STUDIES ON SOME TOPICS IN PRODUCT GRAPHS

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By

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To

My Parents

Certificate

This is to certify that the thesis entitled '**Studies on some topics in product graphs**' submitted to the Cochin University of Science and Technology by Ms. Chithra M.R. for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

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Declaration

I, Chithra M.R., hereby declare that this thesis entitled '**Studies on some topics in product graphs**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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Gurur Saakshat Parahbrahma Thasmai Sree Gurave Namah'
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STUDIES ON SOME TOPICS IN PRODUCT GRAPHS

Contents

1 Introduction **1**

1.1 Basic definitions 7

1.2 Notations 17

1.3 Basic properties and theorems 19

1.4 A survey of results 22

1.5 Summary of the thesis 26

1.6 List of publications 35

2 Diameter variability of the product graphs **37**

2.1 Diameter variability of the Cartesian product of graphs 38

2.2 Diameter variability of the strong product of graphs 54

2.3 Diameter variability of the lexicographic product of graphs 67

3 Diameter vulnerability of the product graphs 85

3.1 Diameter vulnerability of the product graphs . . . 86

3.2 Diameter vulnerability of some graph classes . . . 107

3.3 Wide Diameter of the lexicographic product of graphs 117

4 Component factors of the product graphs 121

4.1 Component factors of the Cartesian product of graphs 122

4.2 Path factors of Hypercubes and Hamming graphs 135

5 Domination criticality in the Cartesian product

of graphs **139**

5.1 Domination critical graphs 140

5.2 Vertex criticality in grids 155

Concluding Remarks **157**

List of symbols **159**

Bibliography **161**

Index **169**

Chapter 1

Introduction

The origin of graph theory can be traced back to Euler's work on the Königsberg bridges problem. The Swiss mathematician Leonhard Euler presented a paper 'On the solution of a problem relating to the geometry of position' to his colleagues at the Academy of Sciences in St Petersburg on 26 August 1735. This was the Konigsberg bridges problem - find a closed walk that crosses each of the seven bridges of Königsberg exactly once, which led to the discovery of Eulerian graphs [6]. Since then the subject has grown into one of the most inter disciplinary branches in mathematics with a great variety of applications

[5], [56].

The development of graph theory with its applications to electrical networks, flows and connectivity are included in [10] and [26]. Interest in graphs and their applications has grown exponentially in the past two decades, due to the usefulness of graphs as models for computation and optimization [35].

The idea of networks has received much attraction in the past years as it affects many aspects of our lives, such as how we store and retrieve information, communication etc. The Web graph [5], [24] is a real world network which became an active field of study in the last decade. A web graph, W has vertices representing the web pages and the edges correspond to links between the pages. This exciting notion of web graph has applications in different areas. The most famous ranking algorithm, 'Page Rank' was introduced in 1998, for Google's web search algorithm [11].

'Scale free network' is a network characterized by a 'power law degree distribution'. The construction of scale free graph is based on its adjacency matrix. Many critical infrastructure systems such as internet, railroads, gas pipeline systems etc have been shown to be scale free [64]. Spectral properties of complex networks are also studied [54]. A large number of biological networks such as metabolic reaction networks, gene regulatory network, food networks between species in an ecosystem have been studied in [27].

In any branch of mathematics we try to get new structures from the given structures. In graph theory also many interesting classes of graphs are obtained by combining graphs in several ways such as join, union, product etc.

'Graph products' are viewed as a convenient method to describe the structure of a graph in terms of its factors. There are three products - Cartesian, strong and lexicographic product which have many applications and theoretical interpretations. These products have the property that projection into at least one factor is a weak homomorphism. For this reason the three standard products are most extensively studied and have the widest range of applications. When dealing with product graphs, one of the main source of reference is the book by R. Hammack et al. [36].

An interconnection network may be modeled by a simple graph whose vertices represent components of the network and the edges represent physical communication links. A basic feature of a network is that its components are connected by physical communication links to transmit information according to some pattern. Many graph theoretic techniques can be used to study the efficiency and reliability of a network, as discussed in [41], [50] and [66]. For designing large-scale interconnection networks, the product graph operation is an important method to obtain large graphs from smaller ones, with a number of parameters that can be calculated from the corresponding parameters of the factor graph.

The distance and diameter of a graph play significant roles in analyzing the efficiency of an interconnection network. The diameter is often taken as a measure of efficiency, when studying the potential effects of link failures on the performance of a communication network, especially for networks with maximum time-delay or signal degradation. In fact, most of the graph products are interconnection networks and a good network must be hard to disrupt and the transmissions must remain connected even if some vertices or edges fail. In order to improve or increase the efficiency of message transmission we need to minimize the diameter of a graph. However, there are nice interconnection networks, such as butterfly networks, honeycomb networks [41], which are not product graphs.

In the design of an interconnection network, another fundamental consideration is the reliability of the network, which is characterized by the vertex connectivity and the edge connectivity of the network. If some processors or links are faulty, the information cannot be transmitted by these links and the efficiency of network will be affected. These problems deal with how the remaining processors can still communicate with a reasonable efficiency. In terms of graphs, this problem is modeled in the literature as the vulnerability of the diameter. These notions have received much research attention in the past years due to its applications in networks [66].

For routing problems in interconnection networks it is important to find the shortest containers between any two vertices, since the w-wide diameter gives the maximum communication delay when there are up to $w-1$ faulty nodes in a network modeled by a graph. The concept of 'wide diameter' was introduced by Hsu [41] to unify the concepts of diameter and connectivity.

The concept of ‘domination’ has attracted interest due to its wide applications in many real world situations [38]. A connected dominating set serves as a virtual backbone of a network and it is a set of vertices that helps in routing [17].

In this thesis, we make an earnest attempt to study some of these notions in graph products. This include, the diameter variability, the diameter vulnerability, the component factors and the domination criticality.

1.1 Basic definitions

The basic notations, terminology and definitions are from [4], [13], [37], [38], [43], [65] and the basic results are from [42], [43], and [36].

Definition 1.1.1. A **graph** $G = (V, E)$ consists of a non-empty collection of points, V called its **vertices** and a set of unordered pairs of distinct vertices, E called its **edges**. The unordered pair of vertices $\{u, v\} \in E$ are called the **end vertices** of the edge $e = \{u, v\}$. In that case, the vertex u is said to be **adjacent** to the vertex v. Two edges e and e' are said to be **incident** if they have a common end vertex. The **neighborhood** of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent to u. The **closed neighborhood** of a vertex u is $N[u] = N(u) \cup \{u\}$. $|V|$ is called the **order** of G, denoted by n or $n(G)$ and $|E|$ is called the **size** of G, denoted by m or $m(G)$. A graph G is **totally disconnected** if it has no edges.

Definition 1.1.2. The number of vertices adjacent to a vertex v is called the **degree** of the vertex, denoted by $deg(v)$.

A vertex of degree zero is an **isolated vertex** and of degree one is called a **pendant vertex**. A vertex of degree $(n-1)$ is called a **universal vertex**. The maximum and the minimum degree of vertices are denoted by $\Delta(G)$ and $\delta(G)$ respectiively. G is regular if $\Delta(G) = \delta(G)$. It is **k-regular**, if $deg(v) = k$ for every vertex $v \in V(G)$.

Definition 1.1.3. A graph G is **isomorphic** to a graph H if there exists a bijection $\phi : V(G) \to V(H)$ such that u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. If G is isomorphic to H, we write $G \cong H$.

Definition 1.1.4. A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is a **spanning subgraph** of G if $V(H) = V(G)$. A subgraph H is called an **induced subgraph** of G if each edge of G having its ends in $V(H)$ is also an edge of H. The subgraph of G induced by H is denoted by $< H >$.

Definition 1.1.5. A $v_0 - v_k$ **walk** in a graph G is a finite list $v_0, e_1, v_1, e_2, v_2, ..., e_k, v_k$ of vertices and edges such that for $1 \leqslant i \leqslant k$, the edge e_i has end vertices v_{i-1} and v_i. If the vertices $v_0, v_1, ..., v_k$ of the above walk are distinct, then it is called a **path**. A path from the vertex u to the vertex v is called a $\mathbf{u-v}$ **path**. A path on n vertices is denoted by P_n. If in addition $v_k = v_0$ and $k = n$ then it is called a **cycle** of length n, C_n. If the edges $e_1, e_2, ..., e_k$ of the walk are distinct, it is called a **trail**. A graph G is **connected** if for every $u, v \in V$, there exists a $u - v$ path. If G is not connected, then it is **disconnected**. A connected acyclic graph is called a **tree**.

Definition 1.1.6. The **distance** between two vertices u and v of a connected graph G, denoted by $d(u, v)$, is the length of a shortest $u - v$ path in G. The **eccentricity** of a vertex u, $e(u) = max\ \{d(u, v)/v \in V(G)\}$. The **radius**, $r(G)$ and the **diameter**, diam(G) are respectively the minimum and the maximum of the vertex eccentricities. For a vertex $u \in V(G)$, if there exists a vertex $v \in V(G)$ such that $d(u, v) =$ diam(G), v is then called a **diametral vertex** of u.

Definition 1.1.7. The **complete graph** K_n is a graph of order n in which each pair of distinct vertices is joined by an edge. A **clique** is a maximal complete subgraph.

Definition 1.1.8. A graph G is **bipartite** if the vertex set can be partitioned into two non-empty sets U and U' such that every edge of G has one end vertex in U and the other in U'. A bipartite graph in which each vertex of U is adjacent to every vertex of U' is called a **complete bipartite graph**.

Definition 1.1.9. Let G be a graph. The **complement** of G, denoted by G^c, is the graph with the same vertex set as G and any two vertices are adjacent in G^c if they are not adjacent in G. K_n^c is called a **totally disconnected** graph.

Definition 1.1.10. For a graph G, a subset V' of $V(G)$ is a k**-vertex cut** of G if the number of components in $G - V'$ is greater than that of G and $|V'| = k$. The **vertex connectivity** of a graph G, $\kappa(G)$, is the least number of vertices whose deletion from G increases the number of components of G. A graph G is k**-connected**, if $\kappa(G) \geqslant k$. A vertex v of G is a **cut vertex** of G if $\{v\}$ is a vertex cut of G. The **edge connectivity** of a graph G, $\kappa'(G)$, is the least number of edges whose deletion from G increases the number of components of G.

Definition 1.1.11. A set $S \subseteq V(G)$ of vertices in a graph G is called a **dominating set,** if every $v \in V(G)$ is either an element of S or is adjacent to an element of S. The domination number of a graph G, $\gamma(G)$, is the minimum cardinality of a dominating set in G. A dominating set S is a connected dominating set if $< S >$ is a connected subgraph of G and the corresponding domination number is the **connected domination number,** $\gamma_c(G)$.

Illustration:

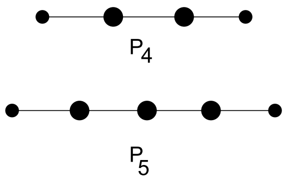

Fig 1.1: $\gamma(P_4) = \gamma_c(P_4) = 2$ and $\gamma(P_5) = 2$, $\gamma_c(P_5) = 3$.

Definition 1.1.12. [25],[38] **Edge critical graphs** are graphs in which domination number decreases upon the addition of any missing edge while **vertex critical graphs** are graphs in which domination number decreases when any vertex is removed. A graph G is k - γ - edge critical if $\gamma(G) = k$ and $\gamma(G+e) < k$ for each $e \notin E(G)$ and G is k - γ - vertex critical if $\gamma(G) = k$ but for each vertex $v \in V(G)$, $\gamma(G - v) < k$. Also, G is k - γ_c - edge critical if $\gamma_c(G) = k$ and $\gamma_c(G + e) < k$ for each $e \notin E(G)$ and G is k - γ_c - vertex critical if $\gamma_c(G) = k$ but for each vertex $v \in V(G)$, $\gamma_c(G - v) < k$.

Illustration:

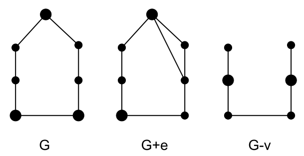

Fig 1.2: $\gamma(G) = 3$, $\gamma(G + e) = 2$ and $\gamma(G - v) = 2$.

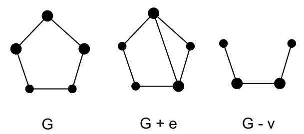

Fig 1.3: $\gamma_c(G) = 3$, $\gamma_c(G + e) = 2$ and $\gamma_c(G - v) = 2$.

Definition 1.1.13. [13] A graph G is **diameter minimal** if $\mathrm{diam}(G - e) > \mathrm{diam}(G)$ for any $e \in E(G)$ and G is **diameter maximal** if $\mathrm{diam}(G + e) < \mathrm{diam}(G)$ for any $e \notin E(G)$.

Illustration:

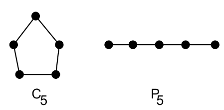

Fig 1.4: C_5 is a diameter minimal graph and P_5 is a diameter maximal graph.

Definition 1.1.16. [36] The **lexicographic product** of two graphs G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ and the two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent if either $u_1 - u_2 \in E(G)$ or $u_1 = u_2$ and $v_1 - v_2 \in E(H)$.

Illustration:

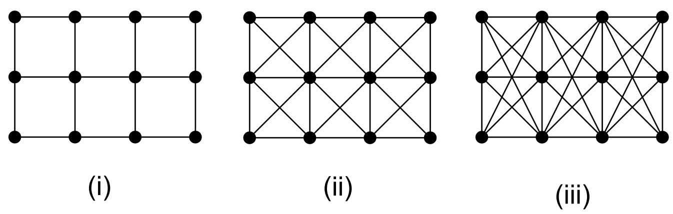

Fig 1.6: (i) $P_4 \Box P_3$ (ii) $P_4 \boxtimes P_3$ (iii) $P_4 \circ P_3$.

Definition 1.1.17. [43] Let $G * H$ be any of the graph products. For any vertex $g \in G$, the subgraph of $G * H$ induced by $\{g\} \times V(H)$ is called the $\mathbf{H-layer}$ at g and denoted by ${}^{g}H$. For any vertex $h \in H$, the subgraph of $G * H$ induced by $V(G) \times \{h\}$ is called the $\mathbf{G-layer}$ at h and denoted by G^h.

Definition 1.1.18. [43] A **hypercube** of dimension n,

denoted by Q_n, is the graph whose vertex set consists of all 0-1 vectors $(v_1, v_2, ..., v_n)$, where two vertices are adjacent if they differ in precisely one coordinate.
Equivalently, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geqslant 2$.

Definition 1.1.19. [43] A graph G is a **Hamming graph** if there exist integers $k, n_1, n_2, n_3, ..., n_{k-1}, n_k$ such that
$G \cong K_{n_1} \square K_{n_2} \square ... \square K_{n_k}$, the vertex set of G is the set of k-tuples $(i_1, i_2, ..., i_k)$, where $i_j \in \{1, 2, ..., n_j\}$ and two k-tuples are adjacent if they differ in exactly one coordinate.

Definition 1.1.20. [46] Let A be a family of connected graphs. If a graph G has a spanning subgraph H such that each component of H is in A then H is called an A**-factor** or **component factor** of G.

Illustration:

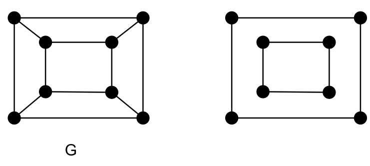

Fig 1.7: A graph G with a C_4-factor.

Note: G has a $\{K_2, P_4, K_{1,3}\}$ - factor also.

Definition 1.1.21. [41] For every integer w: $1 \leqslant w \leqslant \delta(G)$, a w-**container** between any two distinct vertices u and v of G is a set of w internally vertex disjoint paths between them. Let $C_w(u,v)$ denote a w-container between u and v. In $C_w(u,v)$, the parameter w is the **width** of the container. The **length** of the container is the longest path in $C_w(u,v)$. The w-**wide diameter** of G, $D_w(G)$ is the minimum number l such that there is a $C_w(u,v)$ of length l between any pair of distinct vertices u and v.

Illustration:

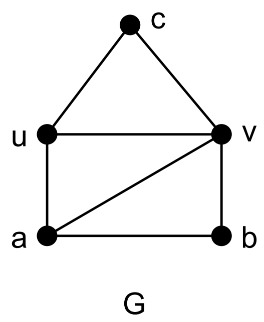

Fig 1.8: For the graph G, $C_2(u,v)$ are $\{u-c-v, u-a-b-v\}$, $\{u-c-v, u-a-v\}$, $\{u-v, u-a-b-v\}$, $\{u-v, u-a-v\}$, $\{u-v, u-c-v\}$ and $D_2(G) = 3$.

1.2 Notations

The diameter of a graph can be affected by the addition or the deletion of some edges. The following notations are used to describe the **diameter variability** [63].

$D^{-k}(G)$: The minimum number of edges to be added to G to decrease the diameter of G by (at least) k, where $k \geqslant 1$.
$D^{k}(G)$: The minimum number of edges to be deleted from G to increase the diameter of G by (at least) k, where $k \geqslant 1$.
$D^{0}(G)$: The maximum number of edges to be deleted from G without an increase in the diameter of G.

Illustration:

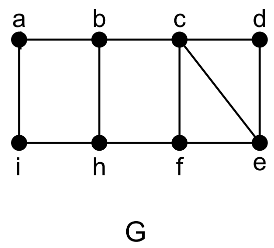

Fig 1.9: $D^{-1}(G) = 1$ (by adding the edge $d - f$).
$D^{1}(G) = 1$ (by deleting the edge $a - b$).
$D^{0}(G) = 3$, (by deleting the edges $a - i$, $c - e$, and $d - e$).

