left\{ (v_i(x) - w_i(x))/s_i(x) \right\} \right].$$

If the right inequality is not true then the constant α is positive. Further, due to the continuity, there exists a point $x_o \in D$ and a number $j \in \{1, \dots, m-1\}$ with the properties

$$v_j(x_o) - w_j(x_o) - \alpha\, s_j(x_o) = 0 \quad \text{and}$$

$$v_i(x) - w_i(x) - \alpha\, s_i(x) \le 0, \ i = 0(1)m, \ x \in \bar{D}.$$

Therefore the function $v_j(x) - w_j(x) - \alpha\, s_j(x)$ attains its maximum at $x = x_o$. Hence

$$v_j'(x_o) - w_j'(x_o) - \alpha\, s_j'(x_o) = 0,$$

$$v_j''(x_o) - w_j''(x_o) - \alpha\, s_j''(x_o) \le 0.$$

Therefore at $x = x_o$,

$$\begin{aligned} 0 &\leq P_j \underset{\sim}{w} - P_j \underset{\sim}{v} \\ &= P_j \underset{\sim}{w} - [-v_j'' + f_j(x_o, v_j'(x_o), \underset{\sim}{v}(x_o))] \\ &\leq P_j \underset{\sim}{w} - [-w_j'' - \alpha s_j'' + f_j(x_o, w_j'(x_o) + \alpha s_j'(x_o), \\ &\qquad \underset{\sim}{w}(x_o) + \alpha \underset{\sim}{s}(x_o))] \\ &\leq P_j \underset{\sim}{w} - P_j(\underset{\sim}{w} + \alpha \underset{\sim}{s}). \end{aligned}$$

It is a contradiction to the condition (2.2). Hence the theorem is proved.

The following theorem corresponds to IVPs for first order ODEs and can be proved by the method adopted in the above theorem.

Theorem 2.3.

Consider the IVP

$$(2.5) \quad \begin{cases} P_i \underset{\sim}{v} := v_i' - f_i(t, \underset{\sim}{v}) = 0, \; t \in J_o, \\ R_i \underset{\sim}{v} := v_i(0) = A_i, \; i = 1(1)m-1, \; A_i \in R, \end{cases}$$

where $\underset{\sim}{v} = (v_o, \ldots, v_m)$, $v_i \in U := C^1(J_o) \cap C(J)$.

Assume

(i) $\underset{\sim}{f} = \underset{\sim}{f}(t, \underset{\sim}{z})$ is quasimonotone increasing in $\underset{\sim}{z}$ in the sense of Definition 2.1;

(ii) $\underset{\sim}{v}$ is a solution of the IVP (2.5);

(iii) there exists a 'test function' $\underset{\sim}{s} : J \to R^{m+1}$ with the properties $\underset{\sim}{s} > 0$ on J, that is, $s_i(x) > 0$ on J, $s_i \in U$, $i = 0(1)m$,

$$(2.6) \qquad P_i\,(\underset{\sim}{z} + \alpha\,\underset{\sim}{s}) - P_i\,\underset{\sim}{z} > 0, \; i = 1(1)m-1,$$

for every positive real number α and $\underset{\sim}{z}$, $z_i \in U$, $i = 0(1)m$.

Then for every $\underset{\sim}{u} = (u_o, \ldots, u_m)$, $\underset{\sim}{w} = (w_o, \ldots, w_m)$, $u_i, w_i \in U$, $i = 0(1)m$, the following implication is true.

$$(2.7) \quad \left\{ \begin{array}{lllll} u_o(t) & \leq & v_o(t) & \leq & w_o(t) \\ u_m(t) & \leq & v_m(t) & \leq & w_m(t), \; t \in J \\ P_i\,\underset{\sim}{u} & \leq & P_i\,\underset{\sim}{v} & \leq & P_i\,\underset{\sim}{w} \text{ in } J_o \\ R_i\,\underset{\sim}{u} & \leq & R_i\,\underset{\sim}{v} & \leq & R_i\,\underset{\sim}{w}, \; i = 1(1)m-1 \end{array} \right.$$

$$\Longrightarrow$$

$$(2.8) \qquad u_i(t) \leq v_i(t) \leq w_i(t), \; t \in J, \; i = 1(1)m-1.$$

In the IVP (2.5) the functions $v_o(t)$ and $v_m(t)$, in general, are known (data) functions. But in Theorem 2.3 they are treated as if they were unknown. Such a treatment helps one to unify the methods of proof involved in obtaining the error estimates for the MOL.

It is to be noted that the symbol U which was used in (2.1) is used here also. This is done with the purpose to avoid creating new symbols. Further, at any time, only a particular class of DEs, for example, either first order, second order ODEs or PDEs shall be considered. In general U stands for set of all functions n times continuously differentiable in the domain where the DE is defined and (n-1) continuously differentiable on the boundary ∂D. Here n is the highest order of the derivative appearing in the DE.

2. MONOTONICITY THEOREMS FOR DOUBLY INFINITE WEAKLY COUPLED SYSTEMS OF ODEs DEFINED ON BOUNDED INTERVALS.

In this section monotonicity theorems for doubly infinite weakly coupled systems of second order ODEs are given. The proof of Theorem 2.5 (given below) is different from that of given in [3, 32] and the conditions are weaker than that of given by Chandra [33]. Also it is a generalization of the theorems given in [8, 10, 33] for ODEs. These theorems shall be made use of in the discussion of MOL to Cauchy problems for PDEs.

Let $B = \ell^{\infty}(Z,R)$ be the Banach space of all doubly infinite sequences with norm of $\underset{\sim}{u} \in B$ given by $\| \underset{\sim}{u} \| = \sup_{i \in Z} |u_i|$.

Here Z is the set of all integers. If $\underset{\sim}{v}, \underset{\sim}{w} \in B$, then $\underset{\sim}{v} \leq \underset{\sim}{w}$ and $\underset{\sim}{v} < \underset{\sim}{w}$ respectively mean $v_i \leq w_i$, $i \in Z$ and $v_i \leq w_i - \delta$, $i \in Z$, for some positive real number δ.

The continuity and differentiability of the functions $\underset{\sim}{u}(t) : \bar{D} := [a,b] \longrightarrow B$, $\bar{D} \subset R$, can be defined as follows. The function $\underset{\sim}{u}(t)$ is continuous at the point $t_o \in \bar{D}$ (respectively differentiable at the point $t_o \in D := (a,b)$ with the derivative $\underset{\sim}{u}'(t_o)$) if $\| \underset{\sim}{u}(t_o+h) - \underset{\sim}{u}(t_o) \| \longrightarrow 0$ as $h \rightarrow 0$ (respectively $\| 1/h\,(\underset{\sim}{u}(t_o+h) - \underset{\sim}{u}(t_o)) - \underset{\sim}{u}'(t_o) \| \longrightarrow 0$ as $h \longrightarrow 0$).

In the following, for an interval $\bar{D} \subset R$, $C^k(\bar{D})$ stands for the set of all functions k times continuously differentiable in $\bar{D}$.

Definition 2.4.

Consider a Banach-valued function $\underset{\sim}{g}: D' \longrightarrow B$, $D' \subset B$. Then the function $\underset{\sim}{g} = \underset{\sim}{g}(\underset{\sim}{u})$ is quasimonotone decreasing in $\underset{\sim}{u}$ if there exist real positive constants M_j's such that

$$\underset{\sim}{M} = (\ldots, M_{-1}, M_o, M_1, \ldots) \in B \quad \text{and}$$

$$\underset{\sim}{u} \leq \underset{\sim}{v} \Rightarrow g_j(\underset{\sim}{u}) - g_j(\underset{\sim}{v}) \geq M_j(u_j - v_j), \; j \in Z, \; \underset{\sim}{u}, \underset{\sim}{v} \in B.$$

Similarly one can define quasimonotone increasing functions.

Theorem 2.5.

Assume

(i) $\underset{\sim}{f}(x,p,\underset{\sim}{z}) : D \times R \times B \to B$ is quasimonotone decreasing in $\underset{\sim}{z}$ in the sense of Definition 2.4;

(ii) there exists a real number $\delta > 0$ and a 'test function' $\underset{\sim}{s}(x) : \bar{D} \to B$ with the properties

$s_i(x) = s_j(x)$, $i,j \in Z$, $s_i(x) > 0$ on $\bar{D}$, $i \in Z$, $\underset{\sim}{s} \in U := C^2(D) \cap C(\bar{D})$, $\underset{\sim}{s} = (\ldots, s_{-1}, s_o, s_1, \ldots)$,

$$(2.9) \qquad P_i(\underset{\sim}{z}+\alpha_1\underset{\sim}{s}) - P_i \underset{\sim}{z} \geq \alpha_1\delta > 0 \text{ in } D,\ i \in Z,$$

for every positive real number α_1 and $\underset{\sim}{z} \in U$. Here

$$\alpha_1 \underset{\sim}{s} = (\ldots, \alpha_1 s_{-1}, \alpha_1 s_o, \alpha_1 s_1, \ldots) \text{ and}$$

$$P_i \underset{\sim}{z} := -z_i'' + f_i(x, z_i', \underset{\sim}{z}).$$

Then for every $\underset{\sim}{v}, \underset{\sim}{w} \in U$ the implication

$$(2.10) \quad \begin{cases} \underset{\sim}{v}(a) \leq \underset{\sim}{w}(a), & \underset{\sim}{v}(b) \leq \underset{\sim}{w}(b) \\ P_i \underset{\sim}{v} \leq P_i \underset{\sim}{w}, & i \in Z \end{cases}$$

$$\Longrightarrow$$

$$(2.11) \quad \underset{\sim}{v}(x) \leq \underset{\sim}{w}(x),\ x \in D,$$

is true provided there exist positive constants L_j's such that

$$\underset{\sim}{L} = (\ldots, L_{-1}, L_o, L_1, \ldots) \in B$$

and

(2.12) $$f_j(x,p,\underset{\sim}{w}+\beta\underset{\sim}{s}) - f_j(x,p,\underset{\sim}{w}+\eta\underset{\sim}{s}) \geq L_j(\eta-\beta)s_j(x), \ j \in Z, \quad \eta \leq \beta, \ \eta,\beta \in R.$$

Proof

Let $\Phi(x) = \inf\{w_i(x) - v_i(x),\ i \in Z\}$, $x \in \bar{D}$, which is a continuous function on $\bar{D}$ [3, p. 98]. Hence there exists a point $x_o \in \bar{D}$ such that

$$\alpha = - \Phi(x_o)/\ s(x_o) = - \min_{x \in \bar{D}} (\Phi(x)/\ s(x)),$$

where $\underset{\sim}{s} = (\ldots, s_{-1}, s_o, s_1, \ldots)$, $s(x) = s_i(x)$, $i \in Z$. If the assertion of the present theorem, that is, (2.11) is not true, then assume the contrary $\underset{\sim}{v}(x) \nleq \underset{\sim}{w}(x)$ in $\bar{D}$, which in turn implies that the real number α defined above is positive.

Also

$$v_i(x) - [w_i(x) + \alpha s(x)] \leq 0, \ i \in Z.$$

Given $\eta > 0$ such that

$$\eta < \alpha, \ \eta < \alpha\delta/(\max_{x \in \bar{D}} s(x) (\| \underset{\sim}{L} \| + \| \underset{\sim}{M} \|) + \delta)$$

and hence given $\eta s(x_o)$, there exists at least one j with the property

$$v_j(x_o) - w_j(x_o) > \alpha s(x_o) - \eta s(x_o),$$

that is,

$$v_j(x_o) - [w_j(x_o) + (\alpha-\eta)\ s(x_o)] > 0.$$

Further the function $v_j(x) - [w_j(x)+(\alpha-\eta)\ s(x)]$ is negative at $x=a$, $x=b$ and is positive at $x=x_o$. Hence this function attains a positive maximum in D, say at some point $x=x_1 \in D$. Therefore, at $x=x_1$,

$$(2.13) \qquad v_j(x_1) - [w_j(x_1) + (\alpha-\eta)\ s(x_1)] > 0,$$

$$(2.14) \qquad v_j'(x_1) - [w_j'(x_1) + (\alpha-\eta)\ s'(x_1)] = 0,$$

$$(2.15) \qquad v_j''(x_1) - [w_j''(x_1) + (\alpha-\eta)\ s''(x_1)] \leq 0.$$

Now from (2.9), (2.10), (2.14), (2.15) and (2.12) at $x = x_1$

$$\begin{aligned} 0 &\geq P_j \underset{\sim}{w} - P_j(\underset{\sim}{w} + (\alpha-\eta)\ \underset{\sim}{s}) + \delta(\alpha-\eta) \\ &\geq P_j \underset{\sim}{v} - P_j(\underset{\sim}{w} + (\alpha-\eta)\ \underset{\sim}{s}) + \delta(\alpha-\eta) \\ &= -(v_j - w_j + (\eta-\alpha)s)''\ (x_1) + f_j(x_1, v_j', \underset{\sim}{v}) \\ &\quad - f_j(x_1, w_j' + (\alpha-\eta)s_j', \underset{\sim}{w} + (\alpha-\eta)\underset{\sim}{s}) + \delta(\alpha-\eta) \\ &\geq f_j(x_1,v_j',\underset{\sim}{v}) - f_j(x_1,w_j'+(\alpha-\eta)s_j', \underset{\sim}{w}+(\alpha-\eta)\underset{\sim}{s}) + \delta(\alpha-\eta) \\ &\geq f_j(x_1,v_j',\underset{\sim}{v}) - f_j(x_1,w_j'+(\alpha-\eta)s_j',\underset{\sim}{v}) + f_j(x_1,w_j'+(\alpha-\eta)s_j',\underset{\sim}{v}) \end{aligned}$$

$$- \; f_j(x_1, w_j' + (\alpha-\eta)s_j', \underset{\sim}{w} + \alpha\underset{\sim}{s}) + f_j(x_1, w_j' + (\alpha-\eta)s_j', \underset{\sim}{w} + \alpha\underset{\sim}{s})$$

$$- \; f_j(x_1, w_j' + (\alpha-\eta)s_j', \underset{\sim}{w} + (\alpha-\eta)\,\underset{\sim}{s}) + \delta\,(\alpha-\eta)$$

$$\geq -M_j\, \eta s(x_1) - L_j\, \eta s(x_1) + \delta\,(\alpha-\eta) > 0.$$

It is a contradiction and hence the theorem is proved.

The following theorem corresponds to first order ODEs which does not require a separate proof.

Theorem 2.6.

Assume

(i) $\underset{\sim}{f}(t,\underset{\sim}{z}) : J_o \times B \longrightarrow B$, $J_o := (0,T]$, is quasimonotone increasing in $\underset{\sim}{z}$ in the sense of Definition 2.4;

(ii) there exists a real number $\delta > 0$ and a 'test function' $\underset{\sim}{s}(t) : J \longrightarrow B$ such that $s_i(t) = s_j(t)$, $s_i(t) > 0$ on J, $i,j \in$ $\underset{\sim}{s} \in U := C^1(J_o) \cap C(J)$, $J = [0,T]$, $\underset{\sim}{s} = (\ldots, s_{-1}, s_o, s_1, \ldots)$ and

(iii) $P_i\,(\underset{\sim}{z} + \alpha_1 \underset{\sim}{s}) - P_i\,\underset{\sim}{z} \geq \alpha_1 \delta > 0$ in J_o, $i \in Z$

for every $\alpha_1 > 0$ and $\underset{\sim}{z} \in U$. Here $\alpha_1 \underset{\sim}{s} = (\ldots, \alpha_1 s_{-1}, \alpha_1 s_o, \alpha_1 s_1, \ldots)$,

$$P_i\,\underset{\sim}{z} := z_i' - f_i(t, \underset{\sim}{z}).$$

Then for every $\underset{\sim}{v}$, $\underset{\sim}{w} \in U$, the implication

$$(2.16) \qquad \underset{\sim}{v}(0) \le \underset{\sim}{w}(0),\ P_i \underset{\sim}{v} \le P_i \underset{\sim}{w},\ i \in Z$$

$$\Longrightarrow$$

$$(2.17) \qquad \underset{\sim}{v}(t) \le \underset{\sim}{w}(t) \text{ on } J$$

is true provided there exist positive constants L_j's such that $\underset{\sim}{L} = (\ \ldots,\ L_{-1},\ L_0,\ L_1,\ \ldots\ ) \in B$ and

$$(2.18) \qquad f_j(t,\underset{\sim}{w}+\beta\underset{\sim}{s}) - f_j(t,\underset{\sim}{w}+\eta\underset{\sim}{s}) \le L_j(\beta-\eta),\ j\in Z,\ \eta,\beta\in R,\ \eta \le \beta.$$

3. NON-QUASIMONOTONE SYSTEMS.

The purpose of this section is to give a method of constructing a new system of DEs, satisfying the quasi-monotonicity condition, from the original system which does not satisfy this condition. Assumption (i) of Theorem 1.3 is nothing but the quasi-monotonicity condition on the vector-valued function $\underset{\sim}{F}(\underset{\sim}{u})$ of (1.15). When the system (1.15) does not satisfy this condition, Adams and Spreuer [8], following Müller [34], were able to construct the following system which satisfies the quasi-monotonicity condition

$$(2.19) \qquad \begin{cases} \hat{A}_i\, \hat{\underset{\sim}{u}} := -\sup\limits_H F_i(x,(\underset{\sim}{\mu},\underline{u}_i),\underline{u}_{i,j},\underline{u}_{i,jk}, \\ \qquad\qquad \int_{\bar G} K_i(x,t,(\underset{\sim}{\mu},\mu_i))dt) = 0 \text{ on } \bar{G}, \\ \hat{A}_{i+n}\, \hat{\underset{\sim}{u}} := \inf\limits_H F_i(x,(\underset{\sim}{\mu},\bar{u}_i),\ \bar{u}_{i,jk},\ \int_{\bar G} K_i(x,t,(\underset{\sim}{\mu},\mu_i))dt) \\ \qquad\qquad = 0 \text{ on } \bar{G},\ i = 1(1)n, \end{cases}$$

where $(\underset{\sim}{\mu},\underline{u}_i) := (\mu_1,\ldots,\mu_{i-1}, \underline{u}_i, \mu_{i+1}, \ldots, \mu_n)$,

$\hat{u}_1 := -\underline{u}_1,\ldots,\hat{u}_n := -\underline{u}_n,\ \hat{u}_{n+1} := \overline{u}_1,\ldots,\hat{u}_{2n} := \overline{u}_n$,

$H := \left\{\mu_j : \underline{u}_j(t) \le \mu_j(t) \le \overline{u}_j(t),\ j=1(1)n,\ t\in\overline{G}\right\}$.

To illustrate the procedure of constructing the operator $\hat{\underset{\sim}{A}}$ for a given operator $\underset{\sim}{A}$, consider the BVP defined by

$$(2.20)\quad \underset{\sim}{A}\,\underset{\sim}{u} := \underset{\sim}{F} \Longleftrightarrow \begin{cases} P_i\underset{\sim}{u} := -u_i'' + \alpha_i(t)u_i' + \sum_{j=1}^{n} \beta_{ij}(t)u_j = \gamma_i, \quad t\in D, \\ R_i\underset{\sim}{u} := \begin{cases} u_i(a) = A_i \\ u_i(b) = B_i, \quad i = 1(1)n. \end{cases} \end{cases}$$

For this BVP one can adjoin the following BVP which satisfies quasi-monotonicity condition.

$$(2.21)\quad \hat{\underset{\sim}{A}}\,\hat{\underset{\sim}{u}} := \hat{\underset{\sim}{F}} \Longleftrightarrow \begin{cases} \hat{P}_i\,\hat{\underset{\sim}{u}} : = -\hat{u}_i'' + \alpha_i\hat{u}_i' + \beta_{ii}\hat{u}_i + \sum_{\substack{j=1\\ j\neq i}}^{n} (-\beta_{ij}^{+}\hat{u}_{j+n} + \beta_{ij}^{-}\hat{u}_j) \\ \qquad = -\gamma_i, \quad i=1(1)n \\ \hat{P}_{i+n}\hat{\underset{\sim}{u}} : = -\hat{u}_{i+n}'' + \alpha_i\hat{u}_{i+n}' + \beta_{ii}\hat{u}_{i+n} + \sum_{\substack{j=1\\ j\neq i}}^{n} (-\beta_{ij}^{+}\hat{u}_j + \beta_{ij}^{-}\hat{u}_{j+} \\ \qquad = \gamma_i, \\ \hat{R}_i\hat{\underset{\sim}{u}} : = (\hat{u}_i(a),\ \hat{u}_i(b))^T = (-A_i,\ -B_i)^T \\ \hat{R}_{i+n}\hat{\underset{\sim}{u}} : = (\hat{u}_{i+n}(a),\ \hat{u}_{i+n}(b))^T = (A_i, B_i)^T, i=1(1)n \end{cases}$$

where $\hat{\underset{\sim}{u}} = (\hat{u}_1, \ldots, \hat{u}_{2n})$

$$\beta^+_{ij} := \begin{cases} \beta_{ij} & \text{if } \beta_{ij} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\beta^-_{ij} := \beta_{ij} - \beta^+_{ij}, \quad i,j = 1(1)n.$$

It is obvious that if $(u_1, \ldots, u_n)$ is a solution of the BVP(2.20) then $(-u_1, \ldots, -u_n, u_1, \ldots, u_n)$ is a solution of the problem (2.21). This observation is quite useful as it shall be used in Chapters 3-6. The detailed properties of the system (2.19) and its relation with (1.15) are found in [6] and [8].

4. MONOTONICITY THEOREMS FOR ODEs DEFINED ON UNBOUNDED INTERVALS.

In this section monotonicity theorems for weakly coupled finite as well as doubly infinite systems of second order ODEs defined on unbounded intervals are presented. These theorems are proved based on the results given in the following two lemmas.

Lemma 2.7.

Let $g(t)$ be a real-valued continuous function on the real interval $[a,\infty)$ and be positive for some point $t_o \in [a,\infty)$. If $\lim_{t\to\infty} g(t) = \ell$, $-\infty < \ell \leq 0$, then $\sup_{t\in[\alpha,\infty)} g(t)$

is attained in $[a,\infty)$, that is, $\sup_{t\in[a,\infty)} g(t) = g(t^*)$, for some $t^* \in [a,\infty)$. In particular if $g(a) \leq 0$ then $t^* \in (a,\infty)$.

Proof

Given $\varepsilon > 0$ there exists a N such that

$$g(t) < \ell + \varepsilon, \; t > N.$$

Let $m_1 = \max_{t\in[a,N]} g(t)$. Then $g(t) \leq \max\{m_1, \ell + \varepsilon\}$. Hence $\sup_{t\in[a,\infty)} g(t)$ exists. To establish the lemma two cases are considered.

Case i. $(\ell < 0)$

Choose a positive number ε such that $\ell + \varepsilon < 0$. For this ε there exists a M_1 such that

$$g(t) < \ell + \varepsilon < 0, \; t > M_1.$$

Let $m_2 = \max_{t\in[a,M_1]} g(t) = g(t_1) > 0$, for some $t_1 \in [a,M_1]$. Then $g(t) \leq m_2$, $t \in [a, \infty)$. Any number $m_2 - \delta$, for some $\delta > 0$ can not be an upper bound for $g(t)$, $t \in [a, \infty)$. That is, m_2 is the $\sup_{t\in[a,\infty)} g(t)$.

Case ii. ($\ell = 0$)

Since $g(t_o) > 0$, there exists a M_2 such that

$$g(t) \; < \; g(t_o), \; t \; > \; M_2 .$$

Let $m_3 = \max\limits_{t\in[a,M_2]} g(t) = g(t_2) > 0$, for some $t_2 \in [a,M_2]$.

It can be proved that

$$m_3 \;=\; \sup_{t\in[a,\infty)} g(t).$$

Lemma 2.8.

Let $g(t)$ be a real-valued continuous function on a real interval $[a,\infty)$ and be negative for some point $t_o \in [a,\infty)$. If $\lim\limits_{t\to\infty} g(t) = \ell$, $0 \leq \ell < \infty$, then $\inf\limits_{t\in[a,\infty)} g(t)$ is attained in $[a,\infty)$, that is, $\inf\limits_{t\in[a,\infty)} g(t) = g(t^*)$ for some $t^* \in [a,\infty)$. In particular if $g(a) \geq 0$ then $t^* \in (a,\infty)$.

Theorem 2.9.

Consider the finite system of the second order ODEs

$$(2.22) \qquad \underset{\sim}{P}\,\underset{\sim}{u} := -\underset{\sim}{u}'' + \underset{\sim}{f}(t,[\underset{\sim}{u}]',\underset{\sim}{u}) = \underset{\sim}{0}, \; t \in D := (a,\infty),$$

that is,

$$P_i\,\underset{\sim}{u} := -u_i'' + f_i(t,u_i',u_1,\ldots,u_n) = 0, \; i=1(1)n$$

subject to the BCs

$$(2.23) \qquad \underset{\sim}{u}(a) = \underset{\sim}{A}, \quad \underset{\sim}{u}(\infty) = \underset{\sim}{B}.$$

Here $\underset{\sim}{u} = (u_1,\ldots,u_n)^T$, $\underset{\sim}{f}=(f_1,\ldots,f_n)^T$, $\underset{\sim}{A}=(A_1,\ldots,A_n)^T$,

$\underset{\sim}{B} = (B_1,\ldots,B_n)^T$, $\underset{\sim}{u}(\infty) = (u_1(\infty),\ldots, u_n(\infty))^T$,

$u_i(\infty) = \lim\limits_{t\to\infty} u_i(t)$

$u_i \in U := C^2(a,\infty) \cap C[a,\infty)$, $i=1(1)n$.

Assume

(i) $\underset{\sim}{f}$ is quasimonotone decreasing in $\underset{\sim}{u}$, that is, f_i is decreasing with u_j, $i \neq j$, $i,j=1(1)n$;

(ii) there exists a 'test function' $\underset{\sim}{s} = (s_1,\ldots,s_n)$ such that $s_i \in U$, $s_i(t) > 0$ on $\bar{D}: = [a,\infty)$, $\lim\limits_{t\to\infty} s_i(t) = \ell_i$, $0 < \ell_i < \infty$, $i = 1(1)n$,

$$(2.24) \qquad \underset{\sim}{P}\,(\underset{\sim}{u}+\alpha_1\underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{u} > 0,$$

for all positive real number α_1, $\underset{\sim}{u}$, $u_i \in U$.

Then the following implication is true for all

$\underset{\sim}{v}$, $\underset{\sim}{w}$, v_i, $w_i \in U$, $i = 1(1)n$:

(2.25) $\underset{\sim}{P}\,\underset{\sim}{v} \le \underset{\sim}{P}\,\underset{\sim}{w}$, $\underset{\sim}{v}(a) \le \underset{\sim}{w}(a)$, $\underset{\sim}{v}(\infty) \le \underset{\sim}{w}(\infty)$,

that is,

$$P_i\,\underset{\sim}{v} \le P_i\,\underset{\sim}{w},\ v_i(a) \le w_i(a),\ v_i(\infty) \le w_i(\infty),\ i=1(1)n$$

$$\Longrightarrow$$

(2.26) $\underset{\sim}{v}(t) \le \underset{\sim}{w}(t)$, $t \in \bar{D}$.

Proof

Define $\alpha = \max\limits_{i=1(1)n} \left[\sup\limits_{t\in \bar{D}} \left\{ (v_i(t)-w_i(t))/s_i(t) \right\} \right]$.

If the implication of the theorem is not true then by Lemma 2.7,

$$0 < \alpha = (v_k(t^*) - w_k(t^*))/ s_k(t^*),$$

for some $k\in\{1, \ldots, n\}$ and for some $t^* \in D$. Also

(2.27) $\underset{\sim}{v}(t) - \underset{\sim}{w}(t) - \alpha\underset{\sim}{s}(t) \le 0$.

Since the function $v_k(t) - w_k(t) - \alpha s_k(t)$ assumes ~~positive~~ nonnegative maximum at $t = t^*$, the following inequalities hold true at $t=t^*$.

(2.28) $v_k(t) - w_k(t) - \alpha s_k(t) = 0$,

(2.29) $v_k'(t) - w_k'(t) - \alpha s_k'(t) = 0$,

(2.30) $v_k''(t) - w_k''(t) - \alpha s_k''(t) \le 0$.

From (2.24), (2.25), (2.30), (2.29), (2.28), (2.27) and the assumption (i) of the theorem one has

$$\begin{aligned} 0 &> [\, P_k \underset{\sim}{w} - P_k(\underset{\sim}{w} + \alpha \underset{\sim}{s})]_{t=t^*} \\ &\geq [\, P_k \underset{\sim}{v} - P_k(\underset{\sim}{w} + \alpha \underset{\sim}{s})]_{t=t^*} \\ &= -\varepsilon[v_k - w_k - \alpha s_k]''(t^*) - [f_k(t^*, w_k' + \alpha s_k', \underset{\sim}{w} + \alpha \underset{\sim}{s}) - f_k(t^*, v_k', \underset{\sim}{v})] \\ &\geq 0. \end{aligned}$$

It is a contradiction and hence the theorem is proved.

The above theorem shall be now extended to doubly infinite system of ODEs. Consider the BVP for weakly coupled doubly infinite system of second order ODEs

(2.31) $\quad \underset{\sim}{P}\, \underset{\sim}{u} := -\varepsilon \underset{\sim}{u}'' + \underset{\sim}{f}(t, [\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon) = \underset{\sim}{0},$

that is,

$$P_i \underset{\sim}{u} := -\varepsilon u_i'' + f_i(t, u_i', \underset{\sim}{u}, \varepsilon) = 0, \quad t \in D, \; i \in Z$$

subject to the BCs

(2.32) $\quad \underset{\sim}{R}\, \underset{\sim}{u} := \begin{cases} \underset{\sim}{\eta}^1 \underset{\sim}{u}(a,\varepsilon) - \underset{\sim}{\eta}^2 \underset{\sim}{u}'(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \\ \underset{\sim}{u}(\infty,\varepsilon) = \underset{\sim}{B}(\varepsilon), \end{cases}$

where $\underset{\sim}{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)^T$, $\varepsilon > 0$ is a small parameter,

$$\underset{\sim}{f} \quad = \quad (\ldots,\ f_{-1}, f_o, f_1, \ldots)^T,$$

$$\underset{\sim}{A}(\varepsilon) \ = \ (\ldots,\ A_{-1}(\varepsilon),\ A_o(\varepsilon),\ A_1(\varepsilon),\ \ldots)^T,$$

$$\underset{\sim}{B}(\varepsilon) \ = \ (\ldots,\ B_{-1}(\varepsilon),\ B_o(\varepsilon),\ B_1(\varepsilon),\ \ldots)^T,$$

T stands for the transpose, Z is the set of all integers,

$$\underset{\sim}{u}(\infty,\varepsilon) = (\ldots,\ u_{-1}(\infty,\varepsilon),\ u_o(\infty,\varepsilon),\ u_1(\infty,\varepsilon),\ldots),$$

$u_i(\infty,\varepsilon) = \lim_{t\to\infty} u_i(t,\varepsilon)$ uniformly with respect to for all $i \in Z$,

$$\underset{\sim}{u} \in U: \ = C^2(D) \cap C^1(\bar{D}),$$

$$\underset{\sim}{\eta}^1(a,\varepsilon) = (\ldots,\ \eta^1_{-1}(a,\varepsilon),\ \eta^1_o(a,\varepsilon),\ \eta^1_1(a,\varepsilon),\ldots)$$

and etc., $\eta_i^j \geq 0$, $\eta_i^1 + \eta_i^2 > 0$, $i \in Z$, $j = 1,2$.

The following theorem is an extension of the above theorem to the BVPs (2.31)-(2.32).

Theorem 2.10.

Consider the BVP (2.31)-(2.32) and assume

(i) $\underset{\sim}{f}$ is quasimonotone decreasing in $\underset{\sim}{u}$ in the sense of Definition 2.4;

(ii) there exist a positive real number δ_2 and a 'test function' $s(t,\varepsilon)$: $\bar{D} \times I \to R_+$, $R_+: = (0,\infty)$, $I:=(0,\varepsilon_1]$ with the properties

$$(2.33) \qquad \underset{\sim}{P}\,(\underset{\sim}{u} + \alpha_1 \underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{u} \geq \alpha_1 \delta_2 > 0 \text{ in } D,$$

(2.34) $\quad \underset{\sim}{R}\,\underset{\sim}{s} > 0, \quad \underset{\sim}{s} = (\ \ldots, s, s, \ldots\ ), \quad \underset{\sim}{s} \in U$

for every $\underset{\sim}{u} \in U$. Then the implication

(2.35) $\quad \underset{\sim}{P}\,\underset{\sim}{v} \leq \underset{\sim}{P}\,\underset{\sim}{w}, \quad \underset{\sim}{R}\,\underset{\sim}{v} \leq \underset{\sim}{R}\,\underset{\sim}{w}$

$$\Longrightarrow$$

(2.36) $\quad \underset{\sim}{v}(t,\varepsilon) \leq \underset{\sim}{w}(t,\varepsilon), \quad t \in \bar{D}$

is true for all $\underset{\sim}{v}, \underset{\sim}{w} \in U$ provided there exists a constant $L > 0$ such that

(2.37) $\quad \underset{\sim}{f}(t,p,\underset{\sim}{w}+\beta\underset{\sim}{s},\varepsilon) - \underset{\sim}{f}(t,p,\underset{\sim}{w}+\eta\underset{\sim}{s},\varepsilon) \geq L(\eta-\beta)\underset{\sim}{s},$

$$0 < \eta \leq \beta, \quad \eta, \beta \in R_+ .$$

Proof

Define $p(t,\varepsilon) = \inf[(w_i(t,\varepsilon) - v_i(t,\varepsilon)), i \in Z]$, $t \in \bar{D}$, which is a continuous function on $\bar{D}$ and $\lim_{t\to\infty} p(t,\varepsilon)$ exists and the limit is non-negative (because of the uniform convergence and (2.35)).

If the implication of the theorem is not true then assume the contrary $\underset{\sim}{v}(t,\varepsilon) \not\leq \underset{\sim}{w}(t,\varepsilon)$ on $\bar{D}$, which in turn implies that $p(t,\varepsilon)$ is negative for some point in $\bar{D} = [a,\infty)$ and for some $\varepsilon_0 \in I$. Hence for this ε_0, by Lemma 2.~~7~~8, there exists a point $t_0 \in \bar{D}$ such that

$$\alpha_1 = -\inf\,[p(t,\varepsilon_0)/s(t,\varepsilon_0)] = -p(t_0,\varepsilon_0)/s(t_0,\varepsilon_0) > 0.$$

Also

(2.38) $\underset{\sim}{v}(t,\varepsilon_0) - [\underset{\sim}{w}(t,\varepsilon_0) + \alpha_1\underset{\sim}{s}(t,\varepsilon_0)] \leq \underset{\sim}{0}, \ t\in\bar{D}.$

Given $\eta > 0$ such that

(2.39) $\eta > \alpha_1$ and $\eta < \alpha_1\delta_2/ [\max_{t\in\bar{D}} s(t,\varepsilon)(L+M)+\delta_2],$

and hence given $\eta s(t_0, \varepsilon_0)$, there exists at least one j with the property

$$v_j(t_0, \varepsilon_0) - w_j(t_0,\varepsilon_0) > \alpha_1 s(t_0,\varepsilon_0) - \eta s(t_0,\varepsilon_0),$$

that is ,

$$v_j(t_0,\varepsilon_0)-[w_j(t_0,\varepsilon_0)+(\alpha_1-\eta)s(t_0,\varepsilon_0)] > 0.$$

Further the function $v_j(t,\varepsilon_0)-[w_j(t,\varepsilon_0)+(\alpha_1-\eta)s(t,\varepsilon_0)]$ has the limit which is negative as $t\to\infty$. Hence this function attains maximum at some point $t^*\in[a,\infty)$ (Lemma 2.8). The point t^* can either belong to D or $t^* = a$. These two cases shall be considered separately in the following discussion and the theorem is proved by getting a contradiction.

Case i. $(t^*\in D)$

(2.40) $v_j(t^*, \varepsilon_0) - [w_j(t^*,\varepsilon_0)+(\alpha_1-\eta)s\ (t^*,\varepsilon_0)] > 0,$

(2.41) $v_j'(t^*, \varepsilon_0) - [w_j'(t^*,\varepsilon_0)+(\alpha_1-\eta)s'(t^*,\varepsilon_0)] = 0,$

$$(2.42) \qquad v_j''(t^*,\varepsilon_0) - [w_j''(t^*,\varepsilon_0)+(\alpha_1-\eta)s''(t^*,\varepsilon_0)] \leq 0.$$

From (2.33), (2.35), (2.41), (2.42), (2.37), (2.40) and (2.39) the following inequalities hold true at $t = t^*$.

$$\begin{aligned}
0 &\geq P_j\underset{\sim}{w} - P_j(\underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s}) + \delta_2(\alpha_1-\eta) \\
&\geq P_j\underset{\sim}{v} - P_j(\underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s}) + \delta_2(\alpha_1-\eta) \\
&\geq -\varepsilon(v_j-w_j+(\eta-\alpha_1)s)''(t^*) + f_j(t^*,v_j',\underset{\sim}{v},\varepsilon_0) \\
&\quad - f_j(t^*,(w_j+(\alpha_1-\eta)s_j)', \underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s},\varepsilon_0)+\delta_2(\alpha_1-\eta) \\
&\geq f_j(t^*,v_j',\underset{\sim}{v},\varepsilon_0)-f_j(t^*,w_j'+(\alpha_1-\eta)s_j', \underset{\sim}{v}, \varepsilon_0) \\
&\quad + f_j(t^*,w_j'+(\alpha_1-\eta)s_j',\underset{\sim}{v},\varepsilon_0)-f_j(t^*,w_j'+(\alpha_1-\eta)s_j',\underset{\sim}{w}+\alpha_1\underset{\sim}{s},\varepsilon_0) \\
&\quad + f_j(t^*,w_j'+(\alpha_1-\eta)s_j', \underset{\sim}{w} + \alpha_1\underset{\sim}{s}, \varepsilon_0) \\
&\quad - f_j(t^*,w_j'+(\alpha_1-\eta)s_j', \underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s}, \varepsilon_0)+ \delta_2(\alpha_1-\eta) \\
&\geq - M\eta s(t^*,\varepsilon_0) - L\eta s(t^*,\varepsilon_0) + \delta_2(\alpha_1-\eta) > 0.
\end{aligned}$$

It is a contradiction.

Case ii. $(t^* = a)$

At $t = t^*$

$$(2.43) \qquad v_j(t^*,\varepsilon_0)-[w_j(t^*,\varepsilon_0)+(\alpha_1-\eta)\, s(t^*,\varepsilon_0)] > 0,$$

$$(2.44) \qquad v_j'(t^*,\varepsilon_0)-[w_j'(t^*,\varepsilon_0)+(\alpha_1-\eta)s'(t^*,\varepsilon_0)] \leq 0.$$

From (2.33), (2.35), (2.43) and (2.44)

$$
\begin{aligned}
0 &> R_j \underset{\sim}{w} - R_j(\underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s}\,) \\
&\geq R_j \underset{\sim}{v} - R_j(\underset{\sim}{w} + (\alpha_1-\eta)\underset{\sim}{s}\,) \\
&= \eta_j^1\,[v_j(a,\varepsilon) - (w_j(a,\varepsilon) + (\alpha_1-\eta)s\,(a,\varepsilon))\,] \\
&\quad - \eta_j^2\,[v_j'(a,\varepsilon) - (w_j'(a,\varepsilon)+(\alpha_1-\eta)s'(a,\varepsilon))\,] \\
&\geq 0.
\end{aligned}
$$

It is a contradiction. Hence the theorem is proved.

