SOME PROBLEMS IN DISCRETE FUNCTION THEORY

STUDIES IN THE GEOMETRY OF THE DISCRETE PLANE

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

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JULY - 1985

CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Sri.A.Vijayakumar under the guidance of Prof. Wazir Hasan Abdi and myself in the Department of Mathematics and Statistics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

A. Vijayakumar 15/7/85

(A. VIJAYAKUMAR)

CONTENTS

ACKNOWLEDGEMENT

I am very much indebted to Dr.Wazir Hasan Abdi, former Professor and Head, Department of Mathematics and Statistics, University of Cochin, who led me to the field of research and has been a source of encouragement throughout my research work.

I owe a great deal to Dr.T.Thrivikraman, Professor and Head, Department of Mathematics and Statistics, University of Cochin, my supervising teacher, for his constant care and unsparing help, during my research career. Many discussions I have had with him were very much helpful in finalising this thesis.

I am thankful to Dr.T.V.S. Jagannathan, School of Mathematics, Madurai Kamaraj University for his keen interest in my work and also for many valuable suggestions.

I shall always remember with pleasure, the help and co-operation extended by my colleagues Jacob, Sunanda, Pramod, Mercy, Padmaja, Jessy and Jacob M.J., during the preparation of this thesis.

I thank all my teachers and non teaching staff of the department for their help during various occasions and to Mr. Jose for typing the thesis.

Finally, I place on record my gratitude to C.S.I.R. for awarding research fellowships from 31st January 1981.

Needless to say, blessings from my parents, brothers and sisters were with me throughout.

SYNOPSIS

In this thesis, an attempt is made to study some geometric properties of the discrete plane $H = \{(q^m x_o, q^n y_o);\ m, n \in Z$, the set of integers$\}$ where (x_o, y_o) is a fixed point in the first quadrant of the complex plane, $x_o \neq o$, $y_o \neq o$, and $q \in (o,1)$ is fixed. This discrete plane was first considered by Harman ('A discrete analytic theory for geometric difference functions' Ph.D. thesis, University of Adelaide (1972)) to develop the theory of q-analytic functions. The theory was a consequence of attempts made by Isaacs, Duffin, Abdullaev etc. since 1941, to evolve a discrete analytic function theory analogous to the classical complex analytic function theory. These theories are free from the classical notion of continuity. Recently, concepts like discrete bianalytic functions, q-monodiffric functions (Velukutty K.K., 'Some problems of discrete function theory', Ph.D. thesis, University of Cochin (1982)) and discrete pseudoanalytic functions (Mercy K Jacob, 'A study of discrete pseudoanalytic functions', Ph.D. thesis, University of Cochin (1983)) have been introduced and studied in detail. All such theories are function theoretic in nature.

This motivated us to study the geometric aspects of the discrete plane H. We have introduced and investigated, the notion of metric on H, discrete analogues of some classical geometric concepts, transformations on H, characterisation of certain special types of transformations, group theoretic and discrete analytic properties of these transformations, discrete analogue of convexity and related concepts. This study, hence will initiate the development of discrete geometry of H.

In chapter 1, we have given the basic principle of discretization, a sketch of the development of discrete analytic function theory, a brief description of geometry of a space and also the summary of results established in this thesis.

In chapter 2, using the concept of discrete curve given by Harman, we define the distance between any two points of H. The distance function d assumes non negative integral values and we call (H,d) the discrete holometric space. We define the notion of domain in H and obtain a metric characterisation of it. Also, bounds for the diameter of any domain is obtained.

We then consider the discrete analogues of segments and circles which are termed, D-linear sets and r-sets respectively. We prove that, the intersection of two D-linear sets is also D-linear, but not the union. We also obtain a necessary and sufficient condition for a subset of H to be D-linear. For r-sets, formulae for the number of points on it and in its interior are found. Defining notions of contact set, intersection, discrete annulus etc. for two r-set, some results are established. We also bring out some contrasts with the Euclidean case. We then consider the intersection of discrete Pythagorean type in analogy with the orthogonal intersection of circles and some properties are obtained.

One of the most important concepts in the development of any geometry is that of a transformation. In chapter 3, we consider bijective mappings of H onto itself called D-transformations. Special transformations like D-translation and D-isometry are defined and studied. Some results obtained seem to be interesting, to mention one, D-isometries map domains onto domains. We define the D-linear transformations and characterise them.

The D-transformations that take r-sets onto r-sets have also been studied in detail. In this case, we need only consider the transformations between r-sets of equal radii, in order to maintain the bijective nature of the D-transformation. It is found that these special type of transformations form a finite, non abelian, solvable, nilpotent group. In the last section of this chapter, discrete analyticity properties of these transformations have been investigated. The geometry developed here, could be used in the analysis done by earlier authors like Harman. The guidelines are provided in this section.

The notion of convexity outside the framework of linear spaces, has been extensively studied. In the first two sections of chapter 4, we define D-convexity for subsets of H and obtain a sufficient condition for a domain to be not D-convex. Also, concepts like D-kernel and D-convex hull are considered and some characterisation theorems are obtained.

In the next section, we present some results obtained in the course of the investigation, which we feel are interesting, although not directly along the main line of thought in the thesis. These include, a matrix representation of domains, wherein we associate a matrix for domains, whose entries are the distance between points of it and the notion of metric content for subsets of H, which is the sum of elements in the upper (lower) triangular part of the distance matrix associated with the subset. The notion of E-set analogues to the ellipse is also considered.

Finally, we give suggestions for further study. We hope that, the theory of discrete integration developed by Harman could be applied to the D-transformations and obtain some more properties. Also, a combinatorial geometry could be developed on H analogous to the combinatorial geometry of the Euclidean plane.

CHAPTER 1

INTRODUCTION

This thesis is an attempt to initiate the development of a discrete geometry of the discrete plane $H = \{(q^m x_o, q^n y_o);\ m,n \in Z$ - the set of integers$\}$, where $q \in (o,1)$ is fixed and (x_o, y_o) is a fixed point in the first quadrant of the complex plane, $x_o, y_o \neq o$.

The discrete plane was first considered by Harman in 1972, to evolve a discrete analytic function theory for geometric difference functions. We shall mention briefly, through various sections, the principle of discretization, an outline of discrete a alytic function theory, the concept of geometry of space and also summary of work done in this thesis.

1.1. THE PRINCIPLE OF DISCRETIZATION AND DISCRE E MATHEMATICS

Discretization of scientific models dat s back to a very early origin. Dissatisfaction of any scientists on the over emphasis of the continuum structure on the scientific models, and the recognition of the fact that information can be transmitted in discrete forms and that information change in a system can be measured in a discrete manner has stimulated the development of mathematical theories of discrete structures. Attempts to compare the discrete with the continuous, to search for analogies between them, and ultimately to effect their unification were initiated by Zeno and tried by Leibnitz, Newton and others. Thus the discrete mathematics, which deals with finite or countable objects, in which the concept of infinitesimals and consequently that of continuity lacks, became the relevant mathematics for many social, biological and environmental problems. To quote Bell [10], " The whole of mathematical history may be interpreted as a battle of the supremacy between these two concepts ... . But the image of a battle is not wholly appropriate in mathematics at least, as the continuous and the discrete have frequently one another to progress" .

In discrete theory, the limit of a quotient of infinitesimal of the continuum structure is replaced by a quotient of finite quantity and consequently the differential equation by difference equation. In [60], Ruack argued that "the differential character of the principal equations of physics implies that the physical systems are governed by laws which operate with a precision beyond the limits of verification by experiment" . He suggested that more emphasis should be given to the use of difference calculus in physics. Many physicists are reluctant to accept the theory of discrete structures, as the equations of motion are to be recasted in the form of difference equations, whose solutions are difficult to be obtained mathematically. Detailed exposition of the philosophy of the discrete is available in [42], [45], [49], [54], [57] and [60].

The principle of discretization and the study of discrete structures are employed in different branches of mathematics. Construction of models and solving problems associated with discrete arrangement of objects are the concern of the theory of graphs, as described in [34]. In [59], a detailed study of other types of discrete mathematical models, with particular emphasis towards applications, is made. Another context where discrete mathematics comes into picture is the theory of discrete functions, functions with finite domain and co-domain, which find applications in the design of sequential switching circuits, communication theory etc., discussed in [19]. Our attempt here, is just to mention a few branches of mathematics where the concept of discreteness has been effectively used.

The term, discrete function , is used by us in a different context. We shall now consider a brief survey of discrete analytic function theory, a branch of study closely related to the work mentioned in this thesis.

1.2. OUTLINE OF DISCRETE ANALYTIC FUNCTION THEORY

Discrete analytic function theory is concerned with the study of complex valued functions defined only on certain discrete subsets of the complex plane. This branch was originated by Isaacs [43,44] in 1941, as an attempt to evolve a discrete analogue of classical complex analytic function theory. The discrete set that was originally used, was the lattice of Gaussian integers $\{m+in/m,n \in Z\}$. Functions defined on it, satisfying, $f(z+1)-f(z) = \frac{f(z+i)-f(z)}{i}$ were called 'monodiffric functions'.

In 1944, Ferrand [31], introduced the idea of preholomorphic functions' by means of the diagonal quotient equality $\frac{f(z+1+i)-f(z)}{1+i} = \frac{f(z+i)-f(z+1)}{i-1}$ and developed a corresponding discrete analytic function theory. This theory was taken up and further developed by Duffin in 1956, and since then quite a lot of work has been done by numerous authors.

In [22], he initiates the development by considering analogues of Cauchy-Riemann equations, contour integrals, Cauchy's formula, and applying them to the study of operational calculus, Hilbert transforms etc. He, in fact, established a school of discrete function theory and studied its extensions and generalisations to the other discrete subsets. This include Duris [24,25], Rohrer [26] and Kurowski [48], who considered the semi discrete lattice $\{(x,y),\ x \in R,\ y=nh,\ n \in Z\}$, $h > o$ is fixed. In [23], Duffin himself has considered the rhombic lattice to study potential theory. Berzsenyi [11] analysed the theory along the lines of Isaacs and has given a comphrehensive bibliography in [12] and so is Deeter [20]. Zeilberger [74] also has done some important work.

A Russian school mainly led by Abdullaev [2], Babadzanov [3], Silic [62] and recently Mednykh [50] has given considerable contributions to the development of the theory.

All these works and numerous others, not mentioned here, were on the lattice of Gaussian integers. It was in 1972, Harman [35-40] developed a discrete function theory on a different discrete set, $H = \{(q^m x_o, q^n y_o);\ m,n \in Z\}$, $q \in (o,1)$ is fixed and (x_o, y_o) is also fixed. The basic tool in developing the theory was that of q-difference functions. Complex valued functions defined on H satisfying $\frac{f(z)-f(qx,y)}{(1-q)x} = \frac{f(z)-f(x,qy)}{(1-q)iy}$ were called by him, q-analytic functions. The theory of q-analytic functions then deals with q-analytic continuation, discrete line integrals, discrete polynomials, analogues of Cauchy's integral formula, discrete convolution etc, which are closely allied to that of monodiffric functions, and has significant advantages and distinctive differences. He defines a notion of p-analytic function also in [35].

This theory finds its further generalisation in [41] for radial lattice, Bednar et. al. [9], West [72], Velukutty [70,71], Kritikumar [47], Richard [58], Mercy [52] and Thresiamma [68]. Velukutty considered discrete bianalytic functions which are both p-analytic and q-analytic and q-monodiffric functions which satisfy

$$\frac{f(q^{-1}x,y) - f(qx,y)}{(q^{-1}-q)x} = \frac{f(x,q^{-1}y) - f(x,qy)}{(q^{-1}-q)iy} .$$

Mercy has studied the discrete analogue of pseudoanalytic functions and in [68], the discrete basic commutative differential operators. In [46], Khan has mentioned the discrete bibasic analytic functions on the lattice $Q = \{(p^m x_0, q^n y_0);\ m,n \in Z\}$, $p \neq q \neq 1$. Abdi in [1] gives a survey of discrete analytic function theory with particular emphasis on q-analytic function theory.

All the works mentioned so far are function theoretic in nature. This motivated us to study in this thesis, some geometric aspects of the theory. Details of the work have been postponed to Section 4.

1.3. GEOMETRY OF A SPACE

We do not intend to give a detailed exposition of this subject, here. Several authentic books like [7], [28], [30], [53], [73] and many others treat this subject elegantly. We will just mention some important concepts here, based on which we have studied the geometry of the discrete plane.

Since the origin of geometry, geometers classified the geometric properties into two categories. The metric properties, in which the measure of distance and angles intervenes and the descriptive properties in which only the relative positional connection of the geometric elements with respect to one another is concerned. In this thesis, the metric properties of H are studied.

A remark mentioned in [15], " ... any serious student should, at some time, become familiar with the great discovery, made at the end of the last century, that large part of geometry do not depend upon continuity" , makes the study of geometry of the discrete structure, not totally irrelevant.

The ideas propounded by Klein in 1872, treated various geometries as theories of invariants under corresponding groups. In H, we define concepts like domain, D-linear set, r-set and discrete transformation and characterise D-linear transformations. We further characterise the transformations which leave invariant the r-sets with origin as centre and study some group theoretic properties.

1.4.SUMMARY OF RESULTS ESTABLISHED IN THIS THESIS

Of concern in this thesis, is the discrete subset, defined by $H = \left\{(q^m x_o, q^n y_o);\ m,n \in Z\right\}$, $q \in (o,1)$ is fixed and (x_o,y_o) is a fixed point in the first quadrant of the plane, $x_o,y_o \neq o$. Studies in the geometry of the discrete plane start from chapter 2 of this thesis, by first defining a suitable integer valued metric in H. We call H, then the discrete holometric space. We investigate the metric properties of H, introduce analogues of classical geometric concepts, transformations etc. of the Euclidean plane.

In chapter 2, we consider the concepts like discrete curve, path, holometric betweenness, discrete triangular triples, discrete Pythagorean triple, basic set, domain, adjacency of basic sets, index and diameter of domain, D-linear set, r-set etc. D-linear sets and r-sets serve as a reasonable analogue in the discrete case, of line segments and circles of the plane. Some of the important results that are established in this chapter are:

(1) H is a metric space.

(2) Two points $z_1=(q^{m_1}x_o, q^{n_1}y_o)$, $z_2=(q^{m_2}x_o, q^{n_2}y_o) \in H$ form with the origin, a discrete Pythagorean triple if and only if $|m_1n_1|+|m_2n_2| = |(m_1-m_2)(n_1-n_2)| - m_1m_2 - n_1n_2$.

(3) $D = \bigcup_{i=1}^{t} B_i$ is a domain if and only if it is connected and B_i, B_{i+1} are adjacent.

(4) For a domain D of index t, its diameter satisfies, $2 \leq \delta(D) \leq 2t$.

(5) If A,B are two D-linear sets, then $A \cap B$ is also D-linear.

(6) $A = \left\{ z_i = (q^{m_i}x_o, q^{n_i}y_o) \right\}_{i=1}^{t}$ is D-linear if and only if $\left\{m_i\right\}_{i=1}^{t}$ and $\left\{n_i\right\}_{i=1}^{t}$ are monotonic, not necessarily of the same type.

(7) The cardinality of $S_{r_1}(z_1)$, an r-set with centre $z_1 \varepsilon H$ and radius r_1, is $4r_1$ that of its interior is $r_1^2 + (r_1-1)^2$.

Defining concepts like contact, overlapping etc, for two r-sets, we have obtained some results. We have also considered, the intersection of discrete Pythagorean type, analogous to the orthogonal intersection of circles.

In chapter 3, we introduce discrete transformations and define concepts like D-isometry, D-translation and D-linear transformation. We show that the D-isometries map domains onto domains and characterise the D-linear transformations. We further characterise the D-transformations leaving invariant an r-set with origin as centre and show that these transformations form a group. In the last section of this chapter, we check for discrete analyticity, these transformations.

Chapter 4 deals with some concepts of convexity. Using the notion of holometric betweenness, we define D-convex sets. Some other concepts that we discuss in this chapter are D-kernel, D-convex hull, matrix representation and metric content for finite subsets of H, E-sets etc. Some of the results obtained in this chapter are:

(8) Intersection of D-convex sets is also D-convex.

(9) A domain in which there is at least one point of the form $(q^m x_o, q^{-m} y_o)$ or $(q^m x_o, q^m y_o)$ for some $m \in Z$, and which does not contain the basic set associated with at least one point of $P(q^m x_o, q^{-m} y_o)$ or $P(q^m x_o, q^m y_o)$ is not D-convex.

(10) D-kernel of A is A if and only if A is D-convex.

(11) D-convex hull of A, where A is a finite subset of H consisting of points in general position, is a domain.

In Section 3 of this chapter, we associate a distance matrix M for certain subsets of H. We have explicitly written the distance matrix for $S_{r_1}(z_o)$ and domains of the form $D_1 = S(z_o) \cup S(q^{-m} x_o, q^{-m} y_o)$, $m=1,2, \ldots, s$. We note that $M(D_1)$ is singular. An estimate for the metric content of $S_{r_1}(z_o)$ is obtained. Next, we consider E-sets, which are analogues of ellipses and denote it by $E_{p,k}(z_1,z_2)$. For E-sets, we prove

(12) For all admissible values of p, the cardinality of $E_{p,k}(z_1,z_2)$ is 2p. Further, for $E_{p,k}(z_o,z_1)$ where $z_1 = (q^m x_o, y_o)$ for some $m \in Z$, cardinality of Int $E_{p,k}$ is $(k+1) + 2(n-1)\ [n+k]$ where $n = \frac{p-k}{2}$.

We conclude the thesis in the section 5 of Chapter 4, by giving some suggestions for further study.

CHAPTER 2

METRIC PROPERTIES OF THE DISCRETE PLANE+

In this chapter, we discuss certain metric properties of the discrete plane $H = \{(q^m x_o, q^n y_o); m,n \in Z\}$, where Z is the set of integers, $q \in (o,1)$ is fixed and (x_o, y_o) is a fixed point in the first quadrant of the complex plane $x_o, y_o \neq o$. This discrete subset was first considered by Harman [35-40], to develop the theory of q-analytic functions and subsequently by Velukutty [70,71] and Mercy [52], for the study of discrete bianalytic and discrete pseudo-analytic functions respectively.

In section 1, we define the notion of distance between any two points of H, and study concepts like betweenness and discrete Pythagorean triples. In section 2, we consider the notion of domains and obtain a metric characterisation. Also, defining the notion of diameter of any subset of H, we obtain bounds for the diameter of a domain. The discrete analogues of line segments called D-linear sets, its properties and

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\+ Some results of this chapter are contained in the paper " Some metric properties of the geometric lattice " J.Math.Phys.Sci.Vol.17,No.5(1983),445-454 characterisation are discussed in section 3. In section 4, we define r-sets, analogous to the circles in the Euclidean plane and obtain some of its properties.

2.1. THE DISCRETE HOLOMETRIC SPACE

Consider $H = \left\{(q^m x_o, q^n y_o); m,n \in Z\right\}$. q is called the base and $z_o = (x_o, y_o)$, the origin of H. The points $z = (q^m x_o, q^n y_o)$; $m, n \in Z$ are called lattice points and H, the discrete plane.

We consider now some basic concepts.

DEFINITION 2.1.1. Let $z \in H$ and consider $N(z) = \left\{(q^{m+1}x_o, q^n y_o), (q^m x_o, q^{n+1}y_o), (q^{m-1}x_o, q^n y_o), (q^m x_o, q^{n-1}y_o)\right\}$. A discrete curve joining any two points z_1 and $z_t \in H$ is a finite sequence of points of H, $C = \langle z_1, z_2, z_3, \ldots, z_t\rangle$ where $z_{i+1} \in N(z_i)$ for $i = 1,2, \ldots, t-1$. The sequence of points $\langle z_t, z_{t-1}, \ldots, z_3, z_2, z_1\rangle$ is denoted by $-C$.

DEFINITION 2.1.2. A discrete curve joining two given points containing minimum number of lattice points is called a path joining them.

DEFINITION 2.1.3. Consider two points $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$ and $z_2 = (q^{m_2}x_o, q^{n_2}y_o) \in H$. The distance d between z_1 and z_2 is defined as $d(z_1,z_2)=N-1$, where N is the number of lattice points on a path joining them. In fact, $d(z_1,z_2) = |m_1-m_2| + |n_1-n_2|$. These concepts are illustrated in Figure 1.

THEOREM 2.1.4. (H,d) is a metric space.

PROOF. Consider three points z_r, z_s and $z_t \in H$.

(a) $d(z_r,z_s) \geqslant o$. For, $d(z_r,z_s)$ by definition is N-1, where N is the number of points on a path joining z_r and z_s. Clearly it is greater than or equal to zero and equality holds if and only if $N-1 = o$. That is, if and only if $z_r = z_s$.

(b) $d(z_r,z_s) = d(z_s,z_r)$. For, let C be a path joining z_r and z_s with $(\alpha+1)$ points. Then by definition 2.1.1, -C will be a path joining z_s and z_r having the same $(\alpha+1)$ points. So $d(z_r,z_s) = d(z_s,z_r)$.