Vulnerability is a measure of the ability of the system to withstand vertex or edge faults and maximum routing delay. Diameter can be used to evaluate the maximum delay in routing. In this context, the following concepts are studied. The notations used are,

$f(G) = max\{diam(G - S)/S \subseteq V(G), |S| = \kappa(G) - 1\}$ (called **fault diameter** [48]) and

$f'(G) = max\{diam(G - F)/F \subseteq E(G), |F| = \kappa'(G) - 1\}$.

Illustration:

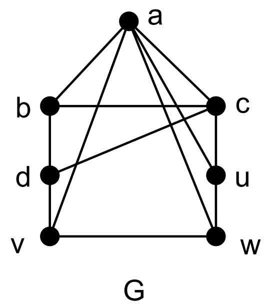

Fig 1.10: diam(G)=2, $\kappa(G) = 3$ and $f(G) = 4$ ($S = \{a, c\}$).
Also, $\kappa'(G) = 3$ and $f'(G) = 3$ ($F = \{u - w, a - w\}$).

1.3 Basic properties and theorems

Product graphs have many interesting algebraic and other properties. The Cartesian product and strong product are commutative and associative. The lexicographic product is associative but not commutative. It is interesting to see that even if the factors G and H of a product graph have a property $'P'$ then it is not necessary that the product $G * H$ also has that property, where $*$ denotes any of the graph products mentioned above. As a case, $C_m \Box C_n$ is non planar and $G \Box H$ need not be Hamiltonian even if both G and H are Hamiltonian.

The Cartesian product is the most prominent graph product. The Cartesian product $G \Box H$ can be obtained from G by substituting a copy H_g of H for any vertex g of G and by joining the corresponding vertices of H_g with H_g' if $g - g' \in E(G)$. The Cartesian product of two connected graphs is a subgraph of both strong and lexicographic product of graphs. Hypercubes and Hamming graphs are important classes of the Cartesian product.

The lexicographic product $G \circ H$ can be obtained from G

by substituting a copy H_g of H for any vertex g of G and then joining all the vertices of H_g with all the vertices of H'_g if $g - g' \in E(G)$.

The following results are of interest to us.

Theorem 1.3.1. *[43] A Cartesian product $G \Box H$ is connected if and only if both factors are connected.*

Theorem 1.3.2. *[43] For any two connected graphs G and H, $diam(G \Box H) = diam(G) + diam(H)$.*

Theorem 1.3.3. *[60] Let G and H be graphs on at least two vertices. Then $\kappa(G \Box H) = min\{\kappa(G)\,|V(H)|\,, \kappa(H)\,|V(G)|\,, \delta(G) + \delta(H)\}$.*

Theorem 1.3.4. *[67] Let G and H be graphs on at least two vertices. Then $\kappa'(G \Box H) = min\{\kappa'(G)\,|V(H)|\,, \kappa'(H)\,|V(G)|\,, \delta(G) + \delta(H)\}$.*

Theorem 1.3.5. *[36] If G and H are connected nontrivial, then $\kappa'(G \Box H) = \kappa'(G) + \kappa'(H)$ if and only if either $\kappa'(G) = \delta(G)$ and $\kappa'(H) = \delta(H)$, or one factor is complete and $\kappa' = 1$ for the other factor.*

Theorem 1.3.6. *[62] For all graphs G and H,*
$\gamma(G \Box H) \leqslant min\{\gamma(G)\,|V(H)|\,, \gamma(H)\,|V(G)|\}$.

Theorem 1.3.7. *[29] For all graphs G and H,*
$\gamma(G\Box H) \geqslant min\{|V(G)|, |V(H)|\}$.

Theorem 1.3.8. *[36] A strong product $G \boxtimes H$ is connected if and only if both factors are connected.*

Theorem 1.3.9. *[36] For any two connected graphs G and H,*
$diam(G \boxtimes H) = max\{diam(G), diam(H)\}$.

Theorem 1.3.10. *[36] Let G and H be connected graphs, at lest one is not complete. Then $\kappa(G \boxtimes H) = min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\}$, where $\ell(G \boxtimes H)$ is the minimum size of a* **7**- *set of $G \boxtimes H$ (if a separating set S has an empty intersection with at least one G - layer and with at least one H - layer, then S is a* **7**- *set of $G \boxtimes H$).*

Theorem 1.3.11. *[36] Let G be not complete. Then*
$\kappa(G \boxtimes K_n) = n\kappa(G)$.

Theorem 1.3.12. *[7] Let G and H be connected graphs. Then*
$\kappa'(G \boxtimes H) = min\{\kappa'(G)(|V(H)| + 2|E(H)|), \kappa'(H)(|V(G)| + 2|E(G)|), \delta(G \boxtimes H)\}$.

Theorem 1.3.13. *[36] A lexicographic product $G \circ H$ is connected if and only if G is connected.*

Theorem 1.3.14. *[36] If G is not complete, then $diam(G \circ H) = diam(G)$ and $diam(K_n \circ G) = 2$*

Theorem 1.3.15. *[36] If G is not complete and H is any graph, then $\kappa(G \circ H) = \kappa(G)\,|V(H)|$.*

Theorem 1.3.16. *[36] For any graph H, $\kappa(K_n \circ H) = \kappa(H) + (n-1)\,|V(H)|$.*

Theorem 1.3.17. *[68] Let G and H be two non-trivial graphs, and G is connected. Then $\kappa'(G) = min\{\kappa'(H_1)n_2^2, \delta(H_2) + \delta(H_1)n_2\}$.*

1.4 A survey of results

This section is a survey of results related to ours.

In [34], Graham and Harary showed that $D^{-1}(Q_n) = 2$, $D^1(Q_n) = n-1$ and $D^0(Q_n) \geqslant (n-3)2^{n-1}+2$. In [12], Bouabdallah et al. obtained the following bound, $(n-2)2^{n-1} - \binom{n}{\lfloor n/2 \rfloor} + 2 \leqslant D^0(Q_n) \leqslant (n-2)2^{n-1} - \lceil 2^n - 1/(2n-1) \rceil + 1$. In [63], J. J. Wang et al. showed that $D^{-1}(C_m \Box C_n) = 2$, for $m \geqslant 12$, $D^1(C_m \Box C_n) = 2$ or 3 and $D^0(C_m \Box C_n) \geqslant \left\{ {mn-2n+1 \atop mn-2n} \right.$ when m is even and odd respectively. This notion is also discussed in [53].

One of the interesting results of diameter minimal graphs of diameter two in [31], is that every graph G can be embedded as an induced subgraph in a diameter minimal graph of diameter two. In [57], Ore O. proved that a graph G is diameter maximal if and only if

(1) G has a unique pair of eccentric peripheral vertices u and v.

(2) the set of vertices at each distance k from u induces a complete graph.

(3) every vertex at distance k is adjacent to every vertex at distance $k+1$.

Also, a disconnected graph is diameter maximal if and only if $G = K_m \cup K_n$.

The problem of determining diameter vulnerability of a graph was proposed by Chung and Garey [23]. The problem is proved to be NP-complete by Schoone et al. [59]. In [58] Peyrat show that $3\sqrt{2t} - 3 < f'(G) \leqslant 3\sqrt{2t} + 4$ where G is a $(t+1)$ - connected graph of diameter 3. In [69] H.X. Ye et al. improves the result of Peyrat and gave a bound as $4\sqrt{2t} - 6 < f'(G) \leqslant max\{59, 5\sqrt{2t} + 7\}$ for $t \geqslant 4$. This notion is also discussed in

[55], [9] and [14]. The concept of fault diameter was introduced by M.S. Krishnamoorthy and B. Krishnamurthy [48]. This notion is also discussed in [32] and [49]. The wide diameter of some networks is studied in [52].

In [2], Ando et al. proved that a connected claw-free graph G with $\delta(G) \geqslant d$ has a path factor having each path of length at least d. Also, they conjectured that a 2-connected claw-free graph G with $\delta(G) \geqslant d$ has a path factor of length at least $3d+2$. In [15], Cada proved the conjecture for line graphs. In [3], Armen et al. showed that a simple (3, 4)-biregular bigraph always has a path factor such that the endpoints of each path have degree three. In [44], Kaneko showed that every cubic graph has a path factor such that each component is a path of length 2,3 or 4. It was shown in [47], that a 2-connected cubic graph has a path factor whose components are paths of length 2 or 3. In [46], Kano et al. proved that if a graph G satisfies iso(G-S) $\leqslant |S|/2$ for all $S \subseteq V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor where iso(G - S) denotes the number of isolated vertices in $G - S$. Some results on different types of path factors can be found in [28], [45], [51]. Hell and Kirkpatrick [39], [40] proved that if A is a graph on at least 3 vertices, then deciding whether G has a

A-factor is NP-complete.

The connected dominating set has attracted interest due to its applications in network routing. In [17], Y.C. Chen and Y.L Syu showed that for an n-dimensional Star graph Q_n and n-dimensional Star graph S_n, the order of minimum connected dominating set (MCDS), $|MCDS(Q_n)| \leqslant 2^{n-2} + 2$ where $n \geqslant 3$ and $|MCDS(S_n)| \leqslant 2(n-1)!$ where $n \geqslant 3$. In [61], Sumner et al. characterized 2 - γ - edge critical graphs and proved that the disconnected 3 - γ - edge critical graphs are the disjoint union of 2 - γ - edge critical graphs and a complete graph. For $k \geqslant 4$, the characterization of connected k - γ - edge critical graphs is not known. In [16], Chen et al. gave a characterization of 2 - γ_c - edge critical graphs. Also, if G is 3 - γ_c - edge critical then either G is isomorphic to C_5 or contains a triangle and that if G is 3 - γ_c - edge critical of even order then G contains a one factor. In [8], Brigham et al. gave a characterization of 2 - γ - vertex critical graphs. But for $k \geqslant 3$, only some properties of k - γ - vertex critical graphs are known and there is no characterization of such graphs. In [30], Flandrin et al. studied some properties of 3 - γ - edge critical graphs and proves that if G is a 3 - γ - edge critical connected graph of order n

with $\delta \geqslant 2$, then G is 1-tough and circumference of G is at least $n-1$. Some properties of 3 - γ_c - vertex critical graphs are discussed in [1]. For $k \geqslant 4$, no characterization of k - γ_c - vertex critical graphs are known. In [33], Goncalves et al. studied the domination number of grids.

We shall discuss these notions in product graphs in this thesis. In this thesis, we consider the graphs H_1, H_2 and denote the $V(H_1) = \{u_1, u_2, ..., u_{n_1}\}$, $V(H_2) = \{v_1, v_2, ..., v_{n_2}\}$ and $V(H_1 * H_2) = \{u_1v_1, u_1v_2, ..., u_{n_1}v_{n_2}\}$ where $* \in \{\Box, \boxtimes, \circ\}$. Also, $|E(H_1)| = m_1$ and $|E(H_2)| = m_2$. Since, $H_1 * K_1 \cong H_1$ we assume that H_1, $H_2 \neq K_1$.

1.5 Summary of the thesis

This thesis entitled '**Studies on some topics in product graphs**' is divided into five chapters including an introductory chapter giving a brief history of graph theory, basic definitions and results which we have used in our work.

In the second chapter the diameter variability of product graphs is studied in detail. The main results in this chapter are:

- ⋆ Let $G \cong H_1 \square H_2$. Then $D^0(G) \geqslant 2$.

- ⋆ Let $G \cong H_1 \square H_2$. Then $D^1(G) = 1$ if and only if H_1 is a complete graph and either H_2 has at least one pair of vertices with exactly one diametral path P and no path of length $\mathrm{diam}(H_2)+1$ which is edge disjoint with P or there exist an edge in H_2 that is on all paths of length $\mathrm{diam}(H_2)$, $\mathrm{diam}(H_2)+1$ between any two diametral vertices in H_2.

- ⋆ Let $G \cong H_1 \square H_2$.

  (a) If both H_1 and H_2 are complete graphs with $n_1, n_2 > 2$, then $D^1(G) = 2$.

  (b) If H_1 is a complete graph and H_2 is a not complete graph, then $D^1(G) \leqslant \delta(H_2)$.

  (c) If both H_1 and H_2 are not complete graphs, then $D^1(G) \leqslant \Delta(G) - 1$.

- ⋆ Let $G \cong H_1 \square H_2$. Then $D^{-1}(G) = 1$ if and only if G is any one of the following graphs where,

  (a) H_1 is a complete graph and H_2 is a not complete graph with $D^{-2}(H_2) = 1$.

(b) H_1 is a not complete graph with a universal vertex or there exist a vertex in H_1 that is on at least one path between any two diametral vertices and H_2 is a not complete graph with $D^{-1}(H_2) = 1$.

- $\star$ Let $G \cong H_1 \boxtimes H_2$. Then $D^0(G) \geqslant 6$.

- $\star$ Let $G \cong H_1 \boxtimes H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,

  (a) both H_1 and H_2 are complete graphs.

  (b) H_1 and H_2 are not complete graphs with $\mathrm{diam}(H_1) = \mathrm{diam}(H_2)$ and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exist an edge in H_1 or H_2 that is on all diametral paths between any two vertices.

- $\star$ Let $G \cong H_1 \boxtimes H_2$. Then $D^1(G) \leqslant \alpha(1 + \delta(H_2))$ where α is the minimum number of edge disjoint paths of length $\mathrm{diam}(H_1)$ between any two vertices in H_1.

- $\star$ Let $G \cong H_1 \boxtimes H_2$ be connected graph. Then $D^{-1}(G) = 1$ if and only if H_2 has a universal vertex and H_1 is a connected graph with $\mathrm{diam}(H_1) \geqslant 4$ and $D^{-2}(H_1) = 1$ when an edge is added between a diametral vertex and any other vertex of H_1 and $D^{-1}(H_1) = 1$ when an edge is added between any two other vertices of H_1.

- ⋆ Let $G \cong H_1 \circ H_2$. Then $D^0(G) \geqslant 3$.

- ⋆ Let $G \cong H_1 \circ H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,
  (a) both H_1 and H_2 are complete graphs.
  (b) $H_1 = K_2$ or a connected graph with diameter two in which there exist at least one pair of adjacent vertices with no path of length two between them and H_2 is a disconnected graph in which there exist at least one component with an isolated vertex.

- ⋆ Let $G \cong H_1 \circ H_2$. Then $D^1(G) \leqslant \alpha\, n_2$ where α is the minimum number of edge disjoint paths of length $\text{diam}(H_1)$ between any two vertices in H_1.

- ⋆ Let $G \cong H_1 \circ H_2$. Then $D^{-1}(G) = 1$ if and only if G is any one of the following graphs where,
  (a) H_2 has a universal vertex and H_1 is a connected graph with $\text{diam}(H_1) \geqslant 4$ and $D^{-2}(H_1) = 1$ when an edge is added between a diametral vertex and any other vertex of H_1.

(b) H_2 is any graph and H_1 is a connected graph with diam$(H_1) \geqslant 4$ and $D^{-1}(H_1) = 1$ when an edge is added between the diametral vertices or between any two other vertices of H_1.

In the third chapter we study the diameter vulnerability of three graph products. Following are some of the results obtained.

- Let $G \cong H_1 \square H_2$, where H_1 is a complete graph and H_2 is a connected graph with $\kappa'(H_2) = \delta(H_2)$. Then $f'(G) = diam(G) + 1$.

- Let $G \cong H_1 \square H_2$ be a connected graph. Then $f'(G) \leqslant max\{f'(H_1) + 2diam(H_2), f'(H_2) + 2diam(H_1)\}$.

- Let $G \cong H_1 \boxtimes H_2$ be a connected graph. Then $f'(G) \leqslant max\{f'(H_1) + diam(H_2), f'(H_2) + diam(H_1)\}$.

- Let $G \cong H_1 \circ H_2$ be a connected graph where $n_1, n_2 \geqslant 3$. Then $f'(G) \leqslant f'(H_1) + diam(H_2)$.

- Let $G \cong H_1 \square H_2$ be a connected graph. Then $f(G) \leqslant max\{f(H_1) + 2diam(H_2), f(H_2) + 2diam(H_1)\}$.

- Let $G \cong H_1 \circ H_2$ be a connected graph. Then
  $f(G) \leqslant max\{f(H_1), f(H_2)\}$.

- Let $G \cong H_1 \boxtimes H_2$ be a connected graph. Then
  $f(G) \leqslant max\{f(H_1) + diam(H_2), f(H_2) + diam(H_1)\}$.

- For any two connected graphs H_1 and H_2,
  Wide diameter $(H_1 \circ H_2)$ = Wide diameter (H_1).

The fourth chapter is the study of the component factors of the product graphs. Some of the results obtained are:

$\bowtie$ Let $G \cong H_1 \square H_2$ be a connected graph where $|H_1| = n_1$ and $|H_2| = n_2$. Then G has a C_4-factor if and only if G is any one of the following graphs where,
((I) H_1 or H_2 has a C_4-factor.
(II) both H_1 and H_2 have no C_4-factor and,
(a) both H_1 and H_2 are complete graphs with n_1, n_2 even and $n_1, n_2 \not\equiv 0 \ mod \ 4$.
(b) H_1 is a complete graph with n_1 even and H_2 is a not complete graph with n_2 even, has at least one vertex with at most one pendant vertex attached to it and has a $\{K_{1,1}\}$-factor.

(c) H_1 and H_2 are not complete graphs with n_1, n_2 even, both have at least one vertex with at most one pendant vertex attached to it and have a $\{K_{1,1}\}$-factor.

$\bowtie$ Let $G \cong K_{n_1} \Box K_{n_2}$ where $n_1, n_2 \geqslant 2$. Then G has a $\{K_{1,2}, C_4\}$-factor.

$\bowtie$ Let $G \cong K_{n_1} \Box H_2$ be a connected graph where H_2 is a not complete graph. Then G has a $\{K_{1,1}, K_{1,2}, C_4\}$-factor.

$\bowtie$ Let $G \cong H_1 * H_2$ where $* \in \{\Box, \boxtimes, \circ\}$ and H_1, H_2 are connected graphs. Then G has a $\{K_{1,n}, C_4\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2.