5. MONOTONICITY THEOREMS FOR PARABOLIC DEs.

This section presents a monotonicity theorem for weakly coupled finite systems of parabolic PDEs defined on unbounded regions. The notations and definitions used in this section are taken from [3]. For the sake of completeness a few of them are given below.

Definition 2.11.

Let D be an open set in R^m defined as

$$D:=\left\{x = (x_1,\ldots,x_m)\in R^m/\ x_i > 0,\ i=1(1)m\right\}.$$

For real interval (0,T) the set G is defined as G=(0,T)xD.

The boundary and closed hull of D or G are denoted respectively by ∂D and $\bar{D}$ or ∂G and $\bar{G}$. The sets $\partial_o G$, $\partial_1 G$ and $\partial_2 G$ stand for

$$\partial_o G: = \{0\} \times \bar{D}, \ \partial_1 G: = (0,T] \times \partial D, \ \partial_2 G: = \{T\} \times D.$$

Also the sets G_p and R_p are given by

$$G_p: = G \cup \partial_2 G, \quad R_p: = \partial G - \partial_2 G.$$

Consider the parabolic system defined by

(2.45) $\partial u_i/\partial t + F_i(t,x,\underset{\sim}{u},u_{i,j},u_{i,jk}) = 0, \ i=1(1)n,$

where $\underset{\sim}{u} = (u_1, \ldots, u_n)$, $\underset{\sim}{x} = (x_1, \ldots, x_m) \in D \subset R^m$, $u_{i,j}: = \partial u_i/\partial x_j$, $u_{i,jk}: = \partial^2 u_i/\partial x_j \partial x_k$, $i=1(1)n$, $j,k=1(1)m$.

Definition 2.12.

For any two symmetric m x m matrices $r = (r_{ij})$ and $\bar{r} = (\bar{r}_{ij})$ the inequality $r \leq \bar{r}$ holds if and only if the matrix $\bar{r}-r$ is positive semi-definite, that is, $r \leq \bar{r}$ if and only if

$$\sum_{i,j=1}^{m} (\bar{r}_{ij}-r_{ij}) \alpha_i \alpha_j \geq 0 \text{ for arbitrary real } \alpha_j.$$

Lemma 2.13.

Let $g(t,x)$ be a real-valued continuous function defined on $\bar{G}$ and be positive for some point $(t_o, x_o) \in \bar{G}$, $\| x \|_e < \infty$ ($\| \cdot \|_e$ is the Euclidean norm). If $\lim_{\|x\|_e \to \infty} g(t,x) = \ell(t)$ uniformly with respect to t, $-\infty < \ell(t) \leq 0$, then $\sup_{(t,x) \in \bar{G}} g(t,x)$ is attained in $\bar{G}$.

Proof

This lemma can be proved in the similar lines of Lemma 2.7.

The Theorem 2.9 can be generalized to the IBVP for finite system of parabolic DEs:

$$(2.46) \begin{cases} P_i \underset{\sim}{u} := \partial u_i/\partial t + F_i(t,x,\underset{\sim}{u},u_{i,j},u_{i,jk}) = f_i(t,x), (t,x) \in G_p \\ R_i \underset{\sim}{u} := \begin{cases} u_i(t,x) = g_i(t,x), \ (t,x) \in R_p, \\ \lim_{\|x\|_e \to \infty} u_i(t,x) = G_i(t) \text{ uniformly with} \end{cases} \end{cases}$$

respect to $t \in [0,T]$,

where

(i) $f_i \in C(G_p)$, $u_i \in U := C^{(1,2)}\, G_p \cap C(\bar{G})$;

(ii) for each fixed i, F_i is monotone decreasing in the matrix $(u_{i,jk})$.

Theorem 2.14.

Consider the IBVP (2.46) and assume

(i) $\underset{\sim}{F} = (F_1, \ldots, F_n)$ is quasimonotone decreasing in $\underset{\sim}{u} = (u_1, \ldots, u_n)$ in the sense that each F_i, $i=1(1)n$, is monotone decreasing with u_j, $j \neq i$, $i,j=1(1)n$;

(ii) there exists a 'test function' $\underset{\sim}{s} = (s_1, \ldots, s_n)$ such that $s_i(t,x) > 0$ on $\bar{G}$, $s_i \in U$, $\lim_{\|x\|_e \to \infty} s_i(t,x)$ exists uniformly with respect to t and the limit is positive,

(2.47) $\quad \underset{\sim}{P}(\underset{\sim}{u}+\alpha\underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{u} > 0, \; \alpha\,\underset{\sim}{s} = (\alpha s_1, \ldots, \alpha s_n),$

for every positive number α and $\underset{\sim}{u}$, $u_i \in U$, $i = 1(1)n$.

Then for all $\underset{\sim}{v}$, $\underset{\sim}{w}$, v_i, $w_i \in U$, $i=1(1)n$, the implication

(2.48) $\quad \underset{\sim}{P}\,\underset{\sim}{v} \leq \underset{\sim}{P}\,\underset{\sim}{w}, \; \underset{\sim}{R}\,\underset{\sim}{v} \leq \underset{\sim}{R}\,\underset{\sim}{w}$

$$\Longrightarrow$$

(2.49) $\quad \underset{\sim}{v}(t,x) \leq \underset{\sim}{w}(t,x), \; (t,x) \in \bar{G}$

is true.

Proof

Define $\alpha = \max_{i=1(1)n} \; \sup_{(t,x)\in\bar{G}} \{(v_i(t,x)-w_i(t,x))/s_i(t,x)\}$

If the implication of the theorem is not true then by Lemma 2.13 $\alpha = (v_\ell(t^*,x^*)-w_\ell(t^*,x^*))/s_\ell(t^*,x^*) > 0$ for some $(t^*,x^*) \in G_p$ and for some $\ell \in \{1, \ldots, n\}$.

Also

$$(2.50) \qquad \underset{\sim}{v}(t,x) - \underset{\sim}{w}(t,x) \leq \alpha\underset{\sim}{s}(t,x) \text{ on } \overline{G} .$$

Since the function $v_\ell - w_\ell - \alpha s_\ell$ attains maximum at (t^*,x^*) then one has at (t^*, x^*),

$$(2.51) \quad \begin{cases} v_\ell - w_\ell - \alpha s_\ell = v_{\ell,j} - w_{\ell,j} - \alpha s_{\ell,j} = 0, \\ \partial(v_\ell - w_\ell - \alpha s_\ell)/\partial t \geq 0, \\ v_{\ell,jk} - w_{\ell,jk} - \alpha s_{\ell,jk} \leq 0. \end{cases}$$

Therefore at (t^*, x^*)

$$\begin{aligned} 0 &> P_\ell \underset{\sim}{w} - P_\ell(\underset{\sim}{w}+\alpha\underset{\sim}{s}) \geq P_\ell \underset{\sim}{v} - P_\ell(\underset{\sim}{w}+\alpha\underset{\sim}{s}) \\ &= \partial(v_\ell - w_\ell - \alpha s_\ell)/\partial t + F_\ell(t^*,x^*,\underset{\sim}{v},v_{\ell,j},v_{\ell,jk}) \\ &\quad - F_\ell(t^*,x^*,\underset{\sim}{w}+\alpha\underset{\sim}{s}, w_{\ell,j}+\alpha s_{\ell,j}, w_{\ell,jk}+\alpha s_{\ell,jk}) \\ &\geq 0, \text{ by (2.47), (2.48), (2.50) and (2.51).} \end{aligned}$$

It is a contradiction and hence the theorem is proved.

If the functions involved in (2.46) are independent of t then the problem reduces to elliptic problem. That is, the above theorem is also applicable to elliptic equations. The proof of the above theorem is different from that of given in [3]. Also it is to be noted that if any one of the conditions (γ_1)-(γ_4) of [3, p.306] is satisfied for each f_i one can prove the existence of a 'test function'. For the sake of simplicity only the Dirichlet type BCs are considered. Other types of BCs can also be treated in a similar way. Following the method of proof of monotonicity theorem given in Section 3, one can generalize the above theorem to doubly infinite systems of integro-differential equations defined on unbounded regions with appropriate BCs.

Chapter 3

SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS IN BANACH SPACE-I*

In this chapter, error estimates for solutions and/or their derivatives of the following BVPs (3.1)-(3.5) are obtained in order to study the asymptotic behaviour of solutions and/or their derivatives. The BVPs consist of infinite weakly coupled systems of ODEs

$$(3.1)\qquad \underset{\sim}{P}\,\underset{\sim}{u} := -\varepsilon\,\underset{\sim}{u}'' + \underset{\sim}{f}(t,[\underset{\sim}{u}]',\underset{\sim}{u},\varepsilon) = \underset{\sim}{0},$$

that is,

$$(3.2)\qquad P_i\underset{\sim}{u} := -\varepsilon\,u_i'' + f_i(t,u_i',\underset{\sim}{u},\varepsilon) = 0,\ t\in D:=(a,b),\ i\in Z$$

subject to one of the following BCs

$$(3.3)\qquad \underset{\sim}{R}\,\underset{\sim}{u} := (\underset{\sim}{u}(a),\ \underset{\sim}{u}(b))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T,$$

$$(3.4)\qquad \underset{\sim}{R}\,\underset{\sim}{u} := (\underset{\sim}{u}(a),\underset{\sim}{u}'(b))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T,$$

$$(3.5)\qquad \underset{\sim}{R}\,\underset{\sim}{u} := (\underset{\sim}{u}(a)-\varepsilon\,\underset{\sim}{u}'(a),\ \underset{\sim}{u}'(b))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T,$$

where

$$\underset{\sim}{u} = (\ \ldots,\ u_{-1},\ u_o,\ u_1,\ \ldots)^T,$$

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* The material of this chapter is to appear in the Journal Bullettino U.M.I.(7) 1-B (1987).

$$\underset{\sim}{A}(\varepsilon) = (\ldots, A_{-1}(\varepsilon), A_0(\varepsilon), A_1(\varepsilon), \ldots)^T,$$

$$A_i(\varepsilon) = A_i(0) + \mathcal{O}(\varepsilon)$$

$$\underset{\sim}{B}(\varepsilon) = (\ldots, B_{-1}(\varepsilon), B_0(\varepsilon), B_i(\varepsilon), \ldots)^T,$$

$$B_i(\varepsilon) = B_i(0) + \mathcal{O}(\varepsilon),\ i \in Z,$$

ε is a small parameter such that $\varepsilon \in I = (0, \varepsilon_1]$.

Several authors obtained some results on the asymptotic behaviour of solutions of singularly perturbed IVPs, BVPs for single second order ODE [15, 17, 18] and finite systems of second order ODEs [23, 26, 27]. In this chapter, some of the results of [15, 17, 18, 23, 26] are extended from finite systems to doubly infinite systems. Some results are direct extensions and others are new. Consideration of the above nonlinear BVPs (3.1)-(3.5) is motivated by problems arising when MOL [3] is applied to the SPPs for elliptic and parabolic PDEs defined on unbounded regions. The following theorem is a modified version of Theorem 2.5 for the BVPs (3.1) - (3.5).

Theorem 3.1.

Assume

(i) $\underset{\sim}{f}(t,p,\underset{\sim}{u},\varepsilon) : D \times R \times G \times I \to B$ is quasimonotone decreasing in $\underset{\sim}{u}$ in the sense of Definition 2.4;

(ii) there exists a positive real number δ_2 and a 'test function' $\underset{\sim}{s}(t,\varepsilon) : \overline{D} \times I \rightarrow R_+$ with the properties

$$\underset{\sim}{P}\,(\underset{\sim}{u} + \alpha_1 \underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{u} \geq \alpha_1 \delta_2 > 0 \quad \text{in } D,$$

$$\underset{\sim}{R}\,\underset{\sim}{s} > 0 \text{ on } \partial D: = \overline{D} - D,$$

for every positive real number α_1 and $\underset{\sim}{u} \in U: = C^2(D) \cap C(\overline{D})$. Here R_+ is the set of all positive reals,

$$\underset{\sim}{s} = (\ldots, s(t,\varepsilon), s(t,\varepsilon), \ldots), \ \underset{\sim}{s} \in U,$$

$$\underset{\sim}{P}\,\underset{\sim}{u}: = -\varepsilon \underset{\sim}{u}'' + \underset{\sim}{f}\,(t, [\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon),$$

$$\underset{\sim}{R}\,\underset{\sim}{u}: = (\underset{\sim}{\eta}^1 \underset{\sim}{u}(a) - \underset{\sim}{\eta}^2 \underset{\sim}{u}'(a), \underset{\sim}{\zeta}^1 \underset{\sim}{u}(b) + \underset{\sim}{\zeta}^2 \underset{\sim}{u}'(b)),$$

with

$$\underset{\sim}{\eta}^1 \underset{\sim}{u}(a) = (\ldots, \eta^1_{-1} u_1(a), \eta^1_0 u_1(a), \eta^1_1 u_1(a) \ldots) \text{ etc.;}$$

η^j_i, ζ^j_i, $i \in Z$, $j = 1,2$, are non-negative real constants such that

$$\eta^1_i + \eta^2_i > 0 \quad \text{and} \quad \zeta^1_i + \zeta^2_i > 0, \ i \in Z.$$

Then the implication

$$\left\{\begin{array}{l} \underset{\sim}{P}\,\underset{\sim}{v} \leq \underset{\sim}{P}\,\underset{\sim}{w}, \text{ that is, } P_i\,\underset{\sim}{v} \leq P_i\,\underset{\sim}{w}, \ i \in Z \\ \underset{\sim}{R}\,\underset{\sim}{v} \leq \underset{\sim}{R}\,\underset{\sim}{w}, \text{ that is, } R_i\,\underset{\sim}{v} \leq R_i\,\underset{\sim}{w}, \ i \in Z \end{array}\right.$$

$$\Longrightarrow$$

$$\underset{\sim}{v}(t,\varepsilon) \leq \underset{\sim}{w}(t,\varepsilon) \text{ on } \overline{D}$$

is true for every $\underset{\sim}{v},\underset{\sim}{w} \in U$ provided there exists a positive constant L such that

$$(3.6) \qquad \underset{\sim}{f}(t,p,\underset{\sim}{w} + \beta\underset{\sim}{s}, \varepsilon) - \underset{\sim}{f}(t,p,\underset{\sim}{w} + \eta\underset{\sim}{s},\varepsilon)$$

$$\geq L(\eta-\beta)\, \underset{\sim}{s}(t,\varepsilon), \quad 0 < \eta \leq \beta, \ \eta,\beta \in R_+.$$

Some examples of DEs satisfying the hypotheses of Theorem 3.1 are given now.

Consider the system of linear second order ODEs

$$(3.7) \qquad P_{Li}\, \underset{\sim}{u}: = -\varepsilon u_i'' + \alpha_{ii}(t,\varepsilon)u_i' + \sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon)u_j$$

$$= \gamma_i(t,\varepsilon), \ t \in D,$$

which may be abbreviated as

$$(3.8) \qquad \underset{\sim}{P}_L\, \underset{\sim}{u}: = -\varepsilon\underset{\sim}{u}'' + \underset{\sim}{\alpha}\underset{\sim}{u}' + \underset{\sim}{\beta}\,\underset{\sim}{u} = \underset{\sim}{\gamma},$$

with $\quad \underset{\sim}{\beta} = (\beta_{ij}), \ \underset{\sim}{\alpha} = (\alpha_{ij}), \ \alpha_{ij}=0, \ i\neq j, \ i,j \in Z,$

$$\underset{\sim}{\alpha}\,\underset{\sim}{u}' = (\ldots, \alpha_{11}u_1', \alpha_{22}u_2', \ldots),$$

$$\underset{\sim}{\beta}\,\underset{\sim}{u} = (\ldots, \sum_{j=-\infty}^{\infty} \beta_{1j}u_j, \sum_{j=-\infty}^{\infty} \beta_{2j}u_j, \ldots),$$

$$\underset{\sim}{\gamma} = (\ldots, \gamma_1, \gamma_2, \ldots).$$

Here $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma}$ are assumed to be continuous functions on $\bar{D} \times I$.

In the following, LBVPs I, II and III stand for (3.7) with the BCs (3.3), (3.4) and (3.5) respectively.

Theorem 3.2.

Consider the system (3.7) and assume

(i) $\beta_{ij} \leq 0,\ i \neq j,\ i,j \in Z;$

(ii) $\beta_{ii} \leq M,\ i \in Z,\ M > 0;$

(iii) $\alpha_{ii}(t,\varepsilon) \geq \delta_3 > 0;$

(iv) $\sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon) \geq -\eta,\ i \in Z,\ \eta > 0.$

Then the quasi-monotonicity condition and the condition (3.6) are valid for the system (3.7). Also for $0 < \varepsilon \leq \varepsilon_1$, ε_1 is sufficiently small, there exists a 'test function' for the same system.

Proof

Assumptions (i) and (ii)imply the quasi-monotonicity condition whereas the condition (3.6) results from (iv). Finally using the assumptions (iii) and (iv), it can be shown that the function $\underset{\sim}{s}$ with $s(t,\varepsilon) = e^{k(t-a)}$ is a required 'test function' by a proper choice of $k > o$.

1. ESTIMATES AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS AND/OR THEIR DERIVATIVES OF LBVPs.

In this section different types of estimates for solutions and/or their derivatives of the LBVPs I-III are obtained. In the following m_i, $i=1,2, \ldots$ are positive constants independent of the small parameter ε .

Throughout this section it is assumed that all the conditions of Theorem 3.2 (that is, conditions (i) to (iv)) hold true (except for Theorem 3.7) for the system (3.7).

Theorem 3.3.

Let $\underset{\sim}{u}$ be a solution of the LBVP I, II or III. Then for the LBVPs I and II:

$$\|\underset{\sim}{u}(t,\varepsilon)\| \leq m_1 [\max \left\{\|\underset{\sim}{A}\| , \|\underset{\sim}{B}\|\right\} + \|\underset{\sim}{\gamma}\| / \delta_3] \text{ on } \bar{D}; \tag{3.9}$$

and for the LBVP III:

$$\|\underset{\sim}{u}(t,\varepsilon)\| \leq m_1[\max \left\{\| 2\underset{\sim}{A}\| , \|\underset{\sim}{B}\|\right\} + \|\underset{\sim}{\gamma}\|/\delta_3] \text{ on } \bar{D} \tag{3.10}$$

where $0 < \varepsilon \leq \varepsilon_1$, ε_1 is sufficiently small.

Proof

Because of the assumptions made just before the statement of the present theorem, Theorem 3.1 is applicable to the LBVPs I, II and III.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or II. Define the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = [\max\left\{\|\underset{\sim}{A}\|, \|\underset{\sim}{B}\|\right\} + \|\underset{\sim}{\gamma}\| / \delta_3]e^{k(t-a)} \text{ on } \bar{D}, i \in Z$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon) \text{ on } \bar{D},$$

where k is a positive number yet to be determined.

Then
$$\begin{aligned} P_{Li}\underset{\sim}{y} &= y_i(t,\varepsilon)\,(-\varepsilon k^2 + \alpha_{ii}k + \sum_{j=-\infty}^{\infty} \beta_{ij}) \\ &\geq y_i(t,\varepsilon)\,(-\varepsilon k^2 + \delta_3 k - \eta) \\ &\geq (\|\underset{\sim}{\gamma}\| / \delta_3)\,\delta_3 e^{k(t-a)} \geq \|\underset{\sim}{\gamma}\| \geq \gamma_i = P_{Li}\underset{\sim}{u}, \ i \in Z, \end{aligned}$$

by a proper choice of $k > 1$ and for $0 < \varepsilon \leq \varepsilon_1$, where ε_1 is sufficiently small. That is, for the LBVPs I and II,

(3.11) $\quad \underset{\sim}{P}_L \underset{\sim}{u} \leq \underset{\sim}{P}_L \underset{\sim}{y}$ in D.

For the LBVP I

$$\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon) \text{ at } t = a,b$$

that is,

(3.12) $\quad \underset{\sim}{R}\,\underset{\sim}{u} \leq \underset{\sim}{R}\,\underset{\sim}{y}$ on ∂D,

which with (3.11), by Theorem 3.1, yields

(3.13) $\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon)$ on $\bar{D}$.

In a similar way it can be shown that

(3.14) $\underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon)$ on $\bar{D}$

which with (3.13) establishes (3.9) for the LBVP I where $e^{k(t-a)} \leq e^{k(b-a)} \leq m_1$.

For the LBVP II

$$\underset{\sim}{u}(a,\varepsilon) \leq \underset{\sim}{y}(a,\varepsilon), \quad \underset{\sim}{u}'(b,\varepsilon) \leq \underset{\sim}{y}'(b,\varepsilon)$$

which with (3.11), by Theorem 3.1, yields

(3.15) $\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon)$ on $\bar{D}$.

Similarly one can show, for the LBVP II, that

$$\underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon) \text{ on } \bar{D}$$

which with (3.15) establishes (3.9) for the LBVP II.

To prove the result (3.10) for the LBVP III, define $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = [\max\{\|2\underset{\sim}{A}\|, \|\underset{\sim}{B}\|\} + \|\underset{\sim}{\gamma}\|/\delta_3]e^{k(t-a)/2}, i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon) \text{ on } \bar{D},$$

where k has to be chosen suitably and follow the same procedure described above for the LBVPs I and II to obtain the desired estimates.

Corollary 3.4.

Let $\underset{\sim}{u}$ be a solution of the LBVP II or III with $\sum_{j=-\infty}^{\infty} | \beta_{ij} | \leq \eta_o$, $i \in Z$.

Then for the LBVP II with $\underset{\sim}{A}(\varepsilon) \equiv \underset{\sim}{O} \equiv \underset{\sim}{B}(\varepsilon)$:

(3.16) $\quad \| \underset{\sim}{u}(t,\varepsilon) \| \leq m_1 \| \underset{\sim}{\gamma} \| / \delta_3$,

(3.17) $\quad \| \underset{\sim}{u}'(t,\varepsilon) \| \leq m_2 \| \underset{\sim}{\gamma} \| (1 - e^{-\delta_3(b-t)/\varepsilon})$ on $\bar{D}$,

and for the LBVP III with $\underset{\sim}{B}(\varepsilon) \equiv \underset{\sim}{O}$:

(3.18) $\quad \| \underset{\sim}{u}(t,\varepsilon) \| \leq 2m_1 \| \underset{\sim}{A} \| + m_1 \| \underset{\sim}{\gamma} \| / \delta_3$,

(3.19) $\quad \| \underset{\sim}{u}'(t,\varepsilon) \| \leq (m_2 \| \underset{\sim}{A} \| + m_3 \| \underset{\sim}{\gamma} \|)(1-e^{-\delta_3(b-t)/\varepsilon})$ on $\bar{D}$.

Proof

The system (3.7) may be written as

$$\varepsilon (u_i' e^{q_i(t,\varepsilon)})' = p_i(t,\varepsilon), \ i \in Z,$$

where

$$q_i(t,\varepsilon) = 1/\varepsilon \int_t^b \alpha_{ii}(t,\varepsilon)dt$$

and

$$p_i(t,\varepsilon) = (\sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon)u_j - \gamma_i(t,\varepsilon))e^{q_i(t,\varepsilon)}, i \in Z.$$

Using Theorem 3.3

$$| p_i(t,\varepsilon) | \leq \| \underset{\sim}{\gamma} \| e^{q_i(t,\varepsilon)} [m_1 \eta_o / \delta_3 + 1]$$

which yields, for the LBVP II,

$$| u_i'(t,\varepsilon) | \leq (\| \underset{\sim}{\gamma} \| / \delta_3) (m_1 \eta_o / \delta_3 + 1)$$
$$\times (1 - e^{-\delta_3 (b-t)/\varepsilon}), \quad i \in Z.$$

This proves the result for the LBVP II. Similarly one can establish the estimates (3.18)-(3.19).

Theorem 3.5.

Let $\underset{\sim}{u}$ be a solution of the LBVP I, II or III together with $\underset{\sim}{\gamma} \equiv \underset{\sim}{0} \equiv \underset{\sim}{A}$.

Then for the LBVP I:

$$\| \underset{\sim}{u}(t,\varepsilon) \| \leq \| \underset{\sim}{B} \| e^{-\delta_3 (b-t)/2\varepsilon} \quad \text{on } \bar{D}, \tag{3.20}$$

and for the LBVPs II and III:

$$\| \underset{\sim}{u}(t,\varepsilon) \| \leq (2\varepsilon / \delta_3) \| \underset{\sim}{B} \| e^{-\delta_3 (b-t)/2\varepsilon}, \tag{3.21}$$

$$\| \underset{\sim}{u}'(t,\varepsilon) \| \leq \| \underset{\sim}{B} \| e^{-\delta_3 (b-t)/2\varepsilon} + \varepsilon m_4 \| B \| \times (1 - e^{-\delta_3 (b-t)/2\varepsilon}) \text{ on } \bar{D}. \tag{3.22}$$

Here $0 < \varepsilon \leq \varepsilon_1$, ε_1 is sufficiently small.

Proof

The procedure adopted in the proof of Theorem 3.3 shall be used here. Let $\underset{\sim}{u}$ be a solution of the LBVP I and define the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = \|\underset{\sim}{B}\| e^{-\delta_3(b-t)/2\varepsilon}, \quad i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon), \text{ on } \bar{D}.$$

Then

$$P_{Li}\underset{\sim}{y} = y_i(t,\varepsilon)\left(-\delta_3^2/4\varepsilon + (\delta_3/2\varepsilon)\alpha_{ii} + \sum_{j=-\infty}^{\infty} \beta_{ij}\right)$$

$$\geq y_i(t,\varepsilon)\left(-\delta_3^2/4\varepsilon + \delta_3^2/2\varepsilon + \eta\right) \geq 0 = P_{Li}\underset{\sim}{u}$$

for $0 < \varepsilon \leq \varepsilon_1$ where ε_1 is sufficiently small.

That is,

(3.23) $\quad \underset{\sim}{P}_L \underset{\sim}{u} \leq \underset{\sim}{P}_L \underset{\sim}{y}$ in D.

Also

$$\underset{\sim}{u}(a,\varepsilon) \leq \underset{\sim}{y}(a,\varepsilon) \text{ and } \underset{\sim}{u}(b,\varepsilon) \leq \underset{\sim}{y}(b,\varepsilon)$$

which with (3.23) and by Theorem 3.1, yields

(3.24) $\quad \underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon)$ on $\bar{D}$.

Similarly

(3.25) $\quad \underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon)$ on $\bar{D}$.

Now the estimate for the LBVP I follows from (3.24)-(3.25).

To establish the results for the LBVP II and III the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ are defined as

$$y_i(t,\varepsilon) = (2\varepsilon/\delta_3) \,|\, \underset{\sim}{B} \,|\, e^{-\delta_3(b-t)/2\varepsilon}, \quad i \in Z$$

$$\underset{\sim}{z}\,(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon) \text{ on } \bar{D}.$$

Following the procedure described above one can obtain the required estimates (3.21) and (3.22) for the LBVPs II and III.

Theorem 3.6.

Let $\underset{\sim}{u}$ be a solution of the LBVP I, II or III. Further assume that

$$|\ \alpha'_{ii}(t,\varepsilon)\ |\ \leq\ \delta_4,$$

$$\sum_{j=-\infty}^{\infty} |\ \beta_{ij}(t,\varepsilon)\ | \leq \eta_o, \quad \sum_{j=-\infty}^{\infty} |\beta'_{ij}(t,\varepsilon)\ | \leq \eta_1,$$

$$|\ \gamma_i(t,\varepsilon)| \leq \xi_1 \quad \text{and} \quad |\ \gamma'_i(t,\varepsilon)| \leq \xi_2, \ i \in Z.$$

Then for the LBVP I:

$$\lim_{\varepsilon\to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \quad a \leq t < b \tag{3.26}$$

and for the LBVP II

$$\lim_{\varepsilon\to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \quad a \leq t \leq b, \tag{3.27}$$

$$\lim_{\varepsilon\to 0+} \| \underset{\sim}{u}'(t,\varepsilon) - \underset{\sim}{x}'(t) \| = 0, \quad a \leq t < b. \tag{3.28}$$

Here $\underset{\sim}{x}$ is the solution of the IVP (that is, reduced problem of the LBVP):

$$(3.29)\qquad \begin{cases} \underset{\sim}{\alpha}(t,0)\ \underset{\sim}{x}' + \underset{\sim}{\beta}(t,0)\underset{\sim}{x} = \underset{\sim}{\gamma}(t,0), & a < t \le b, \\ \underset{\sim}{x}(a) \ = \ \underset{\sim}{A}(0). \end{cases}$$

Remark.

From (3.26) - (3.28) it is evident that the solution of the LBVP I itself shows nonuniform behaviour as $\varepsilon \to 0+$ whereas for the LBVPs II and III the nonuniformity occurs only in the derivative of the solution and not in the solution.

Proof

Let $\underset{\sim}{w}$ be a solution of the IVP

$$(3.30)\qquad \begin{cases} \underset{\sim}{\alpha}(t,\varepsilon)\underset{\sim}{w}' + \underset{\sim}{\beta}(t,\varepsilon)\underset{\sim}{w} = \underset{\sim}{\gamma}(t,\varepsilon), & a < t \le b, \\ \underset{\sim}{w}(a,\varepsilon) \ = \ \underset{\sim}{A}(\varepsilon). \end{cases}$$

The assumptions made on the coefficients $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, $\underset{\sim}{\gamma}$ yield the following results [35]:

$$(3.31)\qquad \lim_{\varepsilon \to 0+} \| \underset{\sim}{w}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \ t \in \bar{D},$$

$$(3.32)\qquad \lim_{\varepsilon \to 0+} \| \underset{\sim}{w}'(t,\varepsilon) - \underset{\sim}{x}'(t) \| = 0, \ t \in \bar{D}$$

and

$$(3.33)\qquad \| \underset{\sim}{w}'' \| \le m_6 \text{ on } \bar{D}.$$

The results (3.27) - (3.28) for the LBVP II shall be proved now and the results for the LBVP I and III can be established in a similar manner.

Let $\underset{\sim}{v}$ be a solution of the BVP

$$(3.34)\quad \begin{cases} -\varepsilon \underset{\sim}{v}'' + \underset{\sim}{\alpha}\, \underset{\sim}{v}' + \underset{\sim}{\beta}\, \underset{\sim}{v} = \underset{\sim}{\gamma} - \varepsilon\, \underset{\sim}{w}'' , \quad t \in \bar{D}, \\ \underset{\sim}{v}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \quad \underset{\sim}{v}'(b,\varepsilon) = \underset{\sim}{B}(\varepsilon). \end{cases}$$

Also let $\underset{\sim}{u}$ be a solution of the LBVP II. Then from Corollary 3.4, for $0 < \varepsilon \leq \varepsilon_1$ where ε_1 is sufficiently small,

$$(3.35)\quad \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{v}(t,\varepsilon) \| \leq \varepsilon\, m_7 \quad \text{on } \bar{D}$$

and

$$(3.36)\quad \| \underset{\sim}{u}'(t,\varepsilon) - \underset{\sim}{v}'(t,\varepsilon) \| \leq \varepsilon\, m_8 (1 - e^{-\delta_3 (b-t)/\varepsilon}) \quad \text{on } \bar{D}.$$

Also from Theorem 3.5, for $0 < \varepsilon \leq \varepsilon_2$ where ε_2 is sufficiently small,

$$(3.37)\quad \| \underset{\sim}{v}(t,\varepsilon) - \underset{\sim}{w}(t,\varepsilon) \| \leq (2\varepsilon/\delta_3)\, \| \underset{\sim}{B} - \underset{\sim}{w}'(b,\varepsilon) \| e^{-\delta_3 (b-t)/2\varepsilon} \quad \text{on } \bar{D},$$

$$(3.38)\quad \| \underset{\sim}{v}'(t,\varepsilon) - \underset{\sim}{w}'(t,\varepsilon) \| \leq \varepsilon\, m_9\, \| \underset{\sim}{B} - \underset{\sim}{w}'(b,\varepsilon) \| (1 - e^{-\delta_3 (b-t)/2\varepsilon}) + \| \underset{\sim}{B} - \underset{\sim}{w}'(b,\varepsilon) \| e^{-\delta_3 (b-t)/2\varepsilon} \quad \text{on } \bar{D}.$$

The results for the LBVP II follow from the above estimates and the inequality

$$\| \underset{\sim}{u}(t,\varepsilon)-\underset{\sim}{x}(t) \| \le \| \underset{\sim}{u}(t,\varepsilon)-\underset{\sim}{v}(t,\varepsilon)\| + \|\underset{\sim}{v}(t,\varepsilon)-\underset{\sim}{w}(t,\varepsilon)\| + \| \underset{\sim}{w}(t,\varepsilon)-\underset{\sim}{x}(t) \| .$$

Remark.

In the above theorem one can obtain the following estimate for the LBVP I.

$$(3.39) \qquad \| \underset{\sim}{u}(t,\varepsilon)-\underset{\sim}{w}(t,\varepsilon) \| \le \varepsilon\, m_{10} + \| \underset{\sim}{u}(b,\varepsilon)-\underset{\sim}{w}(b,\varepsilon) \| \times e^{-\delta_3(b-t)/2\varepsilon} \quad \text{on } \bar{D},$$

where $\underset{\sim}{u}$ and $\underset{\sim}{w}$ are respectively solutions of the LBVP I and IVP (3.30). Following the procedure described in the above theorem one can prove the theorem given below.

Theorem 3.7.

Let $\underset{\sim}{u}$ be a solution of the LBVP I with

$$\alpha_{ii}(t,\varepsilon) \le -\delta_5 < 0, \ \delta_5 > 0, \ i \in Z.$$

Assume that all the conditions of Theorem 3.2 (except the condition (iii)) and the conditions of Theorem 3.6 are satisfied for the system (3.7).

Then

$$(3.40) \qquad \lim_{\varepsilon \to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \ a < t \le b,$$

where $\underset{\sim}{x}$ is the solution of the terminal problem

$$(3.41)\quad \begin{cases} \underset{\sim}{\alpha}(t,0)\underset{\sim}{x}' + \underset{\sim}{\beta}(t,0)\underset{\sim}{x} = \underset{\sim}{\gamma}(t,0), & a \leq t < b, \\ \underset{\sim}{x}(b) = \underset{\sim}{B}(0). \end{cases}$$

2. APPLICATIONS TO LBVPs (CONTINUED).

Validity of the conditions (i) and (ii) of Theorem 3.2 (these conditions are nothing but the quasi-monotonicity conditions for the system (3.7)) to the system (3.7) is assumed throughout the previous section and discussed the limiting behaviour of their solutions. In this section the discussion is continued on the asymptotic behaviour but in the absence of the condition (i) of Theorem 3.2. In the following only the LBVP I is discussed and the other LBVPs can be discussed similarly. When the system (3.7) does not satisfy the condition (i) of Theorem 3.2,

set

$$\beta^{+}_{ij} := \begin{cases} \beta_{ij} & \text{if } \beta_{ij} \geq 0 \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta^{-}_{ij} := \beta_{ij} - \beta^{+}_{ij}.$$

Then, following the method described in Section 3 of Chapter 2 of this thesis, one can adjoint the following BVP to the LBVP I:

$$(3.42)\begin{cases} P_{1i}(\underset{\sim}{v},\underset{\sim}{\hat{v}}) := -\varepsilon v_i'' + \alpha_{ii}v_i' + \beta_{ii}v_i + \sum\limits_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+}\hat{v}_j + \beta_{ij}^{-}v_j) = -\gamma_i, & i \in Z, \\ P_{2i}(\underset{\sim}{v},\underset{\sim}{\hat{v}}) := -\varepsilon \hat{v}_i'' + \alpha_{ii}\hat{v}_i' + \beta_{ii}\hat{v}_i + \sum\limits_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+}v_j + \beta_{ij}^{-}\hat{v}_j) = \gamma_i, & i \in Z, \\ R_{1i}(\underset{\sim}{v},\underset{\sim}{\hat{v}}) := (v_i(a,\varepsilon),\ v_i(b,\varepsilon))^T = (-\eta_{1i},\ -\eta_{2i})^T, & i \in Z, \\ R_{2i}(\underset{\sim}{v},\underset{\sim}{\hat{v}}) := (\hat{v}_i(a,\varepsilon),\ \hat{v}_i(b,\varepsilon))^T = (\eta_{1i},\ \eta_{2i})^T, & i \in Z. \end{cases}$$

It can be seen that the system (3.42) satisfies a condition similar to (i) of Theorem 3.2. Also if $\underset{\sim}{u}$ is a solution of the LBVP I then $((\ldots,-u_1,\ -u_o,\ -u_1,\ \ldots),\ (\ldots,u_{-1},u_o,u_1,\ldots))$ is a solution of the above LBVP (3.42).

The following theorem corresponds to Theorem 3.6 in the unrestricted case being discussed above.

Theorem 3.8.

Consider the LBVP I. If the condition (i) of Theorem 3.2 is not satisfied then construct the BVP (3.42). Further assume

(i) $\beta_{ii} \leq M,\quad i \in Z,\quad M > 0;$

(ii) $\alpha_{ii}(t,\varepsilon) \geq \delta_3 > 0;$

and

(iii) $\beta_{ii} + \sum\limits_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+} + \beta_{ij}^{-}) \geq -\eta,\ i \in Z,\ \eta > 0.$

Then also the conclusion (3.26) for LBVP I is valid.

Remark.

Recall that all the conditions of Theorem 3.2 are assumed to be satisfied for the system (3.7) in Theorem 3.6.

Proof

Consider the LBVP I. If the system (3.7) does not satisfy the condition (i) of Theorem 3.2 then introduce the new system (3.42). Following the same procedure as in the proof of Theorem 3.6 the following estimates for the LBVP(3.42) can be obtained:

$$(3.43)\qquad |v_i - x_i| \leq \varepsilon\, m_{10} + \| (\underset{\sim}{v},\hat{\underset{\sim}{v}})(b,\varepsilon) - (\underset{\sim}{x},\hat{\underset{\sim}{x}})(b,\varepsilon) \| \times e^{-\delta_3 (b-t)/2\varepsilon},$$

$$(3.44)\qquad |\hat{v}_i - \hat{x}_i| \leq \varepsilon\, m_{10} + \| (\underset{\sim}{v},\hat{\underset{\sim}{v}})(b,\varepsilon) - (\underset{\sim}{x},\hat{\underset{\sim}{x}})(b,\varepsilon) \| \times e^{-\delta_3 (b-t)/2\varepsilon},$$

where $(\underset{\sim}{v},\hat{\underset{\sim}{v}})$ and $(\underset{\sim}{x},\hat{\underset{\sim}{x}})$ are respectively solutions of the LBVP (3.42) and the IVP

$$(3.45)\qquad \begin{cases} \alpha_{ii}\, x_i' + \beta_{ii}\, x_i + \displaystyle\sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+}\, \hat{x}_j + \beta_{ij}^{-}\, x_j) = -\gamma_i, \\ \alpha_{ii}\, \hat{x}_i' + \beta_{ii}\, \hat{x}_i + \displaystyle\sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+}\, x_j + \beta_{ij}^{-}\, \hat{x}_j) = \gamma_i, \\ x_i(a,\varepsilon) = -A_i(\varepsilon),\ \hat{x}_i(a,\varepsilon) = A_i(\varepsilon),\ i \in Z. \end{cases}$$

As pointed out earlier whenever $\underset{\sim}{u}, \underset{\sim}{v}$ are respectively solutions of the LBVP I and the IVP (3.30), then $(-\underset{\sim}{u}, \underset{\sim}{u})$ and $(-\underset{\sim}{v}, \underset{\sim}{v})$ are respectively solutions of the LBVP (3.42) and the IVP (3.45). The conclusion of the present theorem now follows from the estimates (3.43)-(3.44) and the assumptions made in Theorem 3.6.