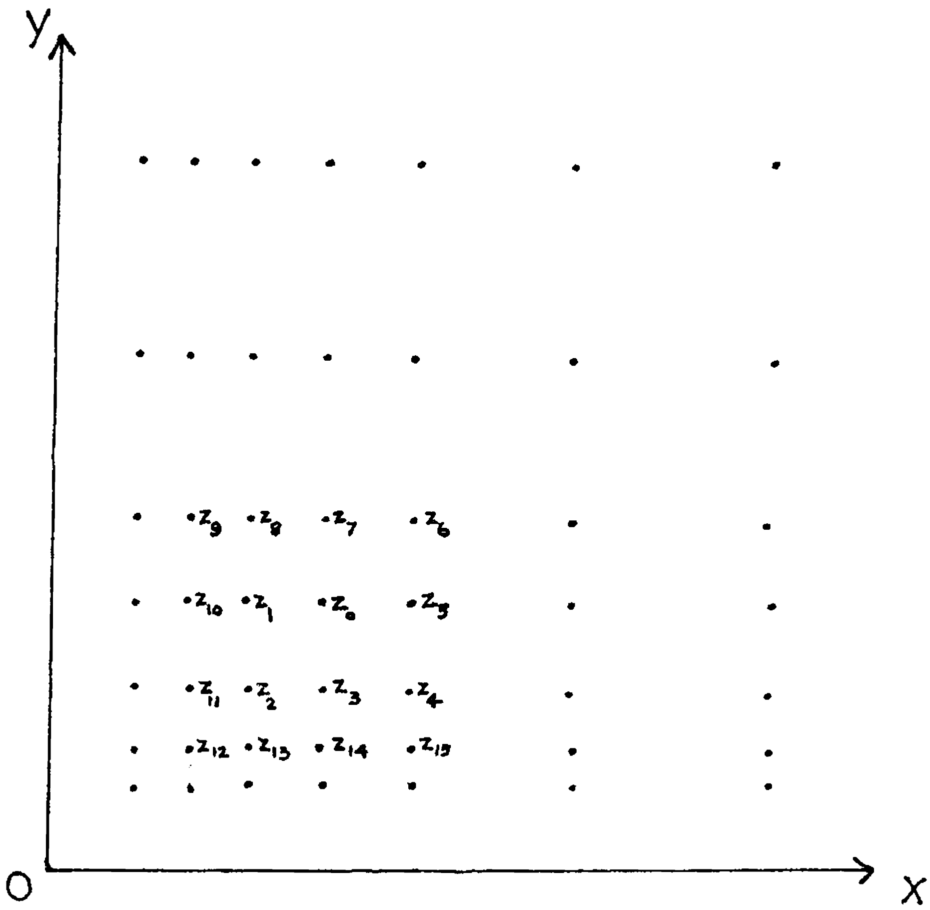


Figure-1

The discrete plane H

z_o - the origin of H, $N(z_2) = \{z_{11}, z_{13}, z_3, z_1\}$.

C_1 = $\langle z_o, z_1, z_{10}, z_{11}, z_2 \rangle$ is a discrete curve joining z_o and z_2.

C_2 = $\langle z_o, z_1, z_2 \rangle$ is a path. $d(z_o, z_2) = 2$.

(c) $d(z_r,z_t) \leq d(z_r,z_s) + d(z_s,z_t)$. For, let $d(z_r,z_s) = \alpha$ and C_1 be a path joining them. So there are $(\alpha+1)$ points on C_1 including z_r and z_s. Now, if C_2 is a path joining z_s and z_t and $d(z_s,z_t) = \beta$, then there are $(\beta+1)$ points on C_2 including z_s and z_t. Now, the curve $C_1+C_2 = \langle z_r, z_{r+1}, z_s, z_{s+1}, \ldots, z_t \rangle$ contains atleast one common point of C_1 and C_2 and hence if $d(z_r,z_t) = \delta$ and C_3 is a path joining them consisting of $(\delta+1)$ points, then $(\delta+1) \leq \alpha+\beta+1$. That is, $\delta \leq \alpha+\beta$.

Thus d satisfies all the conditions of a metric and hence (H,d) is a metric space.

NOTE 2.1.5. By the above theorem, (H,d) is a metric space in which d takes only integral values and so is a holometric space in the sense of [6]. We call (H,d) the discrete holometric space.

NOTATION. We denote by H, both the discrete plane and the discrete holometric space.

Considering distance as a fundamental concept, Menger [51] has developed a geometry called the distance geometry. One of the important concepts of this geometry is that of betweenness. An exhaustive study of the theory and application of distance geometry is available in Blumenthal [13].

Based on the notion of distance defined above, we shall define in the discrete holometric space H, certain discrete analogues of classical geometric concepts.

DEFINITION 2.1.6. Let $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$, $z_2 = (q^{m_2}x_o, q^{n_2}y_o)$, $z_3 = (q^{m_3}x_o, q^{n_3}y_o)$ be three distinct points of H. z_2 is said to be holometrically between z_1 and z_3 if $d(z_1,z_2) + d(z_2,z_3) = d(z_1,z_3)$. That is, $|m_2-m_1| + |n_2-n_1| + |m_2-m_3| + |n_2-n_3| = |m_1-m_3| + |n_1-n_3|$.

NOTATION. When z_2 satisfies the above definition, we write $B(z_1,z_2,z_3)$.

THEOREM 2.1.7. Consider $z_1,z_2,z_3,z_4 \varepsilon$ H. The holometric betweenness has the following properties.

(1) $B(z_1,z_2,z_3) \Leftrightarrow B(z_3,z_2,z_1)$.

(2) If $B(z_1,z_2,z_3)$ then neither $B(z_1,z_3,z_2)$ nor $B(z_2,z_1,z_3)$.

(3) $B(z_1,z_2,z_3)$ and $B(z_1,z_3,z_4) \Leftrightarrow B(z_1,z_2,z_4)$ and $B(z_2,z_3,z_4)$.

The proof follows directly from the definitions and is omitted.

The ternary relation of holometric betweenness satisfy the above properties of metric betweenness mentioned in [13]. The relation of betweenness for triples on a straight line possesses all the properties of metric betweenness. In addition, it has the property that, if z_o is between z_1,z_2 and z_2 is between z_o,z_3 then z_o is between z_1,z_3 and also z_2 is between z_1,z_3 [53].

But in H, we find that these implications need not always hold. As an example, consider the four points $z_o = (x_o,y_o)$, $z_1 = (qx_o,y_o)$, $z_2 = (x_o,qy_o)$ and $z_3 = (qx_o,qy_o)$. We have then, $B(z_1,z_o,z_2)$, $B(z_o,z_2,z_3)$, but not $B(z_1,z_o,z_3)$ and $B(z_1,z_2,z_3)$.

DEFINITION 2.1.8. Let $z_1, z_2, z_3 \in H$. If it satisfies $d(z_1,z_2) < d(z_1,z_3) + d(z_3,z_2)$ and two other similar inequalities, then the triple (z_1,z_2,z_3) is called a discrete triangular triple. Further, if $d(z_1,z_2) = d(z_2,z_3) = d(z_1,z_3)$ holds true, then it is called a discrete equidistant triple. It is a discrete isodistant triple with respect to z_1 if $d(z_1,z_3) = d(z_1,z_2)$.

DEFINITION 2.1.9. A discrete triangular triple (z_1,z_2,z_3) is said to be a discrete Pythagorean triple with respect to z_1 if $d(z_1,z_2)^2 + d(z_1,z_3)^2 = d(z_2,z_3)^2$.

If z_1,z_2,z_3 are the points given in definition 2.1.6, then the conditions mentioned in definition 2.1.8 can be written as

$$|m_1-m_2| + |n_1-n_2| < |m_1-m_3| + |n_1-n_3| + |m_2-m_3| + |n_2-n_3|$$

and two other similar inequalities for the discrete triangular triple,

$$|m_1-m_2| + |n_1-n_2| = |m_2-m_3| + |n_2-n_3| = |m_1-m_3| + |n_1-n_3|$$

for discrete equidistant triple and $|m_1-m_3| + |n_1-n_3| = |m_1-m_2| + |n_1-n_2|$ for discrete isodistant triple with respect to z_1.

EXAMPLES 2.1.10.

(a) $(q^m x_o, q^{-1} y_o)$, $(q^m x_o, q y_o)$, $(q^{m-1} x_o, y_o)$ form discrete equidistant triples.

(b) $(q^m x_o, q^{-2} y_o)$, $(q^m x_o, q^2 y_o)$, $m \neq o$ form with (x_o, y_o) discrete isodistant triples.

(c) $(q^m x_o, q^{-2} y_o)$, $(q^m x_o, q^3 y_o)$ form with $(q^{m-1} x_o, y_o)$ discrete Pythagorean triples.

THEOREM 2.1.11. Two points $z_1 = (q^{m_1} x_o, q^{n_1} y_o)$ and $z_2 = (q^{m_2} x_o, q^{n_2} y_o) \varepsilon H$ form a discrete Pythagorean triple with respect to the origin if and only if $|m_1 n_1| + |m_2 n_2| = |(m_1 - m_2)(n_1 - n_2)| - m_1 m_2 - n_1 n_2$.

PROOF. By definition, z_1, z_2 form a discrete Pythagorean triple with respect to $z_o \Longleftrightarrow d(z_o, z_1)^2 + d(z_o, z_2)^2 = d(z_1, z_2)^2$.

$$\Longleftrightarrow [\,|m_1| + |n_1|\,]^2 + [\,|m_2| + |n_2|\,]^2 = [\,|m_1 - m_2| + |n_1 - n_2|\,]^2$$

$$\Longleftrightarrow m_1^2 + n_1^2 + 2|m_1 n_1| + m_2^2 + n_2^2 + 2|m_2 n_2|$$

$$= m_1^2 - 2m_1 m_2 + m_2^2 + n_1^2 + n_2^2 - 2n_1 n_2 + 2|(m_1 - m_2)(n_1 - n_2)|$$

$$\Longleftrightarrow |m_1 n_1| + |m_2 n_2| = |(m_1 - m_2)(n_1 - n_2)| - m_1 m_2 - n_1 n_2.$$

Hence the theorem is proved.

2.2. DOMAIN AND ITS PROPERTIES

DEFINITION 2.2.1. Let $z_1 = (q^{m_1}x_o, q^{n_1}y_o) \varepsilon H$.

Then $S(z_1) = \left\{(q^{m_1}x_o, q^{n_1}y_o),\ (q^{m_1+1}, q^{n_1}y_o),\ (q^{m_1+1}x_o, q^{n_1+1}y_o),\right.$
$\left.(q^{m_1}x_o, q^{n_1+1}y_o)\right\}$ is called the basic set associated with z_1.

NOTATION. Basic sets will be denoted by B_1, B_2, B_3 etc.

DEFINITION 2.2.2. A finite union of basic sets is called a region. If a region can be expressed as a union of basic sets, $\bigcup_{i=1}^{t} B_i$ with $B_i \cap B_{i+1} \neq \varphi$, $i = 1,2, \ldots, t-1$, then it is called a domain. The minimum number of basic sets in a domain is called index.

NOTATION. R denotes a region. D, D_1, D_2, D_3 etc will denote domains and $I(D)$, the index of D (See Figure-2).

NOTE 2.2.3. If $\left\{D_i\right\}_{i=1}^{t}$ is a collection of domains with indices n_i, which are pairwise not disjoint, then $\bigcup_{i=1}^{t} D_i$ is also a domain with index $\sum_{i=1}^{t} n_i$. But, the intersection

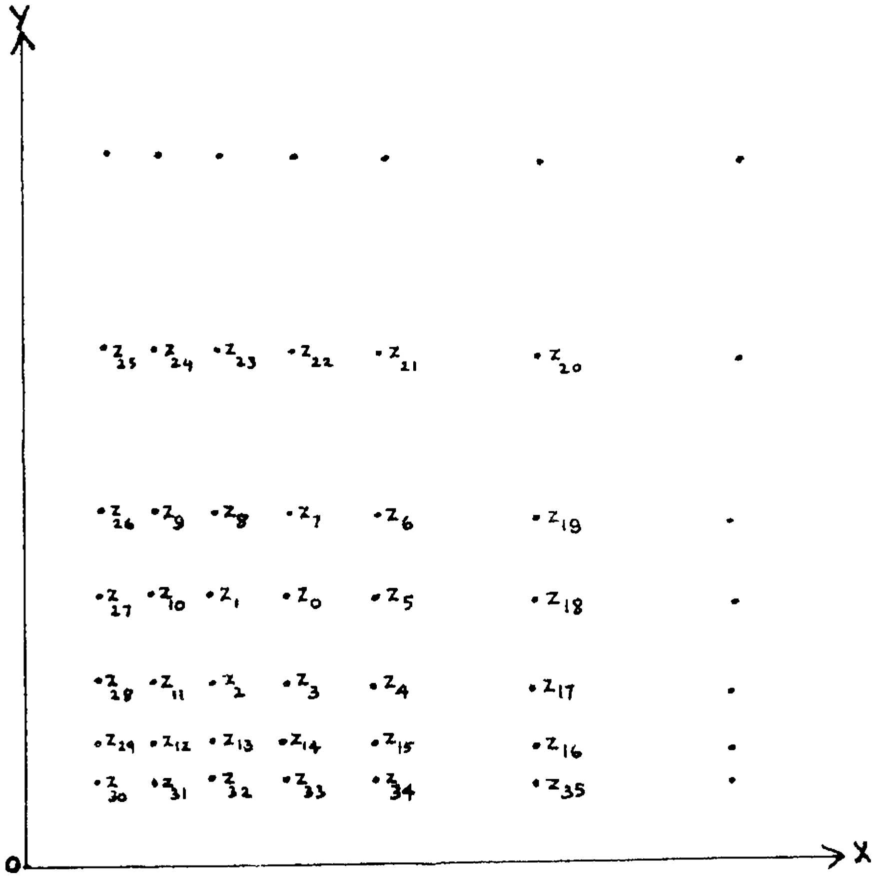


Figure-2

$B_1 = S(z_0) = \{z_0, z_1, z_2, z_3\}$, $B_2 = S(z_{18})$, $B_3 = S(z_{22})$
$B_2 \cup B_3$ is a region. $D_1 = S(z_3)$, $D_2 = S(z_4)$, $B_4 = S(z_{10})$
$B_5 = S(z_{14})$. B_2, B_3, B_4, B_5 are adjacent to B_1. $B_6 = S(z_6)$,
$B_7 = S(z_5)$, $B_8 = S(z_{19})$, $B_9 = S(z_8)$, $B_{10} = S(z_{21})$, $B_{11} = S(z_1)$.
$D_3 = B_1 \cup B_6 \cup B_8$, $D_4 = B_1 \cup B_7 \cup B_2$, $D_5 = B_1 \cup B_6 \cup B_8 \cup B_9 \cup B_{10}$,
$D_6 = B_1 \cup B_6 \cup B_{11}$.

of two domains, need not be a domain. As an example, consider $D_1 = S(z_3)$, $D_2 = S(z_4)$. Then $D_1 \cap D_2 = \{z_3, z_4\}$ which is not a domain (See Figure-2).

DEFINITION 2.2.4. Consider two basic sets B_1 and B_2. Then, $\min \{d(z_1, z_2);\ z_1 \in B_1,\ z_2 \in B_2\}$ is defined as the distance between B_1 and B_2, and written as $d(B_1, B_2)$.

If $B_1 \cap B_2 \neq \varphi$, then clearly $d(B_1, B_2) = o$.

DEFINITION 2.2.5. Two basic sets B_1 and B_2 are adjacent if there are two pairs of points $z_1, z_1' \in B_1$; $z_2, z_2' \in B_2$ such that $d(z_1, z_2) = d(z_1', z_2') = d(B_1, B_2) = 1$.

DEFINITION 2.2.6. Two points $z_1 = (q^{m_1} x_o, q^{n_1} y_o)$ and $z_2 = (q^{m_2} x_o, q^{n_2} y_o)$ are in the same horizontal (vertical) set if $n_1 = n_2$ $(m_1 = m_2)$.

NOTE 2.2.7. If $B_1 = S(z_1)$ and $B_2 = S(z_2)$ are two adjacent basic sets, then z_1 and z_2 belong to the same horizontal or vertical set. Consequently for a given basic set, there are only four basic sets adjacent to it. All these cases are illustrated in Figure-2.

THEOREM 2.2.8. If $D = \bigcup_{i=1}^{t} B_i$ such that B_i, B_{i+1} are adjacent for $i = 1,2, \ldots, t-1$, then D is a domain.

PROOF. Let $D = \bigcup_{i=1}^{t} B_i$ such that B_i, B_{i+1} are adjacent for $i = 1,2, \ldots, t-1$. Then B_{i+1} is such that, it is one among the four possibilities mentioned above. In any case, we can find a basic set (say) B such that $B_i \cap B \neq \varphi$ and $B \cap B_{i+1} \neq \varphi$. Include B also in our collection of basic sets and proceeding like this, D can be expressed as a union of basic sets $\{B_i'\}_{i=1}^{T}$ with $B_i' \cap B_{i+1}' \neq \varphi$ where $T > t$. Hence, D is a domain.

NOTE 2.2.9. Bajaj [8] has defined a subset A of an integer valued metric space to be connected if there do not exist nonempty disjoint sets A_1 and A_2, $A_1 \subset A$, $A_2 \subset A$ such that $A_1 \cup A_2 = A$ and $\min\{d'(x,y) : x \in A_1, y \in A_2\} > 1$. He has also proved that A is connected if and only if given any pair x,y of distinct points in A, there exists points $x = x_1, x_2, \ldots, x_p = y$ such that $d'(x_i, x_{i+1}) = 1$, for $i=1,2, \ldots, p-1$, where d' is the metric in A.

THEOREM 2.2.10. Consider a union of basic sets, $R = \bigcup_{i=1}^{t} B_i$. Then R is connected if and only if $\{B_i\}_{i=1}^{t}$ can be relabelled as $\{B_i'\}$ such that $d(B_i', B_{i+1}') \leq 1$.

PROOF. Let $R = \bigcup_{i=1}^{t} B_i$ be connected. That is, given any two points z and ζ of R, there are points $z = z_1, z_2, \ldots, z_n = \zeta$ such that $d(z_i, z_{i+1}) = 1$, $i = 1,2,\ldots, n-1$. Consider B_1 and choose all other basic sets B_i in $B_2, B_3, \ldots, B_t$ such that $d(B_1, B_i) \leq 1$. By tracing back if necessary at each step to B_1, these basic sets together with B_1 can be relabelled as $B_1', B_2', \ldots, B_t'$ such that $d(B_i', B_{i+1}') \leq 1$, $i = 1,2, \ldots, t-1$. If no such basic sets exist, then every other basic set B_s is such that $d(B_1, B_s) > 1$. Choose one such B_s. So by definition every pair of points $z_1 \in B_1$ and $z_2 \in B_s$ is with $d(z_1, z_2) \geq 2$. For any such pair, we cannot find a sequence of points satisfying the hypothesis and hence the supposition that there are no basic sets with the above property, leads us to a contradiction. Now, in the remaining basic sets of R, if there is at least one basic set B_r' which is at a distance ≤ 1 with atleast one among B_1', B_2', ..., B_t' (say) B_p', then we can similarly relabel the collection of all such basic sets together with those already relabelled, by tracing back if necessary at each step to B_p', such that the distance is less than or equal to 1. Thus, proceeding likewise the basic sets constituting R can be relabelled as $\left\{B_i'\right\}_{i=1}^{t}$ such that $d(B_i', B_{i+1}') \leq 1$.

Converse can be proved easily. As an illustration, consider $R = S(z_{12}) \cup S(z_2) \cup S(z_4) \cup S(z_6)$ (See Figure-2). The basic sets constituting R can be labelled such that the distance between the basic sets is less than or equal to 1. Now, any two points, for example, z_{30} and z_6 of R can be joined by a sequence $\langle z_{30}, z_{29}, z_{12}, z_{11}, z_2, z_3, z_0, z_5, z_6 \rangle$ with distance between consecutive points being 1.

NOTE 2.2.11. In the labelling $\{B_i'\}$ mentioned, if further B_i', B_{i+1}' are adjacent, then R is a domain.

Conversely if R is a domain, then it is connected and there is a labelling $\{B_i'\}$ such that B_i', B_{i+1}' are adjacent for $i = 1,2, \ldots$. Thus we have a metric characterisation of domains.

DEFINITION 2.2.12. Let A be a non empty finite subset of H. Then $\delta(A) = \max_{z_1,z_2 \in A} d(z_1,z_2)$ is defined as the diameter of A.

For domains, we have the following bounds for its diameter.

THEOREM 2.2.13. If D is a domain of index t, then $2 \leq \delta(D) \leq 2t$.

PROOF. If D is of index 1, then it is just a basic set and we have $\delta(D)=2$. Further $\delta(D)$ assumes the value $2t$ when it is a domain of index t associated with points of the form $(q^m x_o, q^m y_o)$, $(q^m x_o, q^{-m} y_o)$, $(q^{-m} x_o, q^m y_o)$ or $(q^{-m} x_o, q^{-m} y_o)$; $m = 1,2, \ldots, t$.

Now, inducting on t, let us assume that the result holds for a domain of index $(t-1)$. That is $\delta(D) \leq 2(t-1)$. Now, a domain of index t is obtained from that of $(t-1)$ by adjoining a basic set, which is of diameter 2. So $\delta(D) \leq 2(t-1)+2$. That is, $\delta(D) \leq 2t$. Hence the result.

Following examples show that there are domains with same index but with different diameters and domains with same diameter and different indices.

EXAMPLES 2.2.13 (See Figure-2)

(1) Let $D_3 = B_1 \cup B_6 \cup B_8$; $D_4 = B_1 \cup B_7 \cup B_2$.
Then, $I(D_3) = I(D_4) = 3$, but $\delta(D_1) = 5$;
$\delta(D_2) = 4$.

(2) Let $D_5 = B_1 \cup B_6 \cup B_9 \cup B_8 \cup B_{10}$, $D_6 = B_1 \cup B_{11} \cup B_6$.
Then $\delta(D_5) = \delta(D_6) = 5$, but $I(D_5) = 5$;
$I(D_6) = 3$.

2.3. D-LINEAR SET AND ITS PROPERTIES

In this section, we shall define the concept of D-linear sets, analogous to the notion of line segments in classical geometry. The property of being a D-linear set is referred to as D-linearity.

DEFINITION 2.3.1. Let A be a finite subset of H. A is said to be D-linear if we can label the points of A as $A = \{z_1, z_2, \ldots, z_n\}$ such that $d(z_1, z_n) = \sum_{i=1}^{n-1} d(z_i, z_{i+1})$. If such a labelling is not possible, we say that A is not D-linear.

NOTE 2.3.2. When we write the D-linear set $A = \{z_1, z_2, \ldots, z_n\}$ we mean that $z_1, z_2, \ldots, z_n$ are in that order in which $d(z_1, z_n) = \sum_{i=1}^{n-1} d(z_i, z_{i+1})$.

EXAMPLES 2.3.3.