$\bowtie$ The hypercube Q_n has a $\{P_4\}$-factor.

$\bowtie$ A Hamming graph has a $\{P_3, P_4\}$-factor.

The domination criticality is discussed in the last chapter. The main results are listed below.

$\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma(G) = 2$ if and only if $H_1 = K_2$ and H_2 is either a C_4 or has a universal vertex.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma_c(G) = \gamma(G) = 2$ if and only if $H_1 = K_2$ and H_2 has a universal vertex.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is 2 - γ - vertex (edge) critical if and only if $G = C_4$.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is 2 - γ_c - vertex (edge) critical if and only if $G = C_4$.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma(G) = 3$ if and only if G is any one of the following graphs where,
  (a) $H_1 = K_3$ or P_3 and H_2 has a universal vertex.
  (b) $H_1 = K_2$ and H_2 has a vertex of degree $n_2 - 2$.
  (c) $H_1 = K_2$ and H_2 has a vertex v_r of degree $n_2 - 3$ and is not adjacent to the vertices v_p and v_q with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$.
  (d) $H_1 = K_3$ or P_3 and $H_2 = C_4$.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma_c(G) = \gamma(G) = 3$ if and only if $H_1 = K_3$ or P_3 and H_2 has a universal vertex.

- $\oplus$ Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is

3 - γ - vertex (edge) critical if and only if $H_1 = H_2 = K_3$.

$\oplus$ Let $G \cong H_1 \square H_2$ be a connected graph. Then G is 3 - γ_c - vertex (edge) critical if and only if $H_1 = H_2 = K_3$.

Some results of this thesis are included in [18] - [22]. The thesis is concluded with some suggestions for further study and a bibliography.

1.6 List of publications

Papers presented

- ✠ "Diameter variability of the Cartesian product of some graphs", IMS Annual Conference, 2009, Kalasalingam University, Krishnankoil, Madurai, December 27-30, 2009.

- ✠ "Component factors of the Cartesian product of graphs", Indo-Slovenia Conference on Graph Theory and Applications, Kerala University, Trivandrum, February 22-24, 2013.

Paper accepted

- ✠ The Diameter Variability of the Cartesian product of graphs, to appear in Discrete Mathematics, Algorithms and Applications.

Papers communicated

- ⋆ Chithra M.R., A. Vijayakumar, Diameter vulnerability of the Cartesian product of graphs.

- ⋆ Chithra M.R., A. Vijayakumar, Component factors of the Cartesian product of graphs.
- ⋆ Chithra M.R., A. Vijayakumar, Domination criticality in product graphs.
- ⋆ Chithra M.R., Manju K. Menon, A. Vijayakumar, Some distance notions in lexicographic product of graphs.
- ⋆ Chithra M.R., A. Vijayakumar, The diameter variability of the product graphs.
- ⋆ Chithra M.R., A. Vijayakumar, The diameter vulnerability of some graph products.

Chapter 2

Diameter variability of the product graphs

The diameter of a graph can be affected by the addition or the deletion of edges. In this chapter we examine the product graphs whose diameter increases (decreases) by the deletion (addition) of a single edge. The problems of minimality and maximality of the product graphs with respect to its diameter are also solved. These problems are motivated by the fact that most of the graph products are good interconnection networks and a good network must be hard to disrupt and the transmis-

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Some results of this chapter are included in the following paper.
1. Chithra M.R., A. Vijayakumar, The Diameter Variability of the Cartesian product of graphs, (to appear in Discrete Mathematics, Algorithms and Applications).

sions must remain connected even if some vertices or edges fail.

2.1 Diameter variability of the Cartesian product of graphs

If both H_1 and H_2 are K_2's, then G is C_4 and the deletion of any edge increases the diam(G).

Theorem 2.1.1. *Let $G \cong H_1 \Box H_2$. Then $D^0(G) \geqslant 2$.*

Proof. We shall prove the theorem by showing that there exist at least two edges in G that can be deleted without an increase in the diam(G) by considering the following three cases.

Case 1: H_1 and H_2 are complete graphs where n_1 or $n_2 > 2$.

Suppose that both n_1, $n_2 > 2$.

Let the two edges $u_iv_p - u_iv_q$ and $u_jv_r - u_xv_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There are paths of length two between u_iv_p, u_iv_q and u_jv_r, u_xv_r in G. Now, consider the vertices whose diametral path contain the deleted edges. The distance between these vertices remains the same, since $\delta(G) \geqslant 4$ there is an alternate path of length diam(G) through the neighbours of the deleted edge. Also, the distance between any two other vertices is not affected by the removal of these two edges.

Suppose that $n_1 = 2$ and $n_2 > 2$.
Let the two edges $u_1v_p - u_1v_q$ and $u_2v_q - u_2v_r$ where $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these two edges. Thus, the diam(G) remains the same.

Case 2: H_1 and H_2 are not complete graphs.

Let the two edges $u_iv_p - u_iv_q$ and $u_jv_r - u_xv_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path $u_iv_p - u_yv_p - u_yv_q - u_iv_q$ of length three between u_iv_p and u_iv_q. Similarly, $d(u_jv_r, u_xv_r) \leqslant 3$. Now, consider the vertices whose diametral path contain the deleted edges. The distance between these vertices remains the same, since $\delta(G) \geqslant 2$ there is an alternate path of length diam(G) through the neighbours of the deleted edge. Thus, the diam(G) remains the same.

Case 3: H_1 is a complete graph and H_2 is a not complete graph.

Let the two edges $u_iv_p - u_jv_p$ and $u_iv_q - u_jv_q$ where $i \neq j \in \{1, 2, ..., n_1\}$ and v_p is not adjacent to v_q in H_2, $p, q \in \{1, 2, ..., n_2\}$, be deleted. There is a path of length at most three between these pairs of vertices. Therefore, $d(u_iv_p, u_iv_q) \leqslant 3$ and $d(u_iv_q, u_jv_q) \leqslant 3$. Also, the distance between any two other vertices is not affected by the removal of these two edges. Thus, the diam(G) remains the same.

Hence, there exist at least two edges in G that can be deleted without an increase in the diam(G). $\square$

Theorem 2.1.2. *Let $G \cong H_1 \square H_2$. Then $D^0(G) = 2$ if and only if G is any one of the graphs shown in Fig 2.1.*

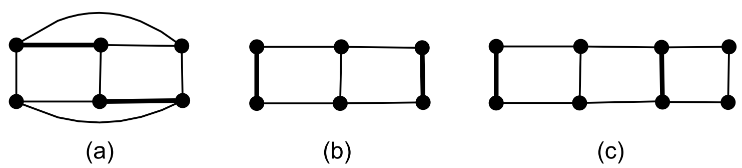

Fig 2.1: The graphs $G : D^0(G) = 2$.

Proof. Suppose that G is any one of the graphs shown in Fig 2.1, then by deleting the bold edges, it is clear that $D^0(G) = 2$.

Conversely suppose that $D^0(G) = 2$. We shall show that G is precisely any one of the graphs in Fig 2.1.
Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

Let $G \cong K_{n_1} \Box K_{n_2}$ where n_1, $n_2 > 2$.

Let the three edges $u_iv_p - u_iv_q$, $u_jv_q - u_jv_r$ and $u_xv_p - u_xv_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path $u_iv_p - u_iv_r - u_iv_q$ of length two between u_iv_p and u_iv_q in G and so $d(u_iv_p, u_iv_q) = 2$. Similarly, $d(u_jv_q, u_jv_r) = d(u_xv_p, u_xv_r) = 2$. Also, the distance between any two other vertices is not affected by the removal of these three edges. Thus, the diam(G) remains the same.

Let $G \cong H_1 \Box H_2$, where H_1 and H_2 are not complete graphs.

Let the three edges $u_iv_p - u_jv_p$, $u_iv_q - u_jv_q$ and $u_av_p - u_av_r$ where $i \neq j \neq a \in \{1, 2, ..., n_1\}$ and v_p is not adjacent to v_q in H_2, $p, q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path of length at most three between these pairs of vertices. Now, $d(u_xv_p, u_yv_p) \leqslant \text{diam}(H_1) + 2$ by a path $u_xv_p - u_xv_r - u_{x+1}v_r -$

... $- u_y v_r - u_y v_p$ where $d(u_x v_p, u_x v_r) = d(u_y v_p, u_y v_r) = 1$ and $d(u_x v_r, u_y v_r) \leqslant \text{diam}(H_1)$. Also, $d(u_x v_q, u_y v_q) \leqslant \text{diam}(H_1) + 2$ and $d(u_a v_w, u_a v_z) \leqslant \text{diam}(H_2) + 2$. Thus, the diam(G) remains the same.

Hence, it is clear that at least one graph (say) H_1 should be a complete graph and H_2 is a not complete graph.

Let $G \cong K_{n_1} \Box H_2$ where $n_1 > 2$.

Let the three edges $u_i v_p - u_j v_p$, $u_j v_q - u_x v_q$ and $u_i v_r - u_x v_r$ where $i \neq j \neq x \in \{1, 2, ..., n_1\}$ and $p \neq q \neq r \in \{1, 2, ..., n_2\}$, be deleted. There is a path $u_i v_p - u_x v_p - u_j v_p$ of length two between $u_i v_p$ and $u_j v_p$ in G. Similarly, $d(u_j v_q, u_x v_q) = d(u_i v_r, u_x v_r) = 2$. Also, the distance between any two other vertices is not affected by the removal of these three edges. Thus, the diam(G) remains the same.

Hence, it follows that $n_1 \leqslant 2$. Now, we will consider the different cases depending on the value of n_2.

Case 1: $G \cong K_2 \Box H_2$ where H_2 is a not complete graph with $n_2 \geqslant 5$.

Suppose that $\text{diam}(H_2) \geqslant 4$.

Consider a pair of diametral vertices v_w to v_z in H_2 where v_l is a vertex in a diametral path between them and is not adjacent to both v_w and v_z. Let the three edges $u_1v_w - u_2v_w$, $u_1v_l - u_2v_l$ and $u_1v_z - u_2v_z$, be deleted. There is a path of length three between these pairs of vertices. Consider the vertex u_1v_w in G. Then u_2v_z, a diametral vertex of u_1v_w is at a distance diam(G) by a path $u_1v_w - u_1v_{w+1} -\ \ldots\ - u_1v_{l-1} - u_2v_{l-1} - u_2v_l -\ \ldots\ - u_2v_z$. Thus, the diam(G) remains the same.

Suppose that $\text{diam}(H_2) = 3$.

Consider a pair of diametral vertices v_w to v_z in H_2 where v_b is a vertex not in any of the diametral path between them in H_2. Let the three edges $u_1v_w - u_2v_w$, $u_1v_z - u_2v_z$ and $u_1v_b - u_2v_b$, be deleted. There is a path of length at most four between these pairs of vertices. Thus, the diam(G) remains the same.

Suppose that $\text{diam}(H_2) = 2$.

Suppose that H_2 has a universal vertex v_p.

Let the three edges $u_1v_q - u_2v_q$, $u_1v_r - u_2v_r$ and $u_1v_l - u_2v_l$ where $q, r, l \neq p$, be deleted. There is a path of length at most three between these pairs of vertices. Thus, the diam(G) remains the same.

Suppose that H_2 does not have a universal vertex and $d(v_w, v_z) = 2$ in H_2.

Let the three edges $u_1v_w - u_2v_w$, $u_1v_z - u_2v_z$ and $u_1v_p - u_1v_q$, be deleted. There is a path of length three between these pairs of vertices in G. Thus, the distance between any two other vertices is at most three.

Case 2: $G \cong K_2 \square K_{n_2}$ where $n_2 \geqslant 5$.

Let the three edges $u_1v_2 - u_1v_3$, $u_1v_2 - u_1v_4$ and $u_1v_2 - u_1v_5$, be deleted. There are paths of length two between these pairs of vertices. Thus, the diam(G) remains the same.

Thus, there exist at least three edges in G that can be deleted without an increase in the diam(G). Hence, it follows that $n_2 \leqslant 4$. Now, by an exhaustive verification of all graphs H_2 with $n_2 \leqslant 4$, it follows that $G \cong K_2 \square K_3$, $K_2 \square P_3$ and $K_2 \square P_4$. $\square$

Theorem 2.1.3. *Let $G \cong H_1 \square H_2$. Then $D^1(G) = 1$ if and only if H_1 is a complete graph and either H_2 has at least one pair of* *vertices with exactly one diametral path P and no path of length $diam(H_2) + 1$ which is edge disjoint with P or there exist an edge in H_2 that is on all paths of length $diam(H_2)$, $diam(H_2)+1$ between any two diametral vertices in H_2.*

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

Suppose that H_1 is a complete graph. If H_2 has a pair of vertices v_w, v_z, with one diametral path P and no path of length diam$(H_2) + 1$ edge disjoint with P, then $v_p - v_q$ be an edge whose deletion increases the diam(H_2). If H_2 has a pair of vertices v_w, v_z, with paths of length diam(H_2), diam$(H_2) + 1$ which are not edge disjoint with each other, then $v_p - v_q$ is a common edge in all these paths. Consider a pair of vertices u_iv_w, u_iv_z in G. Let an edge $u_iv_p - u_iv_q$, be deleted from the path $u_iv_w - u_iv_{w+1}$ ... u_iv_z in G, then the diam(G) increases by a path $u_iv_w - u_jv_w - u_jv_{w+1} - u_jv_{w+2}$ ... $u_jv_z - u_iv_z$ where $d(u_jv_w, u_jv_z) =$ diam(H_2), $d(u_iv_w, u_jv_w) = d(u_iv_z, u_jv_z) = 1$. Also, $d(u_iv_r, u_iv_s) \leqslant$ diam(G) where $r, s \in \{1, 2, ..., n_2\}$. The distance between any two other vertices is not affected by the removal of this edge.

Conversely suppose that $D^1(G) = 1$.
If both H_1 and H_2 are not complete graphs, then at least two edges should be deleted to increase the diam(G).

If H_1 and H_2 are complete graphs with n_1, $n_2 > 2$, there exist two internally vertex disjoint paths of length two between two non adjacent vertices $u_i v_p$ and $u_j v_q$ in G. Thus, at least two edges should be deleted to increase the diam(G).

Hence, it is clear that at least one graph (say) H_1 should be a complete graph and H_2 is a not complete graph.

Suppose that $d(u_i v_w, u_i v_z) = \text{diam}(H_2)$. Let an edge
$u_i v_p - u_i v_q$, be deleted.

If H_2 contains two internally edge disjoint paths, one of length $\text{diam}(H_2)$ and the other of length $\text{diam}(H_2) + 1$ or two internally edge disjoint paths of length $\text{diam}(H_2)$ between v_w and v_z in H_2, then the diam(G) remains the same, since in both the cases there exist an alternate path of length $\text{diam}(H_2) + 1$ or $\text{diam}(H_2)$ between $u_i v_w$ and $u_i v_z$ in G.

If H_2 has paths of length diam(H_2) and diam$(H_2)+1$ between v_w and v_z in H_2, such that all these paths have some edges in common then, the diam(G) remains the same. Since, all these paths does not have a common edge (say) $v_p - v_q$, even if a delete an edge there exist an alternative path of length diam$(H_2)+1$ or diam(H_2) between $u_i v_w$ and $u_i v_z$, without affecting the diam(G).

Hence, either H_2 has at least one pair of vertices with only one diametral path P and no path of length diam$(H_2)+1$ which is edge disjoint with P or there exist an edge in H_2 that is on all paths of length diam(H_2), diam$(H_2) + 1$ between any two diametral vertices in H_2. $\square$

Corollary 2.1.4. *$G \cong H_1 \Box H_2$ is diameter minimal if and only if $H_1 = H_2 = K_2$.*

Proof. If $G = C_4$, then G is diameter minimal.

Conversely suppose that G is diameter minimal. In Theorem 2.1.3 we have characterized the Cartesian product of graphs whose diameter increases by the deletion of a single edge. Hence, we need to prove the theorem only for such Gs.

Let $n_1 > 2$ and $n_2 \geqslant 2$.

Let an edge $u_iv_p - u_jv_p$ where $i, j \in \{1, 2, ..., n_1\}$ and $p \in \{1, 2, ..., n_2\}$, be deleted. There is a path of length two between u_iv_p and u_jv_p in G and the distance between any two other vertices is not affected by the removal of this edge. Thus, the diam(G) remains the same. Therefore, $n_1 = 2$.

Let $n_1 = 2$ and $n_2 > 2$.

Suppose that $d(v_w, v_z) = \text{diam}(H_2)$. Let an edge $u_1v_z - u_2v_z$, be deleted. Then $d(u_1v_z, u_2v_z) = 3 \leqslant \text{diam}(G)$ and the distance between u_1v_w, u_2v_z is diam(G). Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, the diam(G) remains the same. Hence, for a connected graph H_2 with $n_2 > 2$ vertices there exist some $e \in E(G)$ such that $\text{diam}(G - e) < \text{diam}(G)$. Therefore, $n_2 = 2$.

Hence, $H_1 = H_2 = K_2$. $\square$

Theorem 2.1.5. *Let $G \cong H_1 \Box H_2$.*

(a) If both H_1 and H_2 are complete graphs with n_1, $n_2 > 2$, then $D^1(G) = 2$.

(b) If H_1 is a complete graph and H_2 is a not complete graph, then $D^1(G) \leqslant \delta(H_2)$.

(c) If both H_1 and H_2 are not complete graphs, then
$D^1(G) \leqslant \Delta(G) - 1$.

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

(a) H_1 and H_2 are complete graphs with $n_1, n_2 > 2$.

Let the two edges $u_iv_p - u_jv_p$ and $u_iv_q - u_jv_q$ where $i \neq j \in \{1, 2, ..., n_1\}$ and $p \neq q \in \{1, 2, ..., n_2\}$, be deleted. Then $d(u_iv_p, u_jv_q) = 3$ by a path $u_iv_p - u_iv_q - u_xv_q - u_jv_q$. Hence, $D^1(G) = 2$.