3. RESULTS ON QUASILINEAR BVPs.

This section outlines possible extensions of some of the results of Section 1 to the weakly coupled systems of quasilinear ODEs of the form

$$(3.46) \qquad -\varepsilon \underset{\sim}{u}'' + \underset{\sim}{\alpha}(t, \underset{\sim}{u}, \varepsilon)\underset{\sim}{u}' + \underset{\sim}{\beta}(t, \underset{\sim}{u}, \varepsilon)\underset{\sim}{u} = \underset{\sim}{\gamma}(t, \underset{\sim}{u}, \varepsilon), \; a < t < b,$$

where

$$\alpha_{ij}\,(t, \underset{\sim}{u}, \varepsilon) \equiv 0, \; i \neq j,$$

$$\beta_{ij}\,(t, \underset{\sim}{u}, \varepsilon) \leq 0, \; i \neq j, \; i,j \in Z,$$

$$\sum_{\substack{j=-\infty \\ j \neq i}}^{\infty} \left| \beta_{ij}(t, \underset{\sim}{u}, \varepsilon) \right| \leq \eta, \; i \in Z .$$

for some positive real number η and for every $\underset{\sim}{u} \in U$. Now Theorem 3.1 can be reformulated to the system (3.46) as follows.

Theorem 3.9.

Let $\underset{\sim}{u}$ be a solution of the system (3.46) satisfying one of the BCs (3.3)-(3.5). Further assume that there exist a positive real number δ_1 and a function $s(t,\varepsilon)\colon \bar{D} \times I \to R_+$ with the properties

$$\underset{\sim}{s} = (\ldots, s(t,\varepsilon), s(t,\varepsilon), \ldots) \in U,$$

$$\hat{\underset{\sim}{P}}\,\underset{\sim}{s} \geq \delta_1 > 0, \quad \underset{\sim}{R}\,\underset{\sim}{s} \geq \delta_2 > 0, \tag{3.47}$$

where

$$\hat{\underset{\sim}{P}}\,(\;) := -\varepsilon(\;)'' + \underset{\sim}{\alpha}(t,\underset{\sim}{u},\varepsilon)\,(\;)' + \underset{\sim}{\beta}(t,\underset{\sim}{u},\varepsilon)(\;),$$

$\underset{\sim}{R}$ is as in Theorem 3.1.

Then the following implication is true for all $\underset{\sim}{v},\underset{\sim}{w} \in U$:

$$\left\{\begin{array}{l} \hat{\underset{\sim}{P}}\,\underset{\sim}{v} \leq \hat{\underset{\sim}{P}}\,\underset{\sim}{u} \leq \hat{\underset{\sim}{P}}\,\underset{\sim}{w}, \quad \underset{\sim}{R}\,\underset{\sim}{v} \leq \underset{\sim}{R}\,\underset{\sim}{u} \leq \underset{\sim}{R}\,\underset{\sim}{w} \\ \Longrightarrow \\ \underset{\sim}{v} \leq \underset{\sim}{u} \leq \underset{\sim}{w} \text{ on } \bar{D}. \end{array}\right. \tag{3.48}$$

Using this theorem and adopting the same procedure developed for the LBVPs, one can obtain similar estimates for solutions and/or their derivatives for the quasilinear BVPs as done in Section 1. Infact one has to replace $\underset{\sim}{P}$ wherever it appears in Section 1 by $\hat{\underset{\sim}{P}}$ to obtain results for the quasilinear system.

4. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO NONLINEAR BVPs(NBVPs)

In Sections 1-2 the asymptotic behaviour of solutions and/or their derivatives of the LBVPs I-III are discussed. In the present section, guided by the experience with LBVPs, estimates for solutions of the NBVPs (3.1)-(3.5) in terms of the solution of the IVP

$$(3.49)\quad \begin{cases} \underset{\sim}{f}(t, [\underset{\sim}{x}]', \underset{\sim}{x}, 0) = \underset{\sim}{0}, & a < t \leq b, \\ \underset{\sim}{x}(a) = \underset{\sim}{A}(0) \end{cases}$$

are given.

Theorem 3.1 is now reformulated as follows.

Theorem 3.10.

Let $\underset{\sim}{u}$ be a solution of the system (3.1) subject to one of the BCs (3.3)-(3.5). Then the implication

$$(3.50)\quad \begin{cases} \underset{\sim}{P}\,\underset{\sim}{z} \leq \underset{\sim}{P}\,\underset{\sim}{u} \leq \underset{\sim}{P}\,\underset{\sim}{y}, \quad \underset{\sim}{R}\,\underset{\sim}{z} \leq \underset{\sim}{R}\,\underset{\sim}{u} \leq \underset{\sim}{R}\,\underset{\sim}{y} \\ \Longrightarrow \\ \underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon) \quad \text{on } \bar{D} \end{cases}$$

is true for all $\underset{\sim}{z}, \underset{\sim}{y} \in U$ provided

(i) there exist a positive real number δ_2 and a 'test function' $s(t,\varepsilon): \bar{D} \times I \rightarrow R_+$ with the properties

$$(3.51)\quad \begin{cases} \underset{\sim}{P}\,(\underset{\sim}{y} + \alpha_1\underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{y} \geq \alpha_1\delta_2 > 0, \\ \underset{\sim}{P}\,\underset{\sim}{z} - \underset{\sim}{P}\,(\underset{\sim}{z} - \alpha_1\underset{\sim}{s}) \geq \alpha_1\delta_2 > 0 \text{ in } D, \\ \underset{\sim}{R}\,\underset{\sim}{s} > 0 \text{ on } \partial D,\ \underset{\sim}{s} = (\ldots, s(t,\varepsilon), s(t,\varepsilon), \ldots) \in U \end{cases}$$

for every positive real number α_1;

(ii) there exists a positive constant M such that

$$\underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]', \underset{\sim}{u},\varepsilon)-\underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]',\underset{\sim}{y}+\alpha_1\underset{\sim}{s},\varepsilon)$$

$$\geq M(\underset{\sim}{u} - (\underset{\sim}{y} + \alpha_1\underset{\sim}{s}))$$

and

$$\underset{\sim}{f}(t, [\underset{\sim}{u}]', \underset{\sim}{z} - \alpha_1\underset{\sim}{s},\varepsilon) - \underset{\sim}{f}(t, [\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon)$$

$$\geq M(\underset{\sim}{z} - (\underset{\sim}{u} + \alpha_1\underset{\sim}{s})),$$

whenever $\underset{\sim}{u} \leq \underset{\sim}{y} + \alpha_1\underset{\sim}{s}$, $\underset{\sim}{z} \leq \underset{\sim}{u} + \alpha_1\underset{\sim}{s}$ and for all positive number α_1 and η with $\alpha_1-\eta > 0$;

(iii) there exists a positive constant L such that

$$\underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]', \underset{\sim}{y}+\alpha_1\underset{\sim}{s},\varepsilon)-\underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]',\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s},\varepsilon)$$

$$\geq -L\eta\underset{\sim}{s} \leq \underset{\sim}{f}(t,[\underset{\sim}{u}]', \underset{\sim}{u}+\eta\underset{\sim}{s},\varepsilon)-\underset{\sim}{f}(t,[\underset{\sim}{u}]',\underset{\sim}{u},\varepsilon)$$

for every positive number α_1 and η with $\alpha_1-\eta > 0$. The proof of this theorem is the same as that of Theorem 3.1.

Theorem 3.11.

Let $\underset{\sim}{u}$ be a solution of the system (3.1) subject to the BC (3.3). Let $\underset{\sim}{x}$ be the solution of the IVP (3.49) such that $\underset{\sim}{x} \in C^2(D)$ and $\| \underset{\sim}{f}(t,[\underset{\sim}{x}]', \underset{\sim}{x}, \varepsilon) \| = O(\varepsilon)$. Further assume that the conditions (i) and (ii) given below and the conditions (ii) and (iii) of Theorem 3.10 hold good for the functions $\underset{\sim}{y}$, $\underset{\sim}{z}$ which are defined as

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{w},\ \underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{w},\quad \underset{\sim}{w} = (\ldots, w, w, \ldots),$$

$$w(t,\varepsilon) = \varepsilon\, m_3 e^{k(t-a)} + \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{x}(b) \|\, e^{-\delta_2(b-t)/2\varepsilon}$$

where k, δ_2 are pcsitive constants,

(i) there exist constants m_1 and m_2 such that

$$\begin{aligned}\underset{\sim}{f}(t,[\underset{\sim}{y}+\alpha_1\underset{\sim}{s}]', \underset{\sim}{y} + \alpha_1\underset{\sim}{s}, \varepsilon) &- \underset{\sim}{f}(t,[\underset{\sim}{y}]', \underset{\sim}{y}, \varepsilon) \\ &\geq \alpha_1 m_1 \underset{\sim}{s}' - \alpha_1 m_2 \underset{\sim}{s} \\ &\leq \underset{\sim}{f}(t,[\underset{\sim}{z}]', \underset{\sim}{z},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{z}-\alpha_1\underset{\sim}{s}]', \underset{\sim}{z}-\alpha_1\underset{\sim}{s},\varepsilon)\end{aligned}$$

for all positive constants α_1 and $\underset{\sim}{s} = (\ldots, s, s, \ldots)$,

$$s(t) = e^{k(t-a)},\ k \text{ is a positive constant;}$$

(ii) there exist positive constants δ_2 and η such that

$$\begin{aligned}\underset{\sim}{f}(t,[\underset{\sim}{x}+\underset{\sim}{w}]', \underset{\sim}{x}+\underset{\sim}{w},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{x}]', \underset{\sim}{x},\varepsilon) &\leq \delta_2\underset{\sim}{w}' - \eta\underset{\sim}{w} \\ &\geq \underset{\sim}{f}(t,[\underset{\sim}{x}]', \underset{\sim}{x},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{x}-\underset{\sim}{w}]', \underset{\sim}{x}-\underset{\sim}{w},\varepsilon)\end{aligned}$$

for $0 < \varepsilon \leq \varepsilon_1$, where ε_1 is sufficiently small,

then

(3.52) $\quad \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| \le w(t,\varepsilon)$ on $\bar{D}$,

that is,

(3.53) $\quad \lim_{\varepsilon \to 0} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \ a \le t < b.$

Proof

It will be shown that the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ defined in the statement of this theorem and the function $\underset{\sim}{s}$ defined in the hypothesis (i) satisfy the conditions of Theorem 3.10. Then the estimate (3.52) follows.

$$\begin{aligned} P_i(\underset{\sim}{y}+\alpha_1\underset{\sim}{s}) - P_i\,\underset{\sim}{y} &= -\varepsilon\alpha_1 s_i'' + [f_i(t, y_i'+\alpha_1 s_i', \underset{\sim}{y}+\alpha_1\underset{\sim}{s}, \varepsilon) \\ &\qquad - f_i(t, y_i', \underset{\sim}{y}, \varepsilon)] \\ &\ge -\varepsilon\alpha_1 s_i'' + \alpha_1 m_1 s_i' - \alpha_1 m_2 s_i \\ &= \alpha_1(-\varepsilon k^2 + m_1 k - m_2)\, e^{k(t-a)} \ge \alpha_1 \delta_2 > 0, \ i \in Z, \end{aligned}$$

by a proper choice of k and for sufficiently small $\varepsilon > 0$.

Further

$$\begin{aligned} P_i\,\underset{\sim}{y} &= P_i(\underset{\sim}{x}+\underset{\sim}{w}) - P_i\,\underset{\sim}{x} + P_i\,\underset{\sim}{x} \\ &= P_i\underset{\sim}{x} - \varepsilon w_i'' + [f_i(t, x_i' + w_i', \underset{\sim}{x} + \underset{\sim}{w}, \varepsilon) - f_i(t, x_i', \underset{\sim}{x}, \varepsilon)] \\ &\ge -\varepsilon m_4 - \varepsilon w'' + \delta_2 w' - \eta w \end{aligned}$$

$$\geq \varepsilon m_3 e^{k(t-a)}[-m_4 - \varepsilon k^2 + k\delta_2 - \eta]$$

$$+ \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{x}(b) \| [(-\delta_2^2/4\varepsilon) + (\delta_2^2/2\varepsilon) - \eta]$$

$$\geq \quad 0 = P_i \underset{\sim}{u}, \; i \in Z, \text{ for sufficiently small } \varepsilon.$$

Also

$$\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon) \text{ at } t = a,b \text{ by a proper choice of } m_3.$$

That is, the following inequalities are established:

$$\underset{\sim}{P}(\underset{\sim}{y} + \alpha_1 \underset{\sim}{s}) - \underset{\sim}{P}\underset{\sim}{y} \geq \alpha_1 \delta_2 > 0,$$

$$\underset{\sim}{P}\underset{\sim}{y} \geq \underset{\sim}{P}\underset{\sim}{u}, \quad \underset{\sim}{R}\underset{\sim}{y} \geq \underset{\sim}{R}\underset{\sim}{u}, \text{ for all positive constant } \alpha_1.$$

Hence, by Theorem 3.10, the following inequality is obtained:

$$\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon) = \underset{\sim}{x}(t) + \underset{\sim}{w}(t,\varepsilon) \text{ on } \bar{D}.$$

Similarly one can show that

$$\underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon) \text{ on } \bar{D}.$$

Hence the theorem is proved.

Following the procedures discussed in this chapter one can obtain many more results for nonlinear systems as discussed in the case of scalar equation in [15]. Also existence theorems may be obtained, following the method given in [17, 18]. The main objective of this chapter is only to discuss the asymptotic behaviour of solutions and/or their derivatives.

Chapter 4

SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS IN BANACH SPACE-II

The objective of this chapter is to study the asymptotic behaviour of solutions and/or their derivatives of the following BVPs for weakly coupled doubly infinite systems of second order ODEs with a small parameter multiplying the highest derivative. The study is carried out by obtaining proper estimates for solutions and their derivatives. The BVPs consist of doubly infinite system of second order ODEs

$$(4.1) \qquad \underset{\sim}{P}\,\underset{\sim}{u}: = -\varepsilon \underset{\sim}{u}'' + \underset{\sim}{f}(t,[\underset{\sim}{u}]',\underset{\sim}{u},\varepsilon)=\underset{\sim}{0}, \quad t \in D: = (a,\infty),$$

that is,

$$P_i\underset{\sim}{u}: = -\varepsilon u_i'' + f_i(t,u_i', \ldots, u_o, u_1, u_2, \ldots, \varepsilon) = 0, i \in Z$$

subject to one of the BCs:

$$(4.2) \qquad \underset{\sim}{R}\,\underset{\sim}{u}: = (\underset{\sim}{u}(a,\varepsilon),\ \underset{\sim}{u}(\infty,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T,$$

$$(4.3) \qquad \underset{\sim}{R}\,\underset{\sim}{u}: = (-\underset{\sim}{u}'(a,\varepsilon),\ \underset{\sim}{u}(\infty,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T,$$

$$(4.4) \qquad \underset{\sim}{R}\,\underset{\sim}{u}: = (\underset{\sim}{u}(a,\varepsilon) - \varepsilon \underset{\sim}{u}'(a,\varepsilon),\ \underset{\sim}{u}(\infty,\varepsilon))^T = (\underset{\sim}{A}(\varepsilon),\ \underset{\sim}{B}(\varepsilon))^T.$$

The study is carried out by making use of Theorem 2.10. This chapter is divided into two sections the first of which treats linear equations whereas second one considers nonlinear equations.

1. LINEAR SYSTEMS.

This section treats linear systems of second order ODEs of the form

$$\underset{\sim}{P}_{Li}\,\underset{\sim}{u}: = -\varepsilon u_i'' + \alpha_{ii}(t,\varepsilon)\, u_i' + \sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon) u_j$$

$$= \gamma_i(t,\varepsilon), \quad t \in D$$

subject to the BCs (4.2)-(4.4). The linear system can be abbreviated as

$$(4.5) \qquad \underset{\sim}{P}_L\,\underset{\sim}{u}: = -\varepsilon\underset{\sim}{u}'' + \underset{\sim}{\alpha}\,\underset{\sim}{u}' + \underset{\sim}{\beta}\,\underset{\sim}{u} = \underset{\sim}{\gamma},$$

with $\underset{\sim}{\beta} = (\beta_{ij}),\ \underset{\sim}{\alpha} = (\alpha_{ij}),\ \ \alpha_{ij} = 0,\ i \neq j,\ i,j \in Z,$

$$\underset{\sim}{\alpha}\,\underset{\sim}{u}' = (\ldots,\ \alpha_{11}\, u_1',\ \alpha_{22}\, u_2',\ \ldots\ )^T,$$

$$\underset{\sim}{\beta}\,\underset{\sim}{u} = (\ \ldots,\ \sum_{j=-\infty}^{\infty} \beta_{ij}\, u_j,\ \ldots\ )^T,$$

$$\underset{\sim}{\gamma} = (\ \ldots,\ \gamma_1,\ \gamma_2,\ \ldots\ )^T.$$

Here it is assumed that $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, $\underset{\sim}{\gamma}$ are continuous and bounded on $\overline{D} \times I$.

In the following, LBVPs I, II and III stand for (4.5) with BCs (4.2)-(4.4) respectively.

Theorem 4.1.

Consider the LBVPs I-II and assume

(i) $\beta_{ij} \leq 0, \quad i \neq j,$

(ii) $\beta_{ii} \leq M, \quad M > 0,$

(iii) $\alpha_{ii}(t,\varepsilon) \leq -\delta_3 < 0$ and

(iv) $\sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon) \geq \delta_4 > 0, \ i,j \in Z.$

Then the quasi-monotonicity condition and the condition (2.37) is valid for the system (4.5). Also there exists a 'test function' satisfying the conditions (2.33) and (2.34) for the LBVPs I and II.

Proof

Assumptions (i) and (ii) imply the quasi-monotonicity condition whereas the condition (2.37) follows from (iv). Finally using the assumptions (iii) and (iv), the function $\underset{\sim}{s} = (\ldots, s, s, \ldots)$ with $s(t,\varepsilon) = 1+e^{-k(t-a)}$, $k > 0$, by a proper choice of k, can be shown to be a 'test function' for the LBVPs I and II.

Theorem 4.2.

Consider the LBVPs I and III and assume that the conditions (i), (ii) and (iv) of Theorem 4.1 and $\alpha_{ii}(t,\varepsilon) \geq \delta_5 > 0$, $i \in Z$, are satisfied. Then the conclusions of the above theorem hold true for the LBVPs I and III.

Proof

It is enough to observe that the function $\underset{\sim}{s} = (\ldots, s, s, \ldots)$ with $s(t,\varepsilon) = 2 - e^{-k(t-a)}$, $k > 0$, by a proper choice of k, is a required 'test function' for the LBVPs I and III.

Different forms of estimates for solutions and/or their derivatives of the LBVPs I-III are discussed in this section. In the following, m_i, i=1,2, ... are positive constants independent of the small parameter ε.

It is assumed that all the conditions of Theorem 4.1 (that is conditions (i)-(iv)) are valid for the following Theorems 4.3-4.6 and Corollary 4.4.

Theorem 4.3.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or II.

Then

(4.6) $$\| \underset{\sim}{u}(t,\varepsilon) \| \leq m_1 \max \left\{ \|\underset{\sim}{A}\|, \| \underset{\sim}{B} \|, \| \underset{\sim}{\gamma} \| \right\}, \; t \in \bar{D}.$$

Proof

Because of the assumptions made just before the statement of this theorem, the implication (2.35)-(2.36) is true for the LBVPs I and II.

Define $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = m_{15} \max\left\{\|\underset{\sim}{A}\|, \|\underset{\sim}{B}\|, \|\underset{\sim}{\gamma}\|\right\}(1+e^{-k(t-a)}), i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon), \ t \in \bar{D}, \quad m_{15} = \max\left\{1, 1/\delta_4\right\}, \text{ where}$$

k is a positive number yet to be determined.

Then

$$P_{Li}\underset{\sim}{y} = \max\left\{\|\underset{\sim}{A}\|, \|\underset{\sim}{B}\|, \|\underset{\sim}{\gamma}\|\right\}[(-\varepsilon k^2 - k\alpha_{ii})e^{-k(t-a)}$$

$$+ \sum_{j=-\infty}^{\infty} \beta_{ij}(1+e^{-k(t-a)})]$$

$$\geq \max\left\{\|\underset{\sim}{A}\|, \|\underset{\sim}{B}\|, \|\underset{\sim}{\gamma}\|\right\}[(-\varepsilon k^2 + k\delta_3)e^{-k(t-a)} + 1]$$

$$\geq \|\underset{\sim}{\gamma}\| \geq \gamma_i = P_{Li}\underset{\sim}{u}, \quad i \in Z,$$

by a proper choice of k.

That is,

(4.7) $\quad \underset{\sim}{P}_L \underset{\sim}{u} \leq \underset{\sim}{P}_L \underset{\sim}{y}$ in D.

It is easy to verify the following inequalities, by a proper choice of k, for the LBVPs I and II:

(4.8) $\quad \underset{\sim}{R}\,\underset{\sim}{u} \le \underset{\sim}{R}\,\underset{\sim}{y}.$

The inequalities (4.7) and (4.8), by Theorem 2.10,yield

(4.9) $\quad \underset{\sim}{u}(t,\varepsilon) \le \underset{\sim}{y}(t,\varepsilon).$

Similarly one can show that

(4.10) $\quad \underset{\sim}{z}(t,\varepsilon) \le \underset{\sim}{u}(t,\varepsilon)$

which with (4.9) establishes (4.6).

Corollary 4.4.

Let $\underset{\sim}{u}$ be a solution of the LBVP II with

$\sum_{j=-\infty}^{\infty} |\beta_{ij}| \le \eta_o$, $i \in Z$. Then

(4.11) $\quad \| \underset{\sim}{u}(t,\varepsilon) \| \le m_1 \|\underset{\sim}{\gamma}\|,$

(4.12) $\quad \|\underset{\sim}{u}'(t,\varepsilon)\| \le m_2 \|\underset{\sim}{\gamma}\| \,(1-e^{-\delta_3(t-a)/\varepsilon}), \; t \in \bar{D}$

provided $\underset{\sim}{A}(\varepsilon) \equiv \underset{\sim}{O} \equiv \underset{\sim}{B}(\varepsilon)$.

Proof

The equation (4.5) may be written as

$$(4.13)\qquad \varepsilon(u_i' \, e^{q_i})' = p_i, \quad i \in Z,$$

where $\quad q_i(t,\varepsilon) = 1/\varepsilon \int_t^{\infty} \alpha_{ii}(t,\varepsilon)dt$

and $\quad p_i(t,\varepsilon) = \left(\sum_{j=-\infty}^{\infty} \beta_{ij}(t,\varepsilon)u_j - \gamma_i\right) e^{q_i(t,\varepsilon)}.$

Integrating (4.13) and using (4.6) the following inequality is obtained

$$|u_i'(t,\varepsilon)| \le m_2 \, \|\underset{\sim}{\gamma}\| \, (1-e^{-\delta_3(t-a)/\varepsilon}).$$

The estimate (4.12) follows from this inequality.

Theorem 4.5.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or II together with $\underset{\sim}{\gamma} \equiv \underset{\sim}{0} \equiv \underset{\sim}{B}$. Then for the LBVP I:

$$(4.14)\qquad \|\underset{\sim}{u}(t,\varepsilon)\| \le \|\underset{\sim}{A}\| \, e^{-\delta_3(t-a)/\varepsilon} \text{ on } \bar{D},$$

and for the LBVP II:

$$(4.15)\qquad \|\underset{\sim}{u}(t,\varepsilon)\| \le (\varepsilon/\delta_3) \, \|\underset{\sim}{A}\| \, e^{-\delta_3(t-a)/\varepsilon},$$

$$(4.16)\qquad \| \underset{\sim}{u}'(t,\varepsilon) \| \le \| \underset{\sim}{A} \| e^{-\delta_3(t-a)/\varepsilon} + \varepsilon m_4 \|\underset{\sim}{A}\|(1-e^{-\delta_3(t-a)/\varepsilon}) \text{ on } \bar{D}.$$

Proof

Let $\underset{\sim}{u}$ be a solution of the LBVP I.

Define functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = \| \underset{\sim}{A} \| e^{-\delta_3(t-a)/\varepsilon}, \quad i \in Z$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon), \quad t \in D.$$

Then

$$P_{Li}\underset{\sim}{y} = y_i(t,\varepsilon)(-\delta_3^2/\varepsilon + \alpha_{ii}(-\delta_3/\varepsilon) + \sum_{j=-\infty}^{\infty} \beta_{ij})$$

$$\ge y_i(t,\varepsilon)(-\delta_3^2/\varepsilon + \delta_3^2/\varepsilon) = 0 = P_{Li}\underset{\sim}{u}, \quad i \in Z.$$

That is,

$$(4.17)\qquad \underset{\sim}{P}_L \underset{\sim}{y} \ge \underset{\sim}{P}_L \underset{\sim}{u} \text{ in } D.$$

Also for the LBVP I

$$(4.18)\qquad \underset{\sim}{R}\,\underset{\sim}{y} \ge \underset{\sim}{R}\,\underset{\sim}{u}$$

which with (4.17) and by Theorem 2.10 yields

$$(4.19)\qquad \underset{\sim}{u}(t,\varepsilon) \le \underset{\sim}{y}(t,\varepsilon) \text{ on } \bar{D}.$$

Similarly

$$(4.20)\qquad \underset{\sim}{z}(t,\varepsilon) \le \underset{\sim}{u}(t,\varepsilon) \text{ on } \bar{D}$$

and hence the estimate for the LBVP I follows from (4.19)-(4.20).

To establish the results for the LBVP II define the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = (\varepsilon/\delta_3)\ \| \underset{\sim}{A} \|\ e^{-\delta_3(t-a)/\varepsilon},\ i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = -\underset{\sim}{y}(t,\varepsilon),\ t \in \bar{D}.$$

Adopting the above procedure and the idea used in Corollary 4.4, estimates for the LBVP II can be established.

Theorem 4.6.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or II with $\sum_{j=-\infty}^{\infty} |\beta_{ij}(t,\varepsilon)| \leq \eta_o$, $\eta_o \in R_+$. Further let $\underset{\sim}{w}$ be the solution of the terminal value problem

$$(4.21)\quad \begin{cases} \underset{\sim}{\alpha}(t,\varepsilon)\underset{\sim}{w}' + \underset{\sim}{\beta}(t,\varepsilon)\underset{\sim}{w} = \underset{\sim}{\gamma}(t,\varepsilon),\ a \leq t < \infty, \\ \underset{\sim}{w}(\infty,\varepsilon) = \underset{\sim}{B}(\varepsilon) \end{cases}$$

with the properties

$$(4.22)\quad \lim_{\varepsilon \to 0+} \| \underset{\sim}{w}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0,\ t \in \bar{D},$$

$$(4.23)\quad \lim_{\varepsilon \to 0+} \| \underset{\sim}{w}'(t,\varepsilon) - \underset{\sim}{x}'(t) \| = 0,\ t \in \bar{D}$$

and

(4.24) $\quad \| \underset{\sim}{w}'' \| \leq m_6$ on D,

where $\underset{\sim}{x}$ is the solution of the terminal value problem

$$(4.25) \quad \begin{cases} \underset{\sim}{\alpha}(t,0)\underset{\sim}{x}' + \underset{\sim}{\beta}(t,0)\underset{\sim}{x} = \underset{\sim}{\gamma}(t,0), & 0 \leq t < \infty \\ \underset{\sim}{x}(\infty) = \underset{\sim}{A}(0). \end{cases}$$

Then for the LBVP I:

$$(4.26) \quad \lim_{\varepsilon \to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \; a < t < \infty \; ;$$

for the LBVP II:

$$(4.27) \quad \lim_{\varepsilon \to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \quad a \leq t < \infty$$

and

$$(4.28) \quad \lim_{\varepsilon \to 0+} \| \underset{\sim}{u}'(t,\varepsilon) - \underset{\sim}{x}'(t) \| = 0, \; a < t < \infty.$$

Remark.

From equations (4.26) - (4.28) it can be seen that the solution of the LBVP I itself shows nonuniform behaviour as $\varepsilon \to 0+$ whereas for the LBVP II the nonuniformity occurs only in the derivative of the solution and not in the solution.

Proof

The results (4.27)-(4.28) for the LBVP II shall be proved now and the result of the LBVP I can be established in a similar manner.

Let $\underset{\sim}{v}$ be a solution of the BVP

$$(4.29)\quad \begin{cases} -\varepsilon \underset{\sim}{v}'' + \underset{\sim}{\alpha}\, \underset{\sim}{v}' + \underset{\sim}{\beta}\, \underset{\sim}{v} = \underset{\sim}{\gamma} - \varepsilon \underset{\sim}{w}'', \quad t \in D, \\ \underset{\sim}{v}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \quad \underset{\sim}{v}(\infty,\varepsilon) = \underset{\sim}{B}(\varepsilon). \end{cases}$$

Also let $\underset{\sim}{u}$ be a solution of the LBVP II. Then from Corollary 4.4 ,

$$(4.30)\quad \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{v}(t,\varepsilon) \| \leq \varepsilon\, m_7 \text{ on } \overline{D},$$

$$(4.31)\quad \| \underset{\sim}{u}'(t,\varepsilon) - \underset{\sim}{v}'(t,\varepsilon) \| \leq \varepsilon\, m_8 (1 - e^{-\delta_3 (t-a)/\varepsilon}) \text{ on } \overline{D}$$

and from Theorem 4.5

$$(4.32)\quad \| \underset{\sim}{v}(t,\varepsilon) - \underset{\sim}{w}(t,\varepsilon) \| \leq (\varepsilon/\delta_3)\, \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| \times e^{-\delta_3 (t-a)/\varepsilon} \text{ on } \overline{D},$$

$$(4.33)\quad \| \underset{\sim}{v}'(t,\varepsilon) - \underset{\sim}{w}'(t,\varepsilon) \| \leq \varepsilon\, m_9\, \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| (1 - e^{-\delta_3 (t-a)/\varepsilon}) + \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| e^{-\delta_3 (t-a)/\varepsilon} \text{ on } \overline{D}.$$

The results for the LBVP II follow from the above estimates and from the inequality

$$\| \underset{\sim}{u} - \underset{\sim}{x} \| \leq \| \underset{\sim}{u} - \underset{\sim}{v} \| + \| \underset{\sim}{v} - \underset{\sim}{w} \| + \| \underset{\sim}{w} - \underset{\sim}{x} \| .$$

Remark.

In the above theorem one can obtain the following estimates for the LBVP I:

$$(4.34) \qquad \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{w}(t,\varepsilon) \| \leq \varepsilon\, m_{10} + \| \underset{\sim}{u}(a,\varepsilon) - \underset{\sim}{w}(\infty,\varepsilon) \| \times e^{-\delta_3(t-a)/\varepsilon} \text{ on } \bar{D},$$

where $\underset{\sim}{u}$ and $\underset{\sim}{w}$ are respectively solutions of the LBVP I and the terminal value problem (4.21).

Theorem 4.7.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or III. Further assume that all the conditions of Theorem 4.2 are satisfied. Then

$$(4.35) \qquad \| \underset{\sim}{u}(t,\varepsilon) \| \leq m_1 \max \left\{ \| \underset{\sim}{A} \|, \| \underset{\sim}{B} \|, \| \underset{\sim}{\gamma} \| \right\}, \; t \in D.$$

Proof

Because of the assumptions made in this theorem the implication (2.35)-(2.36) is true for the LBVP I and III (Refer Theorem 4.2). Define functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = m_{15} \max \left\{ \| \underset{\sim}{A} \|, \| \underset{\sim}{B} \|, \| \underset{\sim}{\gamma} \| \right\} \times (2 + \varepsilon_1 \delta_5 - e^{-\delta_5(t-a)}), \; i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = - \underset{\sim}{y}(t,\varepsilon), \; t \in \bar{D}, \; m_{15} = \max \left\{ 1, 1/\delta_4 \right\}.$$

It can be easily verified that

$$\underset{\sim}{P}_L \underset{\sim}{u} \le \underset{\sim}{P}_L \underset{\sim}{y}, \quad \underset{\sim}{R} \underset{\sim}{u} \le \underset{\sim}{R} \underset{\sim}{y}$$

which by Theorem 2.10 yield

(4.36) $\underset{\sim}{u}(t,\varepsilon) \le \underset{\sim}{y}(t,\varepsilon), \; t \in \bar{D}.$

Similar analysis yields

$$\underset{\sim}{z}(t,\varepsilon) \le \underset{\sim}{u}(t,\varepsilon), \; t \in \bar{D}$$

which with (4.36) establishes (4.35).

Theorem 4.8.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or III with $\underset{\sim}{A} \equiv \underset{\sim}{O} \equiv \underset{\sim}{\gamma}$. Further assume that all the conditions of Theorem 4.2 are satisfied. Then

(4.37) $\| \underset{\sim}{u}(t,\varepsilon) \| \le \| \underset{\sim}{B} \| (1-e^{-\delta_5 (t-a)\varepsilon}).$

Proof

Define functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ as

$$y_i(t,\varepsilon) = \| \underset{\sim}{B} \| (1-e^{-\delta_5 (t-a)\varepsilon}), \; i \in Z,$$

$$\underset{\sim}{z}(t,\varepsilon) = - \underset{\sim}{y}(t,\varepsilon).$$

Then

$$P_{Li}\,\underset{\sim}{y} = \|\underset{\sim}{B}\|\,(\delta_5^2\,\varepsilon^3 + \alpha_{ii}\varepsilon\,\delta_5)\,e^{-\delta_5(t-a)\varepsilon}$$

$$+ \; y_i(t,\varepsilon)\sum_{j=-\infty}^{\infty}\beta_{ij} \geq 0 = \gamma_i = P_{Li}\underset{\sim}{u},\; i \in Z.$$

That is,

$$\underset{\sim}{P}_L\,\underset{\sim}{u} \;\leq\; \underset{\sim}{P}_L\,\underset{\sim}{y} \quad \text{in} \quad D.$$

Also for the LBVPs I and III

$$\underset{\sim}{R}\,\underset{\sim}{u} \;\leq\; \underset{\sim}{R}\,\underset{\sim}{y}.$$

Following the procedure described in Theorem 4.5 the estimate (4.37) can be obtained.

Theorem 4.9.

Let $\underset{\sim}{u}$ be a solution of the LBVP I or III. Assume that all the conditions of Theorem 4.2 are satisfied. Further let $\underset{\sim}{w}$ be the solution of the IVP

$$(4.38)\quad \begin{cases} \underset{\sim}{\alpha}\,(t,\varepsilon)\underset{\sim}{w}' + \underset{\sim}{\beta}(t,\varepsilon)\underset{\sim}{w} = \underset{\sim}{\gamma}(t,\varepsilon), \quad a < t < \infty, \\ \underset{\sim}{w}\,(a,\varepsilon) = \underset{\sim}{A}(\varepsilon) \end{cases}$$

with the properties

$$(4.39)\quad \lim_{\varepsilon \to 0+} \|\,\underset{\sim}{w}(t,\varepsilon) - \underset{\sim}{x}(t)\,\| = 0, \quad t \in \bar{D}$$

and

(4.40) $\quad \| \underset{\sim}{w}'' \| \leq m_6$ in D, $\quad \| \underset{\sim}{w}' (a,\varepsilon) \| \leq m_7,$

where $\underset{\sim}{x}$ is the solution of the IVP

$$(4.41) \quad \begin{cases} \underset{\sim}{\alpha}(t,0)\ \underset{\sim}{x}' + \underset{\sim}{\beta}(t,0)\ \underset{\sim}{x} = \underset{\sim}{\gamma}(t,0), \quad a < t < \infty, \\ \underset{\sim}{x}(a) \quad = \quad \underset{\sim}{A}(0). \end{cases}$$

Then

$$(4.42) \quad \| \underset{\sim}{u}(t,\varepsilon)-\underset{\sim}{x}(t) \| \leq \varepsilon m_{11}+D(\varepsilon) + \| \underset{\sim}{B}(\varepsilon)-\underset{\sim}{w}(\infty,\varepsilon) \| \times (1-e^{-\delta_5(t-a)\varepsilon}).$$

Proof

Following the procedure adopted in Theorem 4.6 the estimate (4.42) can be established.

Note.

The estimate (4.42) exhibits nonuniform convergence as $\varepsilon \to 0$ of the solution $\underset{\sim}{u}(t,\varepsilon)$ to $\underset{\sim}{x}(t)$ at $t = \infty$.

In the earlier discussion the validity of the condition (i) of Theorem 4.1 for the system (4.5) is assumed and the asymptotic behaviour is discussed. This condition is an essential one for coupled systems because it makes the systems to satisfy the quasi-monotonicity condition. Here arises the question: whether asymptotic behaviour can be established in the absence of the condition (i) of Theorem 4.1?.

The answer is affirmative. In the following discussion, for the sake of simplicity, only the LBVP I is considered as other LBVPs can be treated in a similar way. If the system (4.5) does not satisfy the condition (i) of Theorem (4.1) then set,

$$\beta_{ij}^{+}: = \begin{cases} \beta_{ij} & \text{if } \beta_{ij} \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_{ij}^{-}: = \beta_{ij} - \beta_{ij}^{+}.$$

Then, following the method described in Section 3 of Chapter 2 of this thesis, one can adjoint the following BVP to the LBVP I:

$$(4.43) \begin{cases} -\varepsilon \hat{u}''_{2i+1} + \alpha_{ii} \hat{u}'_{2i+1} + \beta_{ii} \hat{u}_{2i+1} + \sum\limits_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+} \hat{u}_{2j} + \beta_{ij}^{-} \hat{u}_{2j+1}) = -\gamma_i \\ -\varepsilon \hat{u}''_{2i} + \alpha_{ii} \hat{u}'_{2i} + \beta_{ii} \hat{u}_{2i} + \sum\limits_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+} \hat{u}_{2j+1} + \beta_{ij}^{-} \hat{u}_{2j}) = \gamma_i, \; i \in Z, \\ (\hat{u}_{2i+1}(a,\varepsilon), \hat{u}_{2i+1}(\infty,\varepsilon)) = (-A_i, -B_i), \\ (\hat{u}_{2i}(a,\varepsilon), \hat{u}_{2i}(\infty,\varepsilon)) = (A_i, B_i), \; i \in Z. \end{cases}$$

It is obvious that the LBVP (4.43) satisfies a condition similar to that of Theorem 4.1. Also if $\underset{\sim}{u}$ is a solution of the LBVP I then $\hat{\underset{\sim}{u}}$ with $\hat{u}_{2i+1} = -u_i$, $\hat{u}_{2i} = u_i$, $i \in Z$, is a solution of the above LBVP (4.43).

The following theorem corresponds to Theorem 4.6 in the unrestricted case being discussed now.

Theorem 4.10.

Consider the LBVP I. If the condition (i) of Theorem 4.1 is not satisfied then construct the LBVP (4.43). Further assume

(i) $\beta_{ii} \leq M > 0;$

(ii) $\alpha_{ii}(t,\varepsilon) \leq -\delta_3 < 0;$

and

(iii) $\beta_{ii} + \sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} (-\beta_{ij}^{+} + \beta_{ij}^{-}) \geq \delta_4 > 0.$

Then also the conclusion of Theorem 4.6 is valid.