(1) $L_1 = \{ z_o, z_1 = (qx_o, y_o),\ z_2 = (q^2x_o, y_o),\ z_3 = (q^3x_o, y_o)\}$ is D-linear.

(2) A path is a D-linear set, but not conversely. $L_2 = \{z_o, z_1 = (q^{-1}x_o, q^{-1}y_o),\ z_2 = (q^{-2}x_o, q^{-2}y_o), z_3 = (q^{-3}x_o, q^{-3}y_o)\}$ is an example of a D-linear set, which is not a path.

(3) The basic set associated with any point (definition 2.2.1) is not D-linear.

NOTE 2.3.4. If $\{z_1, z_2, \ldots, z_n\}$ is D-linear, then every $\{z_{i-1}, z_i, z_{i+1}\}$ is D-linear, for i=2, ..., n-1. However, the converse does not hold as seen in the case of above example (3), where all the three element subsets are D-linear, but not the basic set.

LEMMA 2.3.5. Let $A = \{z_1, z_2, \ldots, z_n\}$ be a D-linear set. If $r < s < t$ $(r,s,t=1,2, \ldots, n)$ then $B(z_r, z_s, z_t)$.

PROOF. Let us suppose that $B(z_r, z_s, z_t)$ does not hold. Then $\alpha(z_r, z_s) + \alpha(z_s, z_t) > d(z_r, z_t)$. Now $\sum_{i=r}^{s-1} d(z_i, z_{i+1}) \geqslant d(z_r, z_s)$ and $\sum_{i=s}^{t-1} d(z_i, z_{i+1}) \geqslant d(z_s, z_t)$ by triangle inequality. So

$$\sum_{i=1}^{n-1} d(z_i, z_{i+1}) > \sum_{i=1}^{r-1} d(z_i, z_{i+1}) + d(z_r, z_t) + \sum_{i=t}^{n-1} d(z_i, z_{i+1})$$

$$> d(z_1, z_n) \text{ by the definition of distance.}$$

Thus, we have a contradiction to the D-linearity of A. Hence the lemma.

THEOREM 2.3.6. If $A = \{z_1, z_2, \ldots, z_n\}$ is D-linear and $B \subset A$, then B is also D-linear.

PROOF. We shall prove the theorem by the method of induction. Let B be that subset of A obtained by deleting the point z_n. So $B = \{z_1, z_2, \ldots, z_{n-1}\}$. As A is D-linear, $d(z_1,z_n) = \sum_{i=1}^{n-1} d(z_i,z_{i+1}) = \sum_{i=1}^{n-2} d(z_i,z_{i+1}) +$

$d(z_{n-1},z_n)$. So

$$\sum_{i=1}^{n-2} d(z_i,z_{i+1}) = d(z_1,z_n) - d(z_{n-1},z_n) \quad .. \qquad (1)$$

Now, by the above lemma, $B(z_1,z_{n-1},z_n)$. Hence

$(1) \Longrightarrow \sum_{i=1}^{n-2} d(z_i,z_{i+1}) = d(z_1,z_{n-1})$. Thus B is D-linear.

Same arguments hold when the deleted point is z_1.

Now, let B be a subset of A obtained by deleting any point z_s other than z_1 and z_n. Then, the conclusion follows by the above lemma. Hence, by induction, it follows that any subset of a D-linear set is also D-linear.

NOTE 2.3.7. It is an easy consequence of the above theorem, that, intersection of two D-linear sets is also D-linear. However, union of two D-linear sets need not be so.

As an example, let $A = \{z_o, z_1=(q^{-1}x_o,y_o),\ z_2=(q^{-2}x_o,y_o),$ $z_3=(q^{-3}x_o,y_o)\}$ and $B=\{z_o, z_4 = (q^{-1}x_o,q^{-1}y_o),$ $z_5=(q^{-2}x_o,q^{-2}y_o),\ z_6 = (q^{-3}x_o,q^{-3}y_o)\}$. Then, A and B are D-linear sets but $A \cup B$ is not.

NOTATIONS. Let m,n be integers.

$$H_1 = \{(q^m x_o, q^n y_o);\ m,n \geqslant o\}$$
$$H_2 = \{(q^m x_o, q^n y_o);\ m < o;\ n \geqslant o\}$$
$$H_3 = \{(q^m x_o, q^n y_o);\ m,n, < o\}$$
$$H_4 = \{(q^m x_o, q^n y_o;\ m \geqslant o;\ n < o\}$$
$$X_1 = \{(q^m x_o, y_o);\ m \geqslant o\}$$
$$Y_1 = \{(x_o,\ q^n y_o);\ n \geqslant o\}$$
$$X_2 = \{(q^n x_o, y_o);\ m < o\}$$
$$Y_2 = \{(x_o, q^n y_o);n < o\}$$

Then, $H = H_1 \cup H_2 \cup H_3 \cup H_4$.

Following theorem gives a characterisation of D-linear sets.

THEOREM 2.3.8. Let $A = \{z_1, z_2, \ldots, z_t\} = \{(q^{m_1}x_o, q^{n_1}y_o),$ $(q^{m_2}x_o, q^{n_2}y_o),\ \ldots,\ (q^{m_t}x_o, q^{n_t}y_o)\}$ be a finite subset of H .

Then A is D-linear if and only if the sequences $\{m_i\}_{i=1}^{t}$ and $\{n_i\}_{i=1}^{t}$ are monotonic, not necessarily of the same type.

PROOF. Suppose that both $\{m_i\}_{i=1}^{t}$ and $\{n_i\}_{i=1}^{t}$ are monotonic increasing. Then $d(z_1, z_t) = |m_t - m_1| + |n_t - n_1|$ $= (m_t - m_1) + (n_t - n_1)$, and $\sum_{i=1}^{t-1} d(z_i, z_{i+1}) = \sum_{i=1}^{t-1} (|m_{i+1} - m_i| + |n_{i+1} - n_i|) = (m_t - m_1) + (n_t - n_1) = d(z_1, z_t)$. Hence A is D-linear.

Similar arguments prove that if m_i and n_i are both monotonic decreasing or one of them is increasing and the other is decreasing, then A is D-linear. Conversely, let us assume that $A = \{z_1, z_2, \ldots, z_t\}$ is a D-linear subset of H. We shall prove the result, by considering various possibilities.

(a) z_1 is the origin and $z_2, z_3, \ldots, z_t$ are in H_1. Since z_1 is the origin, $m_1 = n_1 = o$. Now $d(z_1, z_t) =$ $|m_t| + |n_t| = m_t + n_t$.

Claim: $m_j \geqslant m_i$; $n_j \geqslant n_i$, for every $j \geqslant i=1,2,\ldots,t$. (2)

Suppose not. Then choose $m_k > o$ such that it is the first, where $\geqslant$ in (2) is violated. Then

$$\sum_{i=1}^{t-1} d(z_i, z_{i+1}) = |m_2| + |n_2| + |m_3 - m_2| + |n_3 - n_2| + \cdots +$$

$$|m_{k-1} - m_{k-2}| + |n_{k-1} - n_{k-2}| + |m_k - m_{k-1}| + |n_k - n_{k-1}| + \cdots +$$

$$|m_{t-1} - m_{t-2}| + |n_{t-1} - n_{t-2}| \neq m_t + n_t ,$$ since the term

involving m_{k-1} does not cancel. So we have a contradiction to the initial assumption that A is D-linear. Hence $m_j \geqslant m_i$, $n_j \geqslant n_i$. Similarly, when the D-linear set is wholly contained in H_2, H_3 or H_4, with the origin as the initial point, we have the other three possibilities. If we consider a D-linear set in H_i, $i=1,2,3,4$, with the origin as end point, then also the conclusion follows.

(b) One of the points other than the end points of the D-linear set is the origin.

Only a sketch of the proof will be given. Let $A = \{z_1, z_2, \ldots, z_k, \ldots, z_t\}$ and one of the points (say) z_k, $k \neq 1$, t be the origin. Suppose the points $z_1, z_2, \ldots, z_{k-1}$ are in H_1 and $z_{k+1}, z_{k+2}, \ldots, z_t$ are in H_3.

Then, it can be proved that the m_is and n_is are both monotonic increasing or decreasing. Also, when the points are such that the (k-1) points of A are in H_2 and the remaining in H_4, we have m_is are increasing and n_is decreasing or vice versa. Further, if z_k is the point distinct from origin belonging to $X_1 \cup X_2$, we have that these points are in $H_1 \cup H_4$ or $H_2 \cup H_3$ and if $z_k \varepsilon Y_1 \cup Y_2$ these points are in $H_1 \cup H_2$ or $H_3 \cup H_4$. In both the cases, the conclusion follows similarly.

Finally, let $A = \{z_1, z_2, \ldots, z_k, z_s, \ldots, z_t\}$ If $z_k \varepsilon X_1$, $z_s \varepsilon Y_1$, then $A \subset H_1 \cup H_2 \cup H_4$, if $z_k \varepsilon X_2$, $z_s \varepsilon Y_1$, then $A \subset H_1 \cup H_2 \cup H_3$, if $z_k \varepsilon X_1$, $z_s \varepsilon Y_2$, then $A \subset H_1 \cup H_3 \cup H_4$ and if $z_k \varepsilon X_2$, $z_s \varepsilon Y_2$, then $A \subset H_2 \cup H_3 \cup H_4$. In all these cases conclusion follows. A detailed proof is omitted, being very lengthy. Hence the theorem.

NOTE 2.3.9. D-linear sets play an important role in Chapter 3, while discussing discrete transformations. In that context, a set of points satisfying the conditions of the above theorem is said to be oriented.

2.4. r-SET AND ITS PROPERTIES

In this section, we shall consider a discrete analogue of circles in the Euclidean plane. Due to the discrete nature of the metric and of the plane H, the r-sets have some notable aspects, which are highlighted in this section.

DEFINITION 2.4.1. An r-set with centre $z_1=(q^{m_1}x_o, q^{n_1}y_o) \varepsilon H$ and radius r_1 is defined as, $\{z \varepsilon H : d(z,z_1) = r_1\} = \{(q^m x_o, q^n y_o) \varepsilon H : |m-m_1| + |n-n_1| = r_1\}$. Also, $\{z \varepsilon H : d(z,z_1) < r_1\}$ is called the interior of the r-set.

NOTATIONS. $S_{r_1}(z_1)$-the r-set with centre z_1 and radius r_1. Int $S_{r_1}(z_1)$ - the interior of $S_{r_1}(z_1)$ and $TS_{r_1}(z_1) =$ Int $S_{r_1}(z_1) \cup S_{r_1}(z_1)$.

NOTE 2.4.2. Let us take the centre of the r-set to be the origin and r_1, the radius. Let $X = q^m x_o$, $Y = q^n y_o$. Then $\log X = m \log q + \log x_o$; $\log Y = n \log q + \log y_o$. So

$$m = \frac{\log(X/x_o)}{\log q} \quad ; \; n = \frac{\log(Y/y_o)}{\log q}$$

Hence, the equation of $S_{r_1}(z_o)$ can be written as

$$|\log (X/x_o)| + |\log (Y/y_o)| = r_1 |\log q|$$

In figure 3, the distribution of points of $S_{r_1}(z_o)$, for $r_1 = 1,2,3$, is illustrated.

Now, we shall find a formula for the number of lattice points in the sets $S_{r_1}(z_1)$, Int $S_{r_1}(z_1)$ and $TS_{r_1}(z_1)$ where $z_1 \in H$ and call the number of points on it, their cardinality.

THEOREM 2.4.3. The cardinality of $S_{r_1}(z_1)$, Int $S_{r_1}(z_1)$ and $TS_{r_1}(z_1)$ are $4r_1$, $r_1^2 + (r_1-1)^2$, $r_1^2 + (r_1+1)^2$ respectively.

PROOF. Without loss of generality, let us take the centre of the r-set to be the origin z_o. Then by definition, $S_{r_1}(z_o) = \left\{(q^m x_o, q^n y_o) \in H : |m|+|n| = r_1\right\}$. The points of $S_{r_1}(z_o)$ can be classified into a disjoint union of four sets as

$$L_1 = \left\{(q^{\alpha} x_o, q^{r_1-\alpha} y_o)\right\} , \quad L_2 = \left\{(q^{r_1-\alpha} x_o, q^{-\alpha} y_o)\right\} ,$$

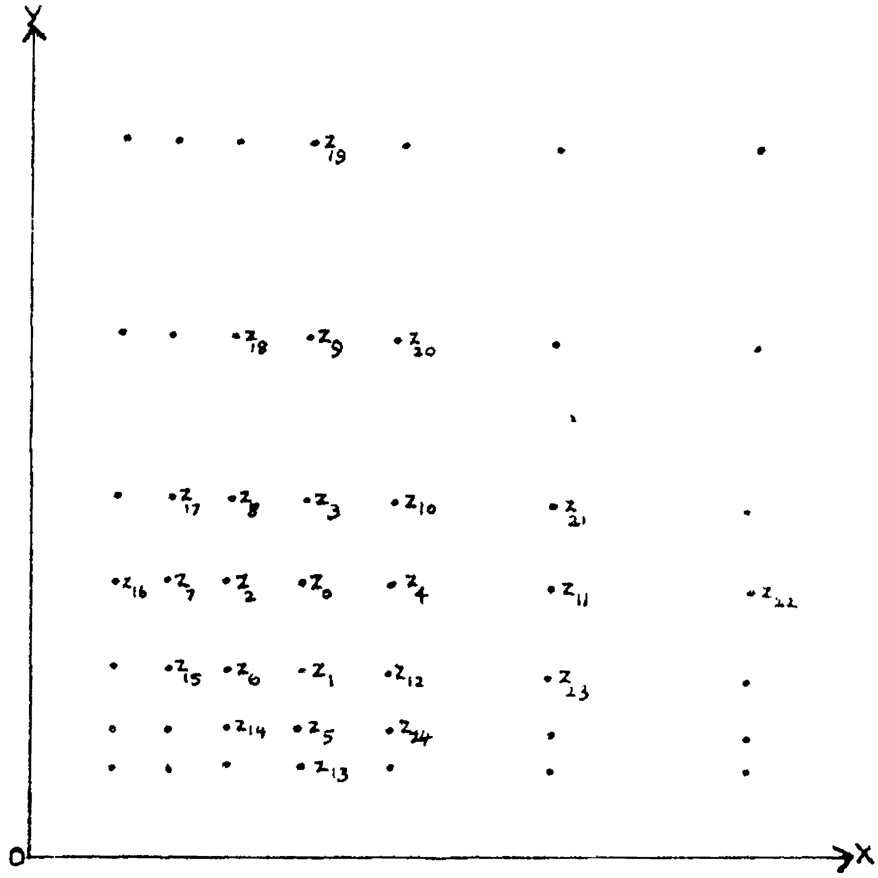


Figure-3

r - sets

$$S_1(z_o) = \{z_1, z_2, z_3, z_4\} .$$

$$S_2(z_o) = \{z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}\} .$$

$$S_3(z_o) = \{z_{13}, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}, z_{19}, z_{20}, z_{21}, z_{22}, z_{23}, z_{24}\} .$$

$$L_3 = \left\{(q^{-\alpha}x_o, q^{-r_1+\alpha}y_o)\right\} \quad , \; L_4 = \left\{(q^{-r_1+\alpha}x_o, q^{\alpha}y_o)\right\}$$

where $\alpha = o,1,2, \ldots, r_1-1$. Then $\bigcup_{i=1}^{4} L_i = S_{r_1}(z_o)$. (In Figure 3, for $S_3(z_o)$, $L_1 = \left\{z_{13}, z_{14}, z_{15}\right\}$, $L_2 = \left\{z_{16}, z_{17}, z_{18}\right\}$, $L_3 = \left\{z_{19}, z_{20}, z_{21}\right\}$, $L_4 = \left\{z_{22}, z_{23}, z_{24}\right\}$). Further, each L_i is D-linear and has r_1 points. So cardinality of $S_{r_1}(z_o)$ is $4r_1$.

Now, Int $S_{r_1}(z_o) = \left\{z \varepsilon H : |m|+|n| < r_1\right\}$. Hence, Int $S_{r_1}(z_o) = S_{r_1-1} \cup S_{r_1-2} \cup S_{r_1-3} \ldots \cup S_1 \cup S_o$ where S_o is the centre. Therefore, cardinality of Int $S_{r_1}(z_1)$ is $4(r_1-1) + 4(r_1-2) + \ldots 4+1$.

$$= \frac{4(r_1-1)r_1}{2} + 1 = 2r_1^2 - 2r_1 + 1$$

$$= (r_1-1)^2 + r_1^2$$

Also, TS_{r_1} = Int $S_{r_1} \cup S_{r_1}$. So, cardinality of TS_{r_1} = Cardinality of Int S_{r_1} + cardinality of $S_{r_1} = (r_1-1)^2 + r_1^2 + 4r_1 = 2r_1^2 + 2r_1 + 1 = (r_1+1)^2 + r_1^2$.

NOTE 2.4.4. It is noted that, there are two D-linear sets $L_1 = \left\{(q^s x_o, y_o)\right\}$ and $L_2 = \left\{(x_o, q^s y_o)\right\}$, $|s| \leq r_1$, with respect to which the points of $S_{r_1}(z_o)$ are distributed symmetrically. These two sets have $(2r_1+1)$ points each and have end points $(q^{r_1} x_o, y_o)$, $(q^{-r_1} x_o, y_o)$ for L_1 and $(x_o, q^{r_1} y_o)$, $(x_o, q^{-r_1} y_o)$ for L_2. Unlike in the case of circles, these are the only two sets with these properties.

THEOREM 2.4.5. For a given $S_r(z_o)$, there is one and only one (r-1) set that is contained in Int $S_r(z_o)$ and there are five (r-1) sets contained in $TS_r(z_o)$.

PROOF. Consider $S_{r_1}(z_o)$. If $S_{r_1-1}(z_1) \subset$ Int $S_{r_1}(z_o)$, then we claim $z_1 = z_o$.

If possible, let $z_1 = (q^{m_1} x_o, q^{n_1} y_o)$, m_1, n_1 not both zero. Then, we show that there are points in $S_{r_1-1}(z_1)$ which are not interior to $S_{r_1}(z_o)$. This is done as follows. Let us first suppose that $z_1 \in H_1$. Then the points of $S_{r_1-1}(z_1)$ lying in H_1 of the form

$$\left\{(q^{\alpha_1+m_1} x_o, q^{r_1-\alpha_1-1+n_1} y_o);\ \alpha_1 = o, 1, 2, \ldots, r_1-2\right\}$$

are not interior to $S_{r_1}(z_o)$. For if $(q^m x_o, q^n y_o)$ is such a point, then

$$|m-m_1| + |n-n_1| = r_1-1 \quad (3)$$

$$|m| + |n| < r_1 \quad (4)$$

(3) is $(m-m_1) + (n-n_1) = r_1-1$, since $m \geq m_1$, $n \geq n_1$ and, (4) is $m+n < r_1$. Now, (3) $\Rightarrow$ $m+n = (r_1+m_1+n_1)-1$

$$\geq r_1 \text{ since } m_1+n_1 \geq 1$$

which contradicts (4). Thus $S_{r_1-1}(z_1) \not\subset \text{Int } S_{r_1}(z_o)$. Hence our claim. Similar argument works when $z_1 \in H_2$, H_3 or H_4. Now, clearly there is one and only one (r_1-1) set centered at z_o and hence there is one and only one (r_1-1) set contained in Int $S_{r_1}(z_o)$.

Now, to prove that there are five (r_1-1) sets contained in $TS_{r_1}(z_o)$. The technique used above can be employed to show that, if $S_{r_1-1}(z_1) \subset TS_{r_1}(z_o)$, then $z_1 \in TS_1(z_o)$ and hence there are five (r_1-1) sets contained in $TS_{r_1}(z_o)$.

NOTE 2.4.6. For a given $S_r(z_o)$, μ, the number of (r-k) sets that are contained in Int $S_r(z_o)$ and ρ, the number of (r-k) sets that are contained in $TS_r(z_o)$, seems to be independent of the value of r. This constant value for some values of k are found to be for $k=1$, $\mu=1$, $\rho=5$, for $k=2$, $\mu=5$, $\rho=13$, for $k=3$, $\mu=13$, $\rho=25$, for $k=4$, $\mu=25$, $\rho=41$ etc. We could however prove onày for the case $k=1$, in the above theorem.

DEFINITION 2.4.7. Consider $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$. They are said to touch each other, if $S_{r_1} \cap S_{r_2} \neq \varphi$ and $TS_{r_1} \cap \text{Int } S_{r_2} = \varphi$. $\{z \in H : z \in S_{r_1} \cap S_{r_2}\}$ is called the contact set.

NOTATION $\quad$ K - the contact set

η - the cardinality of K

NOTE 2.4.8. In the place of the conditions mentioned in the above definition, we could have given $S_{r_1} \cap S_{r_2} \neq \varphi$ and $TS_{r_2} \cap \text{Int } S_{r_1} = \varphi$. We believe that these two are equivalent, however we do not have a proof. Also, for circles in the Euclidean plane, it is true that $TS_{r_1} \cap \text{Int } S_{r_2} = \varphi$ if and only if $\text{Int } S_{r_1} \cap \text{Int } S_{r_2} = \varphi$.

(here TS_r means the closed circular disc). But, for r-sets which are discrete analogue of circles, $TS_{r_1} \cap \text{Int } S_{r_2} = \varphi \Rightarrow \text{Int } S_{r_1} \cap \text{Int } S_{r_2} = \varphi$, but the reverse implication need not hold. For, consider the r-sets $S_2(z_1)$ and $S_2(z_2)$ where $z_1 = (qx_o, y_o)$, $z_2 = (q^{-2}x_o, y_o)$. Then, $\text{Int } S_2(z_1) \cap \text{Int } S_2(z_2) = \varphi$, but $(q^{-1}x_o, y_o) \in TS_2(z_1) \cap \text{Int } S_2(z_2)$. Based on this fact, if we define the overlapping of two r-sets (definition 2.4.14) in terms of the interior, we will get some results which differ from what we have obtained in theorem 2.4.20. However, for obvious reasons we prefer the weaker conditions.

In the following theorem, some formulae for the cardinality of the contact set is obtained, for certain choice of the centres.