(b) H_1 is a complete graph and H_2 is a not complete graph.

Let $d(v_w, v_z) = \text{diam}(H_2)$. Consider a pair of vertices u_iv_w, u_iv_z in G. Let the $\delta(H_2)$ edges $u_iv_q - u_iv_r$, where v_rs are the neighbours of v_q and $r \in \{1, 2, ..., n_2\}$, be deleted. Then, the diam(G) increases by a path $u_iv_w - u_jv_w - u_jv_{w+1} - u_jv_{w+2} ... u_jv_z - u_iv_z$ where $d(u_iv_w, u_jv_w) = 1$, $d(u_iv_z, u_jv_z) = 1$ and $d(u_jv_w, u_jv_z)$ $= \text{diam}(H_2)$. Also, the distance between any two other vertices

is not affected by the removal of these edges.
Hence, $D^1(G) \leqslant \delta(H_2)$, since $deg(v_q) = \delta(H_2)$.

(c)H_1 and H_2 are not complete graphs.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let the edges $u_yv_{z-1} - u_iv_q$ where $i \in \{1, 2, ..., n_1\}$, v_{z-1} is a neighbour of v_z in H_2 and $q \neq z \in \{1, 2, ..., n_2\}$, be deleted. Then the diam(G) increases by a path $u_xv_w - u_{x+1}v_w - \ ... \ - u_yv_w - u_yv_{w+1} - \ ... \ - u_yv_z - u_yv_{z-1}$ where $d(u_xv_w, u_xv_z) = \text{diam}(H_2)$ and $d(u_xv_z, u_yv_{z-1}) = \text{diam}(H_1) - 1$.
Hence, $D^1(G) \leqslant \Delta(G) - 1$, since $deg(u_yv_{z-1}) \leqslant \Delta(G)$. $\square$

Theorem 2.1.6. *Let $G \cong H_1 \square H_2$. Then $D^{-1}(G) = 1$ if and only if G is any one of the following graphs where,*
(a) H_1 is a complete graph and H_2 is a not complete graph with $D^{-2}(H_2) = 1$.
(b) H_1 is a not complete graph with a universal vertex or there exist a vertex in H_1 that is on at least one path between any two diametral vertices and H_2 is a not complete graph with $D^{-1}(H_2) = 1$.

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

(a) Let H_1 be a complete graph and H_2 be a not complete graph with $D^{-2}(H_2) = 1$ and the addition of an edge $v_p - v_q$ in H_2 decreases the diam(H_2) by two. Now, the addition of an edge $u_1v_p - u_1v_q$ in G decreases the diam(G).

(b) Let H_2 be a not complete graph with $D^{-1}(H_2) = 1$ and H_1 has a universal vertex u_i or there exist a vertex u_j in H_1 that is on at least one path between any two diametral vertices. Now, the addition of an edge $u_iv_p - u_iv_q$ or $u_jv_p - u_jv_q$ in G decreases the diam(G).

Conversely suppose that $D^{-1}(G) = 1$.
If both H_1 and H_2 are complete graphs, then diam(G)$= 2$ and the addition of an edge in G will not decrease the diam(G).

Suppose that H_1 is a complete graph.
Consider a pair of diametral vertices u_xv_w, u_yv_z in G and $u_xv_w - u_iv_w - u_iv_{w+1} - ... - u_iv_z - u_yv_z$ is a path between them. Let an edge $u_iv_p - u_iv_q$, be added in G. Then, $d(u_xv_w, u_iv_w) = 1$ and

$d(u_iv_z, u_yv_z) = 1$, since H_1 is a complete graph. Now, consider the distance between the remaining vertices in the diametral path, then the diam(G) decreases by one, only if $d(u_iv_w, u_iv_z)$ $=$ diam$(H_2) - 2$. Hence, to decrease the diam(G) by one, the distance between u_iv_w and u_iv_z should be decreased by two, by the addition of a single edge. Thus, H_2 is a not complete graph with $D^{-2}(H_2) = 1$.

Suppose that $D^{-1}(H_2) = 1$.
Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_iv_p - u_iv_q$, be added in G. If u_i is not a universal vertex of H_1, then a diametral path between them does not contain the edge $u_iv_p - u_iv_q$. Thus, the diam(G) remains the same. Hence, H_1 is a not complete graph with a universal vertex.

Let u_x, u_y and u_s, u_t be the pairs of diametral vertices of H_1 where u_i is a vertex in a diametral path between u_x, u_y and u_i is a vertex not in any of the diametral path between u_s, u_t in H_1. Consider the pairs of diametral vertices u_xv_w, u_yv_z and u_sv_w, u_tv_z in G. Let an edge $u_iv_p - u_iv_q$, be added in G. Then, $d(u_xv_w, u_yv_z)$ = diam(G) - 1, by a path $u_xv_w -$ $u_{x+1}v_w$ $- ... -$ $u_iv_w - u_iv_{w+1}$ $- ... -$ u_iv_z $- ... -$ u_yv_z. Also,

$d(u_s v_w, u_t v_z) = \text{diam}(G)$, since u_i is not in any of the diametral path between u_s and u_t in H_1. Thus, the diam(G) remains the same. Hence, H_1 is a not complete graph with a universal vertex or there exist a vertex in H_1 that is on at least one path between any two diametral vertices. □

Corollary 2.1.7. *There does not exist a graph $G \cong H_1 \square H_2$ such that G is diameter maximal.*

Proof. In Theorem 2.1.6 we have characterized the Cartesian product of graphs whose diameter decreases by the addition of a single edge. Hence, we need to prove the theorem only for such Gs.

Let $d(u_x, u_y) = \text{diam}(H_1)$ and $d(v_w, v_z) = \text{diam}(H_2)$. Consider a pair of diametral vertices $u_x v_w$, $u_y v_z$ in G. Let an edge $u_x v_{w+1} - u_{x+1} v_w$ where u_{x+1} is a neighbour of u_x in H_1 and v_{w+1} is a neighbour of v_w in H_2, be added in G. Then the added edge does not decrease the distance between them in G. Thus, $d(u_x v_p, u_y v_q) = \text{diam}(G)$. Hence, there exist $e \notin E(G)$ such that $\text{diam}(G + e) = \text{diam}(G)$. □

2.2 Diameter variability of the strong product of graphs

If both H_1 and H_2 are complete graphs, then $G \cong H_1 \boxtimes H_2$ is a complete graph and the deletion of any edge increases the diam(G).

Theorem 2.2.1. *Let $G \cong H_1 \boxtimes H_2$. Then $D^0(G) \geqslant 6$.*

Proof. Let $G \cong H_1 \boxtimes H_2$.
Then diam(G)= $max\{diam(H_1), diam(H_2)\}$.

We shall prove the theorem by showing that there exist at least six edges in G that can be deleted without altering the diam(G) by considering the following cases.

Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$.

Case 1: H_1 is a not complete graph and H_2 is any connected graph with $n_2 \geqslant 4$ and diam$(H_2) <$ diam(H_1).

We shall prove that $D^0(G) \geqslant n_1 m_2$.
Let $d(v_w, v_z) = L$ in H_2 by a path $v_w - v_{w+1} - v_{w+2} - ... -$

$v_{z-1} - v_z$. Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let the edges $u_iv_p - u_iv_q$ where $i \in \{1, 2, \ ... \ , n_1\}$ and $p, q \in \{1, 2, \ ... \ , n_2\}$, be deleted. There are paths $u_iv_p - u_{i+1}v_{p+1} - u_iv_{p+2} - ... - u_iv_{q-1} - u_{i+1}v_q - u_iv_q$ or $u_iv_p - u_{i+1}v_{p+1} - u_iv_{p+2} - ... - u_{i+1}v_{q-1} - u_iv_q$ of length $\operatorname{diam}(H_2)+1$ or $\operatorname{diam}(H_2)$ between u_iv_p and u_jv_p when $\operatorname{diam}(H_2)$ is odd or even respectively, where u_{i+1} is a neighbour of u_i in H_1. Also, $d(u_xv_w, u_yv_z) = \operatorname{diam}(G)$ by a path $u_xv_w - u_{x+1}v_w - u_{x+2}v_w - \ ... \ - u_iv_w - \ ... \ - u_{y-2}v_{z-2} - u_{y-1}v_{z-1} - u_yv_z$ where $d(u_xv_w, u_iv_w) = \operatorname{diam}(H_1) - L$, and $d(u_iv_w, u_yv_z) = L$. Thus, the diam(G) remains the same.

Now, we consider $n_2 = 2, 3$.

(a) $G \cong P_3 \boxtimes K_2$.

Let the bold edges in Fig 2.2 be deleted. Then it is clear that $D^0(G) = 6$.

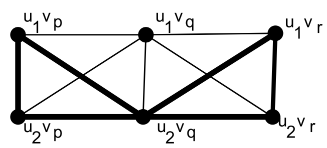

Fig 2.2: $P_3 \boxtimes K_2$.

(b) H_1 is a not complete graph with $n_1 \geqslant 4$ and $H_2 = K_2$.

Consider the three vertices u_p,u_q and u_r in H_1 which form a path P_3. Now, $P_3 \boxtimes K_2$ is a subgraph of G. Let the six bold edges as in Fig 2.2 and an edge $u_1v_s - u_2v_s$, be deleted. There is a path of length two between these pairs of vertices and the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) > 6$.

(c) $G \cong P_3 \boxtimes K_3$, $P_4 \boxtimes P_3$ and $P_4 \boxtimes K_3$.

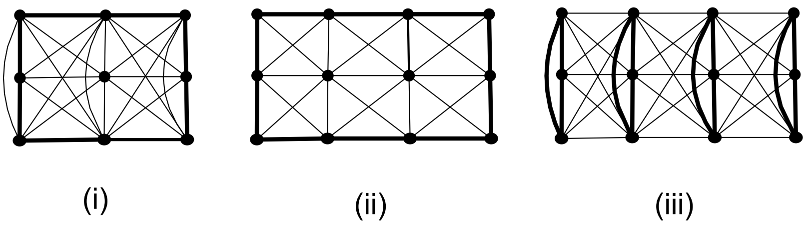

Fig 2.3: (i)$P_3 \boxtimes K_3$ (ii) $P_4 \boxtimes P_3$ (iii) $P_4 \boxtimes K_3$.

From Fig 2.3 it is clear that $D^0(G) > 6$.

(d) H_1 is a not complete graph with $n_1 \geqslant 4$ and $n_2 = 3$.

Let the edges $u_iv_1 - u_jv_1$ and $u_iv_3 - u_jv_3$ where $i, j \in \{1, \dots, n_1\}$, be deleted. There is a path of length three between these pairs of vertices. Also, $d(u_xv_1, u_yv_1) \leqslant \mathrm{diam}(H_1)$ by a path $u_xv_1 - u_{x+1}v_2 - \dots - u_{y-1}v_2 - u_yv_1$. Also, $d(u_iv_3, u_jv_3) \leqslant \mathrm{diam}(H_1)$.

Thus, $D^0(G) > 6$.

Case 2: H_1 and H_2 are connected not complete graphs with $n_1, n_2 \geqslant 4$ and $\text{diam}(H_1) = \text{diam}(H_2)$.

Let $G \cong P_4 \boxtimes P_4$. Then clearly $D^0(G) > 6$.
Consider $G \cong H_1 \boxtimes H_2$. We shall prove that $D^0(G) \geqslant m_1 + m_2$. Suppose that u_x, u_y and v_w, v_z are the pairs of diametral vertices in H_1 and H_2 respectively. Let the edges $u_1v_p - u_1v_q$, $u_iv_1 - u_jv_1$ where $p, q \in \{1, 2, \ \ldots \ , n_2\}$ and $i, j \in \{1, 2, \ \ldots \ , n_1\}$, be deleted. Then, $d(u_1v_w, u_1v_z) = \text{diam}(H_2)$ by a path $u_1v_w - u_2v_{w+1} - u_2v_{w+2} - \ \ldots \ - u_2v_{z-1} - u_1v_z$ and $d(u_xv_1, u_yv_1) = \text{diam}(H_1)$ by a path $u_xv_1 - u_{x+1}v_2 - u_{x+2}v_2 - \ \ldots u_{y-1}v_2 - u_yv_1$. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, the diam(G) remains the same and hence $D^0(G) > 6$.

Next, we consider $n_1 \geqslant 3$ and $n_2 = 3$.
(a) $G \cong P_3 \boxtimes P_3$.
Let the bold edges in Fig 2.4 be deleted. Then it is clear that $D^0(G) > 6$.

(b) $G \cong H_1 \boxtimes P_3$ where $n_1 = 4$.

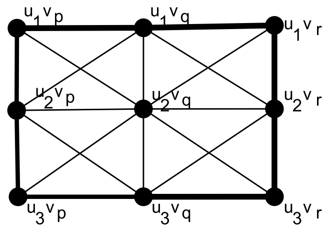

Fig 2.4: $P_3 \boxtimes P_3$.

By an exhaustive verification of all such graphs, it follows that $D^0(G) > 6$.

(c) $G \cong H_1 \boxtimes P_3$ where $n_1 \geqslant 5$.

We shall prove that $D^0(G) \geqslant 2m_1$.

Let the edges $u_pv_1 - u_qv_1$ and $u_pv_3 - u_qv_3$ where $p, q \in \{1, 2, \ldots, n_1\}$, be deleted. Then, $d(u_xv_1, u_yv_1) \leqslant \text{diam}(H_1)$ by a path $u_xv_1 - u_{x+1}v_2 - \ldots - u_{y-1}v_2 - u_yv_1$ and $d(u_xv_3, u_yv_3) \leqslant \text{diam}(H_1)$. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, the diam(G) remains the same and hence $D^0(G) > 6$. □

Corollary 2.2.2. $D^0(G) = 6$ *if and only if* $H_1 = P_3$ *and* $H_2 = K_2$.

Corollary 2.2.3. *Let* $G \cong H_1 \boxtimes H_2$ *where* H_1 *and* H_2 *are connected graphs with diam*$(H_2) <$ *diam*(H_1). *Then* $D^0(G) \geqslant n_1m_2$.

Theorem 2.2.4. *Let $G \cong H_1 \boxtimes H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,*
(a) both H_1 and H_2 are complete graphs.
(b) H_1 and H_2 are not complete graphs with diam(H_1) = diam(H_2) and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exist an edge in H_1 or H_2 that is on all diametral paths between any two vertices.

Proof. Let $G \cong K_{n_1} \boxtimes K_{n_2}$ where $n_1, n_2 \geqslant 2$. Then G is a complete graph and the deletion of any edge increases the diam(G).

Let H_1 and H_2 are not complete graphs with diam(H_1) = diam(H_2) and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exist an edge in H_1 or H_2 that is on all diametral paths between any two vertices. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. Consider a pair of diametral vertices u_xv_w, u_yv_z in G, by a path $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2}$ ... $u_{y-1}v_{z-1} - u_yv_z$. Let an edge $u_xv_w - u_{x+1}v_{w+1}$, be deleted. Then, $d(u_xv_w, u_yv_z) = \text{diam}(G)+1$

by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1} \ldots u_{y-1}v_{z-1} - u_yv_z$ where $d(u_xv_w, u_{x+1}v_{w+1}) = 2$, $d(u_{x+1}v_{w+1}, u_yv_z) = \mathrm{diam}(G) - 1$.

Conversely suppose that $D^1(G) = 1$.

Suppose that H_1 is a not complete graph and H_2 is a complete graph.

Let an edge $u_iv_p - u_iv_q$ or $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_{p+1}$, be deleted. Then $d(u_iv_p, u_iv_q) = d(u_iv_p, u_jv_p) = d(u_iv_p, u_jv_{p+1}) = 2$ by the paths $u_iv_p - u_{i+1}v_q - u_iv_q$, $u_iv_p - u_jv_{p+1} - u_jv_p$ and $u_iv_p - u_iv_{p+1} - u_jv_{p+1}$ respectively. Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, when one factor is a complete graph and the other factor is a not complete graph, a minimum of two edges should be deleted to increase the diam(G). Hence, both the factors should be complete. This proves (a).

Suppose that H_1 and H_2 are not complete graphs with $\mathrm{diam}(H_1) > \mathrm{diam}(H_2)$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G by a path $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2} \ldots u_{y-1}v_{z-1} - u_yv_z$. Let an edge $u_xv_w - u_{x+1}v_{w+1}$, be deleted. Then, $d(u_xv_w, u_yv_z) = \mathrm{diam}(H_2) + 1$

by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_{w+1} \dots u_{y-1}v_{z-1} - u_yv_z$ where $d(u_xv_w, u_{x+1}v_{w+1}) = 2$, $d(u_{x+1}v_{w+1}, u_yv_z) = \mathrm{diam}(H_2) - 1$. Hence, diam($G$) remains the same. Thus, when H_1 and H_2 are not complete graphs with different diameter, at least two edges should be deleted to increase the diam(G).

Suppose that H_1 and H_2 are not complete graphs with $\mathrm{diam}(H_1) = \mathrm{diam}(H_2)$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Since, $\mathrm{diam}(H_1) = \mathrm{diam}(H_2)$, $u_xv_w - u_{x+1}v_{w+1} - u_{x+2}v_{w+2} \dots u_{y-1}v_{z-1} - u_yv_z$ is a shortest path between them in G. Then, the deletion of an edge $u_iv_j - u_{i+1}v_{j+1}$ from this path increases the diam(G) only if either there exist only one diametral path between u_x, u_y in H_1 and v_w, v_z in H_2 or $u_i - u_{i+1}$ is an edge in H_1 that is on all diametral paths between any two vertices in H_1 and $v_j - v_{j+1}$ is an edge in H_2 that is on all diametral paths between any two vertices in H_2. Otherwise, there exist an alternative path of length diam(H_1) between u_xv_w, u_yv_z in G. Hence, H_1 and H_2 are not complete graphs with $\mathrm{diam}(H_1) = \mathrm{diam}(H_2)$ and either H_1 or H_2 have at least one pair of vertices with exactly one diametral path or there exist an edge in H_1 or H_2 that is on all diametral paths between any two vertices. This proves (b).