Proof

If the system (4.5) does not satisfy the condition (i) of Theorem 4.1 then construct the new LBVP (4.43). Following the same procedure given in the proof of Theorem 4.6 the estimate for the LBVP (4.43) can be obtained as

$$(4.44) \quad \| \hat{\underset{\sim}{u}} - \hat{\underset{\sim}{w}} \| \leq \varepsilon m_{10} + \| \hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{w}}(\infty,\varepsilon) \| e^{-\delta_3 (t-a)/\varepsilon},$$

where $\hat{\underset{\sim}{u}}$ and $\hat{\underset{\sim}{w}}$ are respectively the solutions of the

LBVP (4.43) and the IVP

$$(4.45)\begin{cases} \alpha_{ii}\hat{w}'_{2i+1}+\beta_{ii}\hat{w}_{2i+1}+\sum\limits_{\substack{j=-\infty\\ j\neq i}}^{\infty}(-\beta^{+}_{ij}\hat{w}_{2j}+\beta^{-}_{ij}\hat{w}_{2j+1}) = -\gamma_i, \\ \alpha_{ii}\hat{w}'_{2i}+\beta_{ii}\hat{w}_{2i} + \sum\limits_{\substack{j=-\infty\\ j\neq i}}^{\infty}(-\beta^{+}_{ij}\hat{w}_{2j+1}+\beta^{-}_{ij}\hat{w}_{2j}) = \gamma_i, \\ \hat{w}_{2i+1}(a,\varepsilon) = -A_i(\varepsilon),\ \hat{w}_{2i}(a,\varepsilon) = A_i(\varepsilon),\ i\in Z. \end{cases}$$

As pointed out above whenever $\underset{\sim}{u}$, $\underset{\sim}{w}$ are respectively solutions of the LBVP I and the IVP (4.38) then $\hat{\underset{\sim}{u}}$, $\hat{\underset{\sim}{w}}$ are respectively solutions of the LBVP (4.43) and the IVP (4.45). The conclusion of the present theorem now follows from the estimate (4.44) and the assumptions made in Theorem 4.6.

2. NONLINEAR SYSTEMS.

The previous discussion was confined to the asymptotic behaviour of solutions and/or their derivatives of the LBVPs I-III. Guided by the past experience, now the estimates are given for solutions of the nonlinear system (4.1) subject to one of the BCs (4.2)-(4.4) in terms of the IVP

$$(4.46)\begin{cases} \underset{\sim}{f}(t,\ [\underset{\sim}{x}]',\ \underset{\sim}{x},\ 0) = \underset{\sim}{0},\ t\in D, \\ \underset{\sim}{x}(a) = \underset{\sim}{A}(0). \end{cases}$$

Now Theorem 2.10 is reformulated as follows.

Theorem 4.11.

Let $\underset{\sim}{u}$ be a solution of the system (4.1) subject to one of the BCs (4.2)-(4.4). Then the implication

$$(4.47)\quad \left\{\begin{array}{l} \underset{\sim}{P}\,\underset{\sim}{z} \le \underset{\sim}{P}\,\underset{\sim}{u} \le \underset{\sim}{P}\,\underset{\sim}{y},\ \underset{\sim}{R}\,\underset{\sim}{z} \le \underset{\sim}{R}\,\underset{\sim}{u} \le \underset{\sim}{R}\,\underset{\sim}{y} \\ \qquad \Longrightarrow \\ \underset{\sim}{z}(t,\varepsilon) \le \underset{\sim}{u}(t,\varepsilon) \le \underset{\sim}{y}(t,\varepsilon) \quad \text{on } \bar{D} \end{array}\right.$$

is true for all $\underset{\sim}{z},\ \underset{\sim}{y} \in U$ provided the following assumptions are satisfied.

(i) there exist a positive number δ_2 and a 'test function' $s(t,\varepsilon)$: $\bar{D} \times I \rightarrow R+$ with the properties

$$(4.48)\quad \left\{\begin{array}{l} \underset{\sim}{P}\,(\underset{\sim}{y}+\alpha_1\underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{y} \ \ge\ \alpha_1\delta_2 > 0, \\ \underset{\sim}{P}\,\underset{\sim}{z} - \underset{\sim}{P}(\underset{\sim}{z}-\alpha_1\underset{\sim}{s}) \ \ge\ \alpha_1\delta_2 > 0,\ t\in D, \\ \underset{\sim}{R}\,\underset{\sim}{s} > 0, \end{array}\right.$$

for every positive number α_1;

(ii) there exists a positive constant M such that

$$\underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]',\underset{\sim}{u},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]',\underset{\sim}{y}+\alpha_1\underset{\sim}{s},\varepsilon)$$
$$\ge M\,(\underset{\sim}{u}-(\underset{\sim}{y}+\alpha_1\underset{\sim}{s}))$$

and

$$\underset{\sim}{f}(t,[\underset{\sim}{u}]', \underset{\sim}{z} - \alpha_1 \underset{\sim}{s}, \varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon)$$
$$\geq M(\underset{\sim}{z} - (\underset{\sim}{u} + \alpha_1 \underset{\sim}{s})),$$

whenever

$$\underset{\sim}{u} \leq \underset{\sim}{y} + \alpha_1 \underset{\sim}{s}, \quad \underset{\sim}{z} \leq \underset{\sim}{u} + \alpha_1 \underset{\sim}{s}$$

and for all positive number α_1 and η with $\alpha_1 - \eta > 0$;

(iii) there exists a positive constant L such that

$$\underset{\sim}{f}(t, [\underset{\sim}{y} + (\alpha_1 - \eta)\underset{\sim}{s}]', \underset{\sim}{y} + \alpha_1 \underset{\sim}{s}, \varepsilon)$$
$$- \underset{\sim}{f}(t,[\underset{\sim}{y} + (\alpha_1 - \eta)\underset{\sim}{s}]', \underset{\sim}{y} + (\alpha_1 - \eta)\underset{\sim}{s}, \varepsilon)$$
$$\geq - L\eta \underset{\sim}{s} \leq \underset{\sim}{f}(t,[\underset{\sim}{u}]', \underset{\sim}{u} + \eta \underset{\sim}{s}, \varepsilon)$$
$$- \underset{\sim}{f}(t, [\underset{\sim}{u}]', \underset{\sim}{u}, \varepsilon)$$

for every positive number α_1 and η with $\alpha_1 - \eta > 0$.

The proof of this theorem is same as that of Theorem 2.10.

Theorem 4.12.

Let $\underset{\sim}{u}$ be a solution of the system (4.1) subject to one of the BCs (4.2)-(4.4). Further let $\underset{\sim}{x}$ be the solution of the IVP (4.46) such that

$$\| \underset{\sim}{x} \| \leq m_{11} \text{ and } \| \underset{\sim}{f}(t,[\underset{\sim}{x}]', \underset{\sim}{x}, \varepsilon) \| \leq m_{12}\varepsilon .$$

Further assume that the following conditions (i) and (ii) and the conditions (ii) and (iii) of Theorem 4.11 are satisfied for the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ which are defined as

$$\underset{\sim}{y} = \underset{\sim}{x} + \underset{\sim}{w}, \quad \underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{w}, \quad \underset{\sim}{w} = (\ldots, w_1, w_2, \ldots \ ),$$

$$w_i(t,\varepsilon) = \varepsilon\, m_3(1-e^{-\delta_5(t-a)\varepsilon})$$
$$+ \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{x}(\infty) \| (1-e^{-\delta_5(t-a)\varepsilon}),$$

$\delta_5 > 0$, $m_3 > 0$, $i \in Z$.

There exist positive constants δ_4 and δ_5 (independent of m_3) such that

(i) $$\underset{\sim}{f}(t,[\underset{\sim}{y}+\alpha_1\underset{\sim}{s}]', \underset{\sim}{y}+\alpha_1\underset{\sim}{s},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{y}]',\underset{\sim}{y},\varepsilon)$$
$$\geq \alpha_1\delta_5\underset{\sim}{s}' + \alpha_1\delta_4\underset{\sim}{s}$$
$$\leq \underset{\sim}{f}(t,[\underset{\sim}{z}]',\underset{\sim}{z},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{z}-\alpha_1\underset{\sim}{s}]', \underset{\sim}{z}-\alpha_1\underset{\sim}{s},\varepsilon)$$

for all positive constant α_1 and $\underset{\sim}{s}$ with

$$s_i(t,\varepsilon) = 2 - e^{-k(t-a)}, \quad k > 0, \ i \in Z$$

and

(ii) $$\underset{\sim}{f}(t,[\underset{\sim}{x}+\underset{\sim}{w}]', \underset{\sim}{x}+\underset{\sim}{w},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{x}]',\underset{\sim}{x},\varepsilon)$$
$$\geq \delta_5\underset{\sim}{w}' + \delta_4\underset{\sim}{w}$$
$$\leq \underset{\sim}{f}(t,[\underset{\sim}{x}]',\underset{\sim}{x},\varepsilon) - \underset{\sim}{f}(t,[\underset{\sim}{x}-\underset{\sim}{w}]',\underset{\sim}{x}-\underset{\sim}{w},\varepsilon).$$

Then

$$(4.49) \qquad \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| \leq w(t,\varepsilon), \; t \in D,$$

that is,

$$(4.50) \qquad \lim_{\varepsilon \to 0+} \| \underset{\sim}{u}(t,\varepsilon) - \underset{\sim}{x}(t) \| = 0, \quad a \leq t < \infty.$$

Proof

First it will be shown that the function $\underset{\sim}{s}$ defined in the hypothesis (ii) and the functions $\underset{\sim}{y}$, $\underset{\sim}{z}$ satisfy the conditions of Theorem 4.11. Consequently the estimate (4.50) follows.

Now

$$P_i(\underset{\sim}{y}+\alpha_1 \underset{\sim}{s})-P_i \underset{\sim}{y} = -\varepsilon \alpha_1 s_i'' + [f_i(t,y_i'+\alpha_1 s_i',\underset{\sim}{y}+\alpha_1 \underset{\sim}{s},\varepsilon) - f_i(t, y_i', \underset{\sim}{y}, \varepsilon)]$$

$$> -\varepsilon \alpha_1 s_i'' + \alpha_1 \delta_5 s_i' + \alpha_1 \delta_4 s_i$$

$$\geq \alpha_1 \delta_4 \geq 0, \; i \in Z,$$

that is,

$$(4.51) \qquad \underset{\sim}{P}(\underset{\sim}{y}+\alpha_1 \underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{y} > 0.$$

Further

$$\begin{aligned}
P_i \underset{\sim}{y} &= P_i\,(\underset{\sim}{x}+\underset{\sim}{w}) - P_i\,\underset{\sim}{x} + P_i\,\underset{\sim}{x} \\
&= -\varepsilon w_i'' + f_i(t, x_i'+w_i', \underset{\sim}{x}+\underset{\sim}{w}, \varepsilon) \\
&\qquad\qquad -f_i(t, x_i', \underset{\sim}{x}, \varepsilon) + P_i\,\underset{\sim}{x} \\
&\geq -\varepsilon w_i'' + \delta_5\, w_i' + \delta_4\, w_i - \varepsilon\, m_{13} \\
&\geq -\varepsilon w_i'' + \delta_5\, w_i' + \delta_4\, w_i - \varepsilon\, m_{13}(1-e^{-\delta_5 (t-a)\varepsilon}) \\
&\geq 0 = P_i\,\underset{\sim}{u}, \quad i \in Z,
\end{aligned}$$

by a proper choice of m_{13}.

That is,

$$(4.52) \qquad \underset{\sim}{P}\,\underset{\sim}{u} \leq \underset{\sim}{P}\,\underset{\sim}{y}.$$

Also one can show that, by a proper choice of m_{13},

$$(4.53) \quad \begin{cases} \underset{\sim}{y}(a,\varepsilon) \geq \underset{\sim}{u}(a,\varepsilon), \\ \underset{\sim}{y}(\infty,\varepsilon) \geq \underset{\sim}{u}(\infty,\varepsilon). \end{cases}$$

Then inequalities (4.52)-(4.53) yield

$$\underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon).$$

Similarly

$$\underset{\sim}{z}(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon).$$

Hence the proof of the theorem.

The above discussion can be extended to

(i) The BVPs for DEs of the form (4.1) with mixed BCs,

(ii) The turning point BVPs [16];

(iii) The BVPs for strongly coupled systems [38]

and

(iv) The BVPs with more than one small parameter [39-41].

Chapter 5

SINGULARLY PERTURBED INITIAL VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS IN BANACH SPACE*

In this chapter, using the theory of first order ordinary differential inequalities in a Banach space, the asymptotic behaviour of solutions of the following IVP as the small parameter $\varepsilon \to 0$ is established:

$$(5.1)\quad \begin{cases} \underset{\sim}{P}'(\underset{\sim}{x},\underset{\sim}{y}) := \underset{\sim}{x}' - \underset{\sim}{u}(t,\underset{\sim}{x},\underset{\sim}{y},\varepsilon) = \underset{\sim}{0}, \quad t \in D := (a,b], \\ \underset{\sim}{Q}'(\underset{\sim}{x},\underset{\sim}{y}) := \varepsilon \underset{\sim}{y}' - \underset{\sim}{v}(t,\underset{\sim}{x},\underset{\sim}{y},\varepsilon) = \underset{\sim}{0}, \\ \underset{\sim}{R}'(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} \underset{\sim}{x}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon) + O(\varepsilon) \\ \underset{\sim}{y}(a,\varepsilon) = \underset{\sim}{B}(\varepsilon) + O(\varepsilon), \quad \underset{\sim}{x},\underset{\sim}{y} \in \mathfrak{U} = C^1(D) \cap C(\bar{D}). \end{cases} \end{cases}$$

where

$$\underset{\sim}{x} = (\ldots, x_{-1}, x_0, x_1, \ldots)^T, \quad \underset{\sim}{y} = (\ldots, y_{-1}, y_0, y_1, \ldots)^T,$$

$$\underset{\sim}{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)^T, \quad \underset{\sim}{v} = (\ldots, v_{-1}, v_0, v_1, \ldots)^T,$$

$$\underset{\sim}{x}' = (\ldots, x'_{-1}, x'_0, x'_1, \ldots)^T, \quad \underset{\sim}{y}' = (\ldots, y'\ , y'_0, y'_1, \ldots)^T,$$

$$\underset{\sim}{A} = (\ldots, A_{-1}, A_0, A_1, \ldots)^T, \quad \underset{\sim}{B} = (\ldots, B_{-1}, B_0, B_1, \ldots)^T,$$

Z is the set of all integers and $\underset{\sim}{A}(\varepsilon) = \underset{\sim}{A}(0) + O(\varepsilon)$ means $A_i(\varepsilon) = A_i(0) + O(\varepsilon)$, $i \in Z$. The consideration of the above IVP (5.1) is motivated by problems arising when MOL [3, 32] is applied to SPPs for parabolic DEs with a small parameter ε

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* The material of this chapter is to appear in the Indian Journal of Pure and Applied Mathematics.

multiplying the time derivative. A monotonicity theorem in a Banach space which shall be used in the rest of the chapter is given below. Sections 1-2 deal with IVPs for second order linear systems defined in a Banach space and obtain explicit bounds for solutions and their derivatives. In turn, these estimates are used to discuss the asymptotic behaviour of solutions and their derivatives. The results of Section 1 are generalized to cover IVPs for nonlinear systems in Section 3.

The following theorem is only a reformulation of Theorem 2.6 of Chapter 2 of this thesis.

Theorem 5.1.

Let $(\mathbf{x}, \mathbf{y})$ be a solution of the IVP (5.1). Then for every $\underline{\mathbf{x}}, \bar{\mathbf{x}}, \underline{\mathbf{y}}, \bar{\mathbf{y}} \in U := C^1(D) \cap C(\bar{D})$ the implication

$$\left\{\begin{array}{l} \mathbf{P}'(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \le \mathbf{P}'(\mathbf{x}, \mathbf{y}) \le \mathbf{P}'(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \\ \mathbf{Q}'(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \le \mathbf{Q}'(\mathbf{x}, \mathbf{y}) \le \mathbf{Q}'(\bar{\mathbf{x}}, \bar{\mathbf{y}}), \\ \mathbf{R}'(\underline{\mathbf{x}}, \underline{\mathbf{y}}) \le \mathbf{R}'(\mathbf{x}, \mathbf{y}) \le \mathbf{R}'(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \end{array}\right.$$

$$\Longrightarrow$$

$$\underline{\mathbf{x}} \le \mathbf{x} \le \bar{\mathbf{x}}, \quad \underline{\mathbf{y}} \le \mathbf{y} \le \bar{\mathbf{y}}$$

is true provided

(i) $\quad \mathbf{u}(t, \mathbf{p}, \mathbf{q}, \varepsilon) : D \times B^2 \times (0, \varepsilon_0] \longrightarrow B$

is quasimonotone increasing in $\mathbf{p}$ and monotone increasing in $\mathbf{q}$;

$\underset{\sim}{v}(t,\underset{\sim}{p},\underset{\sim}{q},\varepsilon) : D \times B^2 \times (0, \varepsilon_0] \longrightarrow B$ is monotone increasing in $\underset{\sim}{p}$ and quasimonotone increasing in $\underset{\sim}{q}$;

(ii) there exist a positive number δ_1 and a 'test function' $\underset{\sim}{s}(t,\varepsilon): \bar{D} \times (0, \varepsilon_0] \longrightarrow B$ such that

$$s_i(t,\varepsilon) = s_j(t,\varepsilon) > 0, \; i,j \in Z, \; i \neq j, \; \underset{\sim}{s} \in U$$

and

$$(5.2) \quad \begin{cases} \underset{\sim}{P}'(\bar{\underset{\sim}{x}}+\alpha_1\underset{\sim}{s}, \bar{\underset{\sim}{y}}+\alpha_1\underset{\sim}{s}) - \underset{\sim}{P}'(\bar{\underset{\sim}{x}},\bar{\underset{\sim}{y}}) \geq \alpha_1\delta_1 > 0, \\ \underset{\sim}{P}'(\underline{\underset{\sim}{x}}, \underline{\underset{\sim}{y}}) - \underset{\sim}{P}'(\underline{\underset{\sim}{x}} - \alpha_1\underset{\sim}{s}, \underline{\underset{\sim}{y}} - \alpha_1\underset{\sim}{s}) \geq \alpha_1\delta_1 > 0, \\ \underset{\sim}{Q}'(\bar{\underset{\sim}{x}}+\alpha_1\underset{\sim}{s}, \bar{\underset{\sim}{y}}+\alpha_1\underset{\sim}{s}) - \underset{\sim}{Q}'(\bar{\underset{\sim}{x}},\bar{\underset{\sim}{y}}) \geq \alpha_1\delta_1 > 0, \\ \underset{\sim}{Q}'(\underline{\underset{\sim}{x}}, \underline{\underset{\sim}{y}}) - \underset{\sim}{Q}'(\underline{\underset{\sim}{x}}-\alpha_1\underset{\sim}{s}, \underline{\underset{\sim}{y}}-\alpha_1\underset{\sim}{s}) \geq \alpha_1\delta_1 > 0 \end{cases}$$

for every positive real α_1 and for every but fixed ε ;

(iii) there exists a positive constant L such that

$$(5.3) \quad \begin{cases} \underset{\sim}{u}(t,\bar{\underset{\sim}{x}}+\beta\underset{\sim}{s},\bar{\underset{\sim}{y}}+\beta\underset{\sim}{s},\varepsilon) - \underset{\sim}{u}(t,\bar{\underset{\sim}{x}}+\eta\underset{\sim}{s},\bar{\underset{\sim}{y}}+\eta\underset{\sim}{s},\varepsilon) \leq L\underset{\sim}{s}(t)(\beta-\eta), \\ \underset{\sim}{u}(t,\underline{\underset{\sim}{x}}-\eta\underset{\sim}{s},\underline{\underset{\sim}{y}}-\eta\underset{\sim}{s},\varepsilon) - \underset{\sim}{u}(t,\underline{\underset{\sim}{x}}-\beta\underset{\sim}{s},\underline{\underset{\sim}{y}}-\beta\underset{\sim}{s},\varepsilon) \leq L\underset{\sim}{s}(t)(\beta-\eta), \\ \underset{\sim}{v}(t,\bar{\underset{\sim}{x}}+\beta\underset{\sim}{s},\bar{\underset{\sim}{y}}+\beta\underset{\sim}{s},\varepsilon) - \underset{\sim}{v}(t,\bar{\underset{\sim}{x}}+\eta\underset{\sim}{s},\bar{\underset{\sim}{y}}+\eta\underset{\sim}{s},\varepsilon) \leq L\underset{\sim}{s}(t)(\beta-\eta), \\ \underset{\sim}{v}(t,\underline{\underset{\sim}{x}}-\eta\underset{\sim}{s},\underline{\underset{\sim}{y}}-\eta\underset{\sim}{s},\varepsilon) - \underset{\sim}{v}(t,\underline{\underset{\sim}{x}}-\beta\underset{\sim}{s},\underline{\underset{\sim}{y}}-\beta\underset{\sim}{s},\varepsilon) \leq L\underset{\sim}{s}(t)(\beta-\eta) \end{cases}$$

for every η, β such that $0 < \eta \leq \beta$.

Proof

Under the transformation

$z_{2i} = x_i, \quad z_{2i+1} = y_i, \; i \in Z,$

the system (5.1) reduces to a single doubly infinite system. Hence this theorem does not require a separate proof.

To illustrate the above theorem, consider a doubly infinite system of linear second order ODEs

$$(5.4) \qquad \varepsilon \underset{\sim}{x}'' + \underset{\sim}{\alpha}\,\underset{\sim}{x}' + \underset{\sim}{\beta}\,\underset{\sim}{x} = \underset{\sim}{\gamma}, \quad t \in D$$

subject to the ICs

$$(5.5) \qquad \underset{\sim}{x}\,(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \quad \underset{\sim}{x}'(a,\varepsilon) = \underset{\sim}{B}(\varepsilon),$$

where $\underset{\sim}{\alpha} = (\alpha_{ij}), \quad \alpha_{ij} = 0, \quad i \neq j, \quad i,j \in Z,$

$\underset{\sim}{\beta} = (\beta_{ij}), \quad \underset{\sim}{\alpha}\,\underset{\sim}{x}' = (\ldots, \alpha_{11}x_1', \alpha_{22}x_2', \ldots)^T,$

$\underset{\sim}{\beta}\,\underset{\sim}{x} = (\ldots, \sum_{j=-\infty}^{\infty} \beta_{ij}\, x_j, \sum_{j=-\infty}^{\infty} \beta_{2j}\, x_j, \ldots)^T,$

$\underset{\sim}{\gamma} = (\ldots, \gamma_{-1}, \gamma_o, \gamma_1, \ldots)^T,$

and $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$, $\underset{\sim}{\gamma}$ are assumed to be continuous functions in their arguments.

Let $\underset{\sim}{y} = \underset{\sim}{x}'$. Then the IVP (5.4)-(5.5) may be written as

$$(5.6)\quad \begin{cases} P_L\ (\underset{\sim}{x},\ \underset{\sim}{y}) : & = \underset{\sim}{x}'-\underset{\sim}{y} = \underset{\sim}{0} \\ Q_L\ (\underset{\sim}{x},\ \underset{\sim}{y}) : & = \varepsilon \underset{\sim}{y}'+\underset{\sim}{\alpha}\ \underset{\sim}{y} + \underset{\sim}{\beta}\ \underset{\sim}{x} = \underset{\sim}{0},\ t \in D, \\ R_L\ (\underset{\sim}{x},\ \underset{\sim}{y}) : & = \begin{cases} \underset{\sim}{x}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon), \\ \underset{\sim}{y}(a,\varepsilon) = \underset{\sim}{B}(\varepsilon). \end{cases} \end{cases}$$

Theorem 5.2.

The conditions (i)-(iii) of Theorem 5.1 are satisfied for the linear IVP (5.6) provided

(i) $\underset{\sim}{\beta} \leq \underset{\sim}{0}$;

(ii) $\alpha_{ii} \geq \delta_3 > 0, \quad i \in Z$

and

(iii) $\sum_{j=-\infty}^{\infty} \beta_{ij} \geq -L,\ L > 0,\ i \in Z.$

Proof

The assumptions (i) and (ii) yield the condition (i) of Theorem 5.1 whereas the condition (iii) gives the inequalities (5.3). Finally, using the assumptions (ii) and (iii), one can show that the function $\underset{\sim}{s}$, $s_i(t,\varepsilon) = e^{kt/\varepsilon}$, is a required 'test function', by a proper choice of k, for the system (5.6) satisfying the inequalities (5.2).

1. ESTIMATES AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LINEAR SYSTEMS- I.

Necessary estimates are obtained in this section in order to study the asymptotic behaviour of solutions and their derivatives of the IVP (5.4)-(5.5). Since the IVP (5.4)-(5.5) and the IVP (5.6) are equivalent in the sense that every solution of one system is a solution of the other and vice versa, only the problem (5.6) is considered in the following study.

In the rest of this section it is assumed that all the conditions (i)-(iii) of Theorem 5.2 hold true for the IVP (5.6). Consequently the implication of Theorem 5.1 is valid for the IVP (5.6). Further, in the following, m_i, $i = 1,2, \ldots$ stand for real positive constants independent of ε.

Theorem 5.3.

Consider the IVP (5.4)-(5.5) and assume that $\underset{\sim}{A}(\varepsilon) \equiv \underset{\sim}{O} \equiv \underset{\sim}{B}(\varepsilon)$. Then

$$(5.7) \qquad \| \underset{\sim}{x}(t,\varepsilon) \| \leq \| \underset{\sim}{\gamma} \| e^{m(t-a)},$$

$$(5.8) \qquad \| \underset{\sim}{x}'(t,\varepsilon) \| \leq m \| \underset{\sim}{\gamma} \| e^{m(t-a)},$$

for some positive constant m and for every solution $\underset{\sim}{x}$ of (5.4)-(5.5).

Proof

Consider the IVP (5.6) and let $(\underset{\sim}{x}, \underset{\sim}{y})$ be its solution. Define functions $(\underset{\sim}{\underline{x}}, \underset{\sim}{\underline{y}})$ and $(\underset{\sim}{\overline{x}}, \underset{\sim}{\overline{y}})$ as

$$\overline{x}_i = \| \underset{\sim}{\gamma} \| \cdot e^{m(t-a)}, \quad \underset{\sim}{\underline{x}} = - \underset{\sim}{\overline{x}},$$

$$\overline{y}_i = m \| \underset{\sim}{\gamma} \| e^{m(t-a)}, \quad \underset{\sim}{\underline{y}} = - \underset{\sim}{\overline{y}},$$

where m is a positive constant, independent of ε , to be chosen suitably.

Then

$$P_{\ell i}(\underset{\sim}{\overline{x}}, \underset{\sim}{\overline{y}}) : = \overline{x}_i' - \overline{y}_i = 0 = P_{\ell i}(\underset{\sim}{\overline{x}}, \underset{\sim}{\overline{y}}),$$

$$\begin{aligned} Q_{\ell i}(\underset{\sim}{x}, \underset{\sim}{y}) : &= \varepsilon \overline{y}_i' + \alpha_{ii}\overline{y}_i + \sum_{j=-\infty}^{\infty} \beta_{ij} \overline{x}_j \\ &= (\varepsilon m^2 + \alpha_{ii} m + \sum_{j=-\infty}^{\infty} \beta_{ij}) \| \underset{\sim}{\gamma} \| e^{m(t-a)} \\ &\geq (\varepsilon m^2 + \delta_3 m - L) \| \underset{\sim}{\gamma} \| e^{m(t-a)} \\ &\geq \| \underset{\sim}{\gamma} \| \geq \gamma_i = Q_{\ell i}(\underset{\sim}{x}, \underset{\sim}{y}), \end{aligned}$$

by a proper choice of m.

Also

$$\underset{\sim}{\overline{x}}(a, \varepsilon) \geq \underset{\sim}{0} = \underset{\sim}{x}(a, \varepsilon), \quad \underset{\sim}{\overline{y}}(a, \varepsilon) \geq \underset{\sim}{0} = \underset{\sim}{y}(a, \varepsilon).$$

Hence by Theorem 5.1 the following inequalities are obtained.

(5.9) $\quad \underset{\sim}{\underline{x}} \leq \underset{\sim}{\bar{x}}, \quad \underset{\sim}{\underline{y}} \leq \underset{\sim}{\bar{y}}.$

Similar procedure yields

(5.10) $\quad \underset{\approx}{\underline{x}} \leq \underset{\sim}{x}, \quad \underset{\approx}{\underline{y}} \leq \underset{\sim}{y}.$

Hence

(5.11) $\quad | \, x_i(t,\varepsilon) \, | \leq \| \underset{\sim}{\gamma} \| \, e^{m(t-a)}$

and

(5.12) $\quad | \, y_i(t,\varepsilon) \, | \leq m \, \| \underset{\sim}{\gamma} \| \, e^{m(t-a)}.$

Taking supremum in (5.11)-(5.12) and using the fact that $\underset{\sim}{y} = \underset{\sim}{x}'$, the required estimates (5.7)-(5.8) are obtained.

Theorem 5.4.

Consider the IVP (5.4)-(5.5) and assume that $\underset{\sim}{\gamma} \equiv \underset{\sim}{0} \equiv \underset{\sim}{A}$. Then

(5.13) $\quad \| \underset{\sim}{x}(t,\varepsilon) \| \leq \varepsilon \, \| \underset{\sim}{B}(\varepsilon) \| \, m_1 \, [e^{m(t-a)} - e^{-\delta_3 (t-a)/\varepsilon}],$

(5.14) $\quad \| \underset{\sim}{x}'(t,\varepsilon) \| \leq \varepsilon \, \| \underset{\sim}{B}(\varepsilon) \| \, m_2 \, e^{m(t-a)} + \| \underset{\sim}{B}(\varepsilon) \| \, e^{-\delta_3 (t-a)/\varepsilon}$

where $m > 0$, $m\delta_3 - L \geq 0$ and $\underset{\sim}{x}$ is a solution of the IVP(5.4)-(5.5).

Proof

Consider the IVP (5.6) and let $(\tilde{x}, \tilde{y})$ be its solution. Define functions $(\underline{\tilde{x}}, \underline{\tilde{y}})$ and $(\bar{\tilde{x}}, \bar{\tilde{y}})$ as follows:

$$\bar{x}_i = \varepsilon \, \|\tilde{B}(\varepsilon)\| \, [e^{m(t-a)} - e^{-\delta_3(t-a)/\varepsilon}] \, / \, (m\varepsilon+\delta_3),$$

$$\bar{y}_i = \|\tilde{B}(\varepsilon)\| \, [m\,\varepsilon\, e^{m(t-a)} + \delta_3 e^{-\delta_3(t-a)/\varepsilon}](m\varepsilon+\delta_3), i \in Z$$

$$\underline{\tilde{x}} = -\bar{\tilde{x}}, \quad \underline{\tilde{y}} = -\bar{\tilde{y}}.$$

Now

$$P_{\ell i}(\bar{\tilde{x}}, \bar{\tilde{y}}) := \bar{x}_i' - \bar{y}_i = 0 = P_{\ell i}(\tilde{x}, \tilde{y}),$$

$$Q_{\ell i}(\bar{\tilde{x}}, \bar{\tilde{y}}) := \varepsilon \bar{y}_i' + \alpha_{ii} \bar{y}_i + \sum_{j=-\infty}^{\infty} \beta_{ij} \bar{x}_j$$

$$\geq \varepsilon \, \| \tilde{B}(\varepsilon) \| \, e^{m(t-a)} [m\varepsilon^2 + m\delta_3 - L] / (m\varepsilon+\delta_3)$$

$$+ \| \tilde{B}(\varepsilon) \| e^{-\delta_3(t-a)/\varepsilon} [-\delta_3^2 + \delta_3^2 + \varepsilon L]/(m\varepsilon+\delta_3)$$

$$\geq 0 = Q_{\ell i}(\tilde{x}, \tilde{y}), \quad i \in Z.$$

Also

$$\tilde{x}(a,\varepsilon) = \tilde{0} \leq \bar{\tilde{x}}(a,\varepsilon)$$

$$\tilde{y}(a,\varepsilon) = \tilde{B}(\varepsilon) \leq \| \tilde{B}(\varepsilon) \| = \bar{\tilde{y}}(a,\varepsilon)$$

Hence by Theorem 5.1

(5.15) $\quad \tilde{x} \leq \bar{\tilde{x}}, \quad \tilde{y} \leq \bar{\tilde{y}}.$

Similar analysis yields

$$(5.16) \quad \underset{\sim}{\underline{x}} \le \underset{\sim}{x}, \quad \underset{\sim}{\underline{y}} \le \underset{\sim}{y}.$$

Hence

$$(5.17) \quad |\, x_i(t,\varepsilon) \,| \le \varepsilon \,\|\, \underset{\sim}{B}(\varepsilon) \,\|\, m_1 [e^{m(t-a)} - e^{-\delta_3(t-a)/\varepsilon}]$$

and

$$(5.18) \quad |\, y_i(t,\varepsilon) \,| \le \varepsilon \,\|\, \underset{\sim}{B}(\varepsilon) \,\|\, m_2 \; e^{m(t-a)} + \|\underset{\sim}{B}(\varepsilon)\| e^{-\delta_3(t-a)/\varepsilon}.$$

The estimates (5.13) - (5.14) follow from (5.17)-(5.18).

Theorem 5.5.

Let $\underset{\sim}{w}$ be a solution of the IVP

$$(5.19) \quad \underset{\sim}{\alpha}\, \underset{\sim}{w}' + \underset{\sim}{\beta}\, \underset{\sim}{w} = \underset{\sim}{\gamma}, \quad \underset{\sim}{w}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon),$$

with the properties

$$(5.20) \quad \|\, \underset{\sim}{w}'' \,\| \le m_4$$

and

$$(5.21) \quad \lim_{\varepsilon \to 0} \underset{\sim}{w}(t,\varepsilon) = \underset{\sim}{u}(t), \; t \in \bar{D},$$

where $\underset{\sim}{u}$ is the solution of the IVP

$$(5.22) \quad \underset{\sim}{\alpha}(t,0)\underset{\sim}{u}' + \underset{\sim}{\beta}(t,0)\underset{\sim}{u} = \underset{\sim}{\gamma}, \quad \underset{\sim}{u}(a) = \underset{\sim}{A}(0).$$

If $\underset{\sim}{x}$ is a solution of the IVP (5.4)-(5.5) then

$$(5.23)\qquad \|\underset{\sim}{x}(t,\varepsilon) - \underset{\sim}{u}(t)\| \le \varepsilon\, m_3\, e^{m(t-a)} + C(\varepsilon)$$
$$+\varepsilon\, m_4 \,\|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{u}'(a)\|\,[e^{m(t-a)} - e^{-\delta_3(t-a)/\varepsilon}],$$

$$(5.24)\qquad \|\underset{\sim}{x}'(t,\varepsilon) - \underset{\sim}{u}'(t)\| \le \varepsilon\, m_5\, e^{m(t-a)} + D(\varepsilon)$$
$$+\ \varepsilon\, m_6 \|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{u}'(a)\|\, e^{m(t-a)}$$
$$+\ \|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{u}'(a)\|\, e^{-\delta_3(m-a)/\varepsilon},\quad t \in D,$$

where $C(\varepsilon)$, $D(\varepsilon)$ approach zero as $\varepsilon \to 0+$.

Consequently

$$(5.25)\qquad \lim_{\varepsilon \to 0+} \underset{\sim}{x}(t,\varepsilon) = \underset{\sim}{u}(t),\quad t \in \bar{D}$$

and

$$(5.26)\qquad \lim_{\varepsilon \to 0+} \underset{\sim}{x}'(t,\varepsilon) = \underset{\sim}{u}'(t),\quad t \in D.$$

Proof

Let $\underset{\sim}{v}$ be a solution of the IVP

$$\varepsilon\, \underset{\sim}{v}'' + \underset{\sim}{\alpha}\, \underset{\sim}{v}' + \underset{\sim}{\beta}\, \underset{\sim}{v} = \underset{\sim}{\gamma} + \varepsilon\, \underset{\sim}{w}'',\quad t \in D,$$
$$\underset{\sim}{v}(a,\varepsilon) = \underset{\sim}{A}(\varepsilon),\quad \underset{\sim}{v}'(a,\varepsilon) = \underset{\sim}{B}(\varepsilon).$$

Using Theorem 5.3 the following inequalities are obtained.

$$(5.27) \qquad \| \underset{\sim}{x}(t,\varepsilon) - \underset{\sim}{v}(t,\varepsilon) \| \leq \varepsilon\, m_4\, e^{m(t-a)},$$

$$(5.28) \qquad \| \underset{\sim}{x}'(t,\varepsilon) - \underset{\sim}{v}'(t,\varepsilon) \| \leq \varepsilon\, m_5\, e^{m(t-a)}.$$

Also from Theorem 5.4 it follows that

$$(5.29) \qquad \| \underset{\sim}{v}(t,\varepsilon) - \underset{\sim}{w}(t,\varepsilon) \| \leq \varepsilon\, m_6 \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| \times [e^{m(t-a)} - e^{-\delta_3(t-a)/\varepsilon}],$$

$$(5.30) \qquad \| \underset{\sim}{v}'(t,\varepsilon) - \underset{\sim}{w}'(t,\varepsilon) \| \leq \varepsilon\, m_7 \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| e^{m(t-a)} + \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{w}'(a,\varepsilon) \| e^{-\delta_3(t-a)/\varepsilon}.$$

Finally

$$(5.31) \qquad \| \underset{\sim}{x}(t,\varepsilon) - \underset{\sim}{u}(t) \| \leq \| \underset{\sim}{x}(t,\varepsilon) - \underset{\sim}{v}(t,\varepsilon) \| + \| \underset{\sim}{v}(t,\varepsilon) - \underset{\sim}{w}(t,\varepsilon) \| + \| \underset{\sim}{w}(t,\varepsilon) - \underset{\sim}{u}(t) \|$$

and

$$(5.32) \qquad \| \underset{\sim}{x}'(t,\varepsilon) - \underset{\sim}{u}'(t) \| \leq \| \underset{\sim}{x}'(t,\varepsilon) - \underset{\sim}{v}'(t,\varepsilon) \| + \| \underset{\sim}{v}'(t,\varepsilon) - \underset{\sim}{w}'(t,\varepsilon) \| + \| \underset{\sim}{w}'(t,\varepsilon) - \underset{\sim}{u}'(t) \|.$$

The results (5.23)-(5.24) follow immediately from (5.27)-(5.32) and (5.21).

2. ESTIMATES AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF LINEAR SYSTEMS- II.

In the previous section various estimates are obtained which determine the asymptotic behaviour of solutions of the linear system (5.6). These estimates are derived under the assumption that the conditions (i) - (iii) of Theorem 5.2 hold true. Condition (i) is imposed invariably for all coupled systems of DEs. Infact it is the quasi-monotonicity condition for finite systems of DEs. Then if the condition (i) of Theorem 5.2 is not met by the system (5.6) one can still obtain the same limiting behaviour by adjoining a new system of DEs to (5.6) as follows.