NOTE 2.4.9. In the sequel, unless otherwise specified, without loss of generality, we take one of the centres to be the origin.

THEOREM 2.4.10.

(a) If the two r-sets, $S_{r_1}(z_o)$ and $S_{r_2}(z_1)$

$z_1 \in H_1$ touch, then the contact set K is a unique point and is in X_1 if and only if $z_1 \in X_1$.

(b) If the r-sets have equal radii r and $z_1 = (q^{\pm r}x_o, q^{\pm r}y_o)$, then they touch and the dimension of contact η is equal to r+1.

(c) If the r-sets of equal radii r, $S_r(z_o)$ and $S_r(z_1)$, $z_1 \in H_1$ touch and η is equal to r+1, then $z_1 = (q^r x_o, q^r y_o)$.

PROOF (a) Let $z_1 = (q^\alpha x_o, y_o)$ for some $\alpha > o$. Then, first note that $\alpha \geqslant 2$. For, if we take the least possible values for r_1 and r_2, $r_1 = r_2 = 1$, then $S_1(z_o)$ and $S_1(z_1)$ touch means that $d(z_o, z_1) = r_1 + r_2 = 2$, while $z_1 \in X_1$ implies that $z_1 = (q^2 x_o, y_o)$. Now, let $z = (q^m x_o, q^n y_o) \in K$. Then, $|m| + |n| = r_1$, $|m-\alpha| + |n| = r_2$ and $|\alpha| = r_1 + r_2$. That is, $m+n = r_1$, $\alpha - m + n = r_2$ and $\alpha = r_1 + r_2$. So, $n = o$ and hence $z \in X_1$. But $S_{r_1}(z_o)$ has only one point common with H_1. viz. $(q^{r_1} x_o, y_o)$.

Conversely, let the unique point of contact belong to X_1. If possible, let $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$, $n_1 \neq o$ be the centre of the other r-set. Since $z_1 \in H_1$ and $n_1 \neq o$, we have $n_1 > o$. Let $m_1 > n_1$. Then

$(q^{r_1}x_o, y_o) = (q^{m_1+n_1-r_2}x_o, y_o)$ is a point of contact.

In addition, $(q^{m_1-1}x_o, q^{n_1+1-r_2}y_o) = (q^{r_1-1}x_o, qy_o)$ will also be a point of contact. So there are at least two points of contact, contradicting the uniqueness. The case $m_1 < n_1$ can be done using symmetry arguments. Hence $z_1 \in X_1$.

(b) Consider $S_r(z_o)$ and $S_r(z_1)$ where $z_1 = (q^r x_o, q^r y_o)$. By theorem 2.4.3, there are 4r points on $S_r(z_o)$, of which the set of points of the form $(q^{\alpha_1}x_o, q^{r-\alpha_1}y_o)$; $\alpha_1 = o, 1, 2, \ldots, r$, is contained in H_1. The set of points of the form $(q^{r-\alpha_1}x_o, q^{\alpha_1}y_o)$; $\alpha_1 = o, 1, 2, \ldots, r$, is the set of points of $S_r(z_1)$ coinciding with the above set of points. Hence $S_r(z_o) \cap S_r(z_1)$ has (r+1) points and since $TS_r(z_o) \cap \text{Int } S_r(z_1) = \varphi$, they touch. So $\eta = r+1$.

The proof for the cases when $z_1 = (q^{-r}x_o, q^{-r}y_o)$, $(q^r x_o, q^{-r}y_o)$ or $(q^{-r}x_o, q^{\ r}y_o)$ are on similar lines.

(c) Suppose not. Let $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$, both $m_1, n_1 \neq r_1, o$. If $m_1 < n_1$, the point $\xi_1 = (x_o, q^r y_o)$ is a point of $S_r(z_o)$ which is not a point of contact for $S_r(z_1)$, since $d(z_1, \xi_1) = m_1 + n_1 - r \neq r$. If $m_1 > n_1$, $\xi_2 = (q^r x_o, y_o)$ serves the role of ξ_1 and if $m_1 = n_1$ no point of $S_r(z_o)$ is a point of contact. Hence in all cases we reach a contradiction to the hypothesis that $\eta = r+1$. Hence $z_1 = (q^r x_o, q^r y_o)$.

NOTE 2.4.11. In (a) of the above theorem, it is also true that for $z_1 \varepsilon H_1$, K consists of a unique point in Y_1 if and only if $z_1 \varepsilon Y_1$. Further, if $z_1 \varepsilon H_2$ or H_4, then K is a unique point in X_2 or Y_2 if and only if $z_1 \varepsilon X_2$ or Y_2. In [c], it is also true that for $z_1 \varepsilon H_2$ (H_3 or H_4) and η is equal to r+1, then $z_1 = (q^{-r}x_o, q^r y_o)$, $(q^{-r}x_o, q^{-r}y_o)$ or $(q^r x_o, q^{-r}y_o)$. These results have not been proved for the reason that this can be done along similar lines mentioned above.

NOTE 2.4.12. It is seen from the above theorem, that the minimum value of η is 1 and in the case of r-sets of equal radii r, for the proper choice of centres, η assumes the value $(r+1)$ also. It is further noted that for a given r-set $S_{r_1}(z_0)$, we can find an $S_{r_2}(z_1)$ $(r_2 > r_1)$ which has as its contact set any subset of the (r_1+1) points of $S_{r_1}(z_0)$ lying in H_1 (H_2, H_3 or H_4 as the case may be). This observation in its most general case is difficult to be proved. But in the following theorem, we state a particular case.

THEOREM 2.4.13. If $z_1 = (q^{m_1}x_0, qy_0)$, $m_1 \geqslant r_1$, then there exists an $S_{r_2}(z_1)$ for which $K = \left\{ (q^{r_1}x_0, y_0), (q^{r_1-1}x_0, qy_0) \right\}$ and conversely.

DEFINITION 2.4.14. $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$ are said to overlap if $S_{r_1}(z_1) \cap S_{r_2}(z_2) \neq \varphi$ and $TS_{r_1} \cap \text{Int } S_{r_2} \neq \varphi$ (as well as $\text{Int } S_{r_1} \cap TS_{r_2} \neq \varphi$).

NOTATION. When S_{r_1} and S_{r_2} overlap, we denote by I $\left\{ z \in H : z \in S_{r_1} \cap S_{r_2} \right\}$, and by $U = \left\{ z \in H : z \in S_{r_1} \cap \text{Int } S_{r_2} \right\}$

DEFINITION 2.4.15. S_{r_1} and S_{r_2} are seperated if $I=\varphi=U$.

DEFINITION 2.4.16. Consider S_{r_1} and S_{r_2} with $S_{r_1} \cap S_{r_2} \neq \varphi$. Then S_{r_1} is said to be indispensable for S_{r_2} if $\text{Int } S_{r_1} \cap TS_{r_2} = \text{Int } S_{r_1}$ (if $r_1 < r_2$). If $TS_{r_1} \cap \text{Int } S_{r_2} = \text{Int } S_{r_2}$ (if $r_1 > r_2$) then S_{r_2} is said to be indispensable for S_{r_1}.

DEFINITION 2.4.17. Consider S_{r_1} and S_{r_2} with $S_{r_1} \cap S_{r_2} = \varphi$. If $TS_{r_1} \cap \text{Int } S_{r_2} = TS_{r_1}$ $(r_1 < r_2)$ or $TS_{r_1} \cap \text{Int } S_{r_2} = TS_{r_2}$ $(r_1 > r_2)$ then the r-sets are said to form a discrete annulus.

EXAMPLES 2.4.18.

(1) Let $z_1 = (qx_o, y_o)$, $z_2 = (q^{-2}x_o, y_o)$. Then $S_2(z_1)$ and $S_2(z_2)$ overlap and $I = \{(x_o, q^{-1}y_o), (x_o, qy_o)\}$, $U = \{(x_o, y_o), (q^{-1}x_o, y_o)\}$.

(2) Let z_1, z_2, r_1 be as in (1) and $r_2 = 2$, then $S_2(z_1)$, $S_2(z_2)$ are seperated.

(3) Let $z_1 = (qx_o, q^{-3}y_o)$, $z_2 = (q^{-5}x_o, q^{-3}y_o)$, then $S_2(z_1)$ is indispensable for $S_8(z_2)$.

(4) Let $z_1 = (qx_o, y_o)$ and $z_2 = (q^2x_o, y_o)$, then $S_4(z_1)$ and $S_2(z_2)$ form a discrete annulus.

NOTE 2.4.19. If $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$ are either overlapping, seperated or indispensable or if they form a discrete annulus, then $d(z_1, z_2) \leq r_1 + r_2$. Converse need not hold true. As an example, let $z_1 = (qx_o, y_o)$, $z_2 = (q^{-2}x_o, y_o)$, $r_1 = r_2 = 2$. Then $d(z_1, z_2) = 3 < r_1 + r_2 = 4$. But $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$ satisfy none of the above conditions.

THEOREM 2.4.20.

(a) $S_1(z_o)$ and $S_{r_2}(z_1)$ are seperated if and only if $r_2 \leq d(z_o, z_1) - 2$, $S_1(z_o)$ is indispensable for $S_{r_2}(z_1)$ if and only if $r_2 = d(z_o, z_1) + 1$ and they form a discrete annulus if and only if $r_2 \geqslant d(z_o, z_1) + 2$.

(b) If $S_1(z_o)$ and $S_{r_2}(z_1)$ where $z_1 = (q^{m_1}x_o, y_o)$ for some $m_1 \in Z$ overlap, then the cardinality of I is 2.

(c) $S_2(z_o)$ and $S_{r_2}(z_1)$ are seperated if and only if $r_2 \leqslant d(z_o,z_1)-3$, for $r_2 = d(z_o,z_1)$, they overlap and cardinality of I is 2, for $r_2 = d(z_o,z_1) + 2$, S_2 is indispensable for $S_{r_2}(z_1)$ and its cardinality 5. Further for $r_2 \geqslant d(z_o,z_1) + 2$, they form a discrete annulus.

PROOF. Only (a) will be proved here. Proof for (b) and (c) being on similar lines, are omitted.

(a) Consider $S_1(z_o)$ and $S_{r_2}(z_1)$, $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$. Let $r_2 \geqslant d(z_o,z_1)-2 = |m_1| + |n_1| - 2$. If possible, let $(q^m x_o, q^n y_o) \in S_1 \cap S_{r_2}$. Then

$$|m| + |n| = 1 \tag{5}$$

$$|m-m_1| + |n-n_1| \leq |m_1| + |n_1| - 2 \tag{6}$$

Solutions of equations (5) and (6), gives a contradiction. Also $TS_1 \cap \text{Int } S_{r_2} \neq \varphi$ requires $|m'| + |n'| \leq 1$ and $|m'-m_2| + |n'-n_2| \leq |m_2| + |n_2| - 2$ for some $(q^{m'}x_o, q^{n'}y_o) \in H$, which is not possible. So, $S_1 \cap S_{r_2} = \varphi$, $TS_1 \cap \text{Int } S_{r_2} = \varphi$ and hence S_1 and $S_{r_2}(z_1)$ are seperated.

Conversely, suppose if possible $r_2 > d(z_o, z_1)-2$. Consider $(qx_o, y_o) \in S_1$. We have $|1-m_1| + |n_1| > |m_1| + |n_1| -2$. So $(qx_o, y_o) \in S_1 \cap S_{r_2}$ and hence contradicts the hypothesis.

Now, let $r_2 = d(z_o, z_1) + 1 = |m_1| + |n_1| + 1$. Required to prove that $S_1(z_o)$ is indispensable for $S_{r_2}(z_1)$. We have, $(q^{-1}x_o, y_o) \in S_1(z_o)$. Also, $|-1-m_1| + |n_1| = |m_1| + |n_1| + 1$. So, $(q^{-1}x_o, y_o) \in S_{r_2}(z_1)$ also. Thus there is atleast one point (for some choice of z_1 as many as three points) in $S_1 \cap S_{r_2}$. Since Int $S_1 = (x_o, y_o) \in TS_{r_2}$, Int $S_1 \cap TS_{r_2} =$ Int S_1.

Conversely, let $S_1 \cap S_{r_2} \neq \varphi$ and Int $S_1 \cap TS_{r_2} =$ Int S_1. So there exists $(q^m x_o, q^n y_o)$ such that $|m| + |n| = 1$ and $|m-m_1| + |n-n_1| = r_2$. This gives, $r_2 = d(z_o, z_1) + 1$.

Finally, let $r_2 \geqslant d(z_o, z_1)+2 = |m_1| + |n_1| + 2$. If there exists a $(q^m x_o, q^n y_o) \in S_1 \cap S_{r_2}$, then $|m| + |n| = 1$ $|m-m_1| + |n-n_1| = r_2 \geqslant |m_1| + |n_1| + 2$ gives a contradiction. Also, for every $(q^{m'} x_o, q^{n'} y_o) \in TS_1$, $|m'-m_1| + |n'-n_1| < r_2$.

So $TS_{r_1} \cap \text{Int } S_{r_2} = TS_1$. Hence S_1 and S_{r_2} form a discrete annulus.

Conversely, $S_1 \cap S_{r_2} = \varphi$ and $|m'|+|n'| \leq 1$ implies $|m'-m_1| + |n'-n_1| < r_2$ for every $(q^{m'}x_o, q^{n'}y_o) \in TS_1$ yields that $r_2 \geqslant d(z_o, z_1) + 2$.

Thus (a) is proved.

NOTE 2.4.21. In the above theorem, we have proved the results only for certain values of the radii. A more general result in this direction is yet to be obtained.

We shall now consider an analogous notion in the discrete case, of the notion of orthogonal intersection of circles in the Euclidean plane. We recall the definitions 2.1.9 and 2.4.14.

DEFINITION 2.4.22. Let $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$ overlap and consider I. Then, $S_{r_1}(z_1)$ and $S_{r_2}(z_2)$ are said to have discrete Pythagorean type intersection if each point of I forms with z_1, z_2, a discrete Pythagorean triple.

So, for every $z_i \in I$, $d(z_1,z_2)^2 = d(z_1,z_i)^2 + d(z_2,z_i)^2$.

EXAMPLE 2.4.23. $S_3(z_1)$ and $S_4(z_2)$ where $z_1 = (qx_o, q^{-2}y_o)$, $z_2 = (qx_o, q^3y_o)$ have intersection of discrete Pythagorean type.

We conclude this chapter, with the following result.

THEOREM 2.4.24. Consider two r-sets having discrete Pythagorean type intersection and I be their intersection. Then,

(a) centre of each r-set lies outside the other

(b) centres of r-sets belong to the same horizontal or vertical set

(c) the cardinality of I is 2.

PROOF. Proof of (a) is easy.

(b) Let the centres be $z_o, z_1 = (q^{m_1}x_o, q^{n_1}y_o) \in H_1$.

So if $\xi_i = (q^{\alpha_i}x_o, q^{\beta_i}y_o)$ is any point in I, then it is in H_1 or H_2. Let us take it to be in H_1. Then $m_2 > \alpha_i$ and $n_2 > \beta_i$.

Claim: If either of m_2 or n_2 is not zero, then (z_1, ξ_i, z_o) does not form a discrete triangular triple.

For, then $d(z_1, \xi_i) + d(\xi_i, z_o) = |m_1-\alpha_i| + |n_1-\beta_i| + |\alpha_i| + |\beta_i|$

$$= d(z_1, z_o)$$

Thus $B(z_o, \xi_i, z_1)$. Hence, the points of intersection does not form a discrete Pythagorean triple, contradicting the hypothesis. Thus, the centres are in the same horizontal (vertical) set. Similar arguments can be made, when $\xi_i \in H_2$ or when $z_2 \in H_2$, H_3 or H_4.

(c) Let $S_{r_1}(z_o)$ and $S_{r_2}(z_1)$ where $z_1 = (x_o, q^\beta y_o)$; $\beta > o$ be two r-sets having a discrete Pythagorean type intersection. Then, we can express $d(z_o, z_1) = \beta$ as $p(s^2 + t^2)$ for some non negative integers p,s,t. Now by a theorem in [67], $\xi_1 = (q^{m_1}x_o, q^{n_1}y_o) \in H_1$ forms with

z_o and z_1 a discrete Pythagorean triple if and only if $|m_1| + |n_1| = p(s^2-t^2)$ and $|m_1| + |n_1-\alpha| = p.2st$. The combined solution of these two equations, gives the required point to be $\xi_1 = (q^{st-t^2}x_o, q^{s^2-st}y_o)$. By symmetry, $\xi_2 = (q^{t^2-st}x_o, q^{s^2-st}y_o) \varepsilon H_2$ will also be a point, with the properties of ξ_1. There is no loss of generality in assuming $\beta > o$, since the only difference, if we change the centre z_1 to some other part of the horizontal or vertical set, say to X_1, X_2 or Y_2, is that the location of ξ_1 and ξ_2 will be in some other part of H, say in H_1 and H_3 etc. In any case the required cardinality is 2.

COROLLORY 2.4.25.

(1) If two r-sets have discrete Pythagorean type intersection, then the sum of squares of the radii is a perfect square.

(2) Maximum number of r-sets having a discrete Pythagorean type intersection with a given r-set is 4.

NOTE 2.4.26. By corollary (1), there are no r-sets having a discrete Pythagorean type intersection with $S_1(z_o)$. The contrast with the Euclidean plane is obvious.

CHAPTER 3

TRANSFORMATIONS ON THE DISCRETE HOLOMETRIC SPACE+

In this chapter, we introduce the concept of transformations on the discrete plane. We further investigate those, which preserve certain metric relations. Of principal interest are the discrete transformations which preserve distance, domains, r-sets etc and D-linear transformations. These are discussed in sections 1 and 2. In section 3, certain group theoretic properties are investigated. Section 4 deals with discrete analytic properties of these transformations.

3.1. DISCRETE TRANSFORMATIONS

DEFINITION 3.1.1. A bijective mapping of H onto itself is called a D-transformation.

NOTATION: D-transformations will in general be denoted by T, T_1, T_2, T_3 etc.

DEFINITION 3.1.2. A D-transformation T with the property that for every $z_1, z_2 \in H$, $d(z_1, z_2) = d(T(z_1), T(z_2))$ is called a D-isometry.

\+ Some results of this chapter were presented as a paper entitled " Geometry of the discrete plane" in the 50th Session of IMS during February 1985.

DEFINITION 3.1.3. A D-transformation T defined by $T(q^m x_o, q^n y_o) = (q^{m+a} x_o, q^{n+b} y_o)$ where $(q^a x_o, q^b y_o)$ is a fixed point in H is called a D-translation.

EXAMPLES 3.1.4.

(1) $T_1(q^m x_o, q^n y_o) = (q^{-m-1} x_o, q^{n+2} y_o)$ is a D-isometry

(2) $T_2(q^m x_o, q^n y_o) = (q^{m+n} x_o, q^{m-n} y_o)$ is not a D-isometry

(3) $T_3(q^m x_o, q^n y_o) = (q^{m+3} y_o, q^{n+4} y_o)$ is a D-translation

Figures 4 and 5 illustrate the transformations T_1 and T_2. We shall denote by z_o the origin of the image plane also and by $w_1, w_2, \ldots,$ the image of $z_1, z_2, \ldots,$.

THEOREM 3.1.5. All D-translations are D-isometries.