□

Corollary 2.2.5. *$G \cong H_1 \boxtimes H_2$ is diameter minimal if and only if both H_1 and H_2 are complete graphs.*

Theorem 2.2.6. *Let $G \cong H_1 \boxtimes H_2$.*
Then $D^1(G) \leqslant \alpha(1+\delta(H_2))$ where α is the minimum number of edge disjoint paths of length diam(H_1) between any two vertices in H_1.

Proof. Let u_x and u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$. Consider a pair of diametral vertices $u_x v_z$ and $u_y v_z$ in G. Let the edges $u_x v_z - u_q v_z$, $u_x v_z - u_q v_r$ where u_qs are the vertices adjacent to u_x in H_1 and v_rs are the vertices adjacent to v_z in H_2, be deleted. Then, $d(u_x v_z, u_y v_z) = \text{diam}(G) + 1$ by a path $u_x v_z - u_x v_{z+1} - u_{x+1} v_z - \ ... \ - u_{y-1} v_z - u_y v_z$ where $d(u_{x+1} v_z, u_y v_z) = \text{diam}(G) - 1$, $d(u_x v_z, u_{x+1} v_z) = 2$. Also, $d(u_x v_z, u_q v_z) = 2$ and $d(u_x v_z, u_q v_r) = 2$, since there are paths of length two between them.
Thus, $D^1(G) \leqslant \alpha(1 + \delta(H_2))$. □

Theorem 2.2.7. *Let $G \cong H_1 \boxtimes H_2$ be connected graph. Then $D^{-1}(G) = 1$ if and only if H_2 has a universal vertex and H_1 is a* *connected graph with diam*$(H_1) \geqslant 4$ *and* $D^{-2}(H_1) = 1$ *when an edge is added between a diametral vertex and any other vertex of* H_1 *and* $D^{-1}(H_1) = 1$ *when an edge is added between any two other vertices of* H_1.

Proof. Let $G \cong H_1 \boxtimes H_2$ and $\mathrm{diam}(G) = \mathrm{diam}(H_1)$.

Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. Suppose that v_1 is a universal vertex of H_2.

Let $D^{-1}(H_1) = 1$ where $\mathrm{diam}(H_1) \geqslant 4$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_pv_1 - u_qv_1$ where $u_p \neq u_x$, $u_q \neq u_y$, be added in G. Then, $d(u_xv_w, u_yv_z) = \mathrm{diam}(G) - 1$ by a path $u_xv_w - u_{x+1}v_1 - u_{x+2}v_1$ ... $u_{y-1}v_1 - u_yv_z$ where $d(u_xv_w, u_{x+1}v_1) = 1$, $d(u_{x+1}v_1, u_{y-1}v_1) = \mathrm{diam}(G) - 3$ and $d(u_{y-1}v_1, u_yv_z) = 1$.

Consider a pair of diametral vertices u_xv_w, u_yv_z in G. Let an edge $u_xv_1 - u_yv_1$, be added in G. Then, $d(u_xv_w, u_yv_z) = 3$ by a path $u_xv_w - u_xv_1 - u_yv_1 - u_yv_z$.

Suppose that $D^{-2}(H_1) = 1$ where $\text{diam}(H_1) \geqslant 4$.

Consider a pair of diametral vertices $u_x v_w$, $u_y v_z$ in G. Let an edge $u_x v_1 - u_i v_1$ where u_i is a vertex in a diamertal path between u_x and u_y in H_1, be added in G. Then, $d(u_x v_w, u_y v_z) = \text{diam}(G) - 1$ by a path $u_x v_w - u_x v_1 - u_i v_1 - ... - u_{y-1} v_1 - u_y v_z$ where $d(u_x v_w, u_x v_1) = 1$, $d(u_x v_1, u_{y-1} v_1) = \text{diam}(G) - 3$ and $d(u_{y-1} v_1, u_y v_z) = 1$. Thus, the distance between any two vertices in G is at most diam(G)-1.

Conversely suppose that $D^{-1}(G) = 1$. If both H_1 and H_2 are complete graphs, then G is a complete graph. If $\text{diam}(H_1) = 2$, then the addition of a single edge in G will not make G a complete graph. Also, if $\text{diam}(H_1) = 3$, then the addition of a single edge in G will not decrease the $\text{diam}(G)$, since there exist a path of length at least three between any pair of diametral vertices in G. Thus, it is clear that H_1 is a connected graph with $\text{diam}(H_1) \geqslant 4$.

Suppose that H_1 is any connected graph and H_2 is any connected graph without a universal vertex.

Let v_p and v_q be a pair of non adjacent vertices in H_2. Consider a pair of diametral vertices u_xv_q, u_yv_q in G. Let an edge $u_iv_p - u_jv_p$, be added in G. Since v_p is not adjacent to v_q, the diametral path between u_xv_q and u_yv_q does not contain the edge $u_iv_p - u_jv_p$ in G. Hence, to decrease the diam(G), H_2 should contain a universal vertex.

Suppose that H_2 has a universal vertex v_1. Consider a pair of diametral vertices u_xv_w, u_yv_w in G. Let an edge $u_iv_1 - u_jv_1$, be added in G.

Let $i \neq x$, $j \neq y$.
Consider a diametral path $u_xv_w - u_{x+1}v_1 - u_{x+2}v_1 - \ldots - u_{y-1}v_1 - u_yv_w$ between u_xv_w, u_yv_w in G. Then $d(u_xv_w, u_{x+1}v_1) = 1$ and $d(u_{y-1}v_1, u_yv_w) = 1$, since H_2 has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam(G) decreases by one only if $d(u_{x+2}v_1, u_{y-1}v_1) =$[diam($H_1$)-2]-1 = diam($H_1$)-3. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and u_yv_1 should be decreased by one, by the addition of a single edge.

Let $i = x$, $j = y$.
Then, $d(u_xv_w, u_yv_w) = 3$ by a path $u_xv_w - u_xv_1 - u_yv_1 - u_yv_w$, since H_2 has a universal vertex. From the previous case it follows

that diam(G) decreases, only if $d(u_pv_1, u_qv_1) \leqslant \text{diam}(H_1) - 1$. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and u_yv_1 should be decreased by one, by the addition of a single edge.

Now, let $i = x$, $j \neq y$.

Consider a diametral path $u_xv_w - u_xv_1 - u_{x+1}v_1 -$ ... $- u_{y-1}v_1 - u_yv_w$ between u_xv_w, u_yv_w in G. Then $d(u_xv_w, u_xv_1) = 1$ and $d(u_{y-1}v_1, u_yv_w) = 1$, since H_2 has a universal vertex. Now, consider the distance between the remaining vertices in the diametral path. Then, the diam(G) decreases by one, only if $d(u_xv_1, u_{y-1}v_1)$ =[diam(H_1)-1]-2 = diam(H_1)-3. Hence, to decrease the diam(G) by one, the distance between u_xv_1 and $u_{y-1}v_1$ should be decreased by two, by the addition of a single edge. □

Corollary 2.2.8. *There does not exist a graph $G \cong H_1 \boxtimes H_2$ such that G is diameter maximal.*

Proof. In Theorem 2.2.7 we have characterized the strong product of graphs whose diameter decreases by the addition of a single edge. Hence, we need to prove the theorem only for such Gs.

Suppose that H_2 is a not complete graph with a universal vertex and H_1 is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geqslant 4$. Let an edge $u_x v_p - u_x v_q$ be added in G, then the $\text{diam}(G)$ remains the same, since $\text{diam}(G) = \text{diam}(H_1)$.

Suppose that H_2 is a complete graph and H_1 is a connected graph with $D^{-1}(H_1) = 1$ or $D^{-2}(H_1) = 1$ with $\text{diam}(H_1) \geqslant 4$. Let the three vertices u_x, u_s and u_r form a P_3 in H_1. Consider a pair of diametral vertices $u_x v_p, u_y v_p$ in G. Let an edge $u_x v_q - u_r v_p$ where v_q is a neighbour of v_p in H_2, be added. Then the addition of an edge $u_x v_q - u_r v_p$ does not decrease the distance between them in G. Thus, $d(u_x v_p, u_y v_p) = \text{diam}(G)$. Hence, there exist some $e \notin E(G)$ such that $\text{diam}(G + e) = \text{diam}(G)$. $\square$

2.3 Diameter variability of the lexicographic product of graphs

If both H_1 and H_2 are complete graphs, then $G \cong H_1 \circ H_2$ is a complete graph and the deletion of any edge increases the diam(G).

Theorem 2.3.1. *Let $G \cong H_1 \circ H_2$. Then $D^0(G) \geqslant 3$.*

Proof. Let $G \cong H_1 \circ H_2$. Then $\text{diam}(G) = \text{diam}(H_1)$.
We prove the theorem by showing that there exist at least three edges in G that can be deleted without altering the diam(G) by considering the following cases.

Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

Case 1 : H_1 is a complete graph and H_2 is a not complete graph or a disconnected graph with $m_2 \geqslant 1$.

(a) Let $m_2 \geqslant 2$.
We shall prove that $D^0(G) \geqslant n_1 m_2$.
Suppose that $G \cong K_{n_1} \circ H_2$, then diam(G)= 2. Let the edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, \ ... \ , n_1\}$ and $p, q \in \{1, 2, \ ... \ , n_2\}$, be deleted. There are paths $u_i v_p - u_{i+1} v_p - u_i v_q$ of length two between each pair of vertices in G. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geqslant n_1 m_2 \geqslant 4$.

(b) Let $m_2 = 1$.

Suppose that $n_1 = 2$ and $n_2 = 3$.

Let the bold edges in Fig 2.6 be deleted. Then it is clear that $D^0(G) = 3$.

Suppose that $n_1 = 2$ and $n_2 \geqslant 4$.

Let the edges $u_iv_p - u_jv_q$, $u_iv_q - u_jv_q$, $u_iv_q - u_jv_r$ and $u_iv_r - u_jv_q$ where v_q is adjacent to v_p in H_2, be deleted. There are paths $u_iv_p - u_jv_p - u_jv_q$, $u_iv_q - u_jv_p - u_jv_q$, $u_iv_q - u_iv_p - u_jv_r$ and $u_iv_r - u_jv_p - u_jv_q$ of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geqslant 4$.

Suppose that $n_1 = 3$ and $n_2 = 3$.

Let the bold edges in Fig 2.5 be deleted, then it is clear that $D^0(G) > 3$.

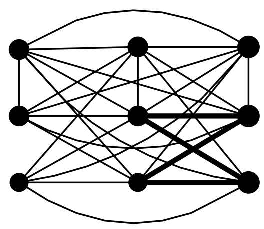

Fig 2.5: $G : D^0(G) > 3$.

Suppose that $n_1 > 3$ and $n_2 \geqslant 3$.
Let the edges $u_iv_1 - u_jv_1$ where $i, j \in \{1, 2, \ ... \ , n_1\}$, be deleted. There are paths $u_iv_1 - u_iv_2 - u_jv_1$ of length two between these pairs of vertices in G. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, $D^0(G) \geqslant 4$.

Case 2 : H_1 is a complete graph and H_2 is a totally disconnected graph.

(a) Let $n_1 = 2$.
Then G has diameter two and the deletion of any edge increases the diam(G).

(b) Let $n_1 \geqslant 3$.
Let the edges $u_iv_1 - u_jv_1$, $u_iv_1 - u_jv_2$, $u_iv_2 - u_jv_2$ and $u_iv_2 - u_jv_1$, be deleted. There are paths $u_iv_1 - u_xv_1 - u_jv_1$, $u_iv_1 - u_xv_2 - u_jv_2$, $u_iv_2 - u_xv_2 - u_jv_2$ and $u_iv_2 - u_xv_1 - u_jv_1$ of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, $D^0(G) \geqslant 4$.

Case 3 : H_1 is a not complete graph and H_2 is a not complete graph or a disconnected graph with $m_2 \geqslant 1$.

(a) Let $n_1 \geqslant 4$.

We shall prove that $D^0(G) \geqslant n_1 m_2$.

Let the edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, \ \dots\ , n_1\}$ and $p, q \in \{1, 2, \ \dots\ , n_2\}$, be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geqslant n_1 m_2 \geqslant 4$.

(b) Let $n_1 = 3$.

Let the edges $u_1 v_i - u_1 v_j$, $u_1 v_j - u_2 v_j$, $u_2 v_j - u_3 v_j$ and $u_3 v_i - u_3 v_j$ where u_2 is adjacent to u_1 and u_3 in H_1, be deleted. There are paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geqslant 4$.

Case 4 : H_1 is a not complete graph and H_2 is a totally disconnected graph.

(a) H_1 is a not complete graph with diameter two in which no two adjacent vertices of H_1 have a path of length two between them.

Then, diam$(G) = 2$ and the deletion of an edge increases the diam(G).

(b) H_1 is a not complete graph with diameter two in which there exist at least one pair of adjacent vertices with a path of length two between them.
Let the edges $u_iv_1 - u_jv_1$, $u_iv_1 - u_jv_2$, $u_iv_2 - u_jv_2$ and $u_iv_2 - u_jv_1$ where there is a path of length two between u_1 and u_2 in H_1, be deleted. There are paths $u_iv_1 - u_xv_1 - u_jv_1$, $u_iv_1 - u_xv_2 - u_jv_2$, $u_iv_2 - u_xv_2 - u_jv_2$ and $u_iv_2 - u_xv_1 - u_jv_1$ of length two between each pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.
Thus, $D^0(G) \geqslant 4$.

(c) $H_1 = P_4$ and $n_2 = 2$.
Let the bold edges in Fig 2.6 be deleted. Then it is clear that $D^0(G) = 3$.

(d) $H_1 = P_4$ and $n_2 > 2$.
Let the edges $u_iv_1 - u_jv_1$ and $u_iv_2 - u_jv_2$ where $i, j \in \{1, 2, 3, 4\}$, be deleted. There are paths of length at most three between these pair of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.

Thus, $D^0(G) > 4$.

(e) Let $\mathrm{diam}(H_1) \geqslant 3$.

We shall prove that $D^0(G) \geqslant m_1$.

Let the edges $u_iv_1 - u_jv_1$ where $i, j \in \{1, 2, \ \dots \ , n_1\}$, be deleted. There are paths of length at most diam(G) between these pairs of vertices. Also, $d(u_xv_1, u_yv_1) = \mathrm{diam}(G)$ by a path $u_xv_1 - u_{x+1}v_2 - u_{x+2}v_2 \dots u_{y-1}v_2 - u_yv_1z$ and the distance between any two other vertices is not affected by the removal of these edges. Thus, $D^0(G) \geqslant m_1 \geqslant 4$.

Hence, $D^0(G) \geqslant 3$. $\square$

Corollary 2.3.2. *$D^0(G) = 3$ if and only if G is any one of the graphs shown in Fig 2.6.*

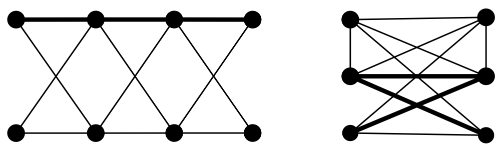

Fig 2.6: The graphs $G : \ D^0(G) = 3$.

Corollary 2.3.3. *Let $G \cong H_1 \circ H_2$ where H_1 and H_2 are connected graphs. Then $D^0(G) \geqslant n_1m_2$.*

Theorem 2.3.4. *Let $G \cong K_{n_1} \circ H_2$ where $n_1 \geqslant 3$.*
Then $D^0(G) = n_2^2 m_1 + n_1 m_2 - (2n_1 n_2 - 3)$.

Proof. Consider a spanning tree T of diameter three, of G as shown in Fig 2.7. From T, let us construct a spanning subgraph H of G having diameter two as follows.

Consider the vertices u_1v_p, u_xv_q where $x \in \{2, 3, ..., n_1\}$ and $p, q \in \{2, 3, ..., n_2\}$. Then, $d(u_1v_p, u_xv_q) = 3$. Let the edges $u_2v_1 - u_xv_p$ where $x \in \{3, 4, ..., n_1\}$ and $p \in \{1, 2, ..., n_2\}$, be added in T. Now, consider the vertices u_1v_p, u_2v_q where $p \in \{2, 3, ..., n_2\}$, then $d(u_1v_p, u_2v_q) = 3 > 2$. Let the edges $u_3v_1 - u_1v_p$ and $u_3v_1 - u_2v_p$ where $p \in \{2, 3, ..., n_2\}$, be added in T.

Let the resulting spanning subgraph of G be denoted by H. Then H has diameter two.
Hence, $D^0(G) \geqslant n_2^2 m_1 + n_1 m_2 - (2n_1 n_2 - 3)$.

Now, to prove the reverse inequality, we proceed as follows. From Corollary 2.3.3 it follows that if the $n_1 m_2$ edges $u_iv_p - u_iv_q$ where $i \in \{1, 2, ...n_1\}$ and $p, q \in \{1, 2, ...n_2\}$ are deleted, then the diam(G) remains the same. Let the edges $u_iv_p - u_jv_p$ where

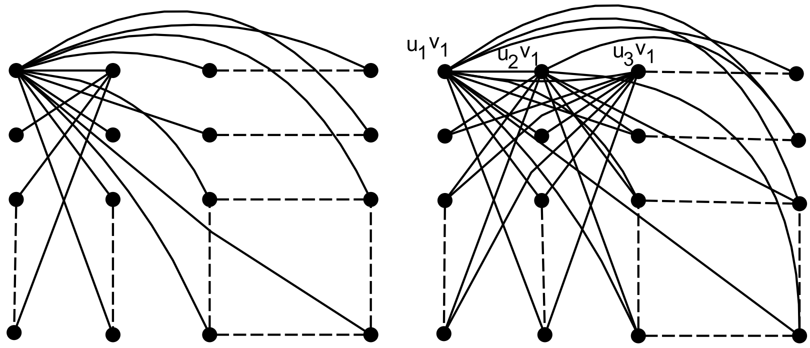

Fig 2.7: A spanning tree T and the spanning subgraph H of G.