Define

$$\beta_{ij}^{+} := \begin{cases} \beta_{ij} & \text{if } \beta_{ij} \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\beta_{ij}^{-} := \beta_{ij} - \beta_{ij}^{+}.$$

Set

$$\hat{x}_{2i} = \bar{x}_i, \quad \hat{x}_{2i+1} = -\underline{x}_i,$$

$$\hat{y}_{2i} = \bar{y}_i, \quad \hat{y}_{2i+1} = -\underline{y}_i.$$

Then, following the method described in Section 3 of Chapter 2 of this thesis one can adjoint the following IVP to the IVP (5.6):

$$(5.33)\begin{cases}\hat{P}_{\ell}(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}}) := \hat{\underset{\sim}{x}}' - \hat{\underset{\sim}{y}} = \underset{\sim}{0}, \\ \hat{Q}_{\ell 2i+1}(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}}) := \varepsilon\,\hat{y}'_{2i+1} + \alpha_{ii}\hat{y}_{2i+1} - \sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} \beta^{+}_{ij}\,\hat{x}_{2j} \\ \qquad\qquad + \sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} \beta^{-}_{ij}\,\hat{x}_{2j+1} = -\gamma_i, \\ \hat{Q}_{\ell 2i}(\hat{\underset{\sim}{x}},\hat{\underset{\sim}{y}}) := \varepsilon\,\hat{y}'_{2i} + \alpha_{ii}\hat{y}_{2i} - \sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} \beta^{+}_{ij}\,\hat{x}_{2j+1} \\ \qquad\qquad + \sum_{\substack{j=-\infty \\ j\neq i}}^{\infty} \beta^{-}_{ij}\,\hat{x}_{2j} = \gamma_i, \\ \hat{x}_{2i+1}(a,\varepsilon) = -A_i(\varepsilon),\ \hat{x}_{2i}(a,\varepsilon) = A_i(\varepsilon), \\ \hat{y}_{2i+1}(a,\varepsilon) = -B_i(\varepsilon),\ \hat{y}_{2i}(a,\varepsilon) = B_i(\varepsilon),\ i\in Z. \end{cases}$$

It can be easily seen that the above system (5.33) satisfy a condition similar to that of (i) of Theorem 5.2. Also if $(\underset{\sim}{x}, \underset{\sim}{y})$ is a solution of the system (5.6) then $(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{y}})$ defined below is a solution of the new system (5.33):

$$\hat{x}_{2i+1} = -x_i,\quad \hat{x}_{2i} = x_i,\quad \hat{y}_{2i+1} = -y_i,\quad \hat{y}_{2i} = y_i,\quad i\in Z.$$

The following theorem corresponds to Theorem 5.5 in the unrestricted case discussed above. It is to be noted that Theorem 5.5 was proved under the validity of the assumptions of Theorem 5.2.

Theorem 5.6.

Consider the IVP (5.4)-(5.5). If the condition (i) of Theorem 5.2 is not satisfied then construct the IVP (5.33). Further assume

(i) $\alpha_{ii} \geq \delta_3 > 0, \quad i \in Z,$

(ii) $\beta_{ii} + \sum_{\substack{j=-\infty \\ j \neq i}}^{\infty} (-\beta_{ij}^{+} + \beta_{ij}^{-}) \geq -L, \; L > 0, \; i \in Z.$

Then also the conclusions (5.25) - (5.26) for the IVP (5.4)-(5.5) are valid.

Proof

Consider the IVP (5.6). If this IVP does not satisfy the condition (i) of Theorem 5.2 then adjoint the IVP (5.33) to it. Following the procedure adopted in the proof of Theorem 5.5 the following estimates for the IVP (5.33) can be obtained.

$$| \hat{x}_i - \hat{u}_i | \leq \varepsilon m_7 e^{m(t-a)} + \varepsilon m_8 \| \hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{u}}'(a) \| \times [e^{m(t-a)} - e^{-\delta_3 (t-a)/\varepsilon}] + D(\varepsilon),$$

$$| \hat{y}_{2i} - \hat{u}_{2i} | \leq \varepsilon m_9 e^{m(t-a)} + \varepsilon m_{10} \| \hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{u}}'(a) \| e^{m(t-a)} + \| \hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{u}}'(a) \| e^{-\delta_3 (t-a)/\varepsilon} + E(\varepsilon),$$

$$| \hat{y}_{2i+1} - \hat{u}_{2i+1} | \leq \varepsilon m_9 e^{m(t-a)} + \varepsilon m_{10} \|\hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{u}}'(a)\| e^{m(t-a)}$$
$$+ \|\hat{\underset{\sim}{B}}(\varepsilon) - \hat{\underset{\sim}{u}}'(a)\| e^{-\delta_3(t-a)/\varepsilon} + F(\varepsilon), \; i \in Z,$$

where $D(\varepsilon)$, $E(\varepsilon)$ and $F(\varepsilon)$ tend to zero as $\varepsilon \to 0_+$ and $\hat{\underset{\sim}{u}}$ is the solution of the IVP

$$(5.34) \begin{cases} \alpha_{ii}(t,0)\, \hat{u}_{2i+1} - \sum\limits_{\substack{j=-\infty \\ j \neq i}}^{\infty} \beta^+_{ij}(t,0)\hat{u}_{2j} + \sum\limits_{\substack{j=-\infty \\ j \neq i}}^{\infty} \beta^-_{ij}(t,0)\hat{u}_{2j+1} = -\gamma \\ \alpha_{ii}(t,0)\, \hat{u}_{2i} - \sum\limits_{\substack{j=-\infty \\ j \neq i}}^{\infty} \beta^+_{ij}(t,0)\hat{u}_{2j+1} + \sum\limits_{\substack{j=-\infty \\ j \neq i}}^{\infty} \beta^-_{ij}(t,0)\hat{u}_{2j} = \gamma_i, \\ \hat{\underset{\sim}{u}}(a) = \hat{\underset{\sim}{A}}(0), \; i \in Z. \end{cases}$$

It can be verified that if $(\underset{\sim}{x}, \underset{\sim}{y})$ and $\underset{\sim}{u}$ are respectively solutions of the IVPs (5.6) and (5.22) then $(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{y}})$ and $\hat{\underset{\sim}{u}}$ defined by

$$\hat{x}_{2i} = x_i, \quad \hat{x}_{2i+1} = -x_i, \quad \hat{y}_{2i} = y_i, \quad \hat{y}_{2i+1} = -y_i,$$
$$\hat{u}_{2i} = u_i, \quad \hat{u}_{2i+1} = -u_i, \quad i \in Z,$$

are respectively solutions of the IVPs (5.33) and (5.34).
These observations complete the proof of the theorem.

3. ESTIMATES AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF NONLINEAR SYSTEMS.

In Section 2 different estimates are obtained for solutions and their derivatives for the linear system (5.4)-(5.5). Guided by the experience with the linear systems, the nonlinear system (5.1) is considered now and estimates for solutions of the same are derived. The estimates thus derived may contain boundary layer terms which explicitly describe the nature of the nonuniform behaviour of solutions of the nonlinear system 5.1 in t and ε.

The reduced problem for the IVP is given by

$$(5.35)\quad \begin{cases} \underset{\sim}{p}' - \underset{\sim}{u}(t, \underset{\sim}{p}, \underset{\sim}{q}, 0) = \underset{\sim}{0}, & t \in D, \\ \underset{\sim}{v}\,(t, \underset{\sim}{p}, \underset{\sim}{q}, 0) = \underset{\sim}{0}, \\ \underset{\sim}{p}\,(a) = \underset{\sim}{A}(0). \end{cases}$$

Further assume

(i) the functions $\underset{\sim}{u}$, $\underset{\sim}{v}$ of (5.1) satisfy the condition (i) of Theorem 5.1;

(ii) there exists a function $\underset{\sim}{Y}$ such that

$$(5.36)\quad \underset{\sim}{v}\,(t, \underset{\sim}{p}, \underset{\sim}{Y}, 0) = \underset{\sim}{0}$$

and the resulting IVP

$$(5.37)\quad \begin{cases} \underset{\sim}{p}' - \underset{\sim}{u}(t, \underset{\sim}{p}, \underset{\sim}{Y}, 0) = \underset{\sim}{0}, \\ \underset{\sim}{p}\,(a) = \underset{\sim}{A}(0) \end{cases}$$

has a unique solution $\underset{\sim}{Y}$ such that

$$(5.38) \qquad \| \underset{\sim}{Y}' \| \leq m_9,$$

$$(5.39) \quad \begin{cases} \underset{\sim}{v}(t, \underset{\sim}{X}+\underset{\sim}{\alpha}, \underset{\sim}{Y}+\underset{\sim}{\beta},\varepsilon) - \underset{\sim}{v}(t,\underset{\sim}{X}+\underset{\sim}{\alpha}, \underset{\sim}{Y},\varepsilon) \leq -K\underset{\sim}{\beta}, \\ \underset{\sim}{v}(t, \underset{\sim}{X}-\underset{\sim}{\alpha}, \underset{\sim}{Y}-\underset{\sim}{\beta},\varepsilon) - \underset{\sim}{v}(t,\underset{\sim}{X}-\underset{\sim}{\alpha}, \underset{\sim}{Y},\varepsilon) \geq K\underset{\sim}{\beta}, \end{cases}$$

where $K > 0$, for every non-negative functions $\underset{\sim}{\alpha}$, $\underset{\sim}{\beta}$;

(iii) for some constant $\ell > 0$ and for every non-negative functions $\underset{\sim}{\alpha}$, $\underset{\sim}{\alpha}^*$, $\underset{\sim}{\beta}$, $\underset{\sim}{\beta}^*$, $\underset{\sim}{\alpha} \leq \underset{\sim}{\alpha}^*$, $\underset{\sim}{\beta} \leq \underset{\sim}{\beta}^*$,

$$(5.40) \quad \begin{cases} \underset{\sim}{u}(t, \underset{\sim}{X}+\underset{\sim}{\alpha}^*, \underset{\sim}{Y}+\underset{\sim}{\beta}^*,\varepsilon) - \underset{\sim}{u}(t,\underset{\sim}{X}+\underset{\sim}{\alpha}, \underset{\sim}{Y}+\underset{\sim}{\beta},\varepsilon) \\ \qquad \leq \ell(\underset{\sim}{\alpha}^*-\underset{\sim}{\alpha} + \underset{\sim}{\beta}^*-\underset{\sim}{\beta}), \\ \underset{\sim}{u}(t, \underset{\sim}{X}-\underset{\sim}{\alpha}, \underset{\sim}{Y}-\underset{\sim}{\beta}, \varepsilon) - \underset{\sim}{u}(t,\underset{\sim}{X}-\underset{\sim}{\alpha}^*, \underset{\sim}{Y}-\underset{\sim}{\beta}^*,\varepsilon) \\ \qquad \leq \ell(\underset{\sim}{\alpha}^*-\underset{\sim}{\alpha} + \underset{\sim}{\beta}^*-\underset{\sim}{\beta}), \end{cases}$$

$$(5.41) \quad \begin{cases} \underset{\sim}{v}(t, \underset{\sim}{X}+\underset{\sim}{\alpha}^*, \underset{\sim}{Y}+\underset{\sim}{\beta}^*,\varepsilon) - \underset{\sim}{v}(t,\underset{\sim}{X}+\underset{\sim}{\alpha}, \underset{\sim}{Y}+\underset{\sim}{\beta},\varepsilon) \\ \qquad \leq \ell(\underset{\sim}{\alpha}^*-\underset{\sim}{\alpha} + \underset{\sim}{\beta}^*-\underset{\sim}{\beta}), \\ \underset{\sim}{v}(t, \underset{\sim}{X}-\underset{\sim}{\alpha}, \underset{\sim}{Y}-\underset{\sim}{\beta}, \varepsilon) - \underset{\sim}{v}(t, \underset{\sim}{X}-\underset{\sim}{\alpha}^*, \underset{\sim}{Y}-\underset{\sim}{\beta}^*,\varepsilon) \\ \qquad \leq \ell(\underset{\sim}{\alpha}^*-\underset{\sim}{\alpha} + \underset{\sim}{\beta}^*-\underset{\sim}{\beta}). \end{cases}$$

Theorem 5.7.

Let $(\underset{\sim}{x}, \underset{\sim}{y})$ and $(\underset{\sim}{X}, \underset{\sim}{Y})$ be respectively solutions of the IVPs (5.1) and (5.35). Further assume

$$(5.42) \qquad \| \underset{\sim}{u}(t, \underset{\sim}{X}, \underset{\sim}{Y},\varepsilon) - \underset{\sim}{u}(t, \underset{\sim}{X}, \underset{\sim}{Y}, 0) \| \leq \varepsilon M_7$$

and

(5.43) $\quad \| \underset{\sim}{v}(t, \underset{\sim}{X}, \underset{\sim}{Y}, \varepsilon) \| \leq \varepsilon\, m_8.$

Then under the above assumptions (i)-(iii) the following inequalities hold.

(5.44) $\quad \| \underset{\sim}{x}(t,\varepsilon) - \underset{\sim}{X}(t) \| \leq \varepsilon\, m_3\, e^{m(t-a)}$

$$+\varepsilon\, m_4 \| \underset{\sim}{B}(\varepsilon)-\underset{\sim}{Y}(a) \| \; [e^{m(t-a)} - e^{-k(t-a)/\varepsilon}],$$

(5.45) $\quad \| \underset{\sim}{y}(t,\varepsilon) - \underset{\sim}{Y}(t) \| \leq \varepsilon\, m_5\, e^{m(t-a)} + \varepsilon\, m_6 \| \underset{\sim}{B}(\varepsilon)-\underset{\sim}{Y}(a) \| \, e^{m(t-a)}$

$$+ \| \underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a) \| \, e^{-k(t-a)/\varepsilon}, \quad m > 0.$$

Proof

Because of the conditions (i) - (iii) stated above, Theorem 5.1 is applicable to the IVP (5.1).

Also

(5.46) $\quad \| \underset{\sim}{A}(\varepsilon) - \underset{\sim}{A}(0) \| \leq \varepsilon\, m_3.$

Consider the vector functions $(\underset{\approx}{x}, \underset{\approx}{y})$ and $(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}})$ defined by

$$\bar{\underset{\sim}{x}} = \underset{\sim}{X} + \underset{\sim}{\eta}, \quad \bar{\underset{\sim}{y}} = \underset{\sim}{Y} + \underset{\sim}{\gamma},$$
$$\underset{\approx}{x} = \underset{\sim}{X} - \underset{\sim}{\eta}, \quad \underset{\approx}{y} = \underset{\sim}{Y} - \underset{\sim}{\gamma},$$

where $\underset{\sim}{\eta} = (\ldots, \eta, \eta, \ldots)$, $\underset{\sim}{\gamma} = (\ldots, \gamma, \gamma, \ldots)$,

$$(5.47)\quad \eta(t,\varepsilon) = \varepsilon\, m_3\, e^{m(t-a)} + \varepsilon\, m_4 \,\|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a)\,\| \times [e^{m(t-a)} - e^{-k(t-a)/\varepsilon}]$$

$$(5.48)\quad \gamma(t,\varepsilon) = \varepsilon\, m_5\, e^{m(t-a)} + \varepsilon\, m_6 \,\|\,\underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a)\,\|\, e^{m(t-a)} + \|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a)\,\|\, e^{-k(t-a)/\varepsilon}.$$

Now

$$\begin{aligned} P_i'(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{y}}) :&= X_i'(t) + \eta' - [u_i(t, \underset{\sim}{X}+\underset{\sim}{\eta}, \underset{\sim}{Y}+\underset{\sim}{\gamma}, \varepsilon) \\ &\quad - u_i(t,\underset{\sim}{X},\underset{\sim}{Y},\varepsilon)] - u_i(t,\underset{\sim}{X},\underset{\sim}{Y},\varepsilon) \\ &> -\varepsilon\, m_7 + \eta' - \ell(\eta+\gamma) \text{ from } (5.42),\ (5.37) \text{ and } (5.40 \\ &\geq \varepsilon\, e^{m(t-a)}\, [-m_7 + mm_3 - \ell m_3 - \ell m_5] \\ &\quad + \varepsilon \|\,\underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a)\,\|\, e^{m(t-a)}\, [mm_4 - \ell m_4 - \ell m_6] \\ &\quad + \|\underset{\sim}{B}(\varepsilon) - \underset{\sim}{Y}(a)\,\|\, e^{-k(t-a)}\, [km_4 + \ell\varepsilon m_4 - \ell] \\ &\qquad \text{by } (5.47) - (5.48),\ i \in Z. \end{aligned}$$

Choose m_3, m_4, m_5, m_6, m_7, m_8 and m such that

$$(5.49)\quad km_4 - \ell > 0,$$

$$(5.50)\quad km_5 - m_8 - m_9 - \ell m_3 > 0,$$

$$(5.51)\quad -\ell m_4 + km_6 > 0,$$

$$(5.52)\quad mm_3 - m_7 - \ell m_3 - \ell m_5 > 0 \text{ and } mm_4 - \ell m_4 - \ell m_6 > 0.$$

Therefore

$$P_i'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}) \geq 0 = P_i'(\underset{\sim}{x}, \underset{\sim}{y}), \; i \in Z,$$

that is,

$$P'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}) \geq 0 = \underset{\sim}{P}'(\underset{\sim}{x}, \underset{\sim}{y})$$

Also

$$\begin{aligned} Q_i'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}) &= \varepsilon Y_i'(t) + \varepsilon\gamma'(t,\varepsilon) \\ &\quad - [v_i(t,\underset{\sim}{X}+\underset{\sim}{\eta}, \underset{\sim}{Y}+\underset{\sim}{\gamma},\varepsilon) - v_i(t,\underset{\sim}{X},\underset{\sim}{Y}+\underset{\sim}{\gamma},\varepsilon) \\ &\quad + v_i(t,\underset{\sim}{X}, \underset{\sim}{Y}+\underset{\sim}{\gamma},\varepsilon) - v_i(t,\underset{\sim}{X},\underset{\sim}{Y},\varepsilon)] - v_i(t,\underset{\sim}{X},\underset{\sim}{Y},\varepsilon) \\ &\geq -\varepsilon m_9 - \varepsilon m_8 + \varepsilon\gamma' - \ell\eta + k\gamma, \\ &\qquad \text{by (5.38), (5.43), (5.37) and (5.41)} \\ &\geq \varepsilon e^{m(t-a)} [-m_9 - m_8 + \varepsilon m m_5 - \ell m_3 + k m_5] \\ &\quad + \varepsilon \| \underset{\sim}{B}(\varepsilon) - Y(a) \| e^{m(t-a)} [\varepsilon m m_6 - \ell m_4 - k m_6] \\ &\quad + \| \underset{\sim}{B}(\varepsilon) - Y(a) \| e^{-k(t-a)/} [-k + \ell\varepsilon m_4 + k], \\ &\qquad \text{by (5.47) and (5.48), } i \in Z. \end{aligned}$$

Using (5.49) - (5.52)

$$Q_i'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}) \geq 0 = Q_i'(\underset{\sim}{x}, \underset{\sim}{y}), \; i \in Z$$

Hence

$$\underset{\sim}{Q}'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}) \geq \underset{\sim}{Q}'(\underset{\sim}{x}, \underset{\sim}{y})$$

Again

$$\begin{aligned} \bar{x}_i(a,\varepsilon) &= X_i(a) + \varepsilon m_3 = A_i(0) + \varepsilon m_3 \\ &\geq A_i(\varepsilon) = x_i(a,\varepsilon), \; i \in Z. \end{aligned}$$

Similarly

$$\bar{y}_i(a,\varepsilon) \geq y_i(a,\varepsilon), \quad i \in Z.$$

That is, the following inequalities are established.

$$(5.53) \begin{cases} \underset{\sim}{P}'(\underset{\sim}{x}, \underset{\sim}{y}) \leq \underset{\sim}{P}'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}), \\ \underset{\sim}{Q}'(\underset{\sim}{x}, \underset{\sim}{y}) \leq \underset{\sim}{Q}'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}), \\ \underset{\sim}{R}'(\underset{\sim}{x}, \underset{\sim}{y}) \leq \underset{\sim}{R}'(\bar{\underset{\sim}{x}}, \bar{\underset{\sim}{y}}), \end{cases}$$

which by Theorem 5.1 yield

$$(5.54) \quad \underset{\sim}{x}(t,\varepsilon) \leq \bar{\underset{\sim}{x}}(t,\varepsilon), \ \underset{\sim}{y}(t,\varepsilon) \leq \bar{\underset{\sim}{y}}(t,\varepsilon), \ t \in \bar{D}.$$

Similar steps yield

$$(5.55) \quad \underline{\underset{\sim}{x}}(t,\varepsilon) \leq \underset{\sim}{x}(t,\varepsilon), \ \underline{\underset{\sim}{y}}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon), \ t \in \bar{D}.$$

The inequalities (5.44)-(5.45) follow from (5.54)-(5.55).

Thus

$$\lim_{\varepsilon \to 0+} \underset{\sim}{x}(t,\varepsilon) = \underset{\sim}{X}(t), \quad t \in \bar{D}$$

$$\lim_{\varepsilon \to 0+} \underset{\sim}{y}(t,\varepsilon) = \underset{\sim}{Y}(t), \quad t \in D.$$

The analysis of the previous section can be carried over to study IVPs for higher order equations of the form

$$\varepsilon \underset{\sim}{y}^{(n)} = \underset{\sim}{f}(t,\underset{\sim}{y},\underset{\sim}{y}^{(1)}, \ldots, \underset{\sim}{y}^{(n-1)}), \ t \in D$$

subject to the ICs

$$\underset{\sim}{y}(a,\varepsilon) = \underset{\sim}{A}^1(\varepsilon), \ldots, \underset{\sim}{y}^{(n-1)}(a,\varepsilon) = \underset{\sim}{A}^n(\varepsilon),$$

where $\varepsilon > 0$ is a small parameter,

$$y^{(n)} = d^n y/ dt^n, \quad \underset{\sim}{A}^i(\varepsilon) = A^i(0) + \mathcal{O}(\varepsilon), \ i \in Z.$$

The method adopted in discussing nonquasimonotone linear systems can be used to discuss nonquasimonotone nonlinear systems.

Chapter 6

ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SINGULARLY PERTURBED PARABOLIC SYSTEMS

Hoppensteadt [36] considered second order quasi-linear parabolic DEs with a small parameter multiplying the time derivative subject to Dirichlet type BCs described on the parabolic boundary of the domain under consideration. He constructed asymptotic expansions for solutions as the small parameter goes to zero. In [31] estimates for solutions of these BVPs are obtained and these estimates in turn, determine the asymptotic behaviour of solutions as the small parameter goes to zero. The DEs considered in [31] are defined only on bounded regions. The present chapter considers such BVPs but now they are defined on unbounded regions. Here also estimates for solutions are obtained in order to discuss the asymptotic behaviour of solutions as the small parameter appearing in the equation goes to zero. The notations and definitions introduced in Section ~~4~~ 5 of Chapter 2 of this thesis are used in the following discussion.

Consider the IBVP for $\underset{\sim}{u} = (u_1, \ldots, u_n)$ defined by

$$
(6.1)\quad \begin{cases} P_i\underset{\sim}{u}: = \varepsilon\,\partial u_i/\partial t + F_i(t,x,\underset{\sim}{u},u_{i,j},u_{i,jk},\varepsilon)=f_i(t,x,\varepsilon), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in G_p, \\ R_i\underset{\sim}{u}: = \begin{cases} u_i(t,x,\varepsilon) = g_i(t,x,\varepsilon) \text{ on } \partial_1 G, \\ u_i(0,x,\varepsilon) = h_i(x,\varepsilon) \quad \text{on } \bar{D}, \\ \lim\limits_{\|x\|_e\to\infty} u_i(t,x,\varepsilon) = G_i(t,\varepsilon), \text{ uniformly with} \end{cases} \end{cases}
$$

respect to $t\in[0,T]$; $i = 1(1)n$, where ε is a small positive parameter and for each fixed i,

(i) $F_i(t,x,\underset{\sim}{u},p,r,\varepsilon)$ is monotone decreasing in r,

(ii) $u_i\in U: = C^{(1,2)}(G_p)\cap C(\bar{G})$, $i=1(1)n$, and

(iii) f_i, g_i, h_i, G_i, $i=1(1)n$ are continuous in the respective domains.

The following definitions of norms shall be used in the rest of this chapter.

(6.2) $\quad \|\underset{\sim}{f}\| : = \sup\,[\,|f_i(t,x,\varepsilon)|: (t,x)\in G_p,\ i=1(1)n]$

(6.3) $\quad \|\underset{\sim}{g}\|' : = \sup\,[\,|g_i(t,x,\varepsilon)|: (t,x)\in \partial_1 G, i=1(1)n]$

(6.4) $\quad \|\underset{\sim}{h}\|'' : = \sup\,[\,|h_i(x,\varepsilon)|: x\in \bar{D},\ i=1(1)n\,]$.

Let $G_o(\varepsilon) = \max\limits_{i=1(1)n,\,t\in[0,T]} G_i(t,\varepsilon)$. The above norms are assumed to be bounded functions of their arguments.

Section 1 deals with linear boundary value problems whereas Section 2 is concerned with nonlinear problems.

1. LINEAR PROBLEMS

In this section a special case of IBVP (6.1) is considered and the asymptotic behaviour of its solution as the small parameter goes to zero is discussed. The linear IBVP for $\underset{\sim}{u}$ is defined by

$$(6.5)\quad \begin{cases} P_{Li}\,\underset{\sim}{u} := \varepsilon\,\partial u_i/\partial t + P_{ei}\underset{\sim}{u} = f_i(t,x,\varepsilon) \text{ in } G_p, \\ R_{Li}\,\underset{\sim}{u} := \begin{cases} u_i(t,x,\varepsilon) = g_i(t,x,\varepsilon) \text{ on } \partial_1 G, \\ u_i(0,x,\varepsilon) = h_i(x,\varepsilon) \quad \text{on } \bar{D}, \\ \lim\limits_{\|x\|_e\to\infty} u_i(t,x,\varepsilon) = G_i(t,\varepsilon),\ t\in[0,T], i=1(1)n, \end{cases} \end{cases}$$

with

$$P_{ei}\underset{\sim}{u} := \sum_{j=1}^{n} a_{ij}(t,x,\varepsilon)u_j + \sum_{k=1}^{m} b_{ik}(t,x,\varepsilon)u_{i,k} - \sum_{j,k=1}^{m} c_{ijk}(t,x,\varepsilon)u_{i,jk},$$
$$i=1(1)n,$$

where the matrix (c_{ijk}), for fixed i, is symmetric and positive definite, the functions a_{ij} to c_{ijk} are continuous and bounded.

Under suitable assumptions the following asymptotic behaviour will be established.

$$(6.6)\quad \lim_{\varepsilon\to 0+} u_i(t,x,\varepsilon) = U_i(t,x) \text{ on } (0,T] \times \bar{D},\ i=1(1)n,$$

where $\underset{\sim}{u}$, $\underset{\sim}{U}$ are respectively solutions of the IBVP (6.5) and its corresponding reduced problem

$$
(6.7)\quad \begin{cases} \sum_{j=1}^{n} a_{ij}(t,x,0)\, u_j + \sum_{k=1}^{m} b_{ik}(t,x,0)u_{i,k} - \sum_{j,k=1}^{m} c_{ijk}(t,x,0)u_{i,jk} \\ \qquad = f_i(t,x,0) \text{ in } G_p, \\ u_i(t,x,0) = g_i(t,x,0) \text{ on } \partial_1 G, \\ \lim_{\|x\|_e \to \infty} u_i(t,x,0) = G_i(t,0),\ t \in [0,T],\ i=1(1)n. \end{cases}
$$

In the following, m_i, $i=1,2,\ldots$ stand for positive constants independent of ε.

In [31], as pointed out earlier, estimates for solutions of linear and nonlinear BVPs for parabolic DEs are obtained. Motivated by this paper a few results on the estimates of solutions for the linear problems (6.5) are obtained.

Theorem 6.1.

Consider the IBVP (6.5) and assume

$$
(6.8)\qquad a_{ij}(t,x,\varepsilon) \leq 0,\ i \neq j,\quad i,j = 1(1)n
$$

and

$$
(6.9)\qquad \sum_{j=1}^{n} a_{ij}(t,x,\varepsilon) \geq \delta > 0,\ i=1(1)n.
$$

Then

$$
(6.10)\qquad |u_i(t,x,\varepsilon)| \leq m_1[\ \|\underset{\sim}{f}\| + \|\underset{\sim}{g}\|' + G_0(\varepsilon)] + \|\underset{\sim}{h}\|'' e^{-t\delta/\varepsilon}, \quad i=1(1)n \text{ on } \overline{G},
$$

where $\underset{\sim}{u}$ is the solution of the IBVP (6.5) and δ is a positive constant independent of ε .

Proof

Define $\underset{\sim}{y} = (y_1, \ldots, y_n)$ as

$$y_i(t,x,\varepsilon) = m_1(2-e^{-m_2 x})\ (\|\underset{\sim}{f}\| + \|\underset{\sim}{g}\|' + G_o) + \|\underset{\sim}{h}\|'' e^{-t\delta/\varepsilon},\quad i=1(1)n.$$

Then

$$(6.11)\ \begin{cases} y_i(t,x,\varepsilon) \geq u_i(t,x,\varepsilon) \text{ on } \partial_1 G \cup \partial_o G, \\ \lim\limits_{\|x\|_e \to \infty} y_i(t,x,\varepsilon) \geq \lim\limits_{\|x\|_e \to \infty} u_i(t,x,\varepsilon),\ t \in \bar{D},\ i=1(1)n, \end{cases}$$

by proper choice of $m_1 \geq 1$.

Also

$$\begin{aligned} P_{Li}\underset{\sim}{y} &= \|\underset{\sim}{h}\|''(-\delta)e^{-t\delta/\varepsilon} + [\sum_{j=1}^{n} a_{ij}(t,x,\varepsilon)]\ \|\underset{\sim}{h}\|''\ e^{-t\delta/\varepsilon} \\ &\quad + m_1(\|\underset{\sim}{f}\| + \|\underset{\sim}{g}\| + G_o)[\{m_2^2 \sum_{j,k=1}^{m} c_{ijk} + m_2 \sum_{k=1}^{m} b_{ik}\}\ e^{-m_2 x} \\ &\quad + (2-e^{-m_2 x}) \sum_{j=1}^{n} a_{ij}\ ] \\ &\geq m_1 \delta \|\underset{\sim}{f}\| \geq \|\underset{\sim}{f}\| \geq P_{Li}\underset{\sim}{u}, \end{aligned}$$

by a proper choice of m_1 and m_2, $i = 1(1)n$.

That is,

$$(6.12)\quad P_{Li}\underset{\sim}{y} \geq P_{Li}\underset{\sim}{u} \text{ in } G_p,\quad i = 1(1)n.$$

Further the function $\underset{\sim}{s} = (s_1,\ldots,s_n)$ with $s_i=(2-e^{-m_3x})$, by a proper choice of m_3 and under the hypotheses of the theorem, is a 'test function' for the IBVP (6.5). Hence Theorem 2.16 is applicable to BVP (6.5) and therefore the inequalities (6.11)-(6.12) yield

$$u_i(t,x,\varepsilon) \le y_i(t,x,\varepsilon) \text{ on } \bar{G},\ i = 1(1)n.$$

Since the BVP is linear

$$(6.13)\qquad |\ u_i(t,x,\varepsilon)\ | \ \le\ y_i(t,x,\varepsilon) \text{ on } \bar{G},\ i = 1(1)n.$$

The estimate (6.10) follows from (6.13) by replacing $2-e^{-m_2x}$ by 2 and then renaming $2m_1$ by m_1. Hence the theorem is proved.

To establish the result (6.6) one needs the following problem for $\underset{\sim}{w}$ defined by

$$(6.14)\begin{cases} \sum_{j=1}^{n} a_{ij}(t,x,\varepsilon)w_j + \sum_{k=1}^{m} b_{ik}(t,x,\varepsilon)w_{i,k} - \sum_{j,k=1}^{m} c_{ijk}(t,x,\varepsilon)w_{i,jk} \\ \qquad\qquad = f_i(t,x,\varepsilon) \text{ in } G_p, \\ w_i(t,x,\varepsilon) = g_i(t,x,\varepsilon) \text{ on } \partial_1 G, \\ \lim\limits_{\|x\|_e\to\infty} w_i(t,x,\varepsilon) = G_i(t,\varepsilon),\ t\in[0,T],\ i=1(1)n. \end{cases}$$

Further it is assumed that there exists a solution $\underset{\sim}{w}$ for the BVP (6.14) which satisfies:

(6.15) $\lim_{\varepsilon \to 0+} w_i(t,x,\varepsilon) = U_i(t,x)$ on $G_p \cup \partial_1 G$,

and

(6.16) $\| \partial \underset{\sim}{w} / \partial t \| \leq m_1 < \infty$.

Theorem 6.2.

Consider the problems (6.5) and (6.14) which satisfy (6.15) and (6.16). Further assume

(6.17) $a_{ij}(t,x,\varepsilon) \leq 0, \quad i \neq j, \ i,j = 1(1)n$

and

(6.18) $\sum_{j=1}^{n} a_{ij}(t,x,\varepsilon) \geq \delta > 0$, $i=1(1)n$, δ is a constant independent of ε. Then

(6.19) $|u_i(t,x,\varepsilon)-U_i(t,x)| \leq \varepsilon m_7 + \|\underset{\sim}{h}-\underset{\sim}{U}(0,x)\|'' e^{-t\delta/\varepsilon} + \gamma_1(\varepsilon)$

where $\gamma_1(\varepsilon) = \gamma_1(t,x,\varepsilon) \to 0+$ on $\overline{G}$ as $\varepsilon \to 0$ and $\underset{\sim}{u}$ and $\underset{\sim}{U}$ are respectively solutions of (6.5) and (6.7).

Proof

If $\underset{\sim}{u}$ and $\underset{\sim}{w}$ are respectively solutions of (6.5) and (6.14), then $\underset{\sim}{u} - \underset{\sim}{w}$ satisfies the IBVP:

$$\begin{cases} \underset{\sim}{P}_L \underset{\sim}{v} := \varepsilon\, \partial \underset{\sim}{v}/\partial t + \underset{\sim}{P}_e \underset{\sim}{v} = -\varepsilon\, \partial \underset{\sim}{w}/\partial t \\ \underset{\sim}{R}_L \underset{\sim}{v} := \begin{cases} \underset{\sim}{v} = \underset{\sim}{0} \quad \text{on} \quad \partial_1 G, \\ \underset{\sim}{v}(0,x,\varepsilon) = \underset{\sim}{h}(x,\varepsilon) - \underset{\sim}{w}(0,x,\varepsilon) \quad \text{on } \overline{D}, \\ \lim_{\|x\|_e \to \infty} \underset{\sim}{v} = \underset{\sim}{0} \quad \text{on } [0,T]. \end{cases} \end{cases}$$

Theorem 6.1 yields

$$|u_i(t,x,\varepsilon) - w_i(t,x,\varepsilon)| \leq \varepsilon\, m_6 \,\| \partial \underset{\sim}{w}/ \partial t \| \quad .$$

$$+ \| \underset{\sim}{h}-\underset{\sim}{w}(0,x,\varepsilon) \|'' \, e^{-t\delta/\varepsilon} \text{ on } \bar{G},\ i=1(1)n.$$

The estimate (6.19) follows from the above estimate (6.15) and the inequality:

$$| u_i(t,x,\varepsilon)-U_i(t,x) | \leq | u_i(t,x,\varepsilon)-w_i(t,x,\varepsilon) |$$

$$+ | w_i(t,x,\varepsilon) - U_i(t,x) |, \ i=1(1)n.$$

Remark.

It is obvious that the asymptotic behaviour (6.6) follows immediately from the estimate (6.19). Also it should be observed that the estimate (6.19) contains the so called boundary layer term having $e^{-t\delta/\varepsilon}$ as a factor which explicitely describes the nature of the nonuniform behaviour of the solution as a function of t and ε.

Example 6.3.

Consider the IBVP

$$(6.20)\quad \begin{cases} \partial W/\partial \tau = \partial^2 W/\partial x^2 - W, & b(\varepsilon\tau) \leq x < \infty,\ 0 < \tau < \infty \\ W(\tau, b(\varepsilon\tau)) = 0,\ W(\tau,\infty) = 0, & 0 \leq \tau < \infty, \\ W(0,x) = W_0(x), & b(0) \leq x < \infty, \end{cases}$$

where $W(\tau,x) = W(\tau,x,\varepsilon)$ and $x=b(\varepsilon\tau) > 0$ is a slowly varying function of τ. Let $t = \varepsilon\tau$, $u(t,x) = W(\tau, x)$. Then the problem (6.20) gets transformed to

$$(6.21) \begin{cases} \varepsilon\, \partial u/\partial t = \partial^2 u/\partial x^2 - u, & b(t) \le x < \infty,\ 0 < t < \infty, \\ u(t, b(t)) = 0 = u(t, \infty), & \\ u(0,x) = W_o(x), & b(0) \le x < \infty. \end{cases}$$

It is easy to verify that all the conditions of Theorem 6.2 are satisfied for the IBVP (6.21)

and hence

$$(6.22) \qquad | u(t,x,\varepsilon) | \le \| W_o(x) \|'' \, e^{-t\delta/\varepsilon}.$$

Theorems 6.1 - 6.2 are proved under the assumption that the quasimonotonicity condition (6.8) is satisfied for the IBVP (6.5). This condition can be relaxed and one can obtain the same asymptotic behaviour (6.6) by introducing an extended auxiliary problem as given below.

Set

$$a^+_{ij} := \begin{cases} a_{ij} & \text{if } a_{ij} \ge 0 \\ 0 & \text{otherwise}, \end{cases}$$

$$a^-_{ij} := a_{ij} - a^+_{ij}.$$

Then the extended auxiliary problem $(\hat{P}_{Li}, \hat{R}_{Li})$ with respect to (P_{Li},R_{Li}) of (6.5) is defined by

$$
(6.23)\quad
\begin{cases}
P_{Li}\ \underset{\sim}{\hat{u}} := \varepsilon\,\partial\hat{u}_i/\partial t + a_{ii}\hat{u}_i - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}[a_{ij}^{+}\hat{u}_{j+n} - a_{ij}^{-}\hat{u}_j] \\
\qquad + \sum\limits_{k=1}^{m} b_{ik}\,\hat{u}_{i,k} - \sum\limits_{j,k=1}^{m} c_{ijk}\,\hat{u}_{i,jk} = -f_i \\
P_{Li+n}\underset{\sim}{\hat{u}} := \varepsilon\,\partial\hat{u}_{i+n}/\partial t + a_{ii}\hat{u}_{i+n} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n}[a_{ij}^{+}\,\hat{u}_j - a_{ij}^{-}\,\hat{u}_{j+n}] \\
\qquad + \sum\limits_{k=1}^{m} b_{ik}\,\hat{u}_{i+n,k} - \sum\limits_{j,k=1}^{m} c_{ijk}\,\hat{u}_{i+n,k} = f_i \\
R_{Li}\ \underset{\sim}{\hat{u}} := \begin{cases} \hat{u}_i(t,x,\varepsilon) = -g_i(t,x,\varepsilon) \text{ on } \partial_1 G \\ \hat{u}_i(0,x,\varepsilon) = -h_i(x,\varepsilon) \quad \text{on } \bar{D} \\ \lim\limits_{\|x\|_e\to\infty} \hat{u}_i(t,x,\varepsilon) = G_o(t) \text{ on } [0,T], \end{cases} \\
R_{L,i+n}\underset{\sim}{\hat{u}} := \begin{cases} \hat{u}_{i+n}(t,x,\varepsilon) = g_i(t,x,\varepsilon) \text{ on } \partial_1 G, \\ u_i(0,x,\varepsilon) = h_i(x,\varepsilon) \text{ on } \bar{D} \\ \lim\limits_{\|x\|_e\to\infty} u_i(t,x,\varepsilon) = -G_o(t) \text{ on } [0,T], \end{cases}
\end{cases}
$$

$$i=1(1)n, \quad \underset{\sim}{\hat{u}} = (\hat{u}_1, \ldots, \hat{u}_{2n}).$$

Then

(i) IBVP (6.23) satisfies quasi-monotonicity condition similar to that of (6.17);

(ii) if $(u_1, \ldots, u_n)$ is a solution of the IBVP (6.5) then $(-u_1, \ldots, -u_n, u_1, \ldots, u_n)$ is a solution of the IBVP (6.23).