PROOF. Let $T : H \longrightarrow H$ be a D-translation. That is, there exists $(q^a x_o, q^b y_o) \in H$ such that $T(q^m x_o, q^n y_o) = (q^{m+a} x_o, q^{n+b} y_o)$ for every $m,n \in Z$. Now let $z_1 = (q^{m_1} x_o, q^{n_1} y_o)$ and $z_2 = (q^{m_2} x_o, q^{n_2} y_o)$ be any two points of H. Then

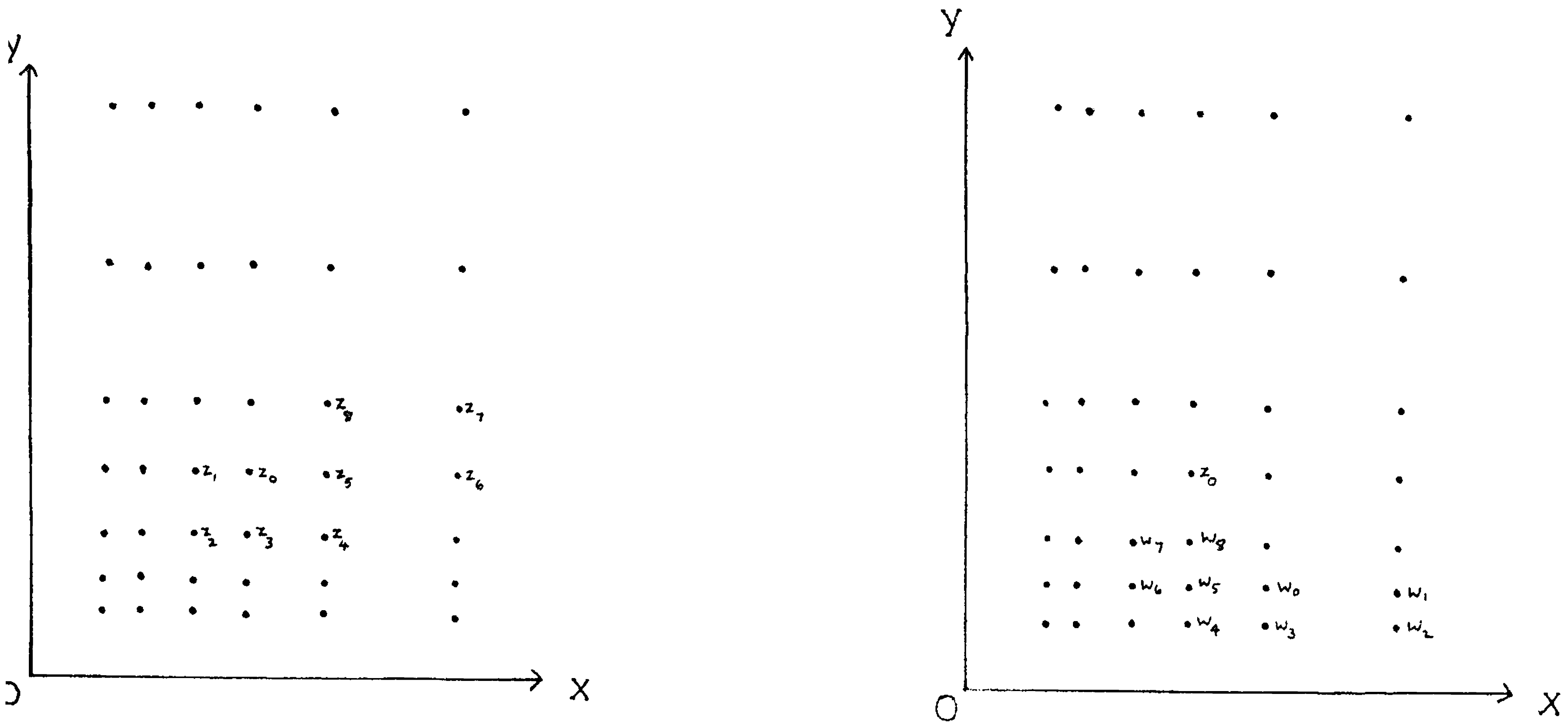
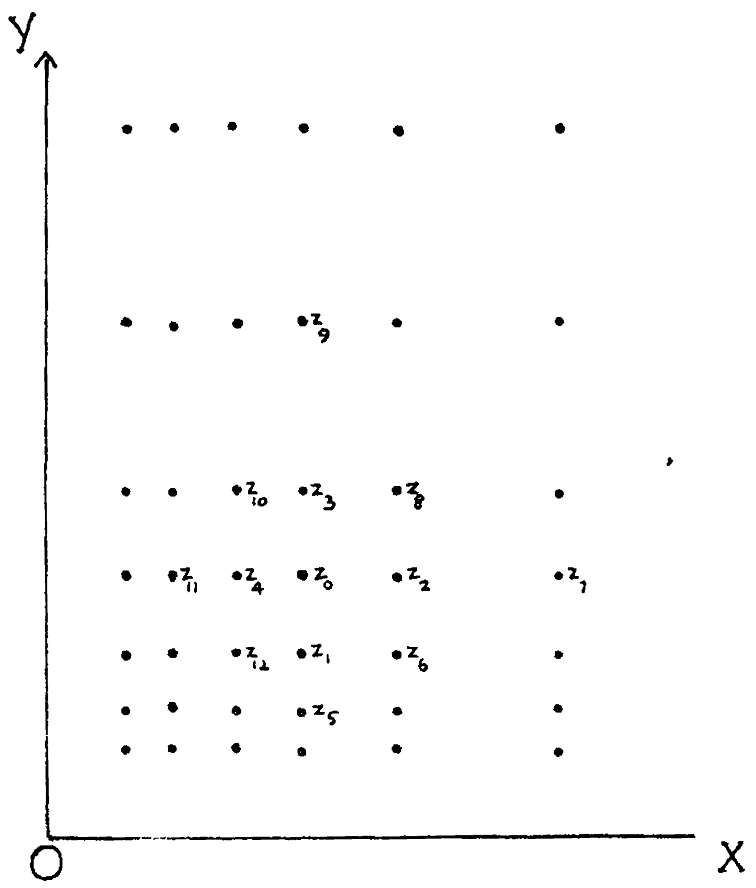


Figure-4

$$T_1(q^m x_o, q^n y_o) = (q^{-m-1} x_o, q^{n+2} y_o)$$

$$z_i \longmapsto w_i \text{ for } i = 0,1,2,\ldots,8.$$



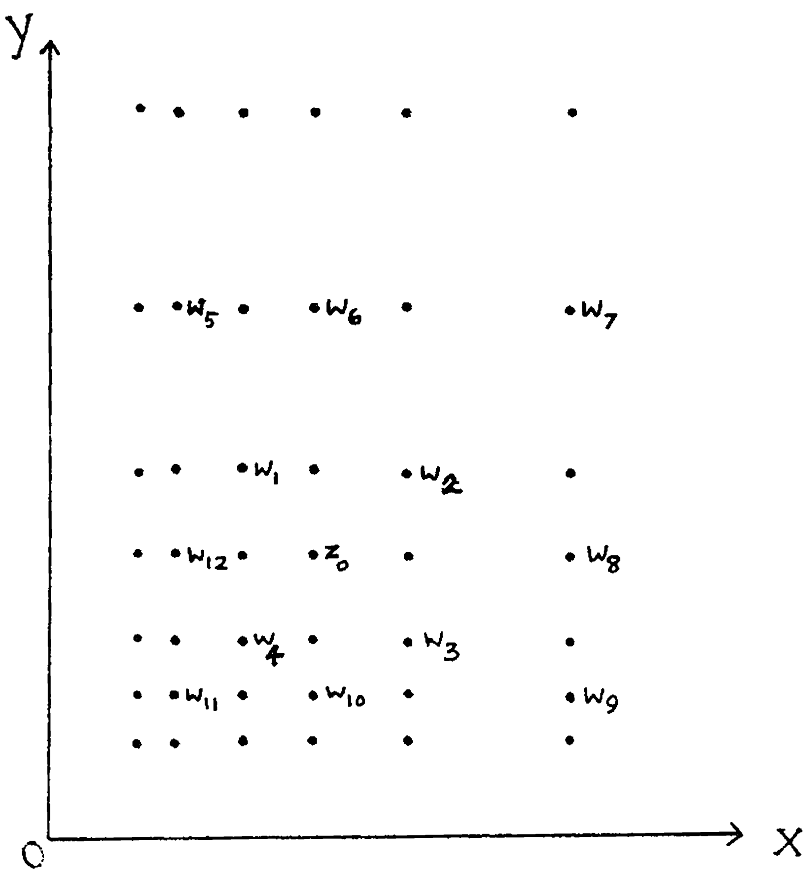


Figure-5

$$T_2(q^m x_o, q^n y_o) = (q^{m+n} x_o, q^{m-n} y_o)$$

$$z_o \longmapsto z_o$$

$$z_i \longmapsto w_i \text{ for } i = 1,2,\ldots,12.$$

$$d(T(z_1),\ T(z_2)) = d((q^{m_1+a}x_o, q^{n_1+b}y_o),\ (q^{m_2+a}x_o, q^{n_2+b}y_o))$$

$$= |(m_2+a)-(m_1+a)| + |(n_2+b)-(n_1+b)|$$

$$= |m_2-m_1| + |n_2-n_1|$$

$$= d(z_1,z_2).$$

Hence T is a D-isometry.

THEOREM 3.1.6. If D is a domain and $T:H \longrightarrow H$ is a D-isometry, then T(D) is also a domain.

PROOF. Let $D = \bigcup_{i=1}^{t} B_i$ be a domain. Then the basic sets B_i and B_{i+1} are adjacent and for any two points z, ξ of D, there are points $z = z_1, z_2, \ldots, \xi = z_n$ in D such that $d(z_i, z_{i+1}) = 1$. Since T is a D-isometry, the points $z_1, z_2, \ldots, z_n \varepsilon D$ will be mapped onto points $w_1, w_2, \ldots, w_n$ with $d(w_i, w_{i+1}) = 1$ and further the adjacency of basic sets will also be preserved. Hence T(D) also is a domain.

3.2. SOME SPECIAL TYPE OF D-TRANSFORMATIONS

In this section, we shall characterise D-linear transformations and the transformations which preserve the property of being an r-set.

DEFINITION 3.2.1. Let L_1 be the D-linear set $\left\{z_i\right\}_{i=1}^{t} = \left\{(q^{m_i}x_o, q^{n_i}y_o)\right\}_{i=1}^{t}$ and L_2 be the D-linear set, $\left\{w_i\right\}_{i=1}^{t} = \left\{(q^{\alpha_i}x_o, q^{\beta_i}y_o)\right\}_{i=1}^{t}$. Consider a D-transformation $T : H \longrightarrow H$ taking L_1 onto L_2. Then

(1) T is called a horizontal reversal (on L_1), if $\left\{m_i\right\}_{i=1}^{t}$ is monotonic increasing (decreasing) implies that $\left\{\alpha_i\right\}_{i=1}^{t}$ is monotonic decreasing (increasing). It is called a vertical reversal (on L_1) if $\left\{n_i\right\}_{i=1}^{t}$ is monotonic increasing (decreasing) implies that $\left\{\beta_i\right\}_{i=1}^{t}$ is monotonic decreasing (increasing).

(2) T is called a horizontal enlargement if $|m_i - m_j| \leqslant |\alpha_i - \alpha_j|$, and a vertical enlargement if $|n_i - n_j| \leqslant |\beta_i - \beta_j|$, for every $j > i = 1, 2, \ldots, t$.

(3) T is called a horizontal contraction if $|m_i-m_j| > |\alpha_i-\alpha_j|$, and a vertical contraction if $|n_i-n_j| > |\beta_i-\beta_j|$ for every $j > i = 1,2, \ldots, t$.

DEFINITION 3.2.2. A D-transformation $T : H \longrightarrow H$ is called a D-linear transformation if it takes D-linear sets onto D-linear sets and is a reversal, enlargement or contraction, on any D-linear set L, horizontally as vertically (not necessarily of the same type).

THEOREM 3.2.3. A D-transformation $T : H \longrightarrow H$ takes the D-linear set L_1 onto the D-linear set L_2 and is a reversal, enlargement or contraction, horizontally as well as vertically (not necessarily of the same type) if and only if $\alpha_i=m_i+p_i$, $\beta_i=n_i+s_i$ where $\left\{(q^{p_i}x_o, q^{s_i}y_o)\right\}_{i=1}^{t}$ is D-linear.

PROOF. Let us suppose that $T : H \longrightarrow H$ is a D-transformation taking L_1 onto L_2 and is a reversal, enlargement or contraction, horizontally as well as vertically, then we have to prove that $\alpha_i = m_i+p_i$ and $\beta_i = n_i+s_i$, where $(q^{p_i}x_o, q^{s_i}y_o)$ is D-linear.

Since, T is a horizontal reversal, we have m_is are monotonic increasing (decreasing) implies that α_is are monotonic decreasing (increasing). Suppose that m_is are increasing and α_is are decreasing. Then $m_j=m_1+\sum_{i=1}^{j-1} a_i$, $a_i \geqslant o$ and $\alpha_j = \alpha_1+\sum_{i=1}^{j-1} c_i$, $c_i \leqslant o$. Further, since T is a D-transformation, we can express α_js and β_js in terms of m_js and n_js as, $\alpha_j=m_j+p_j$ and $\beta_j=n_j+s_j$ where p_j, $s_j \varepsilon Z$.

Therefore, $\alpha_j = m_1 + \sum_{i=1}^{j-1} a_i+p_j$. That is, $\alpha_1 + \sum_{i=1}^{j-1} c_i = m_1 + \sum_{i=1}^{j-1} a_i+p_j$. So for $j > k$ we have

$$p_j-p_k = \sum_{i=k}^{j-1} c_i - \sum_{i=k}^{j-1} a_i \tag{7}$$

$$< o \quad \text{since } c_i \leqslant o \text{ and } a_i \geqslant o.$$

Hence $\left\{p_j\right\}_{j=1}^{t}$ is monotonic decreasing. Now, if m_is are decreasing and α_is are increasing, then p_j-p_k in (7) is greater than zero and consequently $p_j \;{}_{j=1}^{t}$ is monotonic increasing.

Now, if T is a horizontal enlargement, we have as above, an expression (7), where we do not have any condition on the signs of c_is or a_is, but then being a horizontal enlargement, we have, $c_i > a_i$ for every $i = k, \ldots, j-1$ and so $\{p_j\}_{j=1}^{t}$ is monotonic increasing. Further, if T is a horizontal contraction, then $\{p_j\}_{j=1}^{t}$ is monotonic decreasing.

Also, when T is a vertical reversal it can be proved along similar lines that $\{s_j\}_{j=1}^{t}$ is either monotonic decreasing or increasing, when T is a vertical enlargement $\{s_j\}_{j=1}^{t}$ is increasing and when T is a vertical contraction, $\{s_j\}_{j=1}^{t}$ is decreasing. Thus, it is proved that $\{p_j\}$, $\{s_j\}$ are either monotonic increasing or decreasing, not necessarily of the same type. Hence by theorem 2.3.8, it follows that $\left\{(q^{p_i}x_o, q^{s_i}y_o)\right\}_{i=1}^{t}$ is D-linear.

Conversely, suppose that T maps the D-linear set $L_1 = \left\{(q^{m_i}x_o, q^{n_i}y_o)\right\}_{i=1}^{t}$ onto $L_2 = \left\{(q^{\alpha_i}x_o, q^{\beta_i}y_o)\right\}_{i=1}^{t}$

and let $\alpha_i = m_i+p_i$, $\beta_i = n_i+s_i$ where $\left\{(q^{p_i}x_o, q^{s_i}y_o)\right\}_{i=1}^{t}$ is D-linear. Then, required to prove that, $\left\{(q^{\alpha_i}x_o, q^{\beta_i}y_o)\right\}_{i=1}^{t}$ is D-linear and T is a reversal, enlargement or contraction, horizontally as well as vertically. We have

$\left\{(q^{m_i}x_o, q^{n_i}y_o)\right\}_{i=1}^{t}$ and $\left\{(q^{p_i}x_o, q^{s_i}y_o)\right\}_{i=1}^{t}$ are D-linear.

So, let us suppose that m_i, n_i, p_i, q_i, all are monotinic increasing. Then

$$\sum_{i=1}^{t-1} d(w_i, w_{i+1}) = \sum_{i=1}^{t-1} \left(|\alpha_{i+1}-\alpha_i| + |\beta_{i+1}-\beta_i| \right)$$

$$= \sum_{i=1}^{t-1} \left(|m_{i+1}+p_{i+1}-m_i-p_i| + |n_{i+1}+s_{i+1}-n_i-s_i| \right).$$

$$= \sum_{i=1}^{t-1} [(m_{i+1}-m_i)+(p_{i+1}-p_i)+(n_{i+1}-n_i)+(s_{i+1}-s_i)]$$

$$= [(m_t-m_1)+(p_t-p_1)+(n_t-n_1)+(s_t-s_1)]$$

$$= |\alpha_1-\alpha_t| + |\beta_1-\beta_t|$$

$$= d(w_1, w_t). \text{ Thus, } \left\{w_i\right\}_{i=1}^{t} \text{ is D-linear}$$

Now, let us take m_is to be decreasing and p_is to be increasing. So, we have $m_1 > m_2 > m_3 \quad \ldots > m_t$ and $p_1 < p_2 \quad \ldots < p_t$. So from (7), without any restriction on the signs of c_is but a_is $\leq$ o, $c_1 < a_1$, $c_1+c_2 < a_1+a_2$, $\ldots$, $c_1+c_2+c_3 + \ldots \; c_t < a_1+a_2+a_3+ \ldots \; a_t$. Hence, $\alpha_2-\alpha_1 < m_2-m_1 < o$; $\alpha_3-\alpha_2 = c_1+c_2 < a_1+a_2 = m_3-m_2 < o$ etc. That is, α_is are increasing. Similarly, when n_is are decreasing and s_is are increasing, then β_is are increasing and so $\left\{(q^{\alpha_i}x_o, q^{\beta_i}y_o)\right\}$ is D-linear. These are the only typical cases and for all other cases, the result can be proved on similar lines.

Now, if m_is are increasing and p_is are decreasing, then α_is are decreasing and hence T is a horizontal reversal. If m_is are decreasing and p_is are increasing, then T is a horizontal enlargement and if m_is are increasing and p_is are decreasing, then T is a horizontal contraction. Similar conditions imposed on n_is and s_is will prove that T is a vertical reversal, enlargement or contraction.

Thus, converse part also is proved. Hence the theorem. In the following theorem, we characterise D-linear transformations.

THEOREM 3.2.4. A D-transformation $T : H \longrightarrow H$ is a D-linear transformation if and only if $T(q^m x_o, q^n y_o) = (q^\alpha x_o, q^\beta y_o)$, where $\alpha = m + a_m$, $\beta = n + b_n$ and $\{a_i\}_{i=-\infty}^{\infty}$, $\{b_i\}_{i=-\infty}^{\infty}$ are monotonic increasing or decreasing, not necessarily of the same type.

Proof follows from the above theorem, and is omitted.

NOTE 3.2.5. Any D-isometry $T : H \longrightarrow H$ carries D-linear sets to D-linear sets, but not necessarily a D-linear transformation. Converse also is not true.

NOTE 3.2.6. We shall now consider certain transformations which map r-sets onto r-sets. Clearly, D-transformations need not carry r-sets onto r-sets. In the study of transformations of this type, since D-transformations are bijective, we need consider only r-sets of equal radii.

NOTE 3.2.7. A set of points of H satisfying the conditions of theorem 2.3.8 in this context are called oriented set of points.

THEOREM 3.2.8. A D-transformation leaves invariant an r-set with centre at the origin and preserve the centre and orientation of points on it if and only if it is one among the eight transformations belonging to $T^* = \{T_i\}_{i=1}^{8}$ where T_i carries $(q^m x_o, q^n y_o)$ to

$(q^m x_o, q^n y_o)$, $(q^{-m} x_o, q^n y_o)$, $(q^{-m} x_o, q^{-n} y_o)$, $(q^m x_o, q^{-n} y_o)$,

$(q^n x_o, q^m y_o)$, $(q^{-n} x_o, q^m y_o)$, $(q^{-n} x_o, q^{-m} y_o)$ and $(q^n x_o, q^{-m} y_o)$

for i=1,2, ..., 8, respectively.

PROOF. Consider the r-set with origin as centre and radius r_1, $S_{r_1}(z_o)$. It is clear that every transformation in T^* leaves invariant the r-set and preserve the centre. It remains to show that they preserve the orientation of points on the r-set. We know that the $4r_1$ points on $S_{r_1}(z_o)$ can be classified into a disjoint union of four sets as,

$$L_1 = \left\{(q^{\alpha}x_o, q^{r_1-\alpha}y_o)\right\}, \; L_2 = \left\{(q^{r_1-\alpha}x_o, q^{-\alpha}y_o)\right\},$$

$$L_3 = \left\{(q^{-\alpha}x_o, q^{-r_1+\alpha}y_o)\right\} \text{and } L_4 = \left\{(q^{-r_1+\alpha}x_o, q^{\alpha}y_o)\right\}$$

where $\alpha = o,1,2, \ldots, r-1$. It is an easy consequence of the definition that all the L_is are D-linear sets. Each T_i in T^* carries a L_i to some L_j. For example, under T_3, $L_1 \longleftrightarrow L_3$ and $L_2 \longleftrightarrow L_4$. Thus each T_i preserve orientation.

Conversely, if T is a D-transformation which leaves invariant $S_{r_1}(z_o)$, preserving the centre and orientation then $T \in T^*$. For, since the centre has to be preserved, the transformations should be of the form $(q^m x_o, q^n y_o) \longrightarrow (q^{\alpha m}x_o, q^{\beta n}y_o)$; $\alpha, \beta \in Z$. But α, β have to be either +1 or -1, since the transformations are bijective. Hence by definition of $S_{r_1}(z_o)$, it is preserved under a D-transformation only if the transformationis one (1) which keeps m and n fixed, (2) which changes the signs of m and n, or (3) which changes the points as well as signs of m and n. That is, the required transformations are in T^*. Hence the theorem.

We shall now discuss two more situations concerning the transformations of r-sets. They are those (1) which take an r-set with centre origin onto an r-set with centre $(q^a x_o, q^b y_o)$; $a, b \neq o$, and (2) in which an r-set with centre $z_1 = (q^{m_1} x_o, q^{n_1} y_o)$, $m_1, n_1 \neq o$ is mapped onto an r-set with centre $w_1 = (q^{\alpha_1} x_o, q^{\beta_1} y_o)$, $\alpha_1, \beta_1 \neq o$. These two cases exhaust all the possibilities because the transformation which takes an $S_{r_1}(z_1)$ onto $S_{r_1}(w_1)$ maps z_1 to w_1. The result obtained in this direction is a consequence of the above characterization theorem and are considered in the following corollories.

COROLLORIES 3.2.8.

(1) A D-transformation takes an r-set with centre at the origin to an r-set with centre $(q^a x_o, q^b y_o)$, $a, b \neq o$ and preserve the orientation of points on it, if and only if it is one among the transformations belonging to $G = \{g_i\}_{i=1}^{8}$ where g_i carries $(q^m x_o, q^n y_o)$ to $(q^{m+a} x_o, q^{n+b} y_o)$, $(q^{-m+a} x_o, q^{n+b} y_o)$, $(q^{-m+a} x_o, q^{-n+b} y_o)$,

$(q^{m+a}x_o, q^{-n+b}y_o)$, $(q^{n+a}x_o, q^{m+b}y_o)$, $(q^{-n+a}x_o, q^{m+b}y_o)$, $(q^{-n+a}x_o, q^{-m+b}y_o)$ and $(q^{n+a}x_o, q^{-m+b}y_o)$ for $i = 1,2,\ldots,8$, respectively.

(2) An r-set with centre $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$; $m_1, n_1 \neq o$ is mapped to an r-set with centre $w_1 = g^*(z_1) = (q^{\alpha_1}x_o, q^{\beta_1}y_o)$, $\alpha_1, \beta_1 \neq o$ and preserve the orientation of points of it if and only if g^* is one of the transformations belonging to

$G^* = \left\{g_i^*\right\}_{i=1}^{8}$ where g_i^* carries $(q^m x_o, q^n y_o)$ to

$(q^{m+(\alpha_1-m_1)}x_o,\ q^{n+(\beta_1-n_1)}y_o)$, $(q^{-m+(\alpha_1+m_1)}x_o, q^{n+(\beta_1+n_1)}y_o)$,

$(q^{-m+(\alpha_1+m_1)}x_o,\ q^{-n+(\beta_1+n_1)}y_o)$, $(q^{m+(\alpha_1-m_1)}x_o, q^{-n+(\beta_1+n_1)}y_o)$,

$(q^{n+(\alpha_1-m_1)}x_o,\ q^{m+(\beta_1-n_1)}y_o)$, $(q^{-n+(\alpha_1+m_1)}x_o, q^{m+(\beta_1-n_1)}y_o)$,

$(q^{-n+(\alpha_1+m_1)}x_o,\ q^{-m+(\beta_1+n_1)}y_o)$ and $(q^{n+(\alpha_1-m_1)}x_o, q^{-m+(\beta_1+n_1)}y_o)$

for $i = 1,2,\ldots,8$, respectively.