$i, j \in \{1, 2, ...n_1\}$ and $p \in \{1, 2, 3, ...n_2\}$ except $u_1v_1 - u_rv_1$, $u_2v_1 - u_rv_1$ where $r \in \{2, 3, ...n_2\}$, be deleted. There is a path $u_iv_p - u_xv_1 - u_jv_q$ of length two between each pair of verices. Now, let the edges $u_iv_p - u_jv_q$ where $i, j \in \{1, 2, ..., n_1\}$, $p, q \in \{1, 2, ..., n_2\}$ except $u_1v_1 - u_iv_p$, $u_2v_1 - u_jv_p$, $u_3v_1 - u_1v_r$ and $u_3v_1 - u_2v_r$ where $i \in \{2, 3, ...n_2\}$, $j \in \{1, 3, ...n_2\}$, $p \in \{2, 3, ...n_2\}$ and $r \in \{2, 3, ..., n_2\}$, be deleted. There are paths $u_iv_p - u_1v_1 - u_jv_q$, $u_1v_p - u_3v_1 - u_2v_q$ of length two between each pair of verices. In both the cases the diam(G) remains the same.

Thus we have a spanning subgraph H with diameter two as shown in Fig 2.7 and the deletion of any edge from H increases the diam(H). So, $D^0(G) \leqslant n_2^2m_1 + n_1m_2 - (2n_1n_2 - 3)$.

Hence, $D^0(G) = n_2^2 m_1 + n_1 m_2 - (2n_1 n_2 - 3)$. □

Theorem 2.3.5. *Let $G \cong H_1 \circ H_2$ where H_1 and H_2 are connected graphs with diam$(H_2) <$ diam(H_1). Then*
$D^0(G) \geqslant n_2^2 m_1 - (m_1 n_2 + 2m_1 m_2)$.

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$.

Suppose that $d(v_p, v_q) = L$ in H_2 by a path $v_p - v_{p+1} - v_{p+2} - ... - v_{q-1} - v_q$. Consider a pair of diametral vertices $u_x v_p$, $u_y v_q$ in G. Let the $n_1 m_2$ edges $u_i v_p - u_i v_q$ where $i \in \{1, 2, ...n_1\}$ and $p, q \in \{1, 2, ...n_2\}$, be deleted. Then from Corollary 2.3.3 it follows that the diam(G) remains the same. Now, let the $n_2^2 m_1 - (m_1 n_2 + 2m_1 m_2)$ edges $u_i v_p - u_j v_q$ where $i, j \in \{1, 2, ...n_1\}$, $p, q \in \{1, 2, ...n_2\}$, v_ps and v_qs are not adjacent vertices in H_2, be deleted. Then, $d(u_x v_p, u_y v_q) =$ diam(G) by a path $u_x v_p - u_{x+1} v_p - u_{x+2} v_p - ... - u_i v_p - u_{i+1} v_{p+1} - ... - u_{y-2} v_{q-2} - u_{y-1} v_{q-1} - u_y v_q$ where $d(u_x v_p, u_i v_q) =$ diam$(H_1) - L$, and $d(u_i v_p, u_y v_q) = L$. Also, $d(u_i v_w, u_i v_z) =$ diam(H_2) or $d(u_i v_w, u_i v_z) =$ diam$(H_2) + 1$ when the distance between v_w,v_z is even or odd respectively. Thus the diam(G) remains the same.
Hence, $D^0(G) \geqslant n_1 m_2 + n_2^2 m_1 - (n_1 m_2 + m_1 n_2 + 2m_1 m_2) =$

$n_2^2m_1 - (m_1n_2 + 2m_1m_2)$. □

Theorem 2.3.6. *Let $G \cong H_1 \circ H_2$. Then $D^1(G) = 1$ if and only if G is any one of the following graphs where,*

(a) both H_1 and H_2 are complete graphs.

(b) $H_1 = K_2$ or a connected graph with diameter two in which there exist at least one pair of adjacent vertices with no path of length two between them and H_2 is a disconnected graph in which there exist at least one component with an isolated vertex.

Proof. (a) Let $G \cong K_{n_1} \circ K_{n_2}$ where $n_1, n_2 \geqslant 2$. Then the deletion of any edge increases the diam(G).

(b) Suppose that $H_1 = K_2$ and H_2 is a disconnected graph with an isolated vertex v_p, then diam(G)=2. Let an edge $u_iv_p - u_jv_p$, be deleted. There is a path $u_iv_p - u_jv_q - u_iv_q - u_jv_p$ of length three between them.

Let H_1 be a connected graph with diameter two in which the adjacent vertices u_r, u_s have no path of length two between them and H_2 be a disconnected graph with an isolated vertex v_p, then diam(G) $= 2$. Let an edge $u_rv_p - u_sv_p$, be deleted. There is a path $u_rv_p - u_sv_q - u_rv_q - u_sv_p$ of length three between them.

Conversely suppose that $D^1(G) = 1$.
Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$.

Suppose that H_1 is a complete graph and H_2 is any connected graph, then $\mathrm{diam}(G) \leqslant 2$.

Let an edge $u_iv_p - u_iv_q$ or $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. There exist at least two paths of length two between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Thus to increase the diam(G) by one, H_2 should be a complete graph. This proves (a).

Suppose that H_1 is a connected graph.

Let an edge $u_iv_w - u_jv_w$, be deleted. If H_2 is any connected graph, then there exist at least $\kappa(H_2) + 1$ paths $u_xv_w - u_{x+1}v_z \ ... \ u_{y-1}v_z - u_yv_w$ of length $\mathrm{diam}(H_1)$ between u_xv_w and u_yv_w in G where $z \in \{1, 2, \ ... \ , n_2\}$. Thus, when H_2 is a connected graph, at least two edges should be deleted to increase the diam(G). Hence, it is clear that H_2 should be a disconnected graph.

Now, if H_2 is a disconnected graph without an isolated vertex, then there exist at least two paths of length diam(G) between a pair of diametral vertices $u_x v_w$ and $u_y v_w$ in G. Thus, at least two edges should be deleted to increase the diam(G). Hence, H_2 is a disconnected graph in which there exist at least one component with an isolated vertex.

If diam$(H_1) \geqslant 3$, then the deletion of an edge will not increase the diam(G). There is a path of length at most three between each pair of vertices. Hence, H_1 is any connected graph with diam$(H_1) \leqslant 2$.

Let H_1 be a complete graph with $n_1 > 2$.

Since $n_1 > 2$ there exist at least two paths of length two between each pair of vertices in G. Thus, the deletion of an edge from G does not increase the diam(G). Hence, $n_1 = 2$.

Let diam$(H_1) = 2$.

Let an edge $u_i v_p - u_j v_p$, be deleted. Then the diam(G) increases only if u_i and u_j have no path of length two between them in H_1. Otherwise, at least two edges should be deleted to increase the diam(G). Also, the distance between any two other vertices is not affected by the removal of these edges. Hence, H_1 should be a connected graph with diameter two in which there exist at least one pair of adjacent vertices with no path of length two between them.

This proves (b). $\square$

Corollary 2.3.7. *$G \cong H_1 \circ H_2$ is diameter minimal if and only if G is any one of the following graphs where,*

(a) both H_1 and H_2 are complete graphs.

(b) $H_1 = K_2$ or a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1 and H_2 is a totally disconnected graph.

Proof. (a) Let $G = K_{n_1} \circ K_{n_2}$. Then G is diameter minimal.

(b) Suppose that H_1 is a K_2 and H_2 is a totally disconnected graph, then $\text{diam}(G) = 2$. Let an edge $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. Then there is a path $u_iv_p - u_jv_q - u_iv_q - u_jv_p$ or $u_iv_p - u_jv_p - u_iv_q - u_jv_q$ of length three between each pair of vertices. Thus, the deletion of any edge increases the diam(G).

Suppose that H_1 is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1 and H_2 is a totally disconnected graph, then diam(G) $= 2$. Let an edge $u_iv_p - u_jv_p$ or $u_iv_p - u_jv_q$, be deleted. There is a path of length three between these pairs of vertices. Thus, the deletion of any edge increases the diam(G).

Hence, G is diameter minimal.

Conversely suppose that G is diameter minimal. In Theorem 2.3.6 we have characterized the lexicographic product of graphs whose diameter increases by the deletion of a single edge. Hence, we need to prove the theorem only for such Gs.

Let $G \cong K_{n_1} \circ K_{n_2}$. Then, clearly G is diameter minimal.

Suppose that $H_1 = K_2$ and H_2 is a disconnected graph in which there exist at least one component with an isolated vertex.

Let an edge $u_iv_p - u_iv_q$ where v_p, v_q are not isolated vertices in H_2, be deleted. Since v_p, v_q are not isolated vertices there is a path of length two between u_iv_p and u_iv_q in G. Hence, if H_2 contains any pair of adjacent vertices, the deletion of that edge will not increase the diam(G). Thus, H_2 is a totally disconnected graph.

Suppose that H_1 is a connected graph with diameter two in which at least one pair of adjacent vertices have no path of length two between them and H_2 is a disconnected graph in which there exist at least one component with an isolated vertex.

As in the previous case, if H_2 contains any pair of adjacent vertices, the deletion of that edge will not increase the diam(G). Hence, H_2 is a totally disconnected graph.

Let an edge $u_i v_p - u_j v_p$ where the adjacent vertices u_i and u_j have a path of length two in H_1, be deleted. If any two adjacent vertices in H_1 have a path of length two between them, then the deletion of an edge will not increase the diam(G). Thus, H_1 is a connected graph with diameter two in which there is no path of length two between any two adjacent vertices in H_1. □

Theorem 2.3.8. *Let $G \cong H_1 \circ H_2$.*
Then $D^1(G) \leqslant \alpha\ n_2$ where α is the minimum number of edge disjoint paths of length diam(H_1) between any two vertices in H_1.

Proof. Follows from Theorem 2.2.6. □

Theorem 2.3.9. *Let $G \cong H_1 \circ H_2$. Then $D^{-1}(G) = 1$ if and only if G is any one of the following graphs where,*

(a) H_2 has a universal vertex and H_1 is a connected graph with diam$(H_1) \geqslant 4$ and $D^{-2}(H_1) = 1$ when an edge is added between a diametral vertex and any other vertex of H_1.

(b) H_2 is any graph and H_1 is a connected graph with diam$(H_1) \geqslant 4$ and $D^{-1}(H_1) = 1$ when an edge is added between the diametral vertices or between any two other vertices of H_1.

Proof. Follows from Theorem 2.2.7. $\square$

Corollary 2.3.10. *There does not exist a graph $G \cong H_1 \circ H_2$ such that G is diameter maximal.*

Chapter 3

Diameter vulnerability of the product graphs

In the design of an interconnection network, another fundamental consideration is the reliability of the network, which is characterized by the connectivity of the network. If some processors or links are faulty the efficiency of the network may be affected. Vulnerability is a measure of the ability of the system to withstand vertex or edge faults and maximum routing delay. Diameter can be used to evaluate the maximum delay in routing.

Some results of this chapter are included in the following papers.
1.Chithra M.R., A. Vijayakumar, Diameter vulnerability of the Cartesian product of graphs (communicated).
2.Chithra M.R., Manju K. Menon, A. Vijayakumar, Some distance notions in lexicographic product of graphs (communicated).

These problems deal with how the remaining processors can still communicate with a reasonable efficiency [66].

3.1 Diameter vulnerability of the product graphs

Theorem 3.1.1. *Let $G \cong H_1 \square H_2$, where H_1 is a complete graph and H_2 is a connected graph with $\kappa'(H_2) = \delta(H_2)$. Then $f'(G) = diam(G) + 1$.*

Proof. **Case 1** : $G \cong K_{n_1} \square K_{n_2}$.

Then $\kappa'(G) = n_1 + n_2 - 2$ and diam(G) = 2. We shall prove the theorem by considering the following sub cases where the fault occurs on $F \subseteq E(G)$.

(a) Let F be the set of edges of the form $u_i v_p - u_x v_p$ where $i, x \in \{1, 2, \ \dots \ , n_1\}$.

Let the $\kappa'(G) - 1$ edges be deleted from F. There is a path $u_i v_p - u_i v_q - u_j v_q - u_j v_p$ of length three between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges. Hence, diam(G)= 3.

(b) Let F be the set of edges of the form $u_iv_p - u_iv_q$ where $p, q \in \{1, 2, \ \ldots\ , n_2\}$.

Let the $\kappa'(G)-1$ edges be deleted from F. There is a path of length three between these pairs of vertices. Also, the distance between any two other vertices is not affected by the removal of these edges.

(c) Let F be any arbitrary collection of edges.

Consider a pair of non adjacent vertices u_iv_p and u_jv_q in G. Let the $n_1 + n_2 - 3$ edges adjacent to the vertex u_iv_p except u_xv_p, be deleted. Then, $d(u_iv_p, u_jv_q) = 3$ by a path $u_iv_p - u_xv_p - u_xv_q - u_jv_q$ and $d(u_iv_p, u_iv_r) = d(u_iv_p, u_yv_p) = 2$. Also, the distance between any two other vertices is not affected by the removal of these edges.

Hence, $f'(G) = 3$.

Case 2: $G \cong K_{n_1} \square H_2$, where H_2 is a not complete graph.

Then $\kappa'(G) = n_1 - 1 + \kappa'(H_2)$ and diam(G)=1+ diam(H_2). Let v_w, v_z be a pair of diametral vertices in H_2, by a path

$v_w - v_{w+1} - v_{w+2} - \ldots - v_{z-1} - v_z$. We shall prove the theorem by considering the following sub cases.

(a) Let F be the set of edges of the form $u_iv_p - u_xv_p$ where $i, x \in \{1, 2, \ldots, n_1\}$.

Let the $\kappa'(G) - 1$ edges be deleted from F. There is a path of length three between these pairs of vertices. Also, the distance between any two other vertices in G is not affected by the removal of these edges.

(b) Let F be the set of edges of the form $u_iv_p - u_iv_q$ where $p, q \in \{1, 2, \ldots, n_2\}$.

Consider a pair of vertices u_iv_w and u_iv_z in G. Let the $\kappa'(G)-1$ edges be deleted from F. Then, $d(u_iv_w, u_iv_z) = \text{diam(G)}+1$ by a path $u_iv_w - u_jv_w - u_jv_{w+1} - \ldots - u_jv_{z-1} - u_jv_z - u_iv_z$ where $d(u_iv_w, u_jv_w) = d(u_iv_z, u_jv_z) = 1$, $d(u_jv_w, u_jv_z) = \text{diam}(H_2)$. Also, $d(u_iv_w, u_iv_p) = 3$ by a path $u_iv_w - u_jv_w - u_jv_p - u_iv_p$. The distance between any two other vertices is not affected by the removal of these edges.

(c) Let F be any arbitrary collection of edges.

Consider a pair of vertices u_iv_w and u_iv_z in G. Let the $\kappa'(G)-1$ edges adjacent to the vertex u_iv_z except u_jv_z, be deleted. Then, $d(u_iv_w, u_iv_z)$ = diam(G)+1 by a path $u_iv_w - u_jv_w - u_jv_{w+1} - \ ... \ - u_jv_{z-1} - u_jv_z - u_iv_z$ where $d(u_iv_w, u_jv_w) = 1$, $d(u_jv_w, u_jv_z) = \text{diam}(H_2)$ and $d(u_iv_z, u_jv_z) = 1$. Also, $d(u_iv_z, u_iv_p) = 3$ and $d(u_iv_w, u_xv_w) = 3$.

Hence, $f'(G) = diam(G) + 1$. $\square$

Theorem 3.1.2. *Let $G \cong H_1 \square H_2$ be a connected graph. Then $f'(G) \leqslant max\{f'(H_1) + 2diam(H_2), f'(H_2) + 2diam(H_1)\}$.*

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - \ ... \ - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. We shall prove the theorem by considering the following cases.

(a) Let F be the set of edges of the form $u_iv_p - u_iv_q$ where $p, q \in \{1, 2, \ ... \ , n_2\}$.

Consider a pair of vertices u_iv_w, u_iv_z in G. Let the $\kappa'(G)-1$ edges be deleted from F. Then, $d(u_iv_w, u_iv_z) = \text{diam}(H_2) + \ 2$

by a path $u_iv_w - u_jv_w - u_jv_{w+1} - \ \dots \ - u_jv_z - u_iv_z$ where $d(u_iv_w, u_jv_w) = 1$, $d(u_jv_w, u_jv_z) = \text{diam}(H_2)$, $d(u_jv_z, u_iv_z) = 1$. Also, the distance between any two other vertices is not affected by the removal of these edges.

(b) Let F be the set of edges of the form $u_iv_p - u_jv_p$ where $i, j \in \{1, 2, \ \dots \ , n_1\}$.

Consider a pair of vertices u_xv_p, u_yv_p in G. Let the $\kappa'(G)-1$ edges be deleted from F. Then, $d(u_xv_p, u_yv_p) = \text{diam}(H_1) + 2$. Also, the distance between any two other vertices is not affected by the removal of these edges.

(c) Let F be any arbitrary collection of edges.

Consider a pair of diametral vertices u_xv_w and u_yv_z in G. Let the $\kappa'(G) - 1$ edges adjacent to the vertex $u_{y-1}v_z$ except $u_{y-1}v_y$, be deleted. Then, $d(u_xv_w, u_yv_z) =$ diam(G)+1 by a path $u_xv_w - u_{x+1}v_w - \ \dots \ - u_yv_w - \ \dots \ - u_yv_z - u_{y-1}v_z$ where $d(u_xv_w, u_yv_w) = \text{diam}(H_1)$, $d(u_yv_w, u_yv_z) = \text{diam}(H_2)$ and $d(u_yv_z, u_{y-1}v_z) = 1$. Also, $d(u_{y-1}v_z, u_pv_z) = d(u_{y-1}v_z, u_{y-1}v_q) = 3$. Thus, the deletion of $\kappa'(G) - 1$ edges increases the diam(G) by one.