Theorem 6.4.

Consider the IBVP (6.5). If the condition (i) of Theorem 6.1 is not satisfied then construct the system (6.23). Further assume

$$a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} (a_{ij}^{+} - a_{ij}^{-}) \geq \delta > 0, \; i = 1(1)n.$$

Then also the conclusion of Theorem 6.1 is valid.

Proof

The method adopted in the earlier chapters may be made use of to prove this theorem.

2. NONLINEAR EQUATIONS.

Asymptotic behaviour of solutions of the LBVPs are discussed in the previous section. In this section estimates for solutions of the general nonlinear problem (6.1) are obtained in terms of the solution of the problem

$$(6.24)\begin{cases} \underset{\sim}{F}\,(t,x,\underset{\sim}{v},[\underset{\sim}{v}]_x,\ [\underset{\sim}{v}]_{xx},\ 0) = \underset{\sim}{f}\,(t,x,0) \in G_p, \\ \underset{\sim}{v}\,(t,x) = \underset{\sim}{g}\,(t,x,0) \text{ on } \partial_1 G, \\ \lim\limits_{\|x\|_e\to\infty} \underset{\sim}{v}(t,x) = \underset{\sim}{G}(t,0), \text{ uniformly with respect to } t\in[0,T] \end{cases}$$

Following theorem is only a reformulation of Theorem 2.14.

Theorem 6.5.

Let $\underset{\sim}{u}$ be a solution of the system (6.1).

Then the implication

$$(6.25)\begin{cases} \underset{\sim}{P}\,\underset{\sim}{z} \leq \underset{\sim}{P}\,\underset{\sim}{u} \leq \underset{\sim}{P}\,\underset{\sim}{y},\quad \underset{\sim}{R}\,\underset{\sim}{z} \leq \underset{\sim}{R}\,\underset{\sim}{u} \leq \underset{\sim}{R}\,\underset{\sim}{y} \\ \Longrightarrow \\ \underset{\sim}{z}\,(t,\varepsilon) \leq \underset{\sim}{u}(t,\varepsilon) \leq \underset{\sim}{y}(t,\varepsilon) \text{ on } \overline{G} \end{cases}$$

is true for all $\underset{\sim}{z}$, $\underset{\sim}{y}$, z_i, $y_i \in U$, $i=1(1)n$, provided the following assumptions are satisfied.

(i) there exist a positive number δ_2 and a 'test function' $\underset{\sim}{s}(t,x)$ such that $s_i(t,x) > 0$ on $\overline{G}$, $s_i \in U$,

$$\lim_{\|x\|_e\to\infty} s_i(t,x) \text{ exists uniformly with respect to } t$$

and the limit is positive with the properties

$$(6.26)\begin{cases} \underset{\sim}{P}(\underset{\sim}{y}+\alpha_1\underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{y} \geq \alpha_1\delta_2 > 0, \\ \underset{\sim}{P}\,\underset{\sim}{z} - \underset{\sim}{P}(\underset{\sim}{z}-\alpha_1\underset{\sim}{s}) \geq \alpha_1\delta_2 > 0, \; t\in G, \\ \underset{\sim}{R}\,\underset{\sim}{s} > 0, \end{cases}$$

for every positive number α_1;

(ii) there exists a positive constant M such that

$$\underset{\sim}{F}(t,x,\underset{\sim}{u}, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_x, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_{xx}, \varepsilon)$$
$$-\underset{\sim}{F}(t,x,\underset{\sim}{y}+\alpha_1\underset{\sim}{s}, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_x, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_{xx}, \varepsilon)$$
$$\geq M(\underset{\sim}{u} - (\underset{\sim}{y}+\alpha_1\underset{\sim}{s}))$$

and

$$\underset{\sim}{F}(t,x,\underset{\sim}{z} - \alpha_1\underset{\sim}{s}, [\underset{\sim}{u}]_x, [\underset{\sim}{u}]_{xx}) - \underset{\sim}{F}(t,x,\underset{\sim}{u},[\underset{\sim}{u}]_x, [\underset{\sim}{u}]_{xx})$$
$$\geq M(\underset{\sim}{z} - (\underset{\sim}{u}+\alpha_1\underset{\sim}{s})),$$

whenever $\underset{\sim}{u} \leq \underset{\sim}{y}+\alpha_1\underset{\sim}{s}$, $\underset{\sim}{z} \leq \underset{\sim}{u}+\alpha_1\underset{\sim}{s}$ and for all positive number α_1 and η with $\alpha_1-\eta > 0$;

(iii) there exists a positive constant L such that

$$\underset{\sim}{F}(t,x,\underset{\sim}{y}+\alpha_1\underset{\sim}{s}, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_x, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_{xx}, \varepsilon)$$
$$-\underset{\sim}{F}(t,x,\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_x, [\underset{\sim}{y}+(\alpha_1-\eta)\underset{\sim}{s}]_{xx}, \varepsilon)$$
$$\geq - L\eta\underset{\sim}{s} \leq \underset{\sim}{F}(t,x,[\underset{\sim}{u}+\eta\underset{\sim}{s}], [\underset{\sim}{u}]_x, [\underset{\sim}{u}]_{xx}, \varepsilon)$$
$$- \underset{\sim}{F}(t,x,\underset{\sim}{u}, [\underset{\sim}{u}]_x, [\underset{\sim}{u}]_{xx}, \varepsilon)$$

for every positive number α_1 and η with $\alpha_1-\eta > 0$.

The proof of this theorem is same as that of Theorem 2.14.

Theorem 6.6.

Let $\underset{\sim}{u}$ be a solution of the system (6.1). Further let $\underset{\sim}{v}$ be the solution of the problem (6.24) such that

$$\|\underset{\sim}{v}\| < m_{11} \text{ and } \|\underset{\sim}{F}(t,x,\underset{\sim}{v},[\underset{\sim}{v}]_x,[\underset{\sim}{v}]_{xx},\varepsilon)\| \leq m_{12}\varepsilon.$$

Further assume that the following conditions (i) and (ii) and the conditions (ii) and (iii) of Theorem 6.5 are satisfied for the functions $\underset{\sim}{y}$ and $\underset{\sim}{z}$ which are defined as

$$\underset{\sim}{y} = \underset{\sim}{v} + \underset{\sim}{w}, \quad \underset{\sim}{z} = \underset{\sim}{v} - \underset{\sim}{w}, \quad \underset{\sim}{w} = (\ldots, w_1, w_2, \ldots),$$

$$w_i(t,\varepsilon) = \varepsilon\, m_3(1-e^{-\delta_5(t-a)\varepsilon}),\ \delta_5 > 0,\ m_3 > 0,\ i \in Z.$$

There exist positive constants δ_4 and δ_5 (independent of m_3) such that

(i) $\underset{\sim}{F}(t,x,\underset{\sim}{y}+\alpha_1\underset{\sim}{s},\ [\underset{\sim}{y}+\alpha_1\underset{\sim}{s}]_x,\ [\underset{\sim}{y}+\alpha_1\underset{\sim}{s}]_{xx},\ \varepsilon)$

$- \underset{\sim}{F}(t,x,\underset{\sim}{y},\ [\underset{\sim}{y}]_x,\ [\underset{\sim}{y}]_{xx},\ \varepsilon)$

$\geq \alpha_1\delta_4\,\underset{\sim}{s} + \alpha_1\delta_5\,\underset{\sim}{s}_x + \alpha_1\delta_6\,\underset{\sim}{s}_{xx}$

$\leq F(t,x,\underset{\sim}{z},\ [\underset{\sim}{z}]_x,\ [\underset{\sim}{z}]_{xx},\varepsilon) - \underset{\sim}{F}(t,x,\underset{\sim}{z}-\alpha_1\underset{\sim}{s},\ [\underset{\sim}{z}-\alpha_1\underset{\sim}{s}]_x,$

$[\underset{\sim}{z} - \alpha_1\underset{\sim}{s}]_{xx},\varepsilon)$

for all positive constant α_1 and

(ii) $\underset{\sim}{F}(t,x,\underset{\sim}{v}+\underset{\sim}{w}, [\underset{\sim}{v}+\underset{\sim}{w}]_x, [\underset{\sim}{v}+\underset{\sim}{w}]_{xx},\varepsilon) - \underset{\sim}{F}(t,x,\underset{\sim}{v},[\underset{\sim}{v}]_x, [\underset{\sim}{v}]_{xx},\varepsilon)$

$$\geq \delta_4 \underset{\sim}{w} + \delta_5 \underset{\sim}{w}_x + \delta_6 \underset{\sim}{w}_{xx}$$

$$\leq \underset{\sim}{F}(t,x,\underset{\sim}{v},[\underset{\sim}{v}]_x, [\underset{\sim}{v}]_{xx}, \varepsilon)$$

$$-\underset{\sim}{F}(t,x,\underset{\sim}{v}-\underset{\sim}{w}, [\underset{\sim}{v}-\underset{\sim}{w}]_x, [\underset{\sim}{v}-\underset{\sim}{w}]_{xx}, \varepsilon).$$

Then

(6.27) $\quad \| \underset{\sim}{u}(t,x,\varepsilon) - \underset{\sim}{v}(t,x) \| \leq w(t,x,\varepsilon), \; t \in G,$

that is ,

(6.28) $\quad \lim\limits_{\varepsilon \to 0_+} \| \underset{\sim}{u}(t,x,\varepsilon) - \underset{\sim}{v}(t,x) \| = 0, \; t \in G.$

Proof

First it will be shown that the function $\underset{\sim}{s}$ defined in the hypothesis (ii) and the functions $\underset{\sim}{y}$, $\underset{\sim}{z}$ satisfy the conditions of Theorem 6.5. Consequently the estimate (6.28) follows. Now

$$P_i(\underset{\sim}{y}+\alpha_1\underset{\sim}{s}) - P_i\underset{\sim}{y} = \varepsilon\,\alpha_1 s_{it} + [F_i(t,x,\underset{\sim}{y}+\alpha_1\underset{\sim}{s},(y_i+\alpha_i s_i)_x,$$

$$(y_i+\alpha_1 s_i)_{xx},\varepsilon) - F_i(t,x,\underset{\sim}{y},y_{ix},y_{ixx},\varepsilon)]$$

$$> \varepsilon\,\alpha_1 s_{it} + \alpha_1 \delta_4 s_i + \alpha_1 \delta_5 s_{ix} + \alpha_1 \delta_6 s_{ixx}$$

$$\geq \alpha_1 \delta_4 > 0, \; i \in Z,$$

that is,

(6.29) $\quad \underset{\sim}{P}(\underset{\sim}{y} + \alpha_1 \underset{\sim}{s}) - \underset{\sim}{P}\,\underset{\sim}{y} > 0.$

Further

$$\begin{aligned}
P_i \underset{\sim}{y} &= P_i \,[\, \underset{\sim}{v}+\underset{\sim}{w} \,] - P_i \underset{\sim}{v} + P_i \underset{\sim}{v} \\
&= \varepsilon\, w_{it} + F_i(t,\, \underset{\sim}{v}+\underset{\sim}{w},\, (v_i+w_i)_x,\, (v_i+w_i)_{xx},\, \varepsilon) \\
&\quad - F_i(t, \underset{\sim}{v},\, v_{ix},\, v_{ixx}, \varepsilon) + P_i \underset{\sim}{v} \\
&\geq \varepsilon\, w_{it} + \delta_4 w_i + \delta_5 w_{ix} + \delta_6 w_{ixx} - \varepsilon\, m_{13} \\
&\geq \varepsilon\, w_i + \delta_4 w_i + \delta_5 w_{ix} + \delta_6 w_{ixx} - \varepsilon\, m_{13}(1-e^{-\delta_5 (t-a)\varepsilon}) \\
&\geq 0 = P_i \underset{\sim}{u},\ i \in Z,
\end{aligned}$$

by a proper choice of m_{13}.

That is,

(6.30) $\underset{\sim}{P}\,\underset{\sim}{u} \leq \underset{\sim}{P}\,\underset{\sim}{y}.$

Also one can prove that by a proper choice of m_{13},

(6.31) $\underset{\sim}{R}\,\underset{\sim}{u} \leq \underset{\sim}{R}\,\underset{\sim}{y}.$

Then the inequalities (6.30)-(6.31) yield

$$\underset{\sim}{u}(t,x,\varepsilon) \leq \underset{\sim}{y}(t,x,\varepsilon).$$

Similarly

$$\underset{\sim}{z}(t,x,\varepsilon) \leq \underset{\sim}{u}(t,x,\varepsilon).$$

Hence the proof of this theorem.

Other types of BCs can also be considered with above DEs and obtain the similar results.

Chapter 7

METHOD OF LINES FOR ELLIPTIC DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER*

This chapter considers the MOL applied to non-linear elliptic DEs defined on a unit square and a semi-infinite strip when a small parameter multiplies one of the second derivatives. Infact a few results on the error estimates and the convergence of the MOL are obtained to the elliptic BVPs:

(7.1) $$-u_{xx} + f(x,y,u,u_x,u_y,\varepsilon u_{yy}) = 0,$$
$$(x,y) \in G: = (0,1) \times (0,1),$$

(7.2) $$u(x,0) = \phi_1(x), \quad u(x,1) = \phi_2(x), \quad x \in (0,1),$$

(7.3) $$u(0,y) = \psi_1(y), \quad u(1,y) = \psi_2(y), \quad y \in (0,1),$$

and

(7.4) $$-u_{xx} + f(x,y,u,u_x,u_y,\varepsilon u_{yy}) = 0$$
$$(x,y) \in G' : = (0,1) \times (-\infty, \infty),$$

(7.5) $$u(0,y) = \eta_1(y), \quad u(1,y) = \eta_2(y), \quad y \in (-\infty, \infty),$$

(7.6) $$\lim_{|y|\to\infty} u(x,y) = 0, \quad x \in (0,1),$$

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where ε is a small positive parameter; u_x, u_y, u_{xx} and u_{yy} stand respectively for $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$; f is assumed to be monotone decreasing in the last argument.

The monotonicity condition on f with respect to the last argument is known as the ellipticity condition so as to make the DEs (7.1) and (7.4) as the elliptic DEs [3]. In the following the problems (7.1)-(7.3) and (7.4)-(7.6) are respectively called (A) and (A').

The numerical approximation of solutions to PDEs by MOL proceeds by introducing lines in G (or G') parallel to either x-axis or y-axis. Along these lines the derivatives in the normal directions are replaced by finite difference quotients. An approximate solution is then obtained by solving the resulting coupled system of ODEs. The successfulness of the MOL depends on the convergence of the approximate solutions to the corresponding solution. Walter [3], R.C. Thompson [32], etc. discussed the MOL for various PDEs using the theory of differential inequalities. As mentioned in the begining the main objective here is to apply MOL to the problems (A) and (A') and to obtain the error estimates and convergence of MOL with the help of the theory of differential inequalities. It is reasonable to expect, because of the presence of the small parameter ε , that the usual discretization in the MOL is to be modified in order to get satisfactory results.

1. MOL TO THE BVP (A).

The purpose of this section is to point out the difficulties that one faces in the usual discretization of the MOL when applied to the problems of type (A).

Consider the BVP (A) and introduce a mesh size $h= 1/n$, n being a natural number, and denote the grid functions by $v_i(x) = v(x,y_i)$. For an (n+1)- dimensional vector $\underset{\sim}{v}(x) = (v_o, \ldots, v_n)$, the central finite differences corresponding to the first and second derivatives with respect to y are defined as follows:

$$\delta v_i = (v_{i+1} - v_{i-1}) / 2h,$$
$$\delta^2 v_i = (v_{i+1} - 2v_i+v_{i-1}) /h^2, \quad i=1(1)n-1.$$

With the above notations the discrete problem corresponding to the BVP (A) by MOL is

$$(7.7) \quad \begin{cases} - v_i'' + f(x,y_i,v_i,v_i',\delta v_i,\varepsilon\delta^2 v_i) = 0 \\ v_i(0) = \psi_1(y_i), \; v_i(1) = \psi_2(y_i), \; i=1(1)n-1, \end{cases}$$

where

$$v_i'' = d^2v_i/dx^2, \quad v_i' = dv_i/dx,$$
$$v_o(x) = \phi_1(x), \qquad v_n(x) = \phi_2(x).$$

This BVP may be abbreviated as

$$(7.8) \begin{cases} P_i' \underset{\sim}{v} := -v_i'' + g_i'(x, v_i', \underset{\sim}{v}) = 0, \; i=1(1)n-1, \\ R_i' \underset{\sim}{v} := \begin{cases} v_i(0) = \psi_1(y_i) \\ v_i(1) = \psi_2(y_i), \; i=1(1)n-1, \end{cases} \end{cases}$$

where

$$\underset{\sim}{v} = (v_o, \ldots, v_n)$$
$$g_i'(x, v_i', \underset{\sim}{v}) = f(x, y_i, v_i, v_i', \delta v_i, \varepsilon\,\delta^2 v_i),$$
$$i = 1(1)\; n-1.$$

Thompson [32] derived estimates and obtained the convergence of the MOL when applied to elliptic-parabolic DEs making use of the comparison theorems for the BVP for the ODEs. If one applies the same procedure as given in [32] for the problems (A), the following condition has to be satisfied for the discrete problem (7.8):

(7.9) $\quad h \leq 2\alpha\varepsilon / L$

where α, L are positive constants independent of ε such that

(7.10) $\quad f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,q,\varepsilon\bar{r})$
$$\leq -\alpha\varepsilon(r-\bar{r}), \; r \geq \bar{r}$$

and

$$(7.11) \qquad |f(x,y,z,p,q,\varepsilon r)-f(x,y,z,p,\bar{q},\varepsilon r)| \leq L\ |q-\bar{q}|.$$

The condition (7.9) becomes more stringent as the parameter ε becomes very small. It suggests us to look for a modified version of the existing MOL to the problem (A) as well as to the problem (A') because the same problem is encountered with (A') also.

2. MODIFIED VERSION OF THE MOL TO THE BVP (A).

The following discrete problem is associated with the BVP (A).

$$(7.12) \quad \begin{cases} -\ v_i'' + f(x,y_i,v_i,v_i',\delta v_i,\varepsilon\,\sigma(\rho)\delta^2 v_i) = 0, \\ v_i(0) = \psi_1(y_i),\ v_i(1) = \psi_2(y_i),\ i=1(1)n-1, \end{cases}$$

where $\rho = h/\varepsilon$, $v_o(x) = \phi_1(x)$, $v_n(x) = \phi_2(x)$ and $\sigma(\rho)$, a 'fitting factor' will be specified in the following discussion.

The BVP (7.12) may be abbreviated as

$$(7.13) \quad \begin{cases} P_i''\ \underset{\sim}{v} := -v_i'' + g_i(x,v_i',\underset{\sim}{v}) = 0,\ x \in D, \\ R_i''\ \underset{\sim}{v} := \begin{cases} v_i(0) = \psi_1(y_i) \\ v_i(1) = \psi_2(y_i),\ i=1(1)n-1, \end{cases} \end{cases}$$

where

$$\underset{\sim}{v} = (v_o,\ \ldots,\ v_n)$$

$$g_i(x,v_i',\underset{\sim}{v}) = f(x,y_i,v_i,v_i',\delta v_i,\varepsilon\,\sigma(\rho)\,\delta^2 v_i),$$

$$i=1(1)n-1.$$

The following lemma gives sets of conditions which guarantee that $\underset{\sim}{g} = (g_1, \cdots \cdots, g_{n-1})$ of (7.13) would be quasimonotone decreasing in $\underset{\sim}{v}$ in the sense of Definition 2.1.

Lemma 7.1.

The function $\underset{\sim}{g} = (g_1, \cdots \cdots, g_{n-1})$ of (7.13) is quasimonotone decreasing in $\underset{\sim}{v}$ if one of the following sets of conditions are satisfied:

Q_1. $f(x,y,z,p,q,\varepsilon r)$ is independent of p and $\sigma(\rho)=1$.

Q_2. (a) δv_i is the central difference;

(b) $$f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,q,\varepsilon \bar{r}) \leq -\alpha\varepsilon(r-\bar{r}), \quad r \geq \bar{r};$$

(c) $|f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,\bar{q},\varepsilon r)| \leq L|q-\bar{q}|$;

(d) $\sigma(\rho) = (L\rho/2\alpha)/(1-e^{-L\rho/2\alpha})$, or
$\sigma(\rho) = (L\rho/2\alpha) \cot h(L\rho/2\,\alpha)$.

Q_3. (a) δv_i is the forward or backward difference, Q_2(b) and Q_2(c) hold true;

(b) $\sigma(\rho) = (L\rho/\alpha)/(1-e^{-L\rho/\alpha})$, or

$\sigma(\rho) = (L\rho/\alpha) \cot h\,(L\rho/\alpha)$.

Q_4. (a) f is decreasing (respectively increasing) in p and δv_i is the forward (respectively backward) difference;

(b) $\sigma(\rho) = 1$.

Proof

A proof is given for this lemma assuming only the conditions stated in Q_2. Similar proofs can be given considering each of the other conditions stated above. First it can be verified that

(7.14) $\quad \sigma(\rho) \geq L\rho/2\alpha$, that is, $L/2 - \alpha\sigma(\rho)\varepsilon/h \leq 0$.

Next

$$g_i(x,v_i',v_{i+1}+\varepsilon_o,v_i,v_{i-1}) - g_i(x,v_i',v_{i+1},v_i,v_{i-1})$$

$$= f(x,y_i,v_i,v_i',\delta v_i+\varepsilon o/2h,\ \varepsilon(\delta^2 v_i+\varepsilon o/h^2)\ \sigma(\rho))$$

$$- f(x,y_i,v_i,v_i',\delta v_i,\varepsilon\,\sigma(\rho)\delta^2 v_i)$$

$$\leq (L\,\varepsilon o/2h) - \alpha\,\varepsilon\,\sigma(\rho)\ \varepsilon o/h^2$$

$$= \varepsilon o/h\ (L/2 - \alpha\,\varepsilon\,\sigma(\rho)/h\ ) \leq 0.$$

This shows that g_i is decreasing with v_{i+1}. Similarly it can be proved that g_i is decreasing with v_{i-1}.

The BVP (7.13) is consistent with the BVP (A) in the sense that the former problem coincides with the latter when h=0. Further the following estimate is true for $\sigma(\rho)$:

(7.15) $\quad |\varepsilon\,\sigma(\rho) - \varepsilon| \leq Ch$

for some constant C which is independent of ε.

The Theorem 2.2 shall be used in the following to derive error estimates for the modified MOL. Apart from the quasi-monotonicity condition this theorem demands that the BVP (7.13) should possess a 'test function'. A set of conditions under which such a 'test function' exists is given now.

Before presenting these conditions a theorem from [37] which shall be made use of in the further discussion is stated now.

Theorem 7.2.

Consider the nonlinear two point BVP

$$(7.16)\quad \begin{cases} Py := -y'' - f(x,y,y') = 0, \quad x \in D = (a,b), \\ Ry := \begin{cases} y(a) = \eta_1 \\ y(b) = \eta_2, \quad y \in U := C^2(D) \cap C(\bar{D}). \end{cases} \end{cases}$$

Suppose $f(x,y,y')$ satisfies the following conditions:

$$(7.17)\quad | f(x,y,y') - f(x,y,z') | \leq M |y'-z'| ,$$

$$(7.18)\quad | f(x,y,y') - f(x,z,y') | \leq N |y - z|.$$

Then there exists a 'test function' $s \in U$ such that $s(x) > 0$ on $\bar{D}$ and $P(y+\alpha_1 s) - Py > 0$ for every $y \in U$ and for every $\alpha_1 > 0$ provided $b-a < 2\,\alpha[M, N]$

where

$$(7.19)\quad \alpha[M,N]:=\begin{cases} 2/(4N-M^2)^{1/2}\cos^{-1}(M/2N^{1/2}) & \text{if } 4N-M^2>0,\\ 2/(M^2-4N)^{1/2}\cosh^{-1}(M/2N^{1/2}) & \text{if } 4N-M^2<0,\\ & M>0,\ N>0\\ 2/M & \text{if } M^2=4N,\ M>0\\ \alpha & \text{otherwise.}\end{cases}$$

Proof

For every positive number α_1 and $y \in U$,

$$\begin{aligned} P(y+\alpha_1 s)-Py &= -\alpha_1 s''-[f(x,y+\alpha_1 s,\ y'+\alpha_1 s')-f(x,y,y')]\\ &= -\alpha_1 s''-[f(x,y+\alpha_1 s,y'+\alpha_1 s')-f(x,y+\alpha_1 s,y')\\ &\qquad + f(x,y+\alpha_1 s,y') - f(x,y,y')]\\ &\geq -\alpha_1 s'' - M\alpha_1|s'| - N\alpha_1 s \text{ (Using (7.17) and (7.18)).}\end{aligned}$$

The conclusion of the present theorem follows if there exists a function $s(x)$ such that

$s(x) > 0$ on $\bar{D}$, $s \in U$,

$s'' + M|s'| + Ns < 0$ in D.

In the proof of Theorem 2.2 [37, p.145], the existence of such a function $s(x)$ is established.

Lemma 7.3.

Let $f(x,y,z,p,q,\varepsilon r)$ satisfy the Lipschitz conditions

(7.20) $\quad |f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},p,q,\varepsilon r)| \le N|z-\bar{z}|$

and

(7.21) $\quad |f(x,y,z,p,q,\varepsilon r) - f(x,y,z,\bar{p},q,\varepsilon r)| \le M|p-\bar{p}|$,

where M,N are positive constants independent of ε . Then the BVP (7.13) possesses a 'test function' whenever $1 < 2\alpha[M,N]$.

Proof

For every positive number α_1 and from (7.20)-(7.21),

$$
\begin{aligned}
P_i''(\underset{\sim}{v}+\alpha_1\underset{\sim}{s}) - P_i''\underset{\sim}{v} &= -\alpha_1 s_i'' + [g_i(x,v_i'+\alpha_1 s_i', \underset{\sim}{v}+\alpha_1\underset{\sim}{s}) \\
&\qquad - g_i(x,v_i',\underset{\sim}{v})] \\
&= -\alpha_1 s_i'' + f(x,y_i,v_i+\alpha_1 s_i, v_i'+\alpha_1 s_i', \delta v_i, \varepsilon\sigma(\rho)\delta^2 v_i) \\
&\qquad - f(x,y_i,v_i,v_i',\delta v_i,\varepsilon\sigma(\rho)\delta^2 v_i) \\
&\ge -\alpha_1 [s'' + M|s'| + Ns],
\end{aligned}
$$

where $\underset{\sim}{s} = (s_0,\dots \dots, s_n)$ with $s(x) = s_i(x)$, $i=0(1)n$. Then from Theorem 7.2 there exists a function s(x) with the properties

$s(x) > 0$ on $\bar{D}$, $s \in U$,

$s'' + M|s'| + Ns < 0$ in D.

3. ERROR ESTIMATES FOR THE MOL TO THE BVP (A).

In this section error estimates are obtained and the convergence of the modified version of the MOL to the BVP (A) is studied. It is assumed that the BVPs (A) and (7.13) possess solutions.

Theorem 7.4.

Let $\underset{\sim}{v} = (v_o, \ldots \ldots, v_n)$ be a solution of the discrete problem (7.13), $\underset{\sim}{u}$ be a solution of the continuous problem (A) with $\underset{\sim}{u} = (u_o, \ldots \ldots, u_n)$, $u_i(x) = u(x,y_i)$, $i=0(1)n$. It is assumed that f,h and the differences δv_i are defined in such a way that one of the sets of conditions of Lemma 7.1 is satisfied. Further assume that there exists a 'test function' for the BVP (7.13) so that the monotonicity Theorem 2.2 is applicable for this. If

(α) $$|u_y(x,y_i) - \delta u_i| \le \alpha(x),$$

$$|\varepsilon u_{yy}(x,y_i) - \varepsilon\,\sigma(\rho)\delta^2 u_i| \le \varepsilon\,\beta(x) + Kh \text{ in } D,\ i=1(1)n-1,$$

for some positive constant K;

(β) $$f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},\bar{p},\bar{q},\varepsilon\bar{r})$$

$$\ge \omega\,(x, z-\bar{z}, |p-\bar{p}|, |q-\bar{q}|, \varepsilon(r-\bar{r})),\ z \ge \bar{z},\ r \ge \bar{r};$$

(γ) there exists a non-negative function $\rho_1(x)$ satisfying the differential inequality

$$\rho_1'' \le \omega(x, \rho_1, |\rho_1'|, \alpha(x), \varepsilon\beta(x) + Kh),\ x \in D,$$

where $\omega(x,z,p,q,\varepsilon r)$ is decreasing in $|q|$ and r;

then

$$(7.22)\qquad |u(x,y_i) - v_i(x)| \leq \rho_1(x), \quad x \in \bar{D}, \quad i=1(1)n-1.$$

Proof

The estimate (7.22) is equivalent to

$$(7.23)\qquad -\rho_1(x) + v_i(x) \leq u(x,y_i) \leq v_i(x)+\rho_1(x), \quad x \in \bar{D}, \quad i=1(1)n-1.$$

In the following a proof is given for the left inequality of (7.23). The right inequality can be proved similarly.

Consider the BVP (7.13). All the conditions of Theorem 2.2 are satisfied for this problem and therefore apply the same to the functions v_i and $w_i=u_i+\rho_1$, $i=0(1)n$.

Then

$$w_i(0) = u_i(0) + \rho_1(0) = u(0,y_i) + \rho_1(0)$$
$$\geq u(0,y_i) = \psi_1(y_i) = v_i(0),$$

that is,

$$(7.24)\qquad v_i(0) \leq w_i(0), \quad i = 0(1)n.$$

Similarly

$$(7.25)\qquad v_i(1) \leq w_i(1), \quad i=0(1)n.$$

Also

$$w_o(x) = u_o(x) + \rho_1(x) \geq u_o(x) = u(x,0) = \phi_1(x) = v_o(x),$$

that is ,

(7.26) $\quad v_o(x) \leq w_o(x), \quad x \in \bar{D}.$

Similarly

(7.27) $\quad v_n(x) \leq w_n(x), \quad x \in \bar{D}.$

Next it will be shown that

(7.28) $\quad -w_i'' \geq -f(x,y_i,w_i,w_i', \delta w_i, \varepsilon\, \sigma(\rho)\delta^2 w_i), \quad i=1(1)n-1.$

Then the inequalities (7.24)-(7.28) imply

$$P_i \underset{\sim}{v} \leq P_i \underset{\sim}{w}, \quad R_i \underset{\sim}{v} \leq R_i \underset{\sim}{w}, \quad i = 1(1)n-1,$$

$v_o(x) \leq w_o(x)$, $v_n(x) \leq w_n(x)$, $x \in \bar{D}$, which in turn imply, from Theorem 2.2, the left inequality of (7.23). Hence the theorem is proved.

The inequality (7.28) is established as follows:

$$\begin{aligned} -w_i'' &= -u_i'' - \rho_1'' \\ &= -f(x,y_i,u(x,y_i),u_x(x,y_i), u_y(x,y_i), \varepsilon\, u_{yy}(x,y_i)) - \rho_1'' \\ &\geq -f(x,y_i,u(x,y_i),u_x(x,y_i), u_y(x,y_i), \varepsilon\, u_{yy}(x,y_i)) \\ &\quad - \omega(x, \rho_1, |\rho_1'|, \alpha(x), \varepsilon\beta(x) + Kh). \end{aligned}$$

Since $f(x,y,z,p,q,\varepsilon r)$ is decreasing in the last argument, $\omega(x,z,p,q,\varepsilon r)$ is decreasing in $|q|$ and

$$\varepsilon\sigma(\rho)\,\delta^2 u_i \le \varepsilon\, u_{yy} + \varepsilon\beta(x) + Kh,$$

$$\begin{aligned}
&f(x,y_i,u_i+\rho_1,\ u_i' + \rho_1',\ \delta u_i,\ \varepsilon\,\sigma(\rho)\delta^2 u_i\ )\\
&- f(x,y_i,u_i,u_i',u_y(x,y_i),\ \varepsilon\, u_{yy}(x,y_i))\\
&\ge f(x,y_i,u_i+\rho_1,\ u_i'+\rho_1',\ \delta u_i,\ \varepsilon\, u_{yy}+\varepsilon\beta(x)+Kh)\\
&\quad -f(x,y_i,u_i,u_i',u_y(x,y_i),\ \varepsilon u_{yy}(x,y_i))\\
&\ge \omega(x,\rho_1,\ |\rho_1'|,\ |\delta u_i - u_y(x,y_i)|,\ \varepsilon\beta(x)+Kh)\\
&\ge \omega(x,\rho_1,\ |\rho_1'|,\ \alpha(x),\ \varepsilon\beta(x)+Kh).
\end{aligned}$$

Hence

$$- w_i'' \ge -f(x,y_i,u_i+\rho_1,\ u_i'+\rho_1',\ \delta u_i,\ \varepsilon\,\sigma(\rho)\delta^2 u_i),$$

that is ,

$$- w_i'' \ge -f(x,y_i,w_i,w_i',\ \delta w_i,\ \varepsilon\,\sigma(\rho)\delta^2 w_i),\ i=1(1)n-1,$$

which is nothing but the differential inequality (7.28).

Note.

It is to be noted that

$$\begin{aligned}
|\varepsilon\, u_{yy} - \varepsilon\,\sigma(\rho)\delta^2 u_i| &\le |\varepsilon u_{yy} - \varepsilon\delta^2 u_i| + |\varepsilon\delta^2 u_i - \varepsilon\,\sigma(\rho)\ {}^2 u_i|\\
&\le \varepsilon\,\beta(x) + Kh,
\end{aligned}$$

whenever

$$(7.29)\quad \begin{cases} |u_{yy} - \delta^2 u_i| \le \beta(x), \ i=1(1)n-1, \\ |\delta^2 u_i| \le K/c, \ i=1(1)n-1. \end{cases}$$

The method of proof adopted in the above theorem is different from that of given in [3] and [32].

Theorem 7.5.

If under the general hypotheses of Theorem 7.4,

(i) $|u_y(x,y_i) - \delta u_i| \le \alpha(x), \ x \in D;$

(ii) $|\varepsilon u_{yy}(x,y_i) - \varepsilon\sigma(P)\delta^2 u_i| \le \varepsilon\beta(x) + Kh, \ x \in D,$
$i = 1(1)n-1;$

(iii) $f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},\bar{p},\bar{q},\varepsilon\bar{r})$

$$\ge A(x)(z-\bar{z}) + B(x)|p-\bar{p}| - C(x)|q-\bar{q}| - \varepsilon D(x)(r-\bar{r}),$$

where the functions $A(x)$ to $D(x)$ are non-negative, $B(x)$ and $\alpha(x)\,C(x) + D(x)\,(\varepsilon\beta(x)+Kh)$ are continuous in D and integrable over $\bar{D}$;

then

$$(7.30)\quad |u(x,y_i)-v_i(x)| \le \int_x^1 \int_0^t e^{\int_\xi^t B(\eta)d\eta} [C(\xi)\,\alpha(\xi) + D(\xi)\,(\varepsilon\beta(\xi) + Kh)]d\xi\, dt, \quad i=1(1)n-1.$$

Proof

According to Theorem 2.2 it is enough to construct a function $\rho_1(x)$ satisfying the inequalities

$$\rho_1(x) \geq 0 \quad \text{on } \bar{D}$$

and

$$\rho_1'' \leq A(x)\rho_1 + B(x)|\rho_1'| - C(x)\alpha(x) - D(x)(\varepsilon\beta(x)+Kh).$$

One such function $\rho_1(x)$ satisfying the above requirements is got by solving the following DE subject to the conditions $\rho_1(x) \geq 0$, $\rho_1'(x) \leq 0$:

$$\rho_1'' + B(x)\rho_1' = -C(x)\alpha(x) - D(x)(\varepsilon\beta(x)+Kh), \quad x \in D.$$

The solution is given by

$$\rho_1(x) = \int_x^1 \int_0^t e^{\int_\xi^t B(\eta)d\eta}[C(\xi)\ \alpha(\xi) + D(\xi) \times (\varepsilon\,\beta(\xi) + Kh)]\, d\xi\ dt.$$

4. CONVERGENCE OF MOL TO THE BVP (A).

By means of Theorem 7.4 it is possible to obtain results on the convergence of solution of the discrete BVP (7.13) to the solution of the continuous problem (A) as $n \to \infty$. If $u(x,y)$ is a solution of (A), denote the vector function $(u_o(x), \ldots, u_n(x))$ by $\underset{\sim}{u}^n(x)$ with $u_i(x) = u(x,y_i)$, $i=0(1)n$.

For a given value of n the solution of (7.13) is denoted by $\underset{\sim}{v}^n = (v_0, \ldots, v_n)$. The norm $\| \underset{\sim}{w}^n \|$, where $\underset{\sim}{w}^n$ is an (n+1)-dimensional vector function, is defined by

$$(7.31) \qquad \| \underset{\sim}{w}^n \| \doteq \max\{ \mid w_i(x) \mid \; : \; x \in \bar{D},\ i=0(1)n \} .$$

Define the sequence of numbers $\{\alpha_n\}$ and $\{\beta_n\}$ by the equations,

$$(7.32) \qquad \alpha_n = \sup_{x \in \bar{D}} \left\{ \max_{0 \leq i \leq n} |u_y(x,y_i) - \delta u_i| \right\} ,$$

and

$$(7.33) \qquad \beta_n = \sup_{x \in \bar{D}} \left\{ \max_{0 \leq i \leq n} |u_{yy}(x,y_i) - \delta^2 u_i| \right\} .$$

Theorem 7.6.

Let u(x,y) be the solution of the BVP(A) with u, u_y, u_{yy} continuous on $\bar{D} \times \bar{D}$. Suppose one of the conditions of Lemma 7.1 and the condition (iii) of Theorem 7.5 are satisfied. Further suppose there exists a 'test function' for the BVP(7.13). Then

$$\| \underset{\sim}{u}^n - \underset{\sim}{v}^n \| \longrightarrow 0 \text{ as } n \to \infty, \text{ provided}$$

$$| \delta^2 u_i | \leq C_1 \text{ for some positive constant } C_1,$$

$i = 1(1)n-1$.

Proof

The hypotheses on $f(x,y,z,p,q,\varepsilon r)$ imply that

Theorem 7.5 is applicable to the BVP (7.13) for each value of n. Further uniform continuity of u_y and u_{yy} implies that α_n and β_n defined in (7.32)-(7.33) tend to zero as n tends to infinity. The conclusion of the present theorem follows by letting $\alpha(s) \equiv \alpha_n$ and $\beta(s) \equiv \beta_n$ in (7.30) and passing to the limit as $n \to \infty$.

5. ILLUSTRATIONS.

(α) Consider the linear equation

$$au_{xx} + \varepsilon cu_{yy} + du_x + eu_y + fu + g = 0, \tag{7.34}$$

where a, c, d, e, f, g are functions defined on D. If $a(x,y) \leq \bar{a} < 0$ and $c(x,y) \leq \bar{c} < 0$ for all $(x,y) \in \bar{D}$, then Q2(b) is satisfied. If there exists a bounded integrable function h(x) such that $|c/a|$, $|d/a|$, $|e/a| \leq h(x)$ for all $(x,y) \in D$, then Q2(c), (7.17) and (7.18) are satisfied. If the condition on c is weakened to $c(x,y) \leq 0$, then Q2(b) is not satisfied. For example, if $c(x,y) \equiv 0$ then the equation (7.34) becomes a parabolic equation.