3.3. GROUP THEORETIC PROPERTIES OF SOME SPECIAL TYPE OF D-TRANSFORMATIONS

DEFINITION 3.3.1. Let T_1 and T_2 be two D-transformations. Then, we define $T_1 \circ T_2(z) = T_1(T_2(z))$.

THEOREM 3.3.2. (T^*, o) is a finite, non commutative, solvable, nilpotent group.

PROOF. T^* consists of transformations leaving invariant an r-set with centre origin and preserve the centre and orientation of points on it, and by theorem 3.2.8 these are transformations defined by

$$T_1(z) = z,\ T_2(z) = (q^{-m}x_0, q^{n}y_0),\ T_3(z) = (q^{-m}x_0, q^{-n}y_0),$$

$$T_4(z) = (q^{m}x_0, q^{-n}y_0),\ T_5(z) = (q^{n}x_0, q^{m}y_0),\ T_6(z) = (q^{-n}x_0, q^{m}y_0)$$

$$T_7(z) = (q^{-n}x_0, q^{-m}y_0) \text{ and } T_8(z) = (q^{n}x_0, q^{-m}y_0) \text{ where}$$

$z = (q^{m}x_0, q^{n}y_0)$. The transformations satisfy the composition table given in page 75. The transformations satisfy all the group axioms and hence (T^*, o) is a group, which is clearly a finite group. The transformations T_4 and T_5 give a pair of non commuting elements of T^* and hence the group is non abelian. Further by a result in [16], since the order of

$\circ$	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
T_1	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
T_2	T_2	T_1	T_4	T_3	T_6	T_5	T_8	T_7
T_3	T_3	T_4	T_1	T_2	T_7	T_8	T_5	T_6
T_4	T_4	T_3	T_2	T_1	T_8	T_7	T_6	T_5
T_5	T_5	T_8	T_7	T_6	T_1	T_4	T_3	T_2
T_6	T_6	T_7	T_8	T_5	T_2	T_3	T_4	T_1
T_7	T_7	T_6	T_5	T_8	T_3	T_2	T_1	T_4
T_8	T_8	T_5	T_6	T_7	T_4	T_1	T_2	T_3

the group is 8, which is a prime power, it is solvable. Also, the centre of the group is $c = \{T_1, T_3\}$ and T/c is abelian. Hence T is nilpotent. Hence the theorem.

Let us further analyse the properties of the group (T^*, o). It has the following sub-groups.

$s_1 = \{T_1\}$, $s_2 = \{T_1, T_2\}$, $s_3 = \{T_1, T_3\}$, $s_4 = \{T_1, T_4\}$

$s_5 = \{T_1, T_5\}$, $s_6 = \{T_1, T_7\}$, $s_7 = \{T_1, T_2, T_3, T_4\}$,

$s_8 = \{T_1, T_3, T_5, T_7\}$, $s_9 = \{T_1, T_3, T_6, T_8\}$ and T^*.

Among these subgroups s_7, s_8 and s_9 being of index 2, are normal subgroups.

Further, consider the elements T_5 and T_8. $T_5 \text{ o } T_5 = T_1$ - the identity of T^* and $(T_8)^4 = (T_8)^2 \text{ o } (T_8)^2 = T_3 \text{ o } T_3 = T_1$. Also $(T_5 \text{ o } T_8)^2 = T_2^2 = T_1$. Hence (T^*, o) has the defining relation, " $A^4 = I$; $B^2 = (AB)^2 = I$ " and so T^* is isomorphic to the octic group.

NOTATION. F - the set of D-translations.

THEOREM 3.3.3. (F, o) is an abelian group.

PROOF. Consider any two D-translations, $F_1(q^m x_o, q^n y_o) = (q^{m+a_1} x_o, q^{n+b_1} y_o)$ and $F_2(q^m x_o, q^n y_o) = (q^{m+a_2} x_o, q^{n+b_2} y_o)$ where $(q^{a_1} x_o, q^{b_1} y_o)$ and $(q^{a_2} x_o, q^{b_2} y_o) \varepsilon H$. Now, to prove the result, it is enough if we prove that $F_1 \circ F_2^{-1}$ is also a D-translation. F_2^{-1} the inverse of F_2, is defined by $F_2^{-1}(z) = (q^{m-a_2} x_o, q^{n-b_2} y_o)$. Hence $F_1 \circ F_2^{-1}(z) = (q^{m+a_1-a_2} x_o, q^{n+b_1-b_2} y_o)$ is also a D-translation. Further, (F, o) is isomorphic to the additive group of integers and hence is abelian.

3.4. DISCRETE ANALYTIC PROPERTIES OF D-TRANSFORMATIONS

For complex valued functions defined on H, various notions of discrete analyticity are available in [35] and [70]. Consider $f : H \longrightarrow \mathbb{C}$, where $\mathbb{C}$ is the complex plane. Then

(1) f is q-analytic at $z = (x,y)$ if

$$\theta_x = \frac{f(z)-f(qx,y)}{(1-q)x} \quad \text{and} \quad \theta_y = \frac{f(z)-f(x,qy)}{(1-q)iy} \text{ are equal}$$

(2) f is p-analytic at z if

$$\widehat{\partial}_x = \frac{f(z)-f(px,y)}{(1-p)x} \quad \text{and} \quad \widehat{\partial}_y = \frac{f(z)-f(x,py)}{(1-p)iy} \quad \text{are equal,}$$

where $p = q^{-1}$.

(3) f is bianalytic at z if it is both q-analytic and p-analytic at z.

(4) f is q-monodiffric at z if

$$\frac{f(q^{-1}x,y)-f(qx,y)}{(q^{-1}-q)x} = \frac{f(x,q^{-1}y)-f(x,qy)}{(q^{-1}-q)iy}$$

The first two discrete analyticity is due to Harman [35] and the other two due to Velukutty [70]. We apply these definitions to the D-transformations considered in the previous section. Further, by a theorem in [70], the set of bianalytic functions is a proper subset of the set of q-monodiffric functions.

THEOREM 3.4.1. D-translations are bianalytic if and only if a = b.

PROOF. Consider the D-translation

$g_1(q^m x_o, q^n y_o) = (q^{m+a} x_o, q^{n+b} y_o)$ where $(q^a x_o, q^b y_o) \in H$.

$$\theta_x g_1 = \frac{(q^{m+a} x_o, q^{n+b} y_o) - (q^{m+a+1} x_o, q^{n+b} y_o)}{(1-q) q^m x_o}$$

$$= \frac{q^{m+a} x_o (1-q)}{(1-q) q^m x_o} = q^a$$

$$\theta_y g_1 = \frac{(q^{m+a} x_o, q^{n+b} y_o) - (q^{m+a} x_o, q^{n+b+1} y_o)}{i(1-q) q^n y_o}$$

$$= \frac{q^{n+b} y_o (1-q)}{(1-q) q^n y_o} = q^b$$

Therefore, g_1 is q-analytic $\Longleftrightarrow q^a = q^b \Longleftrightarrow a = b$

Now $$\tilde{\theta}_x g_1 = \frac{(q^{m+a} x_o, q^{n+b} y_o) - (q^{m+a-1} x_o, q^{n+b} y_o)}{(1-q^{-1}) q^m x_o}$$

$$= \frac{q^{m+a} x_o (1-q^{-1})}{(1-q^{-1}) q^m x_o} = q^a$$

$$\tilde{\theta}_y g_1 \quad = \quad q^b$$

So, g_1 is p-analytic $\Longleftrightarrow q^a = q^b \Longleftrightarrow a = b$.

Since g_1 is both p-analytic and q-analytic if and only if $a = b$, the theorem follows.

THEOREM 3.4.2. $g_2 : H \longrightarrow H$ defined by $g_2(q^m x_o, q^n y_o) = (q^{-m+a} x_o, q^{n+b} y_o)$ is bianalytic at the points of the form $(q^{\frac{a-b}{2}} x_o, q^n y_o)$; $\frac{a-b}{2} \in Z$.

PROOF. $g_2(q^m x_o, q^n y_o) = (q^{-m+a} x_o, q^{n+b} y_o)$

$$\theta_x g_2 = \frac{(q^{-m+a} x_o, q^{n+b} y_o) - (q^{-m+1+a} x_o, q^{n+b} y_o)}{(1-q) q^m x_o} = q^{-2m+a}$$

$$\theta_y g_2 = q^b.$$

Hence, $\theta_x = \theta_y \quad \Longleftrightarrow \quad q^{-2m+a} = q^b \Longleftrightarrow m = \frac{a-b}{2}$.

Thus, g_2 is q-analytic at all points of the form $(q^{\frac{a-b}{2}} x_o, q^n y_o)$; $\frac{a-b}{2} \in Z$.

$$\breve{\theta}_x g_2 = \frac{(q^{-m+a}x_o, q^{n+b}y_o) - (q^{-m-1+a}x_o, q^{n+b}y_o)}{(1-q^{-1})q^m x_o} = q^{-2m+a}$$

$$\breve{\theta}_y g_2 = q^b$$

Hence, $\breve{\theta}_x = \breve{\theta}_y$ if and only if $m = \frac{a-b}{2}$. Hence g_2 is bianalytic at all points of the form $(q^{\frac{a-b}{2}} x_o, q^n y_o)$.

THEOREM 3.4.3. The D-transformation g_3 defined by $g_3(q^m x_o, q^n y_o) = (q^{-m+a}x_o, q^{-n+b}y_o)$ is bianalytic at points of the form $(q^m x_o, q^n y_o)$ such that $m-n = \frac{a-b}{2}$.

PROOF. $\theta_x g_3 = q^{-2m+a}$, $\theta_y g_3 = q^{-2n+b}$. Therefore,

$$\theta_x = \theta_y \iff q^{-2m+a} = q^{-2n+b} \iff m-n = \frac{a-b}{2}.$$

Also, $\breve{\theta}_x = \breve{\theta}_y \iff m-n = \frac{a-b}{2}$. Hence g_3 is bianalytic at points of the form $(q^m x_o, q^n y_o)$ such that $m-n = \frac{a-b}{2} \in Z$.

THEOREM 3.4.4. The D-transformation $g_4(q^m x_o, q^n y_o) = (q^{m+a}x_o, q^{-n+b}y_o)$ is bianalytic at points of the form $(q^m x_o, q^{\frac{b-a}{2}} y_o)$; $\frac{b-a}{2} \in Z$.

EXAMPLES 3.4.5.

(1) $g_2(q^m x_o, q^n y_o) = (q^{-m+4}x_o, q^{n+6}y_o)$ is bianalytic at points $(q^{-1}x_o, q^n y_o)$, $n \in Z$.

(2) $g_3(q^m x_o, q^n y_o) = (q^{-m+1}x_o, q^{-n+5}y_o)$ is bianalytic at points $(q^m x_o, q^{m+2}y_o)$; $m \in Z$.

(3) $g_4(q^m x_o, q^n y_o) = (q^{m+2}x_o, q^{-n-8}y_o)$ is bianalytic at points $(q^m x_o, q^{-5}y_o)$, $m \in Z$.

NOTE 3.4.6.

(1) Since bianalytic functions are q-monodiffric also, the transformations considered above are q-monodiffric in the respective set of points. Also, the discrete analyticity of the D-translations do not impose any condition on m and n and hence defines an entire function subject to the only condition that $a = b$.

(2) Consider the q-analyticity of $g_5(q^m x_o, q^n y_o) = (q^{n+a}x_o, q^{m+b}y_o)$. We have $\theta_x g_5 = i\,\dfrac{q^b y_o}{x_o}$ and $\theta_y g_r = \dfrac{q^a x_o}{iy_o}$. So $0_x = 0_y$ if and only if $q^{2a}x_o^2 = q^{2b}y_o^2$. The condition on q is undesirable from the point of view of the theory considered so far. Similarly, for g_6, g_7 and g_8. Hence the only transformations, among those mentioned in Cor.3.2.8(1) of interest for discrete analyticity, are g_1, g_2, g_3 and g_4.

CHAPTER 4

SOME OTHER PROPERTIES OF THE DISCRETE HOLOMETRIC SPACE+

Theory of convexity outside the framework of linear spaces has been extensively studied by various authors. Convexity in metric spaces, based on the notion of betweenness was first considered by Menger [51]. For details see Blumenthal [13]. A survey of various other notions of convexity is available in [18]. Notion of convexity for finite dimensional normed linear spaces was studied by Aleksandrov et. al. [5], Soltan P.S.[63,64], Boltjanski [14]. Later on, this notion was further extended and generalised by German L.F. et. al. [32], Soltan V.P. [65,66] etc. for ordinary connected graph [34], using its natural metric and by Sampath Kumar [61], using the concept of a path in a graph. Dooley [21], Narang [56], Ahuja [4], Danzer [17] and many others also have made significant contributions to the development of convexity theory in metric spaces.

In the first two sections of this chapter, we study some convexity concepts in the discrete holometric space, using the notion of holometric betweenness.

+ Some results of this chapter was presented as a paper entitled 'Some characterisation theorems for the discrete holometric space' in the 54th Session of National Academy of Sciences, India,at Madurai during October 1984.

In sections 1 and 2, notions of D-convexity, D-kernel and D-convex hull etc. are considered and some of its properties are investigated. In the next section, we have presented some results obtained in the course of the investigation which we feel are interesting, although not directly along the main line of thought in the thesis. These include, a matrix representation of domains, along the lines of [27] and [33], and the notion of metric content for subsets of H. We have considered an analogue of ellipses also, called E-sets and some of its properties are investigated in section 4.

We conclude the thesis with section 5 of this chapter in which some suggestions for further study are also mentioned.

4.1. D-CONVEXITY

We recall the definition of holometric betweenness (Definition 2.1.6) and the notation $B(z_1,z_2,z_3)$ as given in chapter 2.

DEFINITION 4.1.1. Let A be a subset of H. A is said to be D-convex if for every $z_1,z_2 \in A$, $[z_1,z_2] = \left\{ z \in H : B(z_1,z,z_2) \right\} \subseteq A$.

In particular, if A is a domain satisfying the above conditions, we call it a D-convex domain. We take the empty set to be D-convex.

EXAMPLES 4.1.2.

(1) $A_1 = \{z_o, z_1 = (qx_o, y_o),\ z_2 = (q^2x_o, y_o),$
$z_3 = (q^3x_o, y_o)\}$ is a D-convex set.

(2) The basic set associated with any point is D-convex.

(3) $S_1(z_o)$ is not D-convex. As, $z_1 = (qx_o, y_o)$,
$z_2 = (x_o, qy_o) \in S_1(z_o)$ and $z_3 = (qx_o, qy_o) \in [z_1, z_2]$,
but $z_3 \notin S_1(z_o)$.

NOTATION. For any $z = (q^mx_o, q^ny_o) \in H$, $P(z) = \{(q^mx_o, q^ny_o),$
$(q^{m+1}x_o, q^ny_o),\ (q^mx_o, q^{n+1}y_o)\}$.

THEOREM 4.1.3. The intersection of two D-convex sets is also D-convex.

THEOREM 4.1.4. If a domain, in which there is at least one point of the form (q^mx_o, q^my_o), $m \in Z$, is D-convex, then it contains the basic set associated with atleast one point of $P(q^mx_o, q^my_o)$.

PROOF. Let $D = \bigcup_{i=1}^{t} S(z_i)$ be the domain and

$z_1 = (q^\alpha x_o, q^\alpha y_o)$, $\alpha \neq o$, ε Z, be a point such that $S(z_1) \subset D$. Now, $P(z_1) = \{(q^\alpha x_o, q^\alpha y_o), (q^{\alpha+1} x_o, q^\alpha y_o), (q^\alpha x_o, q^{\alpha+1} y_o)\}$. Let us assume, for the sake of argument, without loss of generality, that D contains the basic set associated with the origin.

Case 1. Let D does not contain the basic set associated with $z_2 = (q^\alpha x_o, q^{\alpha+1} y_o) \; \varepsilon \; P(z_1)$. Then $z_2, z_3 = (x_o, qy_o)$ is a pair of points of D for which points between them are not in D. For,

$d(z_2, z_3) = 2|\alpha| = \begin{cases} -2\alpha, & \alpha < o \\ 2\alpha, & \alpha > o \end{cases}$. If $\alpha < o$, take $z_4 = (q^\alpha x_o, q^{\alpha+2} y_o)$.

Then $d(z_3, z_4) = |\alpha| + |\alpha+1| = -2\alpha-1$, $d(z_4, z_2) = 1$. Therefore,

$d(z_3, z_4) + d(z_4, z_2) = -2\alpha = d(z_3, z_2)$. Hence, $z_4 \; \varepsilon \; [z_2, z_3]$, but $z_4 \notin D$. For $\alpha > o$, take $z_4 = (q^{\alpha-1} x_o, q^{\alpha+1} y_o)$. Then $d(z_3, z_4) = 2\alpha-1$, $d(z_4, z_2) = 1$, $d(z_3, z_2) = 2\alpha$. Therefore, $z_4 \; \varepsilon \; [z_2, z_3]$, but $z_4 \notin D$.

Case 2. Let D does not contain the basic set associated with $z_2 = (q^{\alpha+1}x_o, q^{\alpha}y_o) \varepsilon P(z_1)$. Then $z_2 = (q^{\alpha+1}x_o, q^{\alpha}y_o)$, $z_3 = (qx_o, y_o)$ is a pair of points of D for which no point between them is in D. For $\alpha < o$, take $z_4 = (q^{\alpha+2}x_o, q^{\alpha}y_o)$ and for $\alpha > o$, $z_4 = (q^{\alpha+1}x_o, q^{\alpha-1}y_o)$. Arguments are as those in case 1.

NOTE 4.1.5. It can be proved similarly that if a domain in which there is at least one point of the form $(q^m x_o, q^{-m} y_o)$, $m \varepsilon Z$, is D-convex then it contains the basic set associated with at least one point of $P(q^m x_o, q^{-m} y_o)$. The above theorem further illustrates that a finite union of D-convex sets $\left\{D_i\right\}_{i=1}^{t}$ with $D_i \cap D_{i+1} \neq \varphi$, $i = 1, 2, \ldots, t-1$, need not be D-convex.

4.2. D-KERNEL AND D-CONVEX HULL

DEFINITION 4.2.1. Let A be a non empty subset of H. Then $\{z_i \varepsilon A$: for every $z_j \varepsilon A$, all the D-linear sets with z_i and z_j as end points is contained in $A\}$ is called the D-kernel of A.

NOTATION. D-ker(A) — the D-kernel of A.

EXAMPLES 4.2.2. (See Figure-6)

(1) Let $A_1=\{z_o,\ z_1=(qx_o,y_o),\ z_2=(qx_o,qy_o),\ z_3=(x_o,qy_o),\ z_4=(q^{-1}x_o,y_o),\ z_5=(q^{-1}x_o,q^{-1}y_o),\ z_6=(x_o,q^{-1}y_o)\}$. Then D-ker $(A_1) = \{z_o\}$.

(2) Let $A_2 = A_1 \cup \{z_7=(qx_o,q^{-1}y_o)\}$. Then D-ker $(A_2) = \{z_o,z_1,z_6,z_7\}$.

(3) Let $A_3 = A_2 \cup \{z_8=(q^{-1}x_o,qy_o)\}$. Then D-ker $(A_3)=A_3$.

In the above examples, it turns out that D-ker(A_2) is D-convex, A_3 is D-convex and its D-kernel is itself. So we expect the following questions. Is it true that

(i) for any non empty set, its D-kernel is D-convex?

(ii) D-ker (A) = A if and only if A is D-convex?

In the following theorems, it is proved that, answers to both the questions are affirmative.

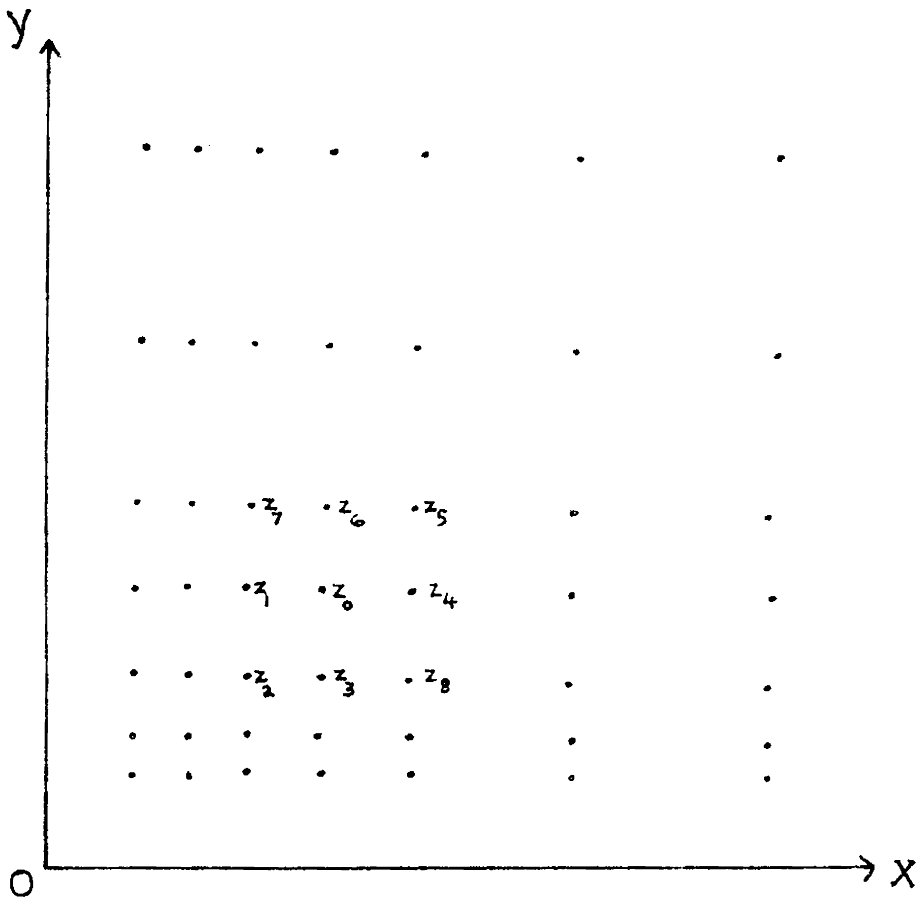


Figure-6

$$A_1 = \{z_0, z_1, z_2, z_3, z_4, z_5, z_6\}.$$

$$A_2 = A_1 \cup \{z_7\}.$$

$$A_3 = A_2 \cup \{z_8\}.$$

THEOREM 4.2.3. For any non empty subset A of H, D-ker(A) is always D-convex.