Now, consider a pair of vertices $u_a v_w$ and $u_b v_w$ in G. Let the edges $u_i v_p - u_j v_p$ where $\{u_i - u_j\}$ is a collection of $\kappa'(H_1)$ edges which form an edge cut of H_1 and $p \in \{1, 2, \ \dots\ , n_2 - 1\}$, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only the $\kappa'(H_1) - 1$ edges, otherwise G becomes disconnected. Then, $d(u_a v_{n_2}, u_b v_{n_2}) \leqslant f'(H_1)$ by a path $u_a v_{n_2} - u_{a+1} v_{n_2} - \ \dots\ - u_{b-1} v_{n_2} - u_b v_{n_2}$, since the deletion of $\kappa(H_1) - 1$ edges from H_1 increases the diam(H_1) to at most $f'(H_1)$. Now,
$d(u_a v_w, u_b v_w) \leqslant f'(H_1) + 2\ \mathrm{diam}(H_2)$ by a path $u_a v_w - u_a v_{w+1} - \dots - u_a v_{n_2} - u_{a+1} v_{n_2} - \ \dots\ - u_b v_{n_2} - u_b v_r - \ \dots\ - u_b v_w$ where $d(u_a v_w, u_a v_{n_2}) \leqslant \mathrm{diam}(H_2)$, $d(u_a v_{n_2}, u_b v_{n_2}) \leqslant f'(H_1)$ and $d(u_b v_{n_2}, u_b v_w) \leqslant \mathrm{diam}(H_2)$. Thus, the deletion of $\kappa'(G) - 1$ edges increases the diam(G) by $f'(H_1) + 2\ \mathrm{diam}(H_2)$.

Similarly, if the $\kappa'(G) - 1$ edges $u_p v_a - u_p v_b$ where $\{v_a - v_b\}$ is a collection of $\kappa'(H_2)$ edges which form an edge cut of H_2 and $p \in \{1, 2, \ \dots\ , n_1\}$, are deleted then, the diam(G) increases by $f'(H_2) + 2\ \mathrm{diam}(H_1)$.

Hence, $f'(G) \leqslant max\{f'(H_1) + 2diam(H_2), f'(H_2) + 2diam(H_1)\}$.

$\square$

Illustration of Theorem 3.1.2

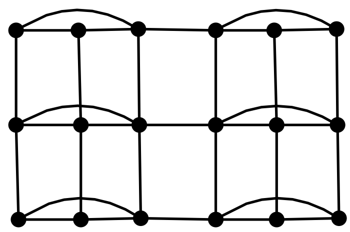

Fig 3.1: A graph G with $f'(G) = f'(H_1) + 2\ \mathrm{diam}(H_2) = 7$.

Theorem 3.1.3. *Let $G \cong H_1 \boxtimes H_2$ be a connected graph. Then $f'(G) \leqslant max\{f'(H_1) + diam(H_2), f'(H_2) + diam(H_1)\}$.*

Proof. Let $G \cong H_1 \boxtimes H_2$ be a connected graph. Then $\kappa'(G) = min\{\kappa'(H_1)(|V(H_2)|+2E(H_2)), \kappa'(H_2)(|V(H_1)|+2E(H_1)), \delta(H_1 \boxtimes H_2)\}$. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. We shall prove the theorem by considering the following cases.

(a) Let F be the set of edges of the form $u_i v_k - u_j v_k$ where $i, j \in \{1, 2, \ ... \ , n_1\}$.

Consider a pair of vertices u_xv_k and u_yv_k in G. Let the $\kappa'(G)-1$ edges be deleted from F. Then, $d(u_xv_k, u_yv_k) = \mathrm{diam}(H_1)$ by a path $u_xv_k - u_{x+1}v_{k+1} - u_{x+2}v_{k+1} - \ ... \ - u_{y-1}v_{k+1} - u_yv_k$.

(b) Let F be the set of edges of the form $u_iv_j - u_iv_k$ where $j, k \in \{1, 2, \ ... \ , n_2\}$.

Consider a pair of vertices u_iv_w and u_iv_z in G. Let the $\kappa'(G)-1$ edges be deleted from F. Then, $d(u_iv_w, u_iv_z) = \mathrm{diam}(H_2)$ by a path $u_iv_w - u_{i+1}v_{w+1} - u_{i+1}v_{w+2} - \ ...u_{i+1}v_{z-1} - u_iv_z$.

(c) Let F be any arbitrary collection of edges.

Consider a pair of diametral vertices u_xv_w and u_yv_w in G. Let the $\kappa'(G) - 1$ edges adjacent to the vertex u_xv_w except u_xv_{w+1}, be deleted. The distance between u_xv_w and u_yv_w increases by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_w - \ ... \ - u_yv_w$ where $d(u_xv_w, u_{x+1}v_w) = 2$ and $d(u_{x+1}v_w, u_yv_w) = \mathrm{diam}(H_1) - 1$. Also, the distance between any two other vertices is not affected by the removal of these edges.

Now, consider a pair of vertices u_av_w and u_bv_w in G. Let the edges $u_iv_r - u_jv_r$, $u_iv_p - u_jv_q$ where $\{u_i - u_j\}$ is a collection of $\kappa'(H_1)$ edges which form an edge cut of H_1 and $r \in \{1, 2, \ ... \ , n_2 - 1\}$,

$q \neq p \in \{1, 2, \ \dots \ , n_2\}$ and v_qs are the vertices adjacent to v_p in H_2, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only the $\kappa'(H_1) - 1$ edges, otherwise G becomes disconnected. Then, $d(u_a v_{n_2}, u_b v_{n_2}) \leqslant f'(H_1)$ by a path $u_a v_{n_2} - u_{a+1} v_{n_2} - \ \dots \ - u_{b-1} v_{n_2} - u_b v_{n_2}$, since the deletion of $\kappa'(H_1) - 1$ edges from H_1 increases the diam(H_1) to at most $f'(H_1)$. Now, $d(u_a v_w, u_b v_w) \leqslant f'(H_1) + \text{diam}(H_2)$ by a path $u_a v_w - u_a v_{w+1} - \dots - u_a v_{n_2} - u_{a+1} v_{n_2} - \ \dots \ - u_{b-1} v_{w+1} - u_b v_w$ where $d(u_a v_w, u_a v_{n_2}) \leqslant \text{diam}(H_2)$ and $d(u_a v_{n_2}, u_b v_w) \leqslant f'(H_1)$. Thus, the distance between any two other vertices is at most $f'(H_1) + \text{diam}(H_2)$.

Similarly, if the $\kappa'(G) - 1$ edges $u_x v_p - u_x v_q$, $u_x v_p - u_i v_r$ where $x \in \{1, 2, \ \dots \ , n_1\}$, u_is are the vertices adjacent to u_x in H_1 and $\{v_p - v_q\}$ is a collection of $\kappa'(H_2)$ edges which form an edge cut of H_2, are deleted then, the diam(G) increases by $f'(H_2) + \text{diam}(H_1)$.

Hence, $f'(G) \leqslant max\{f'(H_1) + diam(H_2), f'(H_2) + diam(H_1)\}$.

$\square$

Illustration of Theorem 3.1.3

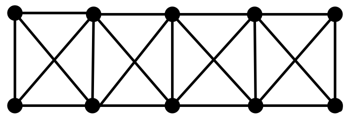

Fig 3.2: A graph G with $f'(G) = max\{f'(H_1) + diam(H_2),$
$f'(H_2) + diam(H_1)\} = 5$.

Theorem 3.1.4. *Let $G \cong H_1 \circ H_2$ be a connected graph with $n_1, n_2 \geqslant 3$. Then $f'(G) \leqslant f'(H_1) + diam(H_2)$.*

Proof. Let $G \cong H_1 \circ H_2$ be a connected graph. Then the $\kappa'(G) = min\{\kappa'(H_1)n_2^2, \delta(H_2)+\delta(H_1)n_2\}$ and diam(G) = diam(H_1). Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. We shall prove the theorem by considering the following cases.

(a) Let F be the set of edges of the form $u_iv_k - u_jv_k$ where $i, j \in \{1, 2, \ ... \ , n_1\}$.

Consider a pair of vertices u_xv_k and u_yv_k in G. Let the

$\kappa'(G)-1$ edges be deleted from F. Then, $d(u_xv_k, u_yv_k) = \text{diam}(G)$ by a path $u_xv_k - u_{x+1}v_{k+1} - u_{x+2}v_{k+1} - ... - u_{y-1}v_{k+1} - u_yv_k$.

(b) Let F be the set of edges of the form $u_iv_j - u_iv_k$ where $j, k \in \{1, 2, \ ... \ , n_2\}$.

Let the $\kappa'(G) - 1$ edges be deleted from F. There is a path $u_iv_j - u_{i+1}v_j - u_iv_k$ of length two between u_iv_j and u_iv_k in G. Thus, the diam(G) remains the same.

(c) Let F be any arbitrary collection of edges.

Consider a pair of diametral vertices u_xv_w and u_yv_w in G. Let the $\kappa'(G)-1$ edges adjacent to the vertex u_xv_w except u_xv_{w+1}, be deleted. Then, $d(u_xv_w, u_yv_w) = \text{diam}(G) + 1$ by a path $u_xv_w - u_xv_{w+1} - u_{x+1}v_w - ... - u_yv_w$ where $d(u_xv_w, u_{x+1}v_w) = 2$ and $d(u_{x+1}v_w, u_yv_w) = \text{diam}(G) - 1$.

Consider a pair of diametral vertices u_xv_w and u_yv_z in G. Since we have already considered the case of the deletion of edges from F of the form $u_iv_k - u_jv_k$ where $i, j \in \{1, 2, \ ... \ , n_1\}$ and $u_iv_j - u_iv_k$ where $j, k \in \{1, 2, \ ... \ , n_2\}$ in (a) and (b) respectively, there will exist at least one edge (say) $u_pv_r - u_qv_r$ for each $r \in \{1, 2, \ ... \ , n_2\}$. Thus, there exist a path $u_xv_w - u_{x+1}v_p -$

$u_{x+2}v_q - \ldots - u_y v_z$ of length diam(G) between $u_x v_w$, $u_y v_z$ in G.

Consider a pair of vertices $u_a v_w, u_b v_w$ in G. Let the edges $u_i v_r - u_j v_r$, $u_i v_p - u_j v_q$ where $\{u_i - u_j\}$ is a collection of $\kappa'(H_1)$ edges which form an edge cut of H_1 and $r \in \{1, 2, \ldots, n_2 - 1\}$, $q \neq p \in \{1, 2, \ldots, n_2\}$, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only the $\kappa'(H_1) - 1$ edges, otherwise G becomes disconnected. Then, $d(u_a v_{n_2}, u_b v_{n_2}) \leqslant f'(H_1)$ by a path $u_a v_{n_2} - u_{a+1} v_{n_2} - \ldots - u_{b-1} v_{n_2} - u_b v_{n_2}$, since the deletion of $\kappa(H_1) - 1$ edges from H_1 increases the diam(H_1) to at most $f'(H_1)$. Now, $d(u_a v_w, u_b v_w) \leqslant f'(H_1) + \mathrm{diam}(H_2)$ by a path $u_a v_w - u_a v_{w+1} - \ldots - u_a v_{n_2} - u_{a+1} v_{n_2} - \ldots - u_{b-1} v_{n_2} - u_b v_w$ where $d(u_a v_w, u_a v_{n_2}) \leqslant \mathrm{diam}(H_2)$ and $d(u_a v_{n_2}, u_b v_w) \leqslant f'(H_1)$. Hence the result. □

Illustration of Theorem 3.1.4

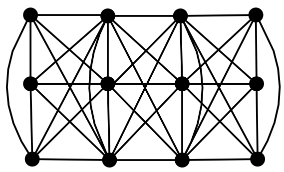

Fig 3.3: A graph G with $f'(G) = f'(H_1) + \mathrm{diam}(H_2) = 4$.

We shall now discuss some results on the notion of fault diameter in the product graphs.

Theorem 3.1.5. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then $f(G) \leqslant max\{f(H_1) + 2diam(H_2), f(H_2) + 2diam(H_1)\}$.*

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. We shall prove the theorem by considering the following cases where the fault occurs on $S \subseteq V(G)$.

(a) Let S be the set of vertices of the form $u_i v_p$ where $p \in \{1, 2, \ ... \ , n_2\}$.

Consider a pair of vertices $u_i v_w$, $u_i v_z$ in G. Let the $\kappa(G) - 1$ vertices from S be deleted. Then, $d(u_i v_w, u_i v_z) = \text{diam}(H_2) + 2$ by a path $u_i v_w - u_j v_w - u_j v_{w+1} - \ ... \ - u_j v_z - u_i v_z$ where $d(u_i v_w, u_j v_w) = 1$, $d(u_j v_w, u_j v_z) = \text{diam}(H_2)$, $d(u_j v_z, u_i v_z) = 1$. Now, $d(u_y v_w, u_z v_w) \leqslant f(H_1)$, since the deletion of the vertex u_i from H_1 increases the diam(H_1) to at most $f(H_1)$. Hence, $f(G) \leqslant max\{diam(H_2) + 2, f(H_1)\}$.

(b) Let S be the set of vertices of the form $u_j v_p$ where

$j \in \{1, 2, \ ... \ , n_1\}$.

Consider a pair of vertices $u_x v_p$, $u_y v_p$ in G. Let the $\kappa(G) - 1$ vertices from S be deleted. Then, $d(u_x v_p, u_y v_p) \leqslant \text{diam}(H_1) + 2$. Now, $d(u_y v_w, u_y v_z) \leqslant f(H_2)$, since the deletion of the vertex v_p from H_2 increases the $\text{diam}(H_2)$ to at most $f(H_2)$. Hence, $f(G) \leqslant max\{diam(H_1) + 2, f(H_2)\}$.

(c) Let S be any arbitrary collection of vertices.

Consider a pair of diametral vertices $u_x v_w$ and $u_y v_z$ in G. Let the $\kappa(G) - 1$ vertices adjacent to the vertex $u_{y-1} v_z$ except $u_y v_z$ in G, be deleted. Then, $d(u_x v_w, u_y v_z) = \text{diam}(G) + 1$ by a path $u_x v_w - u_{x+1} v_w - \ ... \ - u_y v_w - \ ... \ - u_y v_z - u_{y-1} v_z$ where $d(u_x v_w, u_y v_w) = \text{diam}(H_1)$, $d(u_y v_w, u_y v_z) = \text{diam}(H_2)$ and $d(u_y v_z, u_{y-1} v_z) = 1$. Also $d(u_{y-1} v_z, u_p v_z) = d(u_{y-1} v_z, u_{y-1} v_q) = 3$. Thus, the deletion of $\kappa(G) - 1$ vertices increases the diam(G) by one.

Now, consider a pair of vertices $u_p v_w$ and $u_q v_w$ in G. Let the vertices $u_i v_p$ where $\{u_i\}$ is a collection of $\kappa(H_1)$ vertices which form a vertex cut of H_1 and $p \in \{1, 2, 3, ..., n_2 - 1\}$, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only

$\kappa(H_1) - 1$ vertices, otherwise G becomes disconnected. Then, $d(u_p v_{n_2}, u_q v_{n_2}) \leqslant f(H_1)$ by a path $u_p v_{n_2} - u_{p+1} v_{n_2} - \ ... \ - u_{q-1} v_{n_2} - u_q v_{n_2}$, since the deletion of $\kappa(H_1) -\ 1$ vertices from H_1 increases the diam(H_1) to at most $f(H_1)$. Now
$d(u_p v_w, u_q v_w) \leqslant f(H_1) + 2$ diam(H_2) by a path $u_p v_w - u_p v_{w+1} - \ ... \ - u_p v_{n_2} - u_{p+1} v_{n_2} - \ ... \ - u_q v_{n_2} - u_q v_r - \ ... \ - u_y v_w$ where
$d(u_p v_w, u_p v_{n_2}) \leqslant$ diam(H_2), $d(u_p v_{n_2}, u_q v_{n_2}) \leqslant f(H_1)$ and
$d(u_q v_{n_2}, u_q v_w) \leqslant$ diam(H_2).

Similarly, if the $\kappa(G) - 1$ vertices $u_i v_p$ where $i \in \{1, 2, ..., n_1\}$ and $\{v_p\}$ is a collection of $\kappa(H_2)$ vertices which form a vertex cut of H_2, are deleted from G, then the diam(G) increases by $f(H_2)$ + 2 diam(H_1). Hence the result. □

Illustration of Theorem 3.1.5

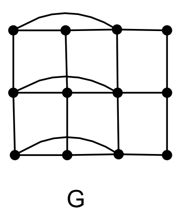

Fig 3.4: A graph G with $f(G) = max\{f(H_1) + 2diam(H_2),$
$f(H_2) + 2diam(H_1)\} = 6$

Theorem 3.1.6. *Let $G \cong H_1 \circ H_2$ be a connected graph. Then $f(G) \leqslant max\{f(H_1), f(H_2)\}$.*

Proof. Let u_x, u_y be a pair of diametral vertices in H_1, by a path $u_x - u_{x+1} - u_{x+2} - ... - u_{y-1} - u_y$ and v_w, v_z be a pair of diametral vertices in H_2, by a path $v_w - v_{w+1} - v_{w+2} - ... - v_{z-1} - v_z$. We shall prove the theorem by considering the following cases.

Case 1: $G \cong K_{n_1} \circ H_2$.

Then diam$(G) = 2$ and $\kappa(G) = (n_1 - 1)n_2 + \kappa(H_2)$.