(β) Consider an almost nonlinear equation

$$au_{xx} + \varepsilon cu_{yy} + du_x + eu_y + f(x,y,u) = 0.$$

If a,b,c,d and e satisfy the conditions given above then this equation satisfy Theorem 7.6.

6. MOL TO THE BVP (A').

Consider the BVP (A') and apply MOL to this as in Section 2 so that one gets the following discrete problem:

$$
(7.35) \quad \begin{cases} -v_i'' + f(x, y_i, v_i, v_i', \delta v_i, \varepsilon\,\sigma(\rho)\delta^2 v_i) = 0, \\ v_i(0) = \eta_1(y_i),\ v_i(1) = \eta_2(y_i)\ ,\ i \in Z,\ x \in D, \end{cases}
$$

where $\rho = h/\varepsilon$, $h > 0$, $y_i = ih$, the differences δv_i can be either central, forward or backward, $\delta^2 v_i$ is the central difference.

In the discrete problem (7.35) the condition (7.6) is not included as this is not directly used in the extension of the results of the previous sections. Some sort of condition of this type is necessary in order that the BVP be properly formulated. Since the existence and uniqueness of solutions are not established here, condition (7.6) is not included in (7.35).

The BVP (7.35) may be conveniently written as

$$
(7.36) \quad \begin{cases} -v_i'' + g_i(x, v_i', \underset{\sim}{v}) = 0\ , \\ v_i(0) = \eta_1(y_i),\ v_i(1) = \eta_2(y_i),\ i \in Z, \end{cases}
$$

where

$$
g_i(x, v_i', \underset{\sim}{v}) = f(x, y_i, v_i, v_i', \delta v_i, \varepsilon\,\sigma(\rho)\delta^2 v_i),\ i \in Z.
$$

Consider the problem (7.36) as a BVP in the Banach space B. Using Theorem 2.5 error estimates for the BVP (7.36) are obtained.

Lemma 7.7.

The function $\underset{\sim}{g} = (\ldots, g_{-1}, g_o, g_1, \ldots)$ in (7.36) is quasimonotone decreasing in $\underset{\sim}{v}$ in the sense of Definition 2.4 if one of the following sets of conditions is satisfied for the function f of the BVP (A'):

Q_1. (a) f is independent of p;

(b) $f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,q,\varepsilon \bar{r})$
$\geq -K\,\varepsilon(r-\bar{r}),\ r \geq \bar{r}$;

(c) δv_i, $\delta^2 v_i$ are the central differences and $\sigma(\rho) = 1$.

Q_2. (a) $|f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,\bar{q},\varepsilon r)| \leq L|q-\bar{q}|$;

(b) $-K\varepsilon(r-\bar{r}) \leq f(x,y,z,p,q,\varepsilon r) - f(x,y,z,p,q,\varepsilon\bar{r})$
$\leq -\alpha\varepsilon(r-\bar{r}),\ r \geq \bar{r}$ and δv_i, $\delta^2 v_i$ are the central differences;

(c) $\sigma(\rho) = (L\rho/2\alpha)/(1-e^{-L\rho/2\alpha})$ or

$\sigma(\rho) = (L\rho/2\alpha) \cot h(L\rho/2\alpha)$, $\rho = h/\varepsilon$.

Q_3. (a) Q_2(a)-Q_2(b) hold true, δv_i is the forward or backward difference and $\delta^2 v_i$ is the central difference;

(b) $\sigma(\rho) = (L\rho/\alpha) / (1-e^{-L\rho/\alpha})$ or

$\sigma(\rho) = (L\rho/\alpha) \cot h(L\rho/\alpha)$, K,L are positive constants.

Comparing the conditions of Lemma 7.1 and Lemma 7.7, it is seen that an extra condition is imposed in Lemma 7.7 rather than in Lemma 7.1. This has been imposed to meet the quasi-monotonicity condition. Following the same procedure of the proof of Lemma 7.1, the above lemma can be proved.

Lemma 7.8.

Let $f(x,y,z,p,q,\varepsilon r)$ satisfy the Lipschitz conditions:

(7.37) $\quad |f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},p,q,\varepsilon r)| \leq N|z-\bar{z}|,$

(7.38) $\quad |f(x,y,z,p,q,\varepsilon r) - f(x,y,z,\bar{p},q,\varepsilon r)| \leq M|p-\bar{p}|,$

where M and N are positive constants independent of ε. Then the problem (7.36) has a 'test function' whenever $1 < 2\,\alpha[M,N]$. Here $\alpha[M,N]$ is given as in (7.19).

To prove this lemma it is enough to construct a 'test function' so that Theorem 2.5 holds. Hence a proof is not needed here.

Theorem 7.9.

Suppose $\underset{\sim}{v} = (\ldots, v_{-1}, v_0, v_1, \ldots)$ is a solution of the BVP (7.36), u is a solution of the BVP (A') with $\underset{\sim}{u} = (\ldots, u_{-1}, u_0, u_1, \ldots)$, $u_i(x) = u(x,y_i)$. It is assumed that f,h and the differences are defined in such a way that one of the conditions of Lemma 7.7 is satisfied. Further assume that there exist constants M,N such that the Lipschitz conditions (7.37) and (7.38) are satisfied. If

(α) $| u_y(x,y_i) - \delta u_i| \leq \alpha(x),$

$|\varepsilon u_{yy}(x,y_i) - \varepsilon\sigma(\rho)\delta^2 u_i| \leq \varepsilon\beta(x)+Kh, \; x\in\bar{D}, \; i\in Z,$

for some positive constant K;

(β) $f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},\bar{p},\bar{q},\varepsilon\bar{r})$

$\geq \omega(x,z-\bar{z}, |p-\bar{p}|, |q-\bar{q}|, \varepsilon(r-\bar{r})), \; z \geq \bar{z}, \quad r \geq \bar{r};$

(γ) there exists a non-negative function $\rho_1(x)$ satisfying the differential inequality

$$\rho_1'' \leq \omega(x, \rho_1, |\rho_1'|, \alpha(x), \varepsilon\beta(x) + Kh)$$

where $\omega(x,z,p,q,\varepsilon r)$ is decreasing in $|q|$ and r;

then

(7.39) $|u(x,y_i) - v_i(x)| \leq \rho_1(x), \; x\in\bar{D}, \; i\in Z.$

Proof

The hypotheses made here guarantee that one could apply Theorem 2.5 for the BVP (7.36). In order to prove the inequality (7.39) it is enough to follow the same procedure given in Theorem 7.4.

Theorem 7.10.

If under the general hypotheses of the above theorem,

(i) $| u_y(x,y_i) - \delta u_i| \leq \alpha(x),\ i \in Z;$

(ii) $| \varepsilon u_{yy}(x,y_i) - \varepsilon \sigma(\rho)\delta^2 u_i| \leq \varepsilon\, \beta(x) + Kh,\ i \in Z,$
for some positive constant K;

(iii) $$f(x,y,z,p,q,\varepsilon r) - f(x,y,\bar{z},\bar{p},\bar{q},\varepsilon\bar{r}) \geq A(x)(z-\bar{z}) + B(x)|p-\bar{p}| - C(x)|q-\bar{q}| - \varepsilon D(x)(r-\bar{r}),$$
$$z \geq \bar{z},\quad r \geq \bar{r},$$

where A(x) to D(x) are non-negative, B(x), $C(x)\alpha(x) + D(x)(\varepsilon\beta(x)+Kh)$ are continuous in D and integrable in $\bar{D}$;

then

$$| u(x,y_i)-v_i(x)| \leq \int_x^t \int_0^t e^{\int_\xi^t B(\eta)d\eta} [C(\xi)\ \alpha(\xi) + D(\xi)\ (\varepsilon\beta(\xi) + Kh)]\ d\xi\ dt,\ i \in Z.$$

The proof of this theorem follows from Theorem 7.9 and Theorem 7.5.

The following theorem gives the convergence for the MOL to the BVP (A'). The proof of this theorem runs along the lines of finite-dimensional case and hence is omitted.

Theorem 7.11.

Let $u(x,y)$ be the solution of the BVP (A'). Further let u_y, u_{yy} be bounded and uniformly continuous on $\overline{G}'$, and $|\delta^2 u_i| \leq C_1$ for some positive constant C_1, $i \in Z$. Suppose that condition (iii) of the above theorem is true, then

$$\sup\{\,|u(x,y_i) - v_i^h(x)|: x \in \overline{D},\ i \in Z\} \to 0 \text{ as } h \to 0,$$

where $\underset{\sim}{v}^h$ is a solution of the BVP (7.36) for a finite h.

This chapter discussed the MOL to the DEs subject to Dirichlet's type BCs. One can consider other types of BCs and discuss the MOL.

Chapter 8

METHOD OF LINES FOR PARABOLIC DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

This chapter considers the MOL applied to non-linear parabolic DEs defined on a rectangular region and an infinite strip. Infact the MOL is applied to the following parabolic BVP and the Cauchy problem for the parabolic DEs respectively:

(8.1) $$u_t - f(t,x,u,u_x, \varepsilon u_{xx}) = 0, \quad (t,x) \in G_p := (0,T] \times (0,a), \quad T < \infty, \quad a < \infty,$$

(8.2) $$u(0,x) = \eta(x), \quad x \in [0, a],$$

(8.3) $$u(t,0) = \eta_0(t), \; u(t,a) = \eta_1(t), \quad t \in [0,T]$$

and

(8.4) $$u_t - f(t,x,u,u_x, \varepsilon u_{xx}) = 0, \quad (t,x) \in G'_p : = (0,T] \times R, \quad R = (-\infty, \infty),$$

(8.5) $$u(0,x) = \eta_2(x), \quad x \in R,$$

where ε is a small positive parameter, $f(t,x,z,p,\varepsilon r)$ is assumed to be monotone increasing in the last argument r which defines the parabolic character of the DEs (8.1) and (8.4) [3].

In the following the IBVP (8.1)-(8.3) and the Cauchy problem (8.4)-(8.5) are respectively called (A) and (A').

1. MOL TO THE IBVP (A).

Aim of this section is to explain the difficulties that one faces when the usual discretization procedure in the MOL is applied to the problems of the type (A).

Discretization of the problem (A) with respect to x yields

$$
(8.6)\quad \begin{cases} v_i'(t) - f(t, x_i, v_i, \delta v_i, \varepsilon\, \sigma(\rho)\, \delta^2 v_i) = 0, \\ v_i(0) = \eta(x_i),\ i = 1(1)n-1, \end{cases}
$$

where $v_i(t) = dv_i/dt$. This IVP may be abbreviated as

$$
(8.7)\quad \begin{cases} P_i\, \underset{\sim}{v} := v_i' - g_i(t, \underset{\sim}{v}) = 0,\ t \in J_o, \\ R_i\, \underset{\sim}{v} := v_i(0) = \eta_i = \eta(x_i),\quad i = 1(1)\ n-1, \end{cases}
$$

where

$$\underset{\sim}{v} = (v_o, \ldots, v_n) \text{ and}$$

$$g_i(t, \underset{\sim}{v}) = f(t, x_i, v_i, \delta v_i, \varepsilon\, \sigma(\rho)\, \delta^2 v_i),\ i=1(1)n-1.$$

The following lemma gives sets of sufficient conditions which guarantee that $\underset{\sim}{g} = (g_1, g_2, \ldots, g_{n-1})$ in the IVP (8.7) would be quasimonotone increasing in the sense of Definition 2.1.

Lemma 8.1.

The function $\underset{\sim}{g}$ of the IVP (8.7) is quasimonotone increasing in $\underset{\sim}{v} = (v_0, \ldots, v_n)$ in the sense of Definition 2.1 if one of the following sets of conditions is satisfied:

Q_1. (a) $f(t,x,z,p,\varepsilon r)$ is independent of p and $\sigma(\rho) = 1$.

Q_2. (a) $f(t,x,z,p,\varepsilon r) - f(t,x,z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r})$;

(b) $|f(t,x,z,p,\varepsilon r) - f(t,x,z,\bar{p},\varepsilon r)| \leq L|p-\bar{p}|$;

where α, L are positive constants independent of ε ;

(c) $\sigma(\rho) = (L\rho/2\alpha) / (1-e^{-L\rho/2\alpha})$ or

$\sigma(\rho) = (L\rho/2\alpha) \coth (L\rho/2\alpha)$;

(d) δv_i is the central difference.

Q_3. (a) δv_i is the forward or backward difference, and Q_2(a) and Q_2(b) hold true;

(b) $\sigma(\rho) = (L\rho/\alpha) / (1-e^{-L\rho/\alpha})$ or

$\sigma(\rho) = (L\rho/\alpha) \coth (L\rho/\alpha)$.

Q_4. (a) f is decreasing (respectively increasing) in p and δv_i is the forward (respectively backward) difference;

(b) $\sigma(\rho) = 1$.

Proof

In the following this lemma is proved by considering, for example, the conditions stated in Q_2. First observation is that $\sigma(\rho)$ satisfies

$$(8.8) \qquad \sigma(\rho) \quad \geq \quad L\rho/2\alpha .$$

Secondly g_i depends only on t, v_{i-1}, v_i, v_{i+1}.

Then

$$
\begin{aligned}
& g_i(t,v_{i-1},v_i,v_{i+1}+\varepsilon_o)-g_i(t,v_{i-1},v_i,v_{i+1}) \\
& \quad = f(t,x_i,v_i,\delta v_i+ \varepsilon_o/2h, \varepsilon\,\sigma(\rho)\ (\delta^2 v_i+\varepsilon_o/h^2)) \\
& \qquad -f(t,x_i,v_i,\delta v_i,\varepsilon\,\sigma(\rho)\delta^2 v_i) \\
& \quad \geq -L\,\varepsilon_o/2h + \alpha\varepsilon\,\sigma(\rho)\ \varepsilon_o/h^2 \\
& \quad = (\alpha\varepsilon\,\varepsilon_o/h^2)(\sigma(\rho)-hL/2\alpha\varepsilon) \geq 0,
\end{aligned}
$$

which shows that g_i is increasing with v_{i+1}. Similarly one can prove that g_i is increasing with v_{i-1}. Hence the proof of this lemma.

The IVP (8.7) is consistent with the problem (A) in the sense that the former coincides with the latter when $h = 0$.

From the proof of the above lemma it appears that the condition on $\sigma(\rho)$ ($Q_2(c)$ and $Q_3(b)$ of Lemma 8.1) may be replaced by the inequality (8.8). This cannot be done because $\sigma(\rho)$ has to satisfy, in the future, the second part of the condition (α) of Theorem 8.3. This in turn, sometimes, insists that the estimate (7.15) has to be satisfied by $\sigma(\rho)$.

The error estimates and hence the convergence to the MOL when applied to the problem (8.1)-(8.3) are obtained using Theorem 2.3. But this theorem demands, apart from the quasi-monotonicity condition on $\underset{\sim}{g}$, that the IVP (8.7) should possesses a 'test function'. A set of conditions under which such a 'test function' exists is given now.

Lemma 8.2.

Let $f(t,x,z,p,\varepsilon r)$ satisfy the one sided Lipschitz condition

$$f(t,x,z,p,\varepsilon r)-f(t,x,\bar{z},p,\varepsilon r) \leq L(z-\bar{z}), \; z \geq \bar{z},$$

where $L > 0$. For the IVP (8.7) the function $\underset{\sim}{s} = (s_o,\ldots\ldots,s_n)$, $s_i(t) = s(t) = e^{\beta t}$, $\beta > L$, $i = 0(1)n$, is a 'test function' provided that δv_i is taken to be the central difference.

Proof

For every $\alpha > 0$ and $\underset{\sim}{v}$, $v_i \in U: = C^1(J_o) \cap C(J)$, $i = 0(1)n$,

$$P_i(\underset{\sim}{v}+\alpha\underset{\sim}{s})-P_i\underset{\sim}{v} = \alpha s_i' - [g_i(t,\underset{\sim}{v}+\alpha\underset{\sim}{s}) - g_i(t,\underset{\sim}{v})]$$

$$= \alpha s_i' - [f(t,x_i,v_i+\alpha s_i, \delta v_i, \varepsilon\sigma(\rho)\delta^2 v_i)$$

$$- f(t,x_i,v_i,\delta v_i,\varepsilon\sigma(\rho)\delta^2 v_i)]$$

$$\geq \alpha s(t) (\beta-L) > 0, \ i = 1(1)n-1.$$

2. ERROR ESTIMATES FOR THE MOL TO THE BVP (A).

In this section the error estimates are obtained and hence studied the convergence of the modified version of the MOL to the problem (A). The existence and the uniqueness of solutions for the problems under consideration are not discussed here and hence it is assumed that both the problems (A) and (8.7) possess unique solutions.

Theorem 8.3.

Let $\underset{\sim}{v} = (v_o, \ldots, v_n)$ be a solution of the IVP (8.7), u be a solution of the problem (A) with $\underset{\sim}{u} = (u_o, \ldots, u_n)$, $u_i(t) = u(t,x_i)$, $x_i = ih$, $i = 0(1)n$. It is assumed that f, h and the difference δv_i are defined in such a way that one of the sets of conditions of Lemma 8.1 is satisfied. Further assume that there exists a 'test function' for the IVP (8.7) so that Theorem 2.3 is applicable to this. If

(α) $\quad | u_x(t,x_i) - \delta u_i | \leq \alpha(t),$

$|\varepsilon u_{xx}(t,x_i) - \varepsilon\sigma(\rho)\delta^2 u_i | \leq \varepsilon\beta(t)+Kh, \ t \in J_o,$

$i=1(1)n-1$, for some positive constant K;

(β) $\quad f(t,x,z,p,\varepsilon r) - f(t,x,\bar{z},\bar{p},\varepsilon\bar{r})$

$\leq \omega(t,z-\bar{z}, |p-\bar{p}|, \varepsilon(r-\bar{r})), \quad z \geq \bar{z}, \ r \geq \bar{r},$

where $\omega(t,z,p,\varepsilon r)$ is defined in $J_o \times R_+^3$, $R_+ := [0, \infty)$, and increasing in $|p|$ and r;

(γ) there exists a non-negative function $\rho_1(t)$ satisfying the differential inequality

(8.9) $\quad \rho_1'(t) > \omega(t,\rho_1(t), \alpha(t), \varepsilon\beta(t)+Kh), \ t \in J_o,$

then

(8.10) $\quad |u(t,x_i) - v_i(t)| \leq \rho_1(t), \ t\in J, \ i=1(1)n-1.$

Proof

The error estimate (8.10) is equivalent to

(8.11) $\quad -\rho_1(t) + v_i(t) \leq u(t,x_i) \leq v_i(t) + \rho_1(t), \ t\in J,$

$i=1(1)n-1.$

In the following a proof is given to establish the left inequality of (8.11). The right inequality can be proved similarly.

All the conditions of Theorem 2.3 are satisfied for the IVP (8.7) and hence it can be applied to the functions $v_i(t)$ and $w_i(t) = u_i(t) + \rho_1(t)$, $i = 0(1)n$.

$$w_i(0) = u_i(0) + \rho_1(0) \geq u(0,x_i) = \eta(x_i) = v_i(0),$$
$$i = 1(1)n-1,$$

that is,

(8.12) $\quad v_i(0) \leq w_i(0)$, $i = 1(1)n-1$.

Also

$$w_o(t) = u_o(t) + \rho_1(t) \geq u_o(t) = u(t,0) = \eta_o(t) = v_o(t),$$

that is,

(8.13) $\quad v_o(t) \leq w_o(t)$, $t \in J$.

Similarly

(8.14) $\quad v_n(t) \leq w_n(t)$, $t \in J$.

If it is possible to establish the differential inequality

(8.15) $\quad w_i' \geq f(t,x_i,w_i,\delta w_i,\varepsilon\sigma(\rho)\delta^2 w_i)$, $i=1(1)n-1$,

then this inequality and (8.12)-(8.14) imply

$$P_i \underset{\sim}{v} \leq P_i \underset{\sim}{w},\quad R_i \underset{\sim}{v} \leq R_i \underset{\sim}{w},\quad i = 1(1)n-1,$$
$$v_o(t) \leq w_o(t),\ v_n(t) \leq w_n(t),$$

which in turn, from Theorem 2.3, imply the left inequality of (8.11).

In the following differential inequality (8.15) shall be established.

$$w_i' = u_i' + \rho_1' = f(t,x_i,u(t,x_i),\ u_x(t,x_i),\ \varepsilon u_{xx}(t,x_i)) + \rho_1'$$

$$\geq f(t,x_i,u(t,x_i),\ u_x(t,x_i),\ \varepsilon u_{xx}(t,x_i))$$

$$+\ \omega(t,\rho_1,\alpha(t),\ \varepsilon\beta(t) + Kh).$$

Since $f(t,x,z,p,\varepsilon r)$ is increasing in the last argument, $\omega(t,z,p,\varepsilon r)$ is increasing in $|p|$ and

$$\varepsilon\,\sigma(\rho)\delta^2 u_i \ \leq\ \varepsilon\, u_{xx}(t,x_i) + \varepsilon\beta(t) + Kh,$$

then

$$f(t,x_i,u_i+\rho_1,\ \delta u_i,\varepsilon\sigma(\rho)\delta^2 u_i) - f(t,x_i,u_i,u_x(t,x_i),$$

$$\varepsilon u_{xx}(t,x_i)\ )$$

$$\leq f(t,x_i,u_i+\rho_1,\ \delta u_i,\ \varepsilon u_{xx}(t,x_i) + \varepsilon\beta(t) + Kh)$$

$$-\ f(t,x_i,u_i,u_x(t,x_i),\ \varepsilon u_{xx}(t,x_i)\ )$$

$$\leq \omega(t,\rho_1,\ |\delta u_i - u_x(t,x_i)|,\ \varepsilon\beta(t) + Kh)$$

$$\leq \omega(t,\rho_1,\ \alpha(t),\ \varepsilon\beta(t) + Kh).$$

Hence

$$w_i' \geq f(t,x_i,u_i+\rho_1,\delta u_i,\varepsilon\sigma(\rho)\delta^2 u_i)$$

$$\geq f(t,x_i,w_i,\delta w_i,\varepsilon\sigma(\rho)\delta^2 w_i),\ i=1(1)n-1.$$

Hence the theorem is proved.

Theorem 8.4.

If, under the general hypotheses of Theorem 8.3,

(i) $|u_x(t,x_i)-\delta u_i| \leq \alpha(t),\ t\in J_o,\ i=1(1)n-1;$

(ii) $|\varepsilon u_{xx}(t,x_i)-\varepsilon\sigma(\rho)\delta^2 u_i| \leq \varepsilon\beta(t)+Kh,\ t\in J_o,\ i=1(1)n-1;$

(iii) $f(t,x,z,p,\varepsilon r)-f(t,x,\bar{z},\bar{p},\varepsilon\bar{r}) \leq A(t)(z-\bar{z}) + B(t)|p-\bar{p}| + \varepsilon C(t)(r-\bar{r}),\ z \geq \bar{z},\ \ r \geq \bar{r},$

where functions $A(t)$, $B(t)\,\alpha(t) + \varepsilon C(t)\beta(t)$ are continuous in J_o and integrable over J;

then

(8.16) $$|u(t,x_i)-v_i(t)| \leq \int_o^t \{B(s)\alpha(s) + \varepsilon C(s)\beta(s) + Kh\} \times e^{\int_s^t A(s)ds}\, ds,\ \text{in } J,\ i=1(1)n-1.$$

If the functions α,β,A,B,C are constants, the above estimate reduces to

(8.17) $$|u(t,x_i)-v_i(t)| \leq (\alpha B+\varepsilon C\beta+Kh)\ (e^{At}-1)/A,\ i\in Z.$$

Proof

According to Theorem 8.3 it is enough to construct a non-negative function $\rho_1(t)$ satisfying the differential inequality

$$\rho_1'(t) \geq A(t)\rho_1 + B(t)\alpha(t) + \varepsilon C(t)\beta(t) + Kh$$

One such function is got by solving the following IVP

$$\rho_1'(t) = A(t)\rho_1 + B(t)\alpha(t) + \varepsilon C(t)\beta(t) + Kh + \varepsilon_o,$$
$$\rho_1(0) = \varepsilon_o, \quad \varepsilon_o > 0$$

and is given by

$$\rho_1(t) = \int_o^t \{B(s)\alpha(s) + \varepsilon C(s)\beta(s) + Kh + \varepsilon_o\} e^{\int_s^t A(r)dr} ds, \text{ in } J.$$

Thus

$$|u(t,x_i) - v_i(t)| \leq \rho_1(t), \ t \in J_o.$$

Letting $\varepsilon_o \to 0$ one gets the required inequality (8.16).

3. CONVERGENCE OF THE MOL TO THE PROBLEM (A).

By means of Theorem 8.4 it is possible to obtain results on the convergence of the solution of the IVP (8.7) to the solution of the problem (A) as the mesh size goes to zero.

If $u(t,x)$ is a solution of the problem (A), denote the vector function $(u_o(t), \ldots, u_n(t))$ by $\underset{\sim}{u}^n(t)$ with $u_i(t) = u(t,x_i)$, $i=0(1)n$. For a given value of n the solution of the IVP (8.7) will be denoted by

$\underset{\sim}{v}^n = (v_o, \ldots, v_n)$. The norm $\| \underset{\sim}{w}^n \|$, where $\underset{\sim}{w}^n$ is an (n+1)-dimensional vector function, is defined by

$$(8.18) \qquad \| \underset{\sim}{w}^n \| = \max\left\{ |w_i(t)| : t \in J,\ i=0(1)n \right\}.$$

Define the numbers α_n and β_n as

$$(8.19) \qquad \alpha_n = \sup_{t \in J} \left\{ \max_{0 \le i \le n} |u_x(t,x_i) - \delta u_i| \right\}$$

and

$$(8.20) \qquad \beta_n = \sup_{t \in J} \left\{ \max_{0 \le i \le n} |u_{xx}(t,x_i) - \delta^2 u_i| \right\}.$$

Theorem 8.5.

Let $u(t,x)$ be the solution of the problem (A) with u, u_x, u_{xx} be continuous on $J \times \bar{J}$. Suppose one of the conditions of Lemma 8.1 and the condition (iii) of Theorem 8.4 are satisfied. Further suppose there exists a 'test function' for the IVP (8.7).

Then $\|\underset{\sim}{u}^n - \underset{\sim}{v}^n\| \longrightarrow 0$ as $n \to \infty$

provided $|\delta^2 u_i| \leq C_1$, $i=1(1)n-1$, for some constant C_1.

Proof

Since it is assumed that one of the conditions of Lemma 8.1 is satisfied and the existence of a 'test function' for the IVP (8.7), the general hypotheses of Theorem 8.3 are valid. Further, since u, u_x, u_{xx} are continuous on J x J and $|\delta^2 u_i| \leq C$, then the conditions (i) and (ii) of Theorem 8.3 are true. The condition (iii) of the same theorem is assumed to be true here. Thus one may apply Theorem 8.4 to the problem (A). The uniform continuity of u_x and u_{xx} implies that α_n and β_n defined in (8.19)-(8.20) tend to zero as n tends to infinity. The conclusion of this theorem follows immediately by letting $\alpha(s) \equiv \alpha_n$ and $\beta(s) \equiv \beta_n$ in (8.16) and passing to the limit as $n \to \infty$.

4. ILLUSTRATIONS.

(α) Consider the linear equation

$$u_t = \varepsilon a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u + d(t,x).$$

If $a(t,x) \geq \alpha > 0$, the coefficients a,b,c,d are continuous in G, then the convergence results given in Theorem 8.5 hold true in this case.

(β) Consider the mildly nonlinear equation

$$u_t = \varepsilon a(t,x)u_{xx} + b(t,x)u_x + c(t,x,u).$$

Theorem 8.5 can be applied to this equation provided $c(t,x,u)$ satisfies the Lipschitz condition

$$c(t,x,z) - c(t,x,\bar{z}) \leq C(z-\bar{z}) \text{ for } z \geq \bar{z}$$

and a,b,c are continuous in $\bar{G}$.

(γ) For the general quasi-linear equation

$$u_t = \varepsilon a(t,x,u,u_x)u_{xx} + b(t,x,u,u_x)$$

one can get convergence only in a small interval $[0,t_0]$ as seen from the following discussion.

Consider the equation

$$\partial u/\partial t = f(t,x,u,u_x,\varepsilon u_{xx})$$

and assume that f is continuous in $\bar{G} \times E^3$ and satisfies local Lipschitz condition.

To each $M > 0$ there corresponds an $L = L(M)$ such that

$$f(t,x,z,p,\varepsilon r)-f(t,x,\bar{z},\bar{p},\varepsilon\bar{r}) \leq L\{(z-\bar{z})+|p-\bar{p}| +\varepsilon(r-\bar{r})\}$$

for $|z|$, $|\bar{z}|$, $|p|$, $|\bar{p}|$, $|r|$, $|\bar{r}| \leq M$ and $z \geq \bar{z}$, $r \geq \bar{r}$.

Further assume that to each M there corresponds an $\alpha = \alpha(M)$ such that

$$f(t,x,z,p,\varepsilon r) - f(t,x,z,p,\varepsilon\bar{r}) \geq \varepsilon\alpha(r-\bar{r}) \text{ for } r \geq \bar{r}$$

and $\quad |z|,\ |p|,\ |r|,\ |\bar{r}| \ \leq\ M.$

Let u be a solution of the problem (A) whose derivative u_{xxxx} is bounded in G, and let M be a constant such that

$$|u|,\ |u_x|,\ |u_{xx}|,\ |u_{xxx}|,\ |u_{xxxx}| \ \leq\ M \text{ in } \bar{G}.$$

Choose $N > M$ and set

$$f_N(t,x,z,p,\varepsilon r) = \varepsilon\alpha r + f(t,x,\phi_N(z),\ \phi_N(p),\ \phi_N(\varepsilon r) - \alpha\phi_N(\varepsilon r)),$$

where $\phi_N(s) = s$ for $|s| \leq N$ and $\phi_N(s) = \pm\ (N+1)$ for $\pm s \geq N+2$

and $\alpha = \alpha(N+1)$. Then the function f_N satisfies the Lipschitz condition (β) of Theorem 8.3 with $L = L(N+1)$, $\alpha = \alpha(N+1)$ and satisfies the condition (iii) of Theorem 8.4 with $A(t)=B(t)=C(t)=L=L(N+1)$ globally, that is, in $\bar{G} \times E^3$.

Then the solution u is a solution of the equation

$$u_t = f_N(t,x,u,u_x,\ \varepsilon u_{xx})$$

and similarly $\underset{\sim}{v}$ is a solution of the corresponding discrete problem A_n for

$$|v_i|,\ |\delta v_i|,\ |\delta^2 v_i| \ \leq\ N, \text{ that is, for small } t.$$

In Theorem 8.4 choose $\alpha(t)$ and $\beta(t)$ as

$$\alpha(t) = Mh^2/24, \quad \beta(t) = Mh^2/6.$$

Then the inequality (8.17) becomes

$$|u(t,x_i)-v_i(t)| \leq [(1+4\varepsilon)Mh^2/24 + Kh/L]\ (e^{Lt}-1)$$

and

$$|\delta v_i-u_x(t,x_i)| \leq |\delta v_i-\delta u_i| + |\delta u_i-u_x|$$

$$\leq [(1+4\varepsilon)\ Mh/12+ K/L]\ (e^{Lt}-1) + Mh^2/24$$

$$|\,\varepsilon\,\sigma(\rho)\,\delta^2 v_i - \varepsilon\, u_{xx}(t,x_i)|$$

$$\leq |\varepsilon\sigma(\rho)\delta^2 v_i-\varepsilon\delta^2 v_i| + \varepsilon|\delta^2 v_i-u_{xx}(t,x_i)|$$

$$\leq Kh + 5\varepsilon/12\ M(e^{Lt}-1) + \varepsilon/6\ Mh^2.$$

Let t_o be a positive number such that the right hand side of these two inequalities are less than or equal to N-M for $0 \leq t \leq t_o$ and small h. Hence from the proof of Theorem 8.5 it is seen that $v_i, \delta v_i$ and $\delta^2 v_i$ are bounded by the absolute value N for $0 \leq t \leq t_o$.

That is, if f satisfies the conditions stated above (local Lipschitz condition and local strong parabolicity) and if u is a solution of the problem (A) with a bounded derivative u_{xxxx}, then there exists a positive t_o such that

$$u(t,x_i) - v_i^n(t) = O(h) \text{ uniformly in } 0 \leq t \leq t_o.$$

The result applies to the quasi-linear equation.

5. MOL TO THE BVP (A').

Consider the Cauchy problem (A') and apply the MOL to this problem so as to get the following discrete problem

$$(8.21)\quad \begin{cases} v_i' - f(t, x_i, v_i, \delta v_i, \varepsilon\sigma(\rho)\delta^2 v_i) = 0, & t \in J_o, \\ v_i(0) = \eta_i, & i \in Z \,. \end{cases}$$

The IVP (8.21) may be conveniently written as

$$(8.22)\quad \begin{cases} v_i' - g_i(t, \underset{\sim}{v}) = 0, & t \in J_o \\ v_i(0) = \eta_i, & i \in Z, \end{cases}$$

where

$$\underset{\sim}{v} = (\ \ldots,\ v_{-1},\ v_o,\ v_1,\ \ldots\ ) \text{ and}$$

$$g_i(t, \underset{\sim}{v}) = f(t, x_i, v_i, \delta v_i, \varepsilon\sigma(\rho)\delta^2 v_i),\ i \in Z.$$

Theorem 2.6 will be made use of to arrive at the error estimates and the convergence of MOL to the Cauchy problem.

Lemma 8.6.

The function $\underset{\sim}{g} = (\ \ldots,\ g_{-1},\ g_o,\ g_1,\ \ldots\ )$ in the IVP (8.22) is quasimonotone increasing in $\underset{\sim}{v}$ in the sense of Definition 2.4 if one of the following sets of conditions is satisfied for the function f of the problem (A'):

Q_1 (a) f is independent of p;

(b) $f(t,x,z,p,\varepsilon r) - f(t,x,z,p,\varepsilon\bar{r}) \leq K\varepsilon(r-\bar{r})$,

for some positive constant K, $r \geq \bar{r}$;

(c) δv_i, $\delta^2 v_i$ are the central differences and $\sigma(\rho)=1$.

Q_2 (a) $|f(t,x,z,p,\varepsilon r) - f(t,x,z,\bar{p},\varepsilon r)| \leq L|p-\bar{p}|$;

(b) $f(t,x,z,p,\varepsilon r) - f(t,x,z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r})$;

(c) $f(t,x,z,p,\varepsilon r) - f(t,x,z,p,\varepsilon\bar{r}) \leq K\varepsilon(r-\bar{r})$,

for some positive constants L, α, K and $r \geq \bar{r}$;

(d) δv_i, $\delta^2 v_i$ are the central differences;

(e) $\sigma(\rho) = (L\rho/2\alpha) / (1-e^{-L\rho/2\alpha})$ or

$\sigma(\rho) = L\rho/2\alpha \coth(L\rho/2\alpha)$, $\rho = h/\varepsilon$.

Q_3 (a) Q_2(a) - Q_2(c) hold true, δv_i is the forward or backward difference and $\delta^2 v_i$ is the central difference ;

(b) $\sigma(\rho) = (L\rho/\alpha) / (1-e^{-L\rho/\alpha})$ or

$\sigma(\rho) = (L\rho/\alpha) / \coth(L\rho/\alpha)$.

Lemma 8.7.

Let $f(t,x,z,p,\varepsilon r)$ of (8.4) satisfies the Lipschitz condition

(8.23) $f(t,x,z,p,\varepsilon r)-f(t,x,\bar{z},p,\varepsilon r) \leq \ell_1(z-\bar{z})$, $\ell_1 > 0$, $z \geq \bar{z}$.

Then the function $\underset{\sim}{s} = (\ldots, s_{-1}, s_o, s_1, \ldots\ )$ with
$s_i(t) = e^{(\ell_1+\delta)t}$, $i \in Z$, δ is any positive real number, is a 'test function'. Also the conditions of the type (2.18) is true for the function $\underset{\sim}{g}$ of the IVP (8.22).

Proof

The first claim follows from

$$\begin{aligned} P_i(\underset{\sim}{u}+\alpha_1\underset{\sim}{s}) - P_i\,\underset{\sim}{u} &= \alpha_1\ s_i' - [g_i(t,\underset{\sim}{u}+\alpha_1\underset{\sim}{s}) - g_i(t,\underset{\sim}{u})] \\ &= \alpha_1\ s_i' - [f(t,x_i,u_i+\alpha_1 s_i,\ \delta u_i, \varepsilon\sigma(\rho)\delta^2 u_i) \\ &\qquad - f(t,x_i,u_i,\delta u_i,\ \varepsilon\,\sigma(\rho)\delta^2 u_i)\ ] \\ &\geq \alpha_1\ s_i' - \ell_1\ \alpha_1\ s_i\ \geq\ \alpha_1\,\delta\ >\ 0. \end{aligned}$$

The proof for the second claim is trivial and hence it is omitted.

Theorem 8.8.

Let $\underset{\sim}{v} = (\ \ldots\ v_{-1},\ v_o,\ v_1,\ \ldots\ )$ be a solution of the IVP (8.22), $\underset{\sim}{u}$ a solution of the problem (A') with $\underset{\sim}{u} = (\ldots,\ u_{-1},\ u_o,\ u_1,\ \ldots)$, $u_i(t) = u(t,x_i)$, $x_i = ih$, $i \in Z$. It is assumed that f, h and hence the differences δv_i are defined in such a way that one of the sets of conditions of Lemma 8.6 is satisfied. Further assume that there exists a constant ℓ_1 such that the Lipschitz condition (8.23) is satisfied. If

(α) $\quad |u_x(t,x_i) - \delta u_i| \le \alpha(t),$

$$|\varepsilon u_{xx}(t,x_i) - \varepsilon\sigma(\rho)\delta^2 u_i| \le \varepsilon\beta(t) + Kh, \ t \in J_0, \ i \in Z,$$

for some positive constant K;

(β) $\quad f(t,x,z,p,\varepsilon r) - f(t,x,\bar{z},\bar{p},\varepsilon\bar{r})$

$$\le \omega(t, z-\bar{z}, |p-\bar{p}|, \varepsilon(r-\bar{r})), \ z \ge \bar{z}, \ r \ge \bar{r},$$

where $\omega(t,z,p,\varepsilon r)$ is defined in $J_0 \times R_+^3$ and increasing in $|p|$ and r;

(γ) $\quad$ there exists a non-negative function $\rho_1(t)$ satisfying the differential inequality

$$\rho_1'(t) > \omega(t, \rho_1(t), \alpha(t), \varepsilon\beta(t)+Kh), \ t \in J_0,$$

then

(8.24) $\quad |u(t,x_i) - v_i(t)| \le \rho_1(t), \ t \in J, \ i \in Z.$

Proof

The assumptions made here guarantee that one could apply Theorem 2.6 for the IVP (8.22). In order to obtain the estimate (8.24) one may follow the same procedure adopted in the proof of Theorem 8.3.

Theorem 8.9.