PROOF. Consider any two points $z_1, z_2 \in$ D-ker(A). That is, for every $z_j \in A$, all the D-linear sets joining z_1 to z_j and z_2 to z_j are contained in A. Required to prove that all the points between z_1 and z_2 are in D-ker(A). That is, if ξ is any such point, then all the D-linear sets joining ξ and z_j, for every $z_j \in A$, is contained in A? Suppose not. That is, there exists atleast one point $\eta \in A$ such that the D-linear set joining ξ and η is not contained in A. That is, there exists a D-linear set (say) $L_1 = \{\xi, \alpha_1, \alpha_2, \ldots, \eta\}$ joining ξ and η of A containing some points not in A. Now ξ is a point between z_1 and z_2. So there exists a D-linear set joining z_1 and ξ viz. $L_2 = \{z_1, \beta_1, \beta_2, \ldots, \xi\}$. Now, $L_3 = \{z_1, \beta_1, \beta_2, \ldots, \xi, \alpha_1, \alpha_2, \ldots, \eta\}$ gives a D-linear set joining z_1 and η (this works since ξ is a point between z_1 and z_2) which is not contained in A and $\eta \in A$, which implies that $z_1 \notin$ D-ker(A). This leads us to a contradiction. Hence the theorem.

THEOREM 4.2.4. Let A be a non empty subset of H. Then, D-ker(A) = A if and only if A is D-convex.

PROOF. Let D-ker(A)=A. Then by the previous theorem A is D-convex. Conversely, let A be D-convex. So by definition, for every $z_i, z_j \varepsilon A$, $\{z \varepsilon H : B(z_i, z, z_j)\} \subseteq A$. Now, D-ker(A) = $\{z_i \varepsilon A$: for every $z_j \varepsilon A$, all the D-linear sets joining z_i and z_j is contained in A$\}$ = A, since A is D-convex. Hence, D-ker(A)=A if and only if A is D-convex.

THEOREM 4.2.5. Let A and B be two D-convex sets. Then, $A \cap B \subset$ D-ker($A \cup B$).

PROOF. Since A and B are D-convex sets, $A \cap B$ is also D-convex. Let $z \varepsilon A \cap B$. To prove that $z \varepsilon$ D-ker($A \cup B$). That is, to prove that for any $z_i \varepsilon A \cup B$ every D-linear set from z to z_i is contained in $A \cup B$. Without loss of generality let $z_i \varepsilon B$. Then, z, $z_i \varepsilon B$ and B is D-convex So the result follows.

NOTE 4.2.6. In the above theorem, the requirement that A and B are D-convex cannot be relaxed. For, consider

$A_1 = \{z_o,\ z_1 = (q^{-1}x_o, q^{-1}y_o),\ z_2 = (q^{-2}x_o, q^{-2}y_o),$
$z_3 = (q^{-2}x_o, q^{-3}y_o)\}$ and $A_2 = \{z_o, z_1, z_2, z_4 = (q^{-3}x_o, q^{-2}y_o)\}$.
Then A_1 and A_2 are not D-convex, $A_1 \cap A_2 = \{z_o, z_1, z_2\}$.
But D-ker$(A_1 \cup A_2) = \varphi$. Also, equality need not hold in the above theorem. As an example, take $A_3 = S(z_o) \cup S(z_5=(q^{-1}x_o, y_o))$ and $A_4 = S(z_6=(q^{-1}x_o, q^{-1}y_o)) \cup S(z_7=(x_o, q^{-1}y_o))$. Then $A_3 \cap A_4 = \{z_o, z_1, z_5\}$ and D-ker$(A_3 \cup A_4) = A_3 \cup A_4$.

DEFINITION 4.2.7. Let $A \subset H$. The intersection of all D-convex sets containing A is called the D-convex hull of A.

NOTATION. D-conv(A) — D-convex hull of A.

EXAMPLE 4.2.8. (See Figure-6)

Let $A = \{z_1, z_3, z_4\}$. Then D-conv(A) = $\{z_o, z_1, z_2, z_3, z_4, z_8\}$.

If A is a subset of the horizontal (vertical) set, then its D-convex hull is also a subset of the horizontal (vertical) set. In the following theorem we take A to be a subset of H such that not all points of it are in the same horizontal (vertical) set and then say that A has points in general position and prove that its D-convex hull is domain.

THEOREM 4.2.9. For a non empty finite subset of H consisting of points in general position, its D-convex hull is a domain.

PROOF. Let A be a non empty finite subset of H consisting of points in general position. Let B=D-conv(A). It is required to prove that B is a domain. By definition, B is the smallest D-convex set containing A. So, for any two points of B all the points holometrically between them are also in B. That is, for any two points in B all points on all paths joining them is in B. That is, for any two points of B, we can find a sequence of points in B with distance between consecutive points of the sequence being 1 and which joins the two points. Hence B is connected. Further, B can be expressed as a union of basic sets B_i with B_i, B_{i+1} adjacent, since B is D-convex and by theorem 4.1.4. Hence, by note 2.2.11. we conclude that B is a domain.

NOTE 4.2.10. By the above theorem, for a finite subset A of H consisting of points in general position, its D-convex hull is a domain. We note that this domain need not be the smallest domain containing A. For example, take $A = \{z_0, z_2, z_5\}$ (See Figure-6), then the smallest domain containing A is $D = \{z_0, z_1, z_2, z_4, z_5, z_6\}$ and $B = \text{D-conv}(A) = D \cup \{z_7, z_8\}$.

4.3. MATRIX REPRESENTATION AND RELATED CONCEPTS

In this section, we shall associate a distance matrix to finite subsets of H, and obtain some properties of those associated with certain special types of domains. The idea of associating distance matrices for digraphs is discussed in [27], [33] and [59]. Also, we define the notion of metric content and some properties are obtained. An estimate for the metric content of an r-set is also found.

DEFINITION 4.3.1. Let A be a non empty finite subset of H consisting of n points, labelled in a definite order as $z_1, z_2, \ldots, z_n$. Then the n x n matrix M(A) where $(i,j)^{th}$ element is the distance between z_i, z_j of A, $i,j = 1,2, \ldots, n$, is called the distance matrix associated with A.

That is,

$$M(A) = \begin{bmatrix} d(z_1,z_1), & d(z_1,z_2), & \dots & d(z_1,z_n) \\ d(z_2,z_1), & d(z_2,z_2), & \dots & d(z_2,z_n) \\ \vdots & \vdots & & \vdots \\ d(z_n,z_1), & d(z_n,z_2), & \dots & d(z_n,z_n) \end{bmatrix}$$

NOTE 4.3.2.

(1) The distance matrix so obtained depends on the way we order the points of A. Any of these matrices will be called the distance matrix of A. Also, whenever we mention the distance matrix of A, we shall mention the order of points of A.

(2) The distance matrix is symmetric, integral matrix with diagonal elements zero.

EXAMPLES 4.3.3.

(1) Consider the basic set associated with a point

$z_1 = (q^{m_1}x_o, q^{n_1}y_o)$, $S(z_1) = \{ z_1, z_2 = (q^{m_1+1}x_o, q^{n_1}y_o),$

$z_3 = (q^{m_1+1}x_o, q^{n_1+1}y_o),\ z_4 = (q^{m_1}x_o, q^{n_1+1}y_o)\}$.

Then, $M(S(z_1)) = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$

The distance matrix of the basic set with the points of it labelled in this way is called the basic matrix.

(2) Consider $S_1(z_1) = \{ z_2 = (q^{m_1+1}x_o, q^{n_1}y_o),$

$z_3=(q^{m_1}x_o, q^{n_1+1}y_o),\ z_4=(q^{m_1-1}x_o, q^{n_1}y_o),\ z_5=(q^{m_1}x_o, q^{n_1-1}y_o)\}$

Then,

$$M(S_1(z_1)) = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix}$$

We shall now write down explicitly the distance matrix associated with $S_{r_1}(z_o)$. It was noted in the proof of theorem 2.4.3. that the $4r_1$ points of $S_{r_1}(z_o)$ can be labelled and classified into a disjoint union of four D-linear sets as,

$$L_1 = \left\{(q^{\alpha}x_o, q^{r_1-\alpha}y_o)\right\} \quad , \quad L_2 = \left\{(q^{r_1-\alpha}x_o, q^{-\alpha}y_o)\right\}$$

$$L_3 = \left\{(q^{-\alpha}x_o, q^{-r_1+\alpha}y_o)\right\}, \quad L_4 = \left\{(q^{-r_1+\alpha}x_o, q^{\alpha}y_o)\right\},$$

where $\alpha = o,1,2, \ldots, r_1-1$.

The order of points of $S_{r_1}(z_o)$ is the natural order of points of L_1, L_2, L_3 and L_4.

Now, $M(S_{r_1}(z_o))$ is a matrix of order $4r_1 \times 4r_1$ which can be partitioned into a 4 x 4 matrix as

$$\begin{bmatrix} M_1 & M_5 & M_9 & M_{15} \\ M_2 & M_6 & M_{10} & M_{14} \\ M_3 & M_7 & M_{11} & M_{15} \\ M_4 & M_8 & M_{12} & M_{16} \end{bmatrix}$$ where each M_k,

$k = 1,2, \ldots, 16$, is a matrix of order $r_1 \times r_1$. Also, these matrices are generated by $m_1, m_2, \ldots, m_{16}$, where

$$m_1 = |i-\alpha_1| + |\alpha_1-i|$$

$$m_2 = |r_1-(\alpha_1+\alpha_2)| + |(\alpha_1-\alpha_2)-r_1|$$

$$m_3 = |\alpha_3+\alpha_1| + |(\alpha_3+\alpha_1) - 2r_1|$$

$$m_4 = |\alpha_4 - (r_1+\alpha_1)| + |\alpha_4+\alpha_1-r_1|$$

$$m_6 = 2|\alpha_2-i|$$

$$m_7 = |\alpha_2-(\alpha_3+r_1)| + |\alpha_3+\alpha_2-r_1|$$

$$m_8 = |\alpha_4+\alpha_2-2r_1| + |\alpha_2+\alpha_4|$$

$$m_{11} = |\alpha_3-i| + |i-\alpha_3|$$

$$m_{12} = |\alpha_4+\alpha_3-r_1| + |\alpha_4-\alpha_3+r_1| ,$$

where i, α_1, α_2, α_3, α_4 varies over o, 1, 2, ..., r_1-1. Due to symmetric nature of the distance matrix, M_5 can be obtained from M_2, M_9 from M_3, etc.

Now, we shall write the matrix for certain special types of domains. Consider the domain $D_1=S(z_o)\cup S(q^{-m}x_o,q^{-m}y_o)$, $m = 1,2, \ldots, s$. The order of points for D_1 is the order of points of the basic matrix for $S(z_o)$ and for $S(q^{-m}x_o,q^{-m}y_o)$, $z_4 = (q^{-1}x_o,y_o)$, $z_5 = (q^{-1}x_o,q^{-1}y_o)$, $z_6 = (x_o,q^{-1}y_o)$ and then $z_7 = (q^{-2}x_o,y_o)$, $z_8 = (q^{-2}x_o,q^{-2}y_o)$ and so on. The distance matrix corresponding to D_1 will be a matrix of order $(3s+4) \times (3s+4)$

given by,

$$M(D_1) = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}, \text{ where } M_1 \text{ is the basic}$$

matrix, M_2 is the 4 x 3s matrix given by,

$$M_2 = \begin{bmatrix} \langle 2j-1, & 2j, & 2j-1 \rangle \\ \langle 2j, & 2j+1, & 2j \rangle \\ \langle 2j+1, & 2j+2, & 2j+1 \rangle \\ \langle 2j, & 2j+1, & 2j \rangle \end{bmatrix}, \text{ the symbol}$$

$\langle 2j-1,\ 2j,\ 2j-1 \rangle$ for $j = 1,2, \ldots, s$ inside M_2 means that the 3s elements in that row is generated by $2j-1$, $2j$, $2j-1$. M_3 can be obtained from M_2 due to symmetry and M_4 is the (3s x 3s) matrix given by,

$$M_4 = \begin{bmatrix} \langle 2|j-k|, & |j-k+1| + |j-k|, & |j-k-1| + |j-k+1| \rangle \\ \langle |j-k| + |j-k+1|, & 2|j-k|, & |j-k| + |j-k-1| \rangle \\ \langle |j-k-1| + |j-k+1|, & |j-k-1| + |j-k|, & 2|j-k| \rangle \end{bmatrix}$$

where $j,k = 1,2, \ldots, s$. The symbol $\langle\ \rangle$ inside M_4 has the same meaning as for M_2. For the matrix $M(D_1)$ we have, THEOREM 4.3.4. $M(D_1)$ is singular.

PROOF. Consider the determinant of $M(D_1)$. To the elements of it, do the following transformations.

(1) R_2-R_4, that is to the elements of second row add (-1) times that of the fourth row.

(2) $R_3 - 2R_4$.

(3) $C_2 + C_4$. Then we have a determinant for which only $(2,4)^{th}$ element is non zero. Expand with respect to that element. Denote by V, the resulting determinant of order $(3s+2) \times (3s+2)$.

(4) For V, $R_1 + R_2$.

(5) Then R_1 of the resulting determinant has all the elements zero. Hence, determinant of $M(D_1)$ is zero. So $M(D_1)$ is singular.

As an explanation for the symbol $\langle\ \rangle$ mentioned earlier, consider the following example.

EXAMPLE 4.3.5. In the above discussion, we take $s = 2$ and consider $D_1=S(z_o) \cup S(q^{-1}x_o,q^{-1}y_o) \cup S(q^{-2}x_o,q^{-2}y_o)$.

Then D_1 is a domain consisting of ten points,

$$D_1 = \{ z_o, z_1=(qx_o,y_o),\ z_2=(qx_o,qy_o),\ z_3=(x_o,qy_o),$$
$$z_4 = (q^{-1}x_o,y_o),\ z_5=(q^{-1}x_o,q^{-1}y_o),\ z_6=(x_o,q^{-1}y_o),$$
$$z_7 = (q^{-2}x_o,q^{-1}y_o),\ z_8=(q^{-2}x_o,q^{-2}y_o),\ z_9=(x_o,q^{-2}y_o)\} .$$

M_2 by formula is given by,

$$M_2 = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 & 3 \\ 2 & 3 & 2 & 4 & 5 & 4 \\ 3 & 4 & 3 & 5 & 6 & 5 \\ 2 & 3 & 2 & 4 & 5 & 4 \end{bmatrix}$$

The symbol $\langle 2j-1,\ 2j,\ 2j-1\rangle$ for $j = 1,2$, giving the first row [1,2,1,3,4,3], $\langle 2j,\ 2j+1,\ 2j\rangle$ for $j = 1,2$, giving the second row [2 3 2 4 5 4] etc. Also M_4, the (6 x 6) matrix is given by,

$$M_4 = \begin{bmatrix} 0 & 1 & 2 & 2 & 3 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 2 & 3 & 2 \\ 2 & 1 & 2 & 0 & 1 & 2 \\ 3 & 2 & 3 & 1 & 0 & 1 \\ 2 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$. To get

the first three rows, fix $j=1$ and $k=1,2$ in the generating elements and fix $j=2$; $k = 1,2$ to get the next three rows.

Finally, we have the (10 x 10) matrix given by,

$$M(D_1) = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 2 & 1 & 3 & 4 & 3 \\ 1 & 0 & 1 & 2 & 2 & 3 & 2 & 4 & 5 & 4 \\ 2 & 1 & 0 & 1 & 3 & 4 & 3 & 5 & 6 & 5 \\ 1 & 2 & 1 & 0 & 2 & 3 & 2 & 4 & 5 & 4 \\ 1 & 2 & 3 & 2 & 0 & 1 & 2 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 & 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 2 & 1 & 0 & 2 & 3 & 2 \\ 3 & 4 & 5 & 4 & 2 & 1 & 2 & 0 & 1 & 2 \\ 4 & 5 & 6 & 5 & 3 & 2 & 3 & 1 & 0 & 1 \\ 3 & 4 & 5 & 4 & 2 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

NOTE 4.3.6.

(1) For the domains $D_2 = S(z_o) \cup S(q^m x_o, q^m y_o)$, $D_3 = S(z_o) \cup S(q^{-m} x_o, q^m y_o)$ and $D_4 = S(z_o) \cup S(q^m x_o, q^{-m} y_o)$, $m = 1, 2, \ldots, s$, their distance matrices are same as that of D_1.

(2) The diameter of all these domains is even.

(3) The distance matrix corresponding to any domain could not be written down explicitly, as we could not enumerate the domains with any number of points. But following facts are noted. Let $N(D)_r$ denote the number of domains containing the basic set associated with the

origin and consisting of r lattice points. Then $N(D)_4=1$ and its distance matrix is the basic matrix. As there are no domains with 5 points, $N(D)_5=0$. $N(D)_6=4$ and points in these domains can be labelled in an order in such a way that all of them have the same distance matrix given by,

$$M(D)_6 = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Now, $N(D)_7$ is also 4 and the points of it can be labelled, so that all the four domains have the same distance matrix. But for r = 8, the situation is different. $N(D)_8 = 14$. They, with reference to Figure-7, are

$$D_1 = \{z_0, z_1, z_2, z_3, z_4, z_5, z_{10}, z_{11}\}$$
$$D_2 = \{z_0, z_1, z_2, z_3, z_7, z_8, z_{13}, z_{14}\}$$
$$D_3 = \{z_0, z_1, z_2, z_3, z_4, z_5, z_{16}, z_{17}\}$$

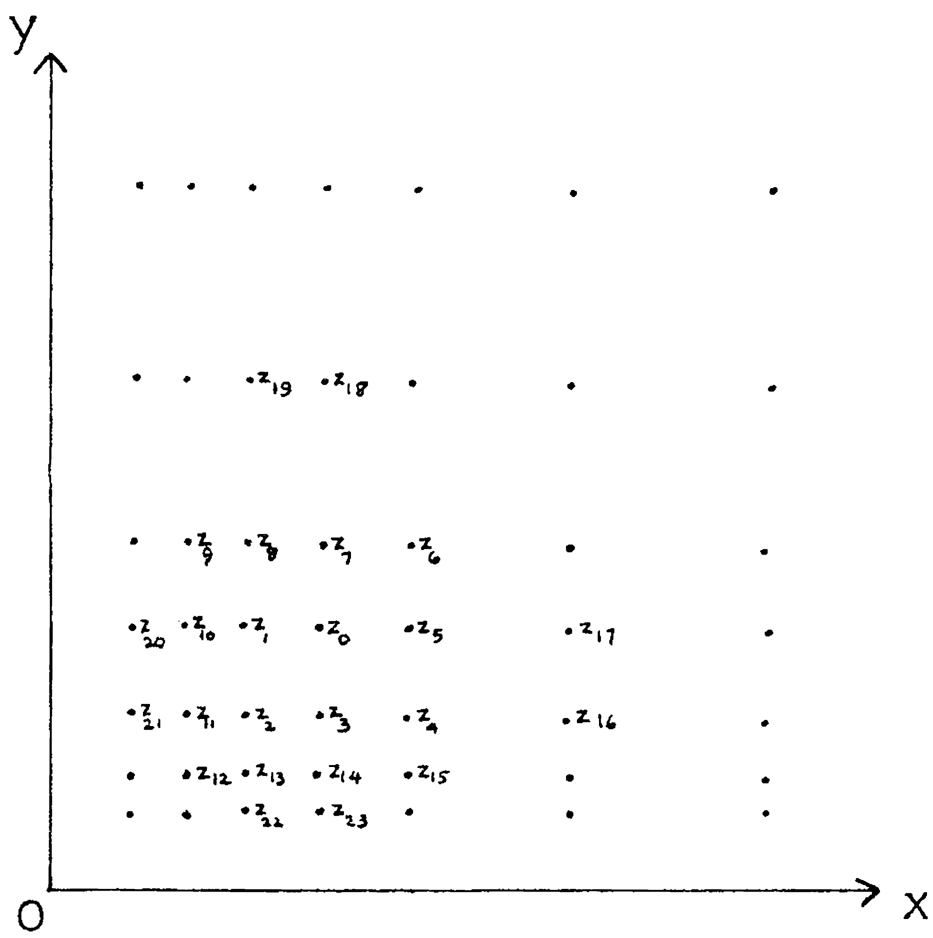

y
O
X
z19
z18
z9
z8
z7
z6
z20
z10
z1
z0
z5
z17
z21
z11
z2
z3
z4
z16
z12
z13
z14
z15
z22
z23

Figure-7

$$D_4 = \{z_o, z_1, z_2, z_3, z_{10}, z_{11}, z_{20}, z_{21}\}$$

$$D_5 = \{z_o, z_1, z_2, z_3, z_7, z_8, z_{18}, z_{19}\}$$

$$D_6 = \{z_o, z_1, z_2, z_3, z_{13}, z_{14}, z_{22}, z_{23}\}$$

$$D_7 = \{z_o, z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$$

$$D_8 = \{z_o, z_1, z_2, z_3, z_4, z_5, z_{14}, z_{15}\}$$

$$D_9 = \{z_o, z_1, z_2, z_3, z_8, z_9, z_{10}, z_{11}\}$$

$$D_{10} = \{z_o, z_1, z_2, z_3, z_7, z_8, z_{10}, z_{11}\}$$

$$D_{11} = \{z_o, z_1, z_2, z_3, z_{10}, z_{11}, z_{12}, z_{13}\}$$

$$D_{12} = \{z_o, z_1, z_2, z_3, z_{10}, z_{11}, z_{13}, z_{14}\}$$

$$D_{13} = \{z_o, z_1, z_2, z_3, z_4, z_5, z_7, z_8\}$$

$$D_{14} = \{z_o, z_1, z_2, z_3, z_4, z_5, z_{13}, z_{14}\} .$$

These domains have essentially two distinct matrices, one typically for D_1 and the other for D_7. They are given by,

$$M(D_1) = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 & 3 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 & 4 & 3 \\ 1 & 2 & 3 & 2 & 1 & 0 & 3 & 4 \\ 2 & 1 & 2 & 3 & 4 & 3 & 0 & 1 \\ 3 & 2 & 1 & 2 & 3 & 4 & 1 & 0 \end{bmatrix} \quad \text{and}$$

$$M(D_7) = \begin{bmatrix} 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 \\ 1 & 2 & 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

However, for larger values of r, the above type of analysis seems to be difficult.