If under the general hypotheses of Theorem 8.8,

(i) $|u_x(t,x_i) - \delta u_i| \leq \alpha(t)$, $t \in J_o$, $i \in Z$;

(ii) $|\varepsilon u_{xx}(t,x_i) - \varepsilon\sigma(\rho)\delta^2 u_i| \leq \varepsilon\beta(t) + Kh$, $t \in J_o$, $i \in Z$;

(iii) $$f(t,x,z,p,\varepsilon r) - f(t,x,\bar{z},\bar{p},\varepsilon\bar{r}) \leq A(t)(z-\bar{z}) + B(t)|p-\bar{p}| + \varepsilon C(t)(r-\bar{r}), \quad z \geq \bar{z},\ r \geq \bar{r},$$

where functions $A(t)$, $B(t)\alpha(t) + \varepsilon C(t)\beta(t)$ are continuous in J_o and integrable over J,

then

$$|u(t,x_i) - v_i(t)| \leq \int_o^t \{B(s)\alpha(s) + \varepsilon C(s)\beta(s) + Kh\} \times e^{\int_s^t A(s)ds} ds, \quad t \in J,\ i \in Z. \tag{8.25}$$

If the functions α, β, A, B, C are constants, the above estimate reduces to

$$|u(t,x_i) - v_i(t)| \leq (\alpha B + \varepsilon C\beta + Kh)\,(e^{At} - 1)/A, \quad i \in Z. \tag{8.26}$$

Following the method of proof of Theorem 8.4 and using Theorem 8.8 one can prove this theorem.

Following theorem gives the convergence for the MOL to the problem (A'). The proof of this theorem runs along the lines of finite dimensional case and hence it is omitted.

Theorem 8.10.

Let $u(t,x)$ be the solution of the problem (A'). Further let u, u_x, u_{xx} be bounded and uniformly continuous on $\bar{G}' = [0,T] \times (-\infty, \infty)$ and $|\delta^2 u_i| \leq C_1$ for some positive constant C_1, $i \in Z$. Suppose that the condition (iii) of Theorem 8.9 holds true, then

$$\sup \left\{ |u(t,x_i) - v_i^h(t)| : t \in J,\ i \in Z \right\} \to 0 \text{ as } h \to 0,$$

where $\underset{\sim}{v}^h$ is a solution of the IVP (8.22) for a finite h.

CONCLUSIONS.

In this chapter, MOL to the DEs subject to the Dirichlet's type BCs is discussed. One can consider other types of BCs and discuss the MOL.

Chapter 9

METHOD OF LINES FOR SYSTEMS OF PARABOLIC DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

This chapter briefly discusses the MOL when apply to weakly coupled systems of two equations of parabolic type. Infact, error estimates and the convergence of the MOL are obtained to the problems:

$$(9.1)\quad \begin{cases} u_t = f(t,x,u,U,u_x,\varepsilon u_{xx}), \\ U_t = F(t,x,u,U,U_x,\varepsilon U_{xx}), \quad (t,x)\in G_p:=(0,T]\times(0,a), \end{cases}$$

subject to the BCs

$$(9.2)\quad \begin{cases} u(0,x) = \eta(x), & U(0,x) = \xi(x) \\ u(t,0) = \eta_0(t), & U(t,0) = \xi_0(t) \\ u(t,a) = \eta_1(t), & U(t,a) = \xi_1(t), \quad t\in(0,T) \end{cases}$$

and the Cauchy problem

$$(9.3)\quad \begin{cases} u_t = f(t,x,u,U,u_x,\varepsilon u_{xx}), \\ U_t = F(t,x,u,U,U_x,\varepsilon U_{xx}), \quad (t,x)\in G_p' := (0,T]\times R, \\ \qquad R = (-\infty,\infty), \end{cases}$$

$$(9.4)\quad u(0,x) = \eta_2(x),\ U(0,x) = \xi_2(x),\ x\in R.$$

Here ε is a small positive parameter, u_t, u_x, U_x, u_{xx}, U_{xx} have the usual meaning and f,F are assumed to be increasing with u_{xx} and U_{xx} respectively. Using the same procedure given in the previous chapters one can obtain the results for the above BVPs. Therefore only the results are stated but not proofs.

Theorem 9.1.

Consider the IVP

$$(9.5)\quad \begin{cases} P_i\,\underset{\sim}{w} := w_i' - H_i(t,\underset{\sim}{w}) = 0, & t \in J_o = (0,T], \\ R_i\,\underset{\sim}{w} := w_i(0) = A_i, & i=1(1)n,\ i=n+2\,(1)2n, \end{cases}$$

where $\underset{\sim}{w} = (w_o, \ldots, w_n, w_{n+1}, \ldots, w_{2n+1})$, $w_i \in U := C^1(J_o) \cap C(J)$.

Assume

(i) $\underset{\sim}{H}(t,\underset{\sim}{z})$ is quasimonotone increasing with $\underset{\sim}{z}$;

(ii) $\underset{\sim}{w}$ is the solution of the IVP (9.5);

(iii) there exists a 'test function' $\underset{\sim}{s}: J \rightarrow R^{2n+2}$ with the properties: $\underset{\sim}{s} > 0$ on J, that is, $s_i(t) > 0$, $s_i \in U$, $i=0(1)2n$ and

$$(9.6)\quad P_i\,(\underset{\sim}{z}+\alpha\underset{\sim}{s}) - P_i\,\underset{\sim}{z} > 0,\ i=1(1)n,\ i=n+2(1)2n,$$

for every positive real number α and $\underset{\sim}{z}$, $z_i \in U$;

Then for every $\underset{\sim}{u} = (u_o, \ldots, u_n, u_{n+1}, \ldots, u_{2n+1})$, $\underset{\sim}{v} = (v_o, \ldots, v_n, v_{n+1}, \ldots, v_{2n+1})$, $u_i, v_i \in U$, the following implication is true:

$$(9.7)\quad \begin{cases} u_o(t) \le w_o(t) \le v_o(t), \\ u_n(t) \le w_n(t) \le v_n(t), \\ u_{n+1}(t) \le w_{n+1}(t) \le v_{n+1}(t), \\ u_{2n+1}(t) \le w_{2n+1}(t) \le v_{2n+1}(t),\ t \in J, \end{cases}$$

$$(9.8)\quad \begin{cases} R_i \underset{\sim}{u} \le R_i \underset{\sim}{w} \le R_i \underset{\sim}{v},\ i = 1(1)n,\ i = n+2(1)2n, \\ P_i \underset{\sim}{u} \le P_i \underset{\sim}{w} \le P_i \underset{\sim}{v}, \\ P_{i+n} \underset{\sim}{u} \le P_{i+n} \underset{\sim}{w} \le P_{i+n} \underset{\sim}{v},\ i=1(1)n \end{cases}$$

$$\Longrightarrow$$

$$z_i(t) \le w_i(t) \le v_i(t) \text{ in } J_o,\ i=1(1)n,\ i=n+2(1)2n.$$

1. MOL TO THE BVP (9.1)-(9.2).

Consider the BVP (9.1)-(9.2) and introduce a mesh size $h = a/n$, n being a natural number and denote the grid functions by $v_i(t) = v(t,x_i)$ and $V_i(t) = V(t,x_i)$. For the $(n-1)$-dimensional vectors $\underset{\sim}{v}(t) = (v_1, \dots, v_{n-1})$ and $\underset{\sim}{V}(t) = (V_1, \dots, V_{n-1})$, the forward differences corresponding to the first and second derivatives with respect to x are defined by

$$\delta v_i = (v_{i+1}-v_i)/h, \quad \delta V_i = (V_{i+1}-V_i)/h$$

$$\delta^2 v_i = (v_{i-1}-2v_i+v_{i+1})/h^2, \quad \delta^2 V_i = (V_{i-1}-2V_i+V_{i+1})/h^2.$$

With these notations the discrete problem corresponding to the BVP (9.1) - (9.2) by MOL is given by

$$(9.9)\quad \begin{cases} v_i' = f(t,\ x_i,\ v_i,\ V_i, \delta v_i, \varepsilon \delta^2 v_i), \\ V_i' = F(t, x_i,\ v_i,\ V_i,\ \delta V_i, \varepsilon \delta^2 V_i), \\ v_i(0) = \eta(x_i), \\ V_i(0) = \mathcal{E}(x_i),\ i = 1(1)n-1. \end{cases}$$

Set

$$w_i = v_i, \quad i = 0(1)n$$

$$w_{i+n+1} = V_i, i = 0(1)n$$

and let

$$\underset{\sim}{w} = (w_o,\ \ldots,\ w_n,\ w_{n+1},\ \ldots,\ w_{2n+1}).$$

Then one gets an equivalent form for the problem (9.9) as

$$(9.10)\quad \begin{cases} P_i \underset{\sim}{w} := w_i' - f(t, x_i, w_i, w_{i+n+1}, \delta w_i, \varepsilon \delta^2 w_i) = 0 \\ P_{i+n}\underset{\sim}{w} := w_{i+n+1}' - F(t, x_i, w_i, w_{i+n+1}, \delta w_{i+n+1}, \\ \qquad\qquad \varepsilon \delta^2 w_{i+n+1}) = 0,\ i=1(1)n-1, \\ R_i \underset{\sim}{w} := w_i(0) + \eta(x_i) \\ R_{i+n+1}\underset{\sim}{w} := w_{i+n+1}(0) = \mathcal{E}(x_i),\ i=1(1)(n-1). \end{cases}$$

The IVP (9.10) may be abbreviated as

$$(9.11)\begin{cases} P_i \underset{\sim}{w} := w_i' - g_i(t,\underset{\sim}{w}) = 0, \\ P_{i+n+1}\underset{\sim}{w} := w_{i+n+1}' - G_i(t,\underset{\sim}{w}) = 0, \quad i=1(1)n-1, \\ R_i \underset{\sim}{w} := w_i(0) = \eta(x_i), \\ R_{i+n+1}\underset{\sim}{w} := w_{i+n+1}(0) = \xi(x_i), \quad i=1(1)n-1 \end{cases}$$

where

$$\underset{\sim}{w} = (w_o, \ldots, w_n, w_{n+1}, \ldots, w_{2n+1}),$$

$$g_i(t,\underset{\sim}{w}) = f(t,x_i,w_i, w_{i+n+1}, \delta w_i, \varepsilon\delta^2 w_i),$$

$$G_i(t,\underset{\sim}{w}) = F(t,x_i,w_i,w_{i+n+1}, \delta w_{i+n+1}, \varepsilon\delta^2 w_{i+n+1}), \quad i=1(1)n-1.$$

If one applies the same procedure as given in Walter [3] to the BVP (9.1)-(9.2) then the following conditions may have to be satisfied for the discrete problem (9.11):

$$(9.12)\qquad h \leq 2\alpha\varepsilon/L$$

where α, L are constants independent of ε such that

$$(9.13)\qquad f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r}),$$

$$(9.14)\qquad F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r}), \quad r \geq \bar{r},$$

and

$$(9.15)\qquad |f(t,x,z,Z,p,\varepsilon r)- f(t,x,z,Z,\bar{p},\varepsilon r)| \leq L\,|p-\bar{p}|,$$

$$(9.16)\qquad |F(t,x,z,Z,p,\varepsilon r)- F(t,x,z,Z,\bar{p},\varepsilon r)| \leq L|p-\bar{p}|.$$

As pointed out in the previous chapters, the condition (9.12) is not feasible especially when ε is very small. Hence a modified version of MOL is necessary.

2. MODIFIED FORM OF THE MOL TO THE BVP (9.1)-(9.2).

The modified discrete problem to be associated with (9.1)-(9.2) is given by

$$(9.17)\quad \begin{cases} v_i' = f(t,x_i,v_i,V_i,\delta v_i,\varepsilon\sigma(\rho)\delta^2 v_i), \\ V_i' = F(t,x_i,v_i,V_i,\delta V_i,\varepsilon\,\sigma(\rho)\delta^2 V_i), \\ v_i(0)=\eta(x_i), \\ V_i(0)=\xi(x_i),\ i=1(1)n-1, \end{cases}$$

where $\rho = h/\varepsilon$, $v_o(t) = \eta_o(t)$, $v_n(t) = \eta_1(t)$, $V_o(t) = \xi_o(t)$, $V_n(t) = \xi_1(t)$ and $\sigma(\rho)$ a 'fitting factor' is specified below. The IVP (9.17) may be abbreviated as

$$(9.18)\quad \begin{cases} \underset{\sim}{P}\,\underset{\sim}{w} := \underset{\sim}{w}' - \underset{\sim}{H}(t,\underset{\sim}{w}) = \underset{\sim}{0},\ t\in J_o, \\ \underset{\sim}{R}\,\underset{\sim}{w} := \underset{\sim}{w}(0) = \underset{\sim}{A}, \end{cases}$$

where

$$H_i = g_i(t,\underset{\sim}{w}) = f(t,x_i,w_i,w_{i+n+1},\delta w_i,\varepsilon\sigma(\rho)\delta^2 w_i),\ i=1(1)n-$$

$$H_i = G_i(t,\underset{\sim}{w}) = F(t,x_i,w_i,w_{i+n+1},\delta w_{i+n+1},\varepsilon\,\sigma(\rho)\delta^2 w_{i+n+1}),$$

$$i=n+2(1)\ 2n,$$

$$\underset{\sim}{H} = (H_1,\dots,H_{n-1},H_{n-2},\dots,H_{2n}),$$

$$\underset{\sim}{w} = (w_o,\dots,w_n,w_{n+1},\dots,w_{2n+1}).$$

The following lemma gives sets of conditions for $\underset{\sim}{H}$ to be quasimonotone increasing in $\underset{\sim}{w}$.

Lemma 9.2.

The function $\underset{\sim}{H}$ of the IVP (9.18) is quasimonotone increasing in $\underset{\sim}{w}$ if f is increasing with V and F is increasing with u and if one of the following sets of conditions is satisfied.

I (a) (i) $f(t,x,z,Z,p,\varepsilon r)$ and $F(t,x,z,Z,p,\varepsilon r)$ are increasing in r and independent of p;

(b) $\sigma(\rho) = 1$.

II (a) (i) $f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r})$, $r \geq \bar{r}$,

(ii) $F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\bar{r}) \geq \alpha\varepsilon(r-\bar{r})$, $r \geq \bar{r}$;

(b) (i) $|f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,\bar{p},\varepsilon r)| \leq L|p-\bar{p}|$,

(ii) $|F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,\bar{p},\varepsilon r)| \leq L|p-\bar{p}|$;

where α, L are positive constants independent of ε;

(c) $\sigma(\rho) = (L\rho/2\alpha)/(1-e^{-L\rho/2\alpha})$ or

$\sigma(\rho) = (L\rho/2\alpha)\coth(L\rho/2\alpha)$, $\rho = h/\varepsilon$;

(d) δv_i and δV_i are the central differences.

III (a) I(a) and I(b) hold and δv_i and δV_i are the forward or backward differences;

(b) $\sigma(\rho) = (L\rho/\alpha) / (1-e^{-L\rho/\alpha})$ or

$\sigma(\rho) = (L\rho/\alpha)\coth(L\rho/\alpha)$.

IV (a) f and F are increasing in r and increasing (respectively decreasing) in p, δv_i and δV_i are the forward (respectively backward) differences;

(b) $\sigma(\rho) = 1$.

The following lemma gives a set of conditions under which a 'test function' as defined in Theorem 9.1 exists for the IVP (9.18).

Lemma 9.3.

Let $f(t,x,z,Z,p,\varepsilon r)$ and $F(t,x,z,Z,p,\varepsilon r)$ satisfy the Lipschitz conditions

$$|f(t,x,z,Z,p,\varepsilon r) - f(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r})|$$
$$\leq L[(z-\bar{z})+(Z-\bar{Z})+|p-\bar{p}|+\varepsilon(r-\bar{r})], \; z \geq \bar{z}, \; Z \geq \bar{Z}, \; r \geq \bar{r},$$

$$|F(t,x,z,Z,p,\varepsilon r) - F(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r})|$$
$$\leq L[(z-\bar{z})+(Z-\bar{Z})+|p-\bar{p}|+\varepsilon(r-\bar{r})], \; z \geq \bar{z}, \; Z \geq \bar{Z}, \; r \geq \bar{r},$$

where L is a positive constant independent of ε. Then the function $\underset{\sim}{s} = (s_0, \ldots, s_{2n+1})$ with $s_i(t) = s(t) = e^{\alpha_2 t}$, $i = 0(1)2n+1$, α_2 is sufficiently large positive number, is a 'test function' for the problem (9.18).

3. ERROR ESTIMATES.

The theorems stated in this section give error estimates and the convergence of the modified MOL to the BVP (9.1)-(9.2).

Theorem 9.4.

Let $\underset{\sim}{v} = (v_o, \ldots, v_n)$, $\underset{\sim}{V} = (V_1, \ldots, V_n)$ be a solution of (9.17), u, U be a solution of (9.1)-(9.2) with $u_i(t) = u(t,x_i)$, $U_i(t) = U(t,x_i)$, $x_i = ih$, $i = O(1)n$. It is assumed that f, F, h, the differences $\delta v_i, \delta V_i, \delta^2 v_i, \delta^2 V_i$ are defined in such a way that one of the sets of conditions of Lemma 9.2 is satisfied. Further assume that there exists a 'test function' for (9.18) so that the monotonicity Theorem 9.1 is applicable for it.

If

$$(9.19)\quad \begin{cases} |\ \partial u/\partial x\ (t,x_i) - \delta u_i| \leq \alpha_1(t) \\ |\ \partial U/\partial x\ (t,x_i) - \delta U_i| \leq \alpha_2(t),\ i=1(1)n, \end{cases}$$

$$(9.20)\quad \begin{cases} |\varepsilon \partial^2 u/\partial x^2(t,x_i) - \varepsilon\,\sigma(P)\delta^2 u_i| \leq \varepsilon\,\beta_1(t) + K_1 h \\ |\varepsilon \partial^2 U/\partial x^2(t,x_i) - \varepsilon\,\sigma(P)\delta^2 U_i| \leq \varepsilon\,\beta_2(t) + K_2 h,\ i=1(1)n, \end{cases}$$

K_1, K_2 are constants.

$$(9.21)\quad \begin{cases} f(t,x,z,Z,p,\varepsilon r) - f(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \\ \qquad \leq \omega_1(t, z-\bar{z}, Z-\bar{Z}, |p-\bar{p}|, \varepsilon(r-\bar{r})),\ z \geq \bar{z},\ Z \geq \bar{Z},\ r \geq \bar{r}, \\ F(t,x,z,Z,p,\varepsilon r) - F(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \\ \qquad \leq \omega_2(t, z-\bar{z}, Z-\bar{Z}, |p-\bar{p}|, \varepsilon(r-\bar{r})),\ z \geq \bar{z},\ Z \geq \bar{Z},\ r \geq \bar{r}, \end{cases}$$

$(t,x,z,Z,p,\varepsilon r)$, $(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \in \mathcal{U}$

$$(9.22)\quad \begin{cases} \rho_1' > \omega_1(t, \rho_1, \rho_2, \alpha_1(t), \alpha\beta_1(t) + K_1 h) \\ \rho_2' > \omega_2(t, \rho_1, \rho_2, \alpha_2(t), \alpha\beta_2(t) + K_2 h) \end{cases}$$

where $\rho_1(0)$, $\rho_2(0) \geq 0$, $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ are defined in J_o, $\omega_1(t,z,Z,p,\varepsilon r)$, $\omega_2(t,z,Z,p,\varepsilon r)$ are defined in $J_o \times \{(z,Z,p,\varepsilon r),\ z \geq 0,\ Z \geq 0,\ p \geq 0,\ r \geq 0\}$ and increasing in p, r,

then

$$(9.23)\quad \begin{cases} |\, u(t,x_i) - v_i(t)| \leq \rho_1(t) \\ |\, U(t,x_i) - V_i(t)| \leq \rho_2(t) \text{ in } J,\ i=1(1)n-1. \end{cases}$$

Theorem 9.5.

If under the general hypothesis of Theorem 9.4,

(i) (a) $|\, u_x(t,x_i) - \delta u_i| \leq \alpha_1(t)$,

(b) $|\, U_x(t,x_i) - \delta U_i| \leq \alpha_2(t)$ in J_o;

(ii) (a) $|\varepsilon u_{xx}(t,x_i) - \varepsilon\sigma(\rho)\delta^2 u_i| \leq \varepsilon\,\beta_1(t) + K_1 h$

(b) $|\varepsilon U_{xx}(t,x_i) - \varepsilon\sigma(\rho)\delta^2 U_i| \leq \varepsilon\,\beta_2(t) + K_2 h$, in J_o;

(iii) (a) $f(t,x,z,Z,p,\varepsilon r) - f(t,x,\bar z,\bar Z,\bar p,\varepsilon\bar r)$

$\leq A_1(t)(u-\bar u) + B_1(t)(U-\bar U) + C_1(t)|p-\bar p| + \varepsilon D_1(t)(r-$

$u \geq \bar u,\quad U \geq \bar U,\quad r \geq \bar r$,

(b) $F(t,x,z,Z,p,\varepsilon r) - F(t,x,\bar z,\bar Z,\bar p,\varepsilon\bar r)$

$\leq A_2(t)(u-\bar u) + B_2(t)(U-\bar U) + C_2(t)|p-\bar p| + \varepsilon D_2(t)(r-$

$u \geq \bar u,\quad U \geq \bar U,\quad r \geq \bar r$,

where the functions $A_i(t)$, $B_i(t)$, $C_i(t)$, $D_i(t)$, $i = 1,2$, are continuous in J, $B_1(t)\,\alpha_1(t) + C_1(t)\,(\varepsilon\,\beta_1(t) + K_1(t)$,

$B_2(t)\alpha_2(t) + C_2(t)(\varepsilon\beta_2(t)+ k_2h)$ are integrable in J_o and $B_1(t)$, $B_2(t)$, $A_1(t)$, $A_2(t)$ are non-negative functions,

then

$$(9.24)\qquad |u(t,x_i) - v_i(t)| \le \int_o^t \{B_1(s)\alpha_1(s)+C_1(s)\ (\varepsilon\,\beta_1(s)+K_1h)\} \times e^{\int_s^t A_1(r)dr}\, ds \text{ in } J,$$

$$(9.25)\qquad |U(t,x_i)-V_i(t)| \le \int_o^t \{B_2(s)\alpha_2(s) + C_2(s)(\varepsilon\beta_2(s)+K_2h)\} \times e^{\int_s^t A_2(r)dr}\, ds \text{ in } J.$$

Further if the functions α_i, β_i, A_i, B_i, C_i, $i=1,2$, are constants then the above inequalities reduce to

$$|\ u(t,x_i)-v_i(t)| \le \{\alpha_1B_1+C_1(\varepsilon\beta_1+K_1h)\}\ (e^{A_1t}-1)/A_1,$$

$$|\ U(t,x_i)-V_i(t)| \le \{\alpha_2B_2+C_2(\varepsilon\beta_2+K_2h)\}\ (e^{A_2t}-1)/A_2.$$

Using Theorem 9.4 one can obtain the results on the convergence of solutions of (9.17) to the solutions of BVP (9.1) - (9.2) as $n\to\infty$. If $u(t,x)$, $U(t,x)$ is a solution of the BVP (9.1)-(9.2) then let $u^n(t)$, $U^n(t)$ be the vector functions $\underset{\sim}{u}^n = (u_1(t), \ldots, u_{n-1}(t)\ )$ and $\underset{\sim}{U}^n = (U_1(t),\ldots,U_{n-1}(t))$ respectively, where $h = a/n$, $x_i = ih$, $u_i(t) = u(t,x_i)$, $U_i(t) = U(t,x_i)$, $i=1(1)n-1$. Solution of (9.17) is denoted by

$\underset{\sim}{v}^n$, $\underset{\sim}{V}^n$, for a given value of n. The norms $\| \underset{\sim}{w}^n \|$ and $\| \underset{\sim}{W}^n \|$ are (n-1)-dimensional vector functions defined by

$$\| \underset{\sim}{w}^n \| = \max\left\{ | w_i(t) | \; : \; t \in J, \; i = 1(1)n-1 \right\}$$

and

$$\| \underset{\sim}{W}^n \| = \max\left\{ | W_i(t) | \; : \; t \in J, \; i = 1(1)n-1 \right\}.$$

Define the sequences of real numbers $\{\alpha_{1n}\}$, $\{\alpha_{2n}\}$, $\{\beta_{1n}\}$ and $\{\beta_{2n}\}$ by

$$\alpha_{1n} = \sup_{t \in J} \left\{ \max_{i=1(1)n-1} | u_x(t,x_i) - \delta u_i | \right\},$$

$$\alpha_{2n} = \sup_{t \in J} \left\{ \max_{i=1(1)n-1} | U_x(t,x_i) - \delta U_i | \right\},$$

$$\beta_{1n} = \sup_{t \in J} \left\{ \max_{i=1(1)n-1} | u_{xx}(t,x_i) - \delta^2 u_i | \right\},$$

$$\beta_{2n} = \sup_{t \in J} \left\{ \max_{i=1(1)n-1} | U_{xx}(t,x_i) - \delta^2 U_i | \right\}.$$

Theorem 9.6.

Let u(t,x), U(t,x) be the solution of (9.1)-(9.2) and let u_x, U_x, u_{xx}, U_{xx} be continuous in $\bar{G}$. Suppose one of the conditions of Lemma 9.2 and the one sided Lipschitz condition (iii) of Theorem 9.5 are satisfied. Further suppose there exists a 'test function' for the problem (9.17). Then

$$\| \underset{\sim}{u}^n - \underset{\sim}{v}^n \| \to 0 \text{ and } \| \underset{\sim}{U}^n - \underset{\sim}{V}^n \| \to 0 \text{ as } n \to \infty.$$

4. MONOTONICITY THEOREM IN BANACH SPACE.

In order to discuss the MOL to the Cauchy problem (9.3) - (9.4) the following monotonicity theorem is needed.

Theorem 9.7.

Assume

(i) $\underset{\sim}{H}(t,\underset{\sim}{v},\underset{\sim}{V}) : J_o \times B^2 \longrightarrow B$ is quasimonotone increasing in $\underset{\sim}{v}$ and $\underset{\sim}{V}$ in the sense of Definition 2.4;

(ii) there exists a real number $\delta > 0$ and a 'test function' $\underset{\sim}{s}(t): J \rightarrow B$ such that $s_i(t) = s_j(t)$, $i,j \in Z$, $s_i(t) > 0$ on J, $i \in Z$, $\underset{\sim}{s} \in U := C^1(J_o) \cap C(J)$, $\underset{\sim}{s} = (\ldots, s_{-1}, s_o, s_1, \ldots)$ and

$$(9.26) \begin{cases} P_i\,(\underset{\sim}{v}+\alpha_1\underset{\sim}{s},\ \underset{\sim}{V}+\alpha_1\underset{\sim}{s}) - P_i(\underset{\sim}{v},\underset{\sim}{V}) \geq \alpha_1\,\delta > 0, \\ P_i\,(\underset{\sim}{v},\ \underset{\sim}{V}) - P_i(\underset{\sim}{v}-\alpha_1\underset{\sim}{s},\ \underset{\sim}{V}-\alpha_1\underset{\sim}{s}) \geq \alpha_1\,\delta > 0, \end{cases}$$

for every positive real α_1 and for every $\underset{\sim}{v}, \underset{\sim}{V} \in U$,

$$P_i\underset{\sim}{W} := \underset{\sim}{W}' - H_i(t,\underset{\sim}{W}) = \underset{\sim}{0}, \text{ where } W_{2i}=v_i,\ W_{2i+1}=V_i,\ i \in Z;$$

there exists a positive constant L such that

$$(9.27) \begin{cases} \underset{\sim}{H}\,(t,\underset{\sim}{\bar{v}}+\beta\underset{\sim}{s},\ \underset{\sim}{\bar{V}}+\beta\underset{\sim}{s}) - \underset{\sim}{H}(t,\underset{\sim}{\bar{v}}+\eta\underset{\sim}{s},\ \underset{\sim}{\bar{V}}+\eta\underset{\sim}{s}) \leq L\underset{\sim}{s}(t)\,(\beta-\eta), \\ \underset{\sim}{H}\,(t,\underset{\sim}{v}-\eta\underset{\sim}{s},\ \underset{\sim}{V}-\eta\underset{\sim}{s}) - \underset{\sim}{H}(t,\underset{\sim}{v}-\beta\underset{\sim}{s},\ \underset{\sim}{V}-\beta\underset{\sim}{s}) \leq L\underset{\sim}{s}(t)\,(\beta-\eta), \end{cases}$$

for every η, β such that $0 < \eta \leq \beta$.

Then for every $\bar{\underset{\sim}{v}}$, $\bar{\underset{\sim}{V}}$, $\underset{\approx}{v}$, $\underset{\approx}{V}$, the implication

$$\begin{aligned} &\underset{\sim}{P}(\underset{\approx}{v}, \underset{\approx}{V}) \leq \underset{\sim}{P}(\underset{\sim}{v}, \underset{\sim}{V}) \leq \underset{\sim}{P}(\bar{\underset{\sim}{v}}, \bar{\underset{\sim}{V}}), \\ &\underset{\sim}{R}(\underset{\approx}{v}, \underset{\approx}{V}) \leq \underset{\sim}{R}(\underset{\sim}{v}, \underset{\sim}{V}) \leq \underset{\sim}{R}(\bar{\underset{\sim}{v}}, \bar{\underset{\sim}{V}}) \\ &\Longrightarrow \\ &\underset{\approx}{v} \leq \underset{\sim}{v} \leq \bar{\underset{\sim}{v}}, \qquad \underset{\approx}{V} \leq \underset{\sim}{V} \leq \bar{\underset{\sim}{V}} \quad \text{on } J \end{aligned}$$

is true provided there exist positive constants L_j s such that $\underset{\sim}{L} = (\ldots, L_{-1}, L_0, L_1, \ldots) \in B$ and

$$H_j(t, \underset{\sim}{W} + \beta \underset{\sim}{s}) - H_j(t, \underset{\sim}{W} + \eta \underset{\sim}{s}) \leq L_j(\beta - \eta), \; j \in Z, \; \eta, \beta \in R, \; \eta \leq \beta. \tag{9.28}$$

5. MOL TO CAUCHY PROBLEM.

Applying the modified line method to the Cauchy problem one arrives at the following IVP

$$\begin{cases} v_i' = f(t, x_i, v_i, V_i, \delta v_i, \varepsilon\sigma(\rho)\delta^2 v_i), \\ V_i' = F(t, x_i, v_i, V_i, \delta V_i, \varepsilon\sigma(\rho)\delta^2 V_i), \quad t \in J_0 = (0, T], \\ v_i(0) = \eta(x_i), \\ V_i(0) = \xi(x_i), \quad i \in Z, \end{cases} \tag{9.29}$$

where $h > 0$, $x_i = ih$, the differences δv_i and δV_i can be either central, forward or backward, $\delta^2 v_i$ and $\delta^2 V_i$ are the central differences and $\sigma(\rho)$, $\rho = h/\varepsilon$, is a 'fitting factor'. This problem can be abbreviated as

$$\begin{cases} \underset{\sim}{W}' = \underset{\sim}{H}(t, \underset{\sim}{W}) \\ \underset{\sim}{W}(0) = \underset{\sim}{A}, \end{cases} \tag{9.30}$$

where

$$\underset{\sim}{H} = (\ldots, H_{-1}, H_o, H_1, \ldots),$$

$$\underset{\sim}{W}' = (\ldots, W'_{-1}, W'_o, W'_1, \ldots),$$

$$\underset{\sim}{W} = (\ldots, W_{-1}, W_o, W_1, \ldots),$$

$$W_{2i} = v_i, \ i \in Z,$$

$$W_{2i+1} = V_i, \ i \in Z.$$

Lemma 9.8.

The functions $\underset{\sim}{H}$ of (9.30) is quasimonotone increasing in $\underset{\sim}{v}$ and $\underset{\sim}{V}$ respectively if f is increasing with U and F is increasing with u and if one of the following sets of conditions is satisfied for the functions f and F of (9.3) - (9.4).

Q_1 (a) $f(t,x,z,Z,p,\varepsilon r)$ and $F(t,x,z,Z,p,\varepsilon r)$ are increasing in r and independent of p;

(b) $f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\bar{r}) \leq K\,\varepsilon(r-\bar{r})$,

$F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\bar{r}) \leq K\,\varepsilon(r-\bar{r})$,

for some positive constant k, $r \geq \bar{r}$,

(c) $\delta v_i, \delta V_i, \delta^2 v_i, \delta^2 V_i$ are the central differences;

(d) $\sigma(\rho) = 1$;

Q_2 (a) $|f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,\bar{Z},\bar{p},\varepsilon r) \leq L[|p-\bar{p}| + |Z-\bar{Z}|]$,

$|F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,\bar{Z},\bar{p},\varepsilon r) \leq L[|p-\bar{p}| + |Z-\bar{Z}|]$,

(b) $f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\bar{r}) \leq \alpha\,\varepsilon(r-\bar{r})$,

$F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\bar{r}) \leq \alpha\,\varepsilon(r-\bar{r})$,

(c) $f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\bar{r}) \leq K\varepsilon(r-\bar{r}),$

$F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\bar{r}) \leq K\varepsilon(r-\bar{r}),$

for some positive constants L, α, K, $r \geq \bar{r}$;

(d) $\delta v_i, \delta V_i, \delta^2 v_i, \delta^2 V_i$ are the central differences;

(e) $\sigma(\rho) = (L\rho/2\alpha)\,(1-e^{-L\rho/2\alpha})$ or

$\sigma(\rho) = (L\rho/2\alpha)\cot h\,(L\rho/2\alpha);$

Q_3 (a) Q_2(a) - Q_2(c) hold, $\delta v_i, \delta V_i$ are the forward or backward differences and $\delta^2 v_i, \delta^2 V_i$ are the central differences

(b) $\sigma(\rho) = (L\rho/\alpha)\,(1-e^{-L\rho/\alpha})$ or

$\sigma(\rho) = (L\rho/\alpha)\cot h\,(L\rho/\alpha).$

Lemma 9.9.

Let $f(t,x,z,Z,p,\varepsilon r)$ and $F(t,x,z,Z,p,\varepsilon\bar{r})$ of (9.3) satisfy the Lipschitz condition

$$(9.31)\quad \begin{cases} f(t,x,z,Z,p,\varepsilon r) - f(t,x,\bar{z},\bar{Z},p,\varepsilon r) \\ \qquad \leq \ell_1[(z-\bar{z}) + (Z-\bar{Z})],\ \ell_1 > 0,\ z \geq \bar{z},\ Z \geq \bar{Z}, \\ F(t,x,z,Z,p,\varepsilon r) - F(t,x,\bar{z},\bar{Z},p,\varepsilon r) \\ \qquad \leq \ell_1[(z-\bar{z}) + (Z-\bar{Z})],\ \ell_1 > 0,\ z \geq \bar{z},\ Z \geq \bar{Z}; \end{cases}$$

Then the function $\underset{\sim}{s} = (\ldots, s_{-1}, s_0, s_1, \ldots)$ with $s_i(t)=e^{\alpha_2 t}$, $i \in Z$, α_2 is sufficiently large positive number is a 'test function' in the sense that it satisfies the condition of the type (9.27) is true for $\underset{\sim}{H}$ of (9.30).

The following theorem gives the error estimates and convergence of the MOL to the Cauchy problem.

Theorem 9.10.

Suppose $\underset{\sim}{v} = (\ldots, v_{-1}, v_o, v_1, \ldots)$, $\underset{\sim}{V} = (\ldots, V_{-1}, V_o, V_1, \ldots) \in U$ are solutions of the problem (9.29), $u, U \in V$ where $u_i(t) \doteq u(t,x_i)$, $U_i(t) = U(t,x_i)$. It is assumed that f, F, h the differences δv_i, δV_i, $\delta^2 v_i$, $\delta^2 V_i$ are defined in such a way that one of the sets of conditions of Lemma 9.8 is satisfied. Further assume that there exists a constant ℓ_1 such that the Lipschitz condition (9.31) is satisfied. If

$$(9.32)\quad \begin{cases} |\, \partial u/\partial x\ (t,x_i) - \delta u_i| \leq \alpha_1(t) \\ |\, \partial U/\partial x\ (t,x_i) - \delta U_i| \leq \alpha_2(t),\ i \in Z, \end{cases}$$

$$(9.33)\quad \begin{cases} |\varepsilon\, \partial^2 u/\partial x^2\ (t,x_i) - \varepsilon\, \sigma(\rho)\, \delta^2 u_i| \leq \varepsilon\, \beta_1(t) + K_1 h \\ |\varepsilon\, \partial^2 U/\partial x^2\ (t,x_i) - \varepsilon\, \sigma(\rho)\, \delta^2 U_i| \leq \varepsilon\, \beta_2(t) + K_2 h, \end{cases}$$

$i=1(1)n$, k_1, k_2 are constants,

$$f(t,x,z,Z,p,\varepsilon r) - f(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \leq \omega_1(t, z-\bar{z}, Z-\bar{Z}, |p-\bar{p}|, \varepsilon(r-\bar{r})$$

$$F(t,x,z,Z,p,\varepsilon r) - F(t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \leq \omega_2(t, z-\bar{z}, Z-\bar{Z}, |p-\bar{p}|, \varepsilon(r-\bar{r})),\ z \geq \bar{z},$$

$$Z \geq \bar{Z},\ r \geq \bar{r},\ (t,x,z,Z,p,\varepsilon r),\ (t,x,\bar{z},\bar{Z},\bar{p},\varepsilon\bar{r}) \in \mathcal{V}$$

$$\rho_1' \geq \omega_1(t, \rho_1, \rho_2, \alpha_1(t), \alpha\beta_1(t) + K_1 h)$$

$$\rho_2' \geq \omega_2(t, \rho_1, \rho_2, \alpha_2(t), \varepsilon\beta_2(t) + K_2 h)$$

where $\rho_1(0), \rho_2(0) \geq 0$, $\alpha_1(t)$, $\alpha_2(t)$, $\beta_1(t)$, $\beta_2(t)$ are defined in J_o, $\omega_1(t,z,Z,p,\varepsilon r)$, $\omega_2(t,z,Z,p,\varepsilon r)$ are defined in $J_o \times \{(z,Z,p,r) : z \geq 0, Z \geq 0, p \geq 0, r \geq 0\}$ and in p and r, then

$$(9.33) \quad \begin{cases} |u(t,x_i) - v_i(t)| \leq \rho_1(t) \\ |U(t,x_i) - V_i(t)| \leq \rho_2(t) \text{ in } J,\ i \in Z. \end{cases}$$

Theorem 9.11.

Let $u(t,x)$, $U(t,x)$ be the solution of (9.3) and let u_x, U_x, u_{xx}, U_{xx} be bounded and uniformly continuous in $\overline{G}$. Suppose there exists two positive constants L and α such that

$$f(t,x,z,Z,p,\varepsilon r) - f(t,x,\overline{z},\overline{Z},\overline{p},\varepsilon\overline{r})$$
$$\leq L\,[(z-\overline{z}) + (Z-\overline{Z}) + |p-\overline{p}| + \varepsilon(r-\overline{r})]$$

$$F(t,x,z,Z,p,\varepsilon r) - F(t,x,\overline{z},\overline{Z},\overline{p},\varepsilon\overline{r})$$
$$\leq L\,[(z-\overline{z}) + (Z-\overline{Z}) + |p-\overline{p}| + \varepsilon(r-\overline{r})],$$

for $z \geq \overline{z}$, $Z \geq \overline{Z}$, $r \geq \overline{r}$ and

$$f(t,x,z,Z,p,\varepsilon r) - f(t,x,z,Z,p,\varepsilon\overline{r}) \geq \alpha(r-\overline{r}),$$
$$F(t,x,z,Z,p,\varepsilon r) - F(t,x,z,Z,p,\varepsilon\overline{r}) \geq \alpha(r-\overline{r}),\ r \geq \overline{r}.$$

Then

$$\sup\{|u(t,x_i) - v_i^h(t)| : t \in J,\ i \in Z\} \to 0 \text{ as } h \to 0$$
$$\sup\{|U(t,x_i) - V_i^h(t)| : t \in J,\ i \in Z\} \to 0 \text{ as } h \to 0$$

where $\underset{\sim}{v}^h$, $\underset{\sim}{V}^h$ is a solution of (9.30).

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