Based on the notion of the distance matrix we consider the following related concept.

DEFINITION 4.3.7. Let $A \subset H$ be finite. Then,

$\mu(A) = \sum_{i<j} d(z_i, z_j)$ for every $z_i, z_j \in A$ is called the the metric content of A.

NOTE 4.3.8. $\mu(A)$ is the sum of the elements in the upper (lower) triangular part of the distance matrix M(A) associated with A.

EXAMPLES 4.3.9.

(1) For the basic set,

$$M(S(z)) = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \text{ and hence } \mu(S(z)) = 8.$$

(2) For the $S_1(z)$,

$$M(S_1(z)) = \begin{bmatrix} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & 2 & 0 \end{bmatrix} \text{ and hence } \mu(S_1(z)) = 12.$$

THEOREM 4.3.10. Let $D = \bigcup_{i=1}^{t} B_i$ be a domain. Then,

$$\mu(D) \geqslant \sum_{i=1}^{t} \mu(B_i).$$

PROOF. Let the domain D be just the basic set B_1. Then $\mu(D) = \mu(B_1) = 8$ and we have the equality. Since $\mu(B_i) = 8$ for $i = 1, 2, \ldots, t$, $\sum_{i=1}^{t} \mu(B_i) = 8t$. Now, suppose that the result holds for a domain D of index (t-1). That is, $\mu(D) \geqslant \sum_{i=1}^{t-1} \mu(B_i) = 8(t-1)$. It is clear that we can obtain a domain of index t from that of (t-1) by the addition of at least one and atmost three points. Let the points of the domain of index (t-1) be $z_1, z_2, \ldots, z_n$. Let z be a point added so as to make the index of the domain to be equal to t. Now, in the metric content of the new domain, the quantity that gets added up is $\beta = d(z_1,z) + d(z_2,z) + \ldots d(z_n,z)$. The result is proved if we prove that $\beta \geqslant 8$. Now, apart from atmost two points among

$z_1, z_2, \ldots, z_n$ of D, $d(z, z_i) > 1$, $i = 1, 2, \ldots, n$. That is, the distance is greater than or equal to 2. So, $\beta \geqslant (n-2)2 \geqslant 8$, since there is no domain with five lattice points, $n \geqslant 6$. Hence the result holds for a domain of index t also and the result is proved.

EXAMPLE 4.3.11. As an illustration of the above theorem consider a domain of index 2. There are 8 domains with index 2, in which 4 has 7 points and 4 has 6 points. With reference to Figure-7, they are $D_1 = S(z_0) \cup S(z_6)$, $D_2 = S(z_0) \cup S(z_8)$, $D_3 = S(z_0) \cup S(z_2)$, $D_4 = S(z_0) \cup S(z_4)$ belonging to the first category and $D_5 = S(z_0) \cup S(z_5)$, $D_6 = S(z_0) \cup S(z_7)$, $D_7 = S(z_0) \cup S(z_1)$, $D_8 = S(z_0) \cup S(z_3)$ belonging to the second. The metric contents are respectively 40 and 25, both greater than $8 \times 2 = 16$.

Attempts were made to find a formula for the metric content of an r-set. Though, we could explicitly write down $M(S_{r_1}(z_0))$, a formula for $\mu\, S_{r_1}(z_0)$ could not be obtained. However, we were successful in obtaining an upper bound for the same, working out along the lines of problems 1 and 2 of [55].

In the problems 1 and 2, the following situation is considered. Let (P:X) denote a set P of n points satisfying a condition X and $f_n(P:X)$ denote the number of different distances determined by (P:X). The conditions X that are considered are, the points are vertices of a strictly convex polygon etc. Several references and a survey of related results including those in [29] are mentioned in [55].

We asked the following question. When the condition X mentioned above is that, the points are in $S_{r_1}(z)$, how many and what are the different distances assumed by points belonging to $S_{r_1}(z)$? An answer obtained is proved in the following lemma.

LEMMA 4.3.12. The number of distinct distances assumed by the points of an r-set with radius r_1 is r_1. The r_1 distinct values are the even integers between o and $2r_1$.

PROOF. Consider $S_{r_1}(z_o)$. We know by theorem 2.4.3. that the points of $S_{r_1}(z_o)$ can be classified into a disjoint union of four D-linear sets L_1, L_2, L_3, L_4. Consider

$$L_1 = \{z_1 = (q^{r_1}x_o, y_o),\ z_2 = (q^{r_1-1}x_o, qy_o),\ \ldots,$$

$z_t = (x_o, q^{r_1}y_o)$. Hence, $d(z_1,z_2) = 2$, $d(z_1,z_3) = 4$, ..., $d(z_1,z_7) = 2r_1$. Thus, there are r_1 distinct distances, they being $2t$ for $t = 1,2, \ldots, r_1$. Also, these values are repeated for any two points belonging to any of the other three D-linear sets L_2, L_3 or L_4. Now, it remains to prove that these are precisely the values taken by the distance function. That is for any two points $z_1, z_2 \in S_{r_1}(z_o)$, $d(z_1,z_2) = 2t$, $t = 1,2, \ldots, r_1$. When these two points are in the same D-linear set, the assertion has been already verified.

Suppose, $z_1 \in L_1$ and $z_2 \in L_2 = \left\{(q^{r_1-\alpha_2}x_o, q^{-\alpha_2}y_o);\ \alpha_2 = o, 1,2, \ldots, r_1-1\right\}$.

Then, $$\beta = d(z_1,z_2) = |\alpha_1-1|- r_1+\alpha_2| + |r_1-\alpha_1+1+\alpha_2|$$
$$= |(\alpha_1+\alpha_2)-(r_1+1)|+|(\alpha_2-\alpha_1) + (r_1+1)|$$

where $\alpha_1 = 1,2, \ldots, r_1$ and $\alpha_2 = o, 1, 2, \ldots, r_1-1$.

Note that $\alpha_1=1$, $\alpha_2=o$ gives $\beta=2r_1$, $\alpha_1=r$, $\alpha_2=o$ gives $\beta=2$. Now, β is easily seen to be always even. Similarly for all other choice of z_1 and z_2. Thus, for any two distinct points of $S_{r_1}(z_o)$, the distance between them will be some

2t, for t = 1,2, ..., r_1. Hence the lemma is proved.

NOTE 4.3.13. Since the distance between any two points of $S_{r_1}(z_o)$ is an even integer, $\mu\, S_{r_1}(z_o)$ is also even.

Using the above lemma, following estimate for the metric content of an r-set is an easy consequence.

THEOREM 4.3.14. $\mu\ S_{r_1}(z) \leq \frac{n}{2}.\binom{n}{2}$ where $n = 4r_1$.

NOTE 4.3.15. Computer evaluation of $\mu\, S_{r_1}(z)$, for values of $r_1 = 1,2, \ldots, 10$ was done, which gave the following values for $\mu\, S_{r_1}(z_o)$. We shall denote $\mu\, S_{r_1}(z_o)$ by μ_{r_1}. We have, $\mu_1 = 12$, $\mu_2 = 88$, $\mu_3 = 292$, $\mu_4 = 688$, $\mu_5 = 1340$, $\mu_6 = 2312$, $\mu_7 = 3668$, $\mu_8 = 5472$, $\mu_9 = 7788$ and $\mu_{10} = 10680$.

4.4. E-SET

In analogy with the notion of ellipses in the usual geometry of the plane, we consider here the notion of E-set. Only very limited study of E-sets could be carried out, due to lack of uniformity of distribution of points of it in comparison with that of r-sets.

DEFINITION 4.4.1. Let p,k be positive integers and z_1, z_2 be two points of H such that $d(z_1,z_2) = k$. Then $\{z \in H : d(z,z_1) + d(z,z_2) = p\}$ is called an E-set with fixed points z_1 and z_2 and $\{z \in H : d(z,z_1) + d(z.z_2) < p\}$ its interior.

NOTATION. $E_{p,k}\,(z_1,z_2)$ or $E_{p,k}$ will denote an E-set and Int $E_{p,k}$, the interior.

$E_{3,1}$, $E_{5,1}$ are illustrated in Figure-8.

LEMMA 4.4.2. If k is odd (even) then $E_{p,k} = \varphi$ for p even (odd).

PROOF. Consider the E-set $E_{p,k}(z_1,z_2)$ where $z_1 = (q^{m_1}x_o, q^{n_1}y_o)$ and $z_2 = (q^{m_2}x_o, q^{n_2}y_o)$. Suppose that $z = (q^m x_o, q^n y_o) \in E_{p,k}$. Then $d(z_1,z_2) = k$ and $d(z,z_1) + d(z,z_2) = p$ implies that

$$|m_1-m_2| + |n_1-n_2| = k \quad \text{and}$$
$$|m-m_1| + |n-n_1| + |m-m_2| + |n-n_2| = p.$$

But there are no values for m,n satisfying simultaneously both these equations when we assume that either p or k is even, and the other is odd and hence $E_{p,k} = \varphi$.

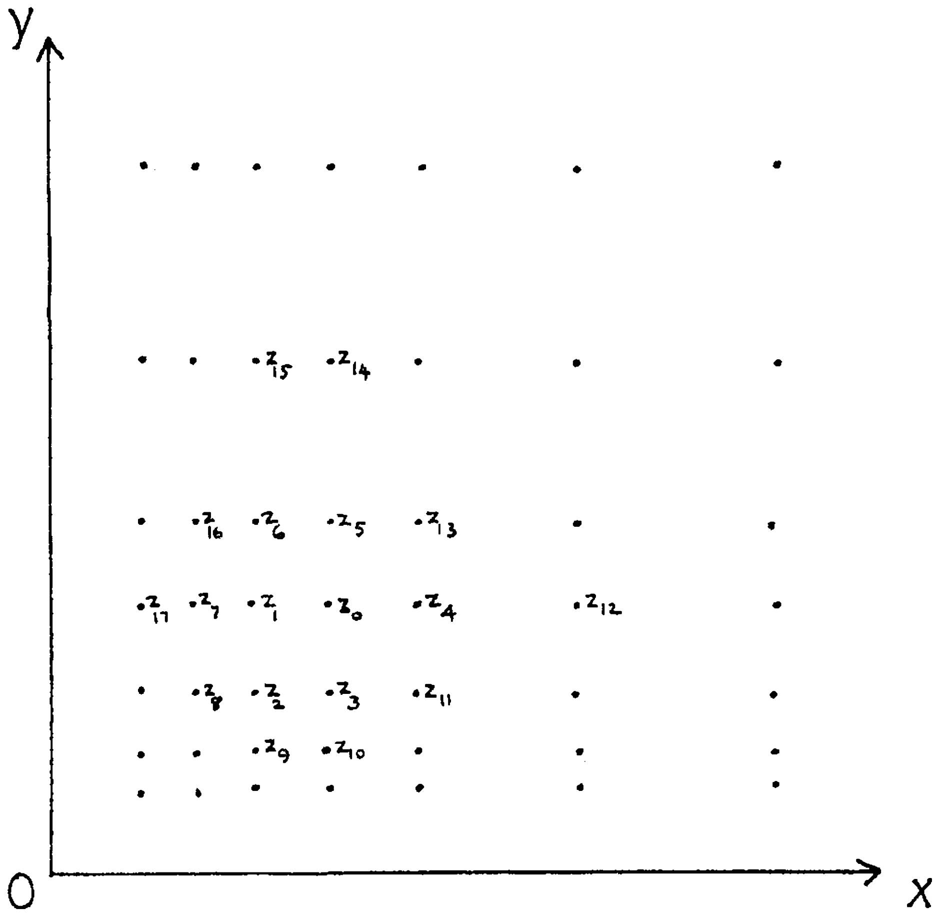


Figure-8
E - sets

$$E_{3,1}(z_0, z_1) = \{z_2, z_3, z_4, z_5, z_6, z_7\} .$$

$$E_{5,1}(z_0, z_1) = \{z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}, z_{16}, z_{17}\} .$$

NOTE 4.4.3. For $p = k = 1$, $E_{1,1}$ consists of just the fixed points. Let $k = 2$. If the fixed points are in the same horizontal (vertical) set, then $E_{2,2}$ has three points and when the fixed points are in the set $\{(q^m x_o, q^m y_o)\}$ or $\{(q^m x_o, q^{-m} y_o)\}$, $m \varepsilon Z$, then there are four points in $E_{2,2}$. Thus the cardinality of E-sets when $p = k$ heavily depends on the location of the fixed points. This situation is illustrated in Figure-9. Here we avoid this situation and assume that $p > k$. Regarding the cardinality of $E_{p,k}$ we have the following theorem.

THEOREM 4.4.4. The cardinality of $E_{p,k}(z_1, z_2)$ is 2 p, if not zero.

THEOREM 4.4.5. Consider $E_{p_1,k_1}(z_o, z_1)$ where $z_1 \varepsilon X_1 = \{(q^m x_o, y_o);\ m \varepsilon Z\}$. Then $E_{p_1,k_1} \cap X_1$ has only two points.

PROOF. Consider $E_{p_1,k_1}(z_o, z_1)$ where $z_1 = (q^{m_1} x_o, y_o)$. Suppose first that m_1 is positive. Then consider the point $\xi_1 = (q^{\frac{p_1+k_1}{2}} x_o, y_o)$. Since E_{p_1,k_1} exist, by lemma 4.4.2. both p_1, k_1 are either odd or even.

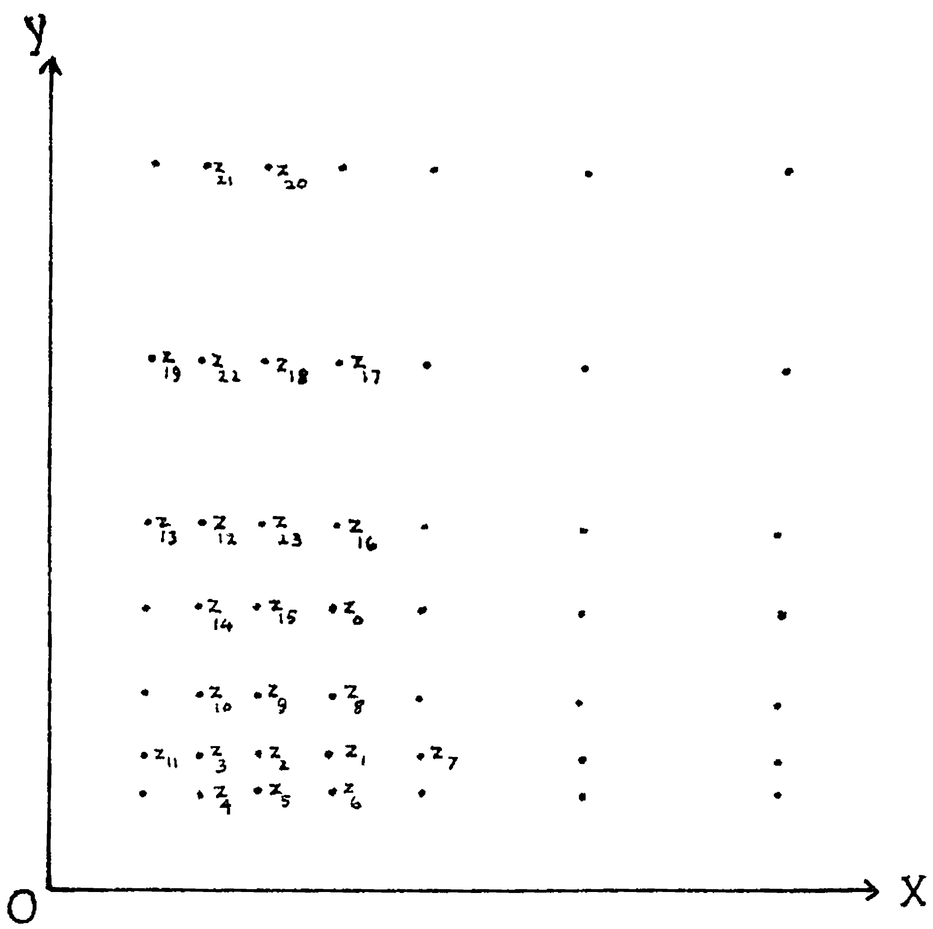


Figure-9

$$E_{2,2}(z_1,z_3) = \{z_1,z_2,z_3\}.$$

$$E_{4,2}(z_1,z_3) = \{z_4,z_5,z_6,z_7,z_8,z_9,z_{10},z_{11}\}.$$

$$E_{2,2}(z_{12},z_{18}) = \{z_{12},z_{23},z_{18},z_{22}\}.$$

$$E_{4,2}(z_{12},z_{18}) = \{z_{13},z_{14},z_{15},z_{16},z_{17},z_{19},z_{20},z_{21}\}.$$

In addition, since both p_1, k_1 are positive, $\frac{p_1+k_1}{2} \in Z^+$.

Now, $\xi_1 \in E_{p_1,k_1}(z_o,z_1)$?

For, $d(z_o, \xi_1) + d(z_1, \xi_1) = \frac{p_1+k_1}{2} - \left|m_1 - \frac{p_1+k_1}{2}\right|$

$= p_1$ since $m_1 = k_1$ and $p_1 > k_1$.

Now, if m_1 is negative, $\xi_2 = (q^{\frac{k_1-p_1}{2}} x_o, y_o)$ satisfies all the above conditions. Thus $\xi_1, \xi_2, \in E_{p_1,k_1} \cap X_1$. Hence the required cardinality is 2.

THEOREM 4.4.6. Consider $E_{p_1,k_1}(z_o,z_1)$ where $z_1 \in X_1$. Then the cardinality of Int E_{p_1,k_1} is $(k_1+1) + 2(n-1)[n+k_1]$, where $p_1 = k_1+2n$.

PROOF. Consider $E_{p_1,k_1}(z_o,z_1)$ where $z_1 = (q^{m_1}x_o,y_o) \in X_1$. Since $d(z_o,z_1) = k_1$, by definition of distance, there is a path joining z_o and z_1 containing (k_1+1) points which are points between z_o, z_1, $\in X_1$. So these (k_1+1) points are interior points of E_{p_1,k_1}. Now, $p_1 = k_1+2n$ for some n. So points on $E_{p,k}$s with fixed points z_o and z_1 where $p < p_1$

will also be interior points of E_{p_1,k_1}. Thus the number of interior points of

$$E_{p_1,k_1} = (k_1+1) + 2 \sum_{i=1}^{n-1} 2(k_1+2i)$$

$$= (k_1+1) + 2(n-1)k_1 + \frac{4n(n-1)}{2}$$

$$= (k_1+1) + 2(n-1)\ [k_1+n] \text{ where } n = \frac{p_1-k_1}{2}.$$

Hence the theorem.

NOTE 4.4.7. Above results remain true when X_1 is replaced by $Y_1 = \left\{(x_o, q^n y_o);\ n \in Z\right\}$, with points being different and the cardinality same. The case when z_1 is any point in H could not be solved. So are the concepts like overlapping etc. considered for r-sets in chapter 2.

4.5. CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER STUDY

This thesis is an attempt to introduce and investigate the analogues of some geometric concepts in the discrete plane H and thereby to initiate the development of a discrete geometry of H. This has been carried out to the extent possible, as follows.

By first defining an integer valued metric on H, and studying some metric properties of it, we considered the notion of domains, D-linear sets, r-sets and their characterisation. Then we introduced the idea of discrete transformations on H. The group theoretic properties of those which leave invariant, the property of an r-set, it's characterisation and discrete analytic properties are also considered. Finally, we discuss some convexity and related concepts for subsets of H. Naturally a metric approach is preferred. We considered a matrix representation of domains, metric content etc. and analogous notion in the discrete case of the concept of ellipses of the classical Euclidean geometry.

The study mentioned in this thesis is far from complete. Several problems are left unanswered, either due to the lack of sufficient tools or due to certain other limitations. Some interesting problems that we have come across during our investigation, solutions of which either have not been tried or could not be obtained, are indicated below.

Our study is mainly focussed on the metric properties of H. Another fundamental concept of the usual geometry is that of angles. Suitable notion of angles and consequently the notion of conformality, it's relation with various discrete analyticity notions can also be considered. Some guidelines in this direction are available in [69].

Applications of discrete transformations to the theory of discrete integration developed in [35] can be attempted. Discrete analogues of periodic functions etc. can be defined in terms of the special types of discrete transformations. Discrete transformations taking D-linear sets onto r-sets and vice versa can be studied. All these taken together can then be an analogue of the classical fractional linear transformations. Transformations which take r-sets onto E-sets can also be looked into.

Among the various generalizations of convexity, we have preferred that due to Menger in [51] and defined D-convexity. Analogues of Helly's theorems and it's relatives of the classical convexity theory can be tried for D-convex sets also. Still different attempt to define convexity in H can be made along the lines mentioned in [17].

Answer to the question, how often can the same distance be realised by points of an r-set, may be helpful in obtaining better estimates for $\mu\, S_{r_1}(z_o)$. Matrix representation and the metric content of any domain can be discussed if a complete enumeration of domains with n lattice points is done. Several problems of combinatorial nature and others related to finite metric spaces mentioned in [55] can be attempted.

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