Consider a vertex u_1v_1 in G. Let the $\kappa(G) - 1$ vertices adjacent to the vertex u_1v_1 except u_iv_r, be deleted. Now, let G' be the subgraph of G obtained after deleting the $\kappa(G) - 1$ vertices, as shown in Fig 3.5 and if v_p is not adjacent to v_q in H_2, then $d(u_1v_p, u_1v_q) = 2$ by the path $u_1v_p - u_iv_r - u_1v_q$, since $u_i \in K_{n_1}$. Thus, the diam(G) remains the same.

Now, let the $\kappa(G) - 1$ vertices adjacent to u_1v_1 except u_1v_s, be deleted. Let G' be the subgraph of G obtained after deleting $\kappa(G) - 1$ vertices. Then, $d(u_1v_p, u_1v_q) \leqslant f(H_2)$, since the dele-

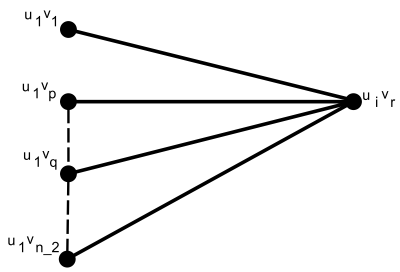

Fig 3.5: A subgraph G' of G.

tion of $\kappa(H_2) - 1$ vertices from H_2 increases the diam(H_2) to at most $f(H_2)$. Thus $f(G) \leqslant f(H_2)$.

Case 2: $G \cong H_1 \circ H_2$ where $\kappa(H_1) = 1$ and $H_1 \neq K_2$.

Then diam(G)=diam(H_1) and $\kappa(G) = \kappa(H_1)\,|V(H_2)| = n_2$. We shall prove the theorem by considering the following sub cases.

(a) Let S be the set of vertices of the form $u_i v_p$ where $p \in \{1, 2, ..., n_2\}$.

Consider a pair of diametral vertices $u_x v_a$, $u_y v_a$ in G. Let the $n_2 - 1$ vertices except $u_i v_{n_2}$ from S, be deleted. Then, $d(u_x v_a, u_y v_a) = \text{diam}(G)$ by a path $u_x v_a - u_{x+1} v_a - ... - u_{i-1} v_a -$

$u_iv_{n_2} - u_{i+1}v_a - ... - u_{y-1}v_a - u_yv_a$. Thus, the diam(G) remains the same.

(b) Let S be the set of vertices of the form u_iv_p where
$i \in \{1, 2, ..., n_1\}$.

Let the $n_2 - 1$ vertices from S, be deleted. Clearly, the distance between any two vertices in G is not affected by the removal of these vertices. Thus, the diam(G) remains the same.

(c) Let S be any arbitrary collection of vertices.

Consider a pair of diametral vertices u_xv_p and u_yv_q in G. Then, $d(u_xv_p, u_yv_q) = \text{diam}(H_1)$ by a path $u_xv_p - u_{x+1}v_a - u_{x+2}v_b - ... - u_yv_q$, since we have already considered the case of the deletion of vertices from S of the form u_iv_p where
$i \in \{1, 2, ..., n_1\}$, there exist at least one vertex (say) u_iv_j for each $j \in \{1, 2, \ ... \ , n_2\}$ and are adjacent to the vertices u_rv_p where $p \in \{1, 2, ..., n_2\}$. Thus, the diam(G) remains the same.

Case 3: $G \cong H_1 \circ H_2$ where $\kappa(H_1) > 1$.

We have $\kappa(G) \geqslant 2n_2$. We shall prove the theorem by considering the following sub cases.

(a) Let S be the set of vertices of the form u_iv_p where $i \in \{1, 2, ..., n_1\}$.

Let $\kappa(G)-1$ vertices from S be deleted. Clearly the distance between any two vertices in G is not affected by the removal of these vertices.

(b) Let S be any arbitrary collection of vertices.

Consider a pair of diametral vertices u_xv_q and u_yv_r in G, then $d(u_xv_q, u_yv_r) = \text{diam}(H_1)$ by a path $u_xv_q - u_{x+1}v_a - u_{x+2}v_b - ... - u_yv_r$, since we have already considered the case of the deletion of the vertices from S of the form u_xv_p where $x \in \{1, 2, ..., n_1\}$, there exist at least one vertex (say) u_iv_j for each $j \in \{1, 2, \ ... \ , n_2\}$ and are adjacent to the vertices u_rv_p where $p \in \{1, 2, 3, ..., n_2\}$. Thus, the diam(G) remains the same.

Now, consider a pair of vertices u_pv_w and u_qv_w in G. Let the vertices u_iv_p where $\{u_i\}$ is a collection of $\kappa(H_1)$ vertices which form a vertex cut of H_1 and $p \in \{1, 2, ..., n_2 - 1\}$, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only the $\kappa(H_1) - 1$ vertices, otherwise G becomes disconnected. Then, $d(u_pv_{n_2}, u_qv_{n_2}) \leqslant f(H_1)$ by a path $u_pv_{n_2} - u_{p+1}v_{n_2} - \ ... \ -$

$u_{q-1}v_{n_2} - u_qv_{n_2}$, since the deletion of $\kappa(H_1) - 1$ vertices from H_1 increases the diam(H_1) to at most $f(H_1)$. Now,
$d(u_pv_w, u_qv_w) \leqslant f(H_1)$ by a path $u_pv_w - u_{p+1}v_{n_2} - ... - u_{q-1}v_{n_2} - u_qv_w$. Thus, $f(G) \leqslant f(H_1)$.

From the above cases, the result follows. □

Illustration of Theorem 3.1.6

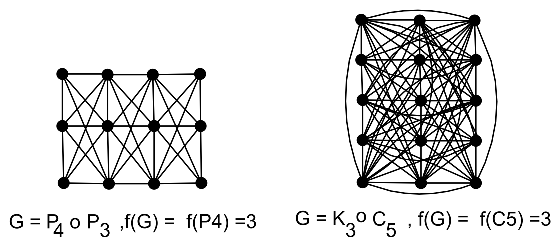

Fig 3.6: Graphs with $f(G) = f(H_1)$ and $f(G) = f(H_2)$.

Theorem 3.1.7. *Let* $G \cong H_1 \boxtimes H_2$ *be a connected graph. Then* $f(G) \leqslant max\{f(H_1) + diam(H_2), f(H_2) + diam(H_1)\}$.

Proof. **Case 1**: $H_1 \boxtimes K_{n_2}$.

Then diam(G) = diam(H_1) and $\kappa(H_1 \boxtimes K_{n_2}) = n_2\kappa(H_1)$. From the Case 2 and Case 3 of Theorem 3.1.6, it follows that $f(G) \leqslant f(H_1)$.

Case 2: $H_1 \boxtimes H_2$ where H_1 and H_2 are not complete graphs.

Suppose that diam(G) = diam(H_1). We shall prove the theorem by considering the following sub cases.

(a) Let S be the set of vertices of the form $u_i v_p$ where $i \in \{1, 2, 3, ..., n_1\}$.

Consider a pair of vertices $u_a v_q$ and $u_b v_q$ in G. Let the $\kappa(G)-1$ vertices from S be deleted. Then, $d(u_a v_q, u_b v_q) \leqslant \text{diam}(H_1)$. Now, $d(u_y v_a, u_y v_b) \leqslant f(H_2)$, since the deletion of the vertex v_p from H_2 increases the diam(H_2) to at most $f(H_2)$. Hence, $f(G) \leqslant max\{diam(H_1), f(H_2)\}$.

(b) Let S be the set of vertices of the form $u_j v_p$ where $p \in \{1, 2, 3, ..., n_2\}$.

Consider a pair of vertices $u_w v_q$ and $u_w v_r$ in G. Let the $\kappa(G)-1$ vertices from S be deleted. Then, $d(u_w v_q, u_w v_r) \leqslant \text{diam}(H_2)$. Now, $d(u_a v_w, u_b v_w) \leqslant f(H_1)$, since the deletion of the vertex u_x from H_1 increases the diam(H_1) to at most $f(H_1)$. Hence, $f(G) \leqslant f(H_1)$.

(c) Let S be any arbitrary collection of vertices.

Now, consider a pair of vertices $u_p v_w$ and $u_q v_w$ in G. Let the vertices $u_i v_p$ where $\{u_i\}$ is a collection of $\kappa(H_1)$ vertices which form a vertex cut of H_1 and $p \in \{1, 2, ..., n_2 - 1\}$, be deleted. From the H_1 - layer at v_{n_2} in G, we delete only the $\kappa(H_1) - 1$ vertices, otherwise G becomes disconnected. Then, $d(u_p v_{n_2}, u_q v_{n_2}) \leqslant f(H_1)$ by a path $u_p v_{n_2} - u_{p+1} v_{n_2} - \; ... \; - u_{q-1} v_{n_2} - u_q v_{n_2}$, since the deletion of $\kappa(H_1) - 1$ vertices from H_1 increases the diam(H_1) to at most $f(H_1)$. Now,
$d(u_p v_w, u_q v_w) \leqslant f(H_1) + \text{diam}(H_2)$, by a path $u_p v_w - u_p v_{w+1} - \; ... \; - u_p v_{n_2} - \; ... \; - u_{q-1} v_r - u_q v_w$ where $d(u_p v_w, u_p v_{n_2}) \leqslant \text{diam}(H_2)$ and $d(u_p v_{n_2}, u_q v_w) \leqslant f(H_1)$. Hence, the deletion of $\kappa(G) - 1$ vertices increases the diam(G) to at most $f(H_1) + \text{diam}(H_2)$.

Similarly, if diam(G)= diam(H_2), then the deletion of $\kappa(G)$− 1 vertices increases the diam(G) by $f(H_2) + \text{diam}(H_1)$. □

3.2 Diameter vulnerability of some graph classes

We shall first discuss the diameter vulnerability in grids.

Theorem 3.2.1. *Let $G \cong P_2 \Box P_{n_2}$ be a grid, where $n_2 \geqslant 2$. Then $f'(G) = diam(G) + 1$.*

Proof. Let $G \cong P_2 \Box P_{n_2}$. Then diam(G) $= n_2$ and $\kappa'(G) = 2$.

Let $n_2 > 2$.

Let an edge $u_1v_p - u_2v_p$ where $p \in \{1, 2, ...n_2\}$, be deleted. Then $d(u_1v_p, u_2v_p) = 3 \leqslant n_2$ by a path $u_1v_p - u_1v_q - u_2v_q - u_2v_p$. Also, the distance between any two other vertices is not affected by the removal of this edge. Thus, the deletion of an edge does not increase the diam(G).

Consider a pair of vertices u_1v_1, $u_1v_{n_2}$ in G. Let an edge $u_1v_p - u_1v_q$, be deleted. There exist a unique path of length $n_2 - 1$ between them in G. Thus, $d(u_1v_1, u_1v_{n_2}) = n_2 + 1$ by a path $u_1v_1 - u_2v_1 - u_2v_2 - \ ... \ - u_2v_{n_2} - u_1v_{n_2}$ where $d(u_1v_1, u_2v_1) = 1$, $d(u_2v_1, u_2v_{n_2}) = n_2 - 1$, $d(u_2v_{n_2}, u_1v_{n_2}) = 1$. Also, $d(u_1v_p, u_1v_q) = 3$ and the distance between any two other vertices is not affected by the removal of this edge. Thus, the deletion of an edge increases the diam(G) by one.

Let $n_2 = 2$. Then, G is C_4 and $f'(G) = 3$.

Hence, $f'(G) = \text{diam}(G) + 1$. $\square$

Note: For $n_2 > 2$, the deletion of a single edge will not change the diameter of G. In this context we consider $g'(G)$ [69]. Consider a connected graph G from which if $\kappa'(G)$ edges are deleted, then the resulting graph G (also denoted by G) is still connected. Then, $g'(G)$ denotes the maximum diameter of a connected graph G obtained when $\kappa'(G)$ edges are deleted from G.

Theorem 3.2.2. *Let $G \cong P_3 \square P_{n_2}$ be a grid, where $n_2 \geqslant 2$. Then $g'(G) = diam(G) + 2$.*

Proof. Let $G \cong P_3 \square P_{n_2}$. Then diam(G) $= n_2 + 1$ and $\kappa'(G) = 2$.

If any two edges are deleted arbitrarily, then the diam(G) remains the same, since there is a path of length three between them in G.

Consider a pair of diametral vertices u_1v_1, $u_3v_{n_2}$ in G. Let the edges $u_3v_{n_2-1} - u_3v_{n_2-2}$ and $u_3v_{n_2-1} - u_2v_{n_2-1}$, be deleted. Then, $d(u_1v_1, u_3v_{n_2-1}) = \text{diam}(G) + 1$ by a path $u_1v_1 - u_1v_2 \ldots u_1v_{n_2} - u_2v_{n_2} - u_3v_{n_2} - u_3v_{n_2-1}$ where $d(u_1v_1, u_1v_{n_2}) = n_2 - 1$ and $d(u_1v_{n_2}, u_3v_{n_2-1}) = 3$. Also, $d(u_3v_{n_2-1}, u_2v_{n_2-1}) = 3$ and

$d(u_3v_{n_2-1}, u_2v_{n_2-2}) = 5$. Thus, the deletion of two edges increases the diam(G) by one.

Consider a pair of vertices u_jv_1, $u_jv_{n_2}$ in G. Let the two edges $u_iv_p - u_iv_q$ and $u_jv_p - u_jv_q$ where u_i is adjacent to u_j in P_3, $j = 1$ or 3, be deleted. Then, $d(u_jv_1, u_jv_{n_2}) = n_2 + 3$ by a path $u_jv_1 - u_iv_1 - u_xv_1 - u_xv_2 \ldots u_xv_{n_2} - u_iv_{n_2} - u_jv_{n_2}$ where $d(u_jv_1, u_xv_1) = 2$, $d(u_xv_1, u_xv_{n_2}) = n_2 - 1$, $d(u_xv_{n_2}, u_jv_{n_2}) = 2$. Also, $d(u_iv_p, u_iv_q) = 3$ and $d(u_jv_p, u_jv_q) = 5$ by a path $u_jv_p - u_iv_p - u_xv_p - u_xv_q - u_iv_q - u_jv_q$. Similarly, $d(u_iv_1, u_iv_{n_2}) \leqslant n_2 + 1$. Thus, the diam(G) increases by two.

Hence, $g'(G) = \text{diam}(G) + 2$. □

Theorem 3.2.3. *Let $G \cong P_{n_1} \Box P_{n_2}$ be a grid, where $n_1, n_2 \geqslant 4$. Then $g'(G) = diam(G) + 1$.*

Proof. Let $G \cong P_{n_1} \Box P_{n_2}$. Then diam(G) $= n_1 + n_2 - 2$ and $\kappa'(G) = 2$.

If any two edges are deleted arbitrarily, then the diam(G) remains the same, since there is a path of length three between them in G.

Let $n_1 > 4$.

Consider the vertices u_jv_1, u_jv_n in G. Let the two edges $u_iv_p - u_iv_q$ and $u_jv_p - u_jv_q$ where u_j is adjacent to u_i in P_{n_1} and $i \neq 1$, $j \neq n_1$, be deleted. Then, $d(u_iv_p, u_iv_q) = 3$ and $d(u_jv_p, u_jv_q) = 3$. If u_j is adjacent to u_i in P_{n_1} and $j = 1$ or n_1, then $d(u_iv_p, u_iv_q) = 3$ and $d(u_jv_p, u_jv_q) = 5$ by a path $u_jv_p - u_iv_p - u_xv_p - u_xv_q - u_iv_q - u_jv_q$. Also, $d(u_jv_1, u_jv_{n_2}) = (n_2 - 1) + 4$ by a path $u_jv_1 - u_iv_1 - u_xv_1 - u_xv_2 - \ ... \ - u_xv_{n_2} - u_iv_{n_2} - u_jv_{n_2}$ where $d(u_jv_{n_2}, u_xv_{n_2}) = d(u_xv_1, u_jv_1) = 2$, $d(u_xv_1, u_xv_{n_2}) = n_2 - 1$ and $d(u_iv_1, u_iv_{n_2}) \leqslant (n_2 - 1) + 2$.
Similarly, if the two edges $u_iv_p - u_jv_p$ and $u_iv_q - u_jv_q$, are deleted then the diam(G) remains the same.

Let $n_1 = 4$.

Consider the vertices u_jv_1, $u_jv_{n_2}$ in G. Let the two edges $u_iv_p - u_iv_q$ and $u_jv_p - u_jv_q$ where u_j is adjacent to u_i in P_{n_1} and $i \neq 1$, $j \neq n_1$, be deleted. Then, $d(u_iv_p, u_iv_q) = 3$ and $d(u_jv_p, u_jv_q) = 3$. If u_j is adjacent to u_i in P_{n_1} and $j = 1$ or n_1, then the diam(G) increases by one. Also, $d(u_jv_1, u_jv_{n_2}) = n_2 + 3$ by a path $u_jv_1 - u_iv_1 - u_xv_1 - u_xv_2 - \ ... \ - u_xv_{n_2} - u_iv_{n_2} - u_jv_{n_2}$ where $d(u_jv_{n_2}, u_xv_{n_2}) = d(u_xv_1, u_jv_1) = 2$ and $d(u_xv_1, u_xv_{n_2}) = n_2 - 1$.

Consider a pair of diametral vertices u_1v_1, $u_{n_1}v_{n_2}$ in G. Let the edges $u_{n_1}v_{n_2-1}-u_{n_1}v_{n_2-2}$, $u_{n_1}v_{n_2-1}-u_{n_1-1}v_{n_2-1}$, be deleted. Then, $d(u_1v_1, u_{n_1}v_{n_2-1})$ = diam(G)+1 by a path $u_1v_1 - u_1v_2 -$ ... $u_1v_{n_2-1} - u_1v_{n_2} - u_2v_{n_2} -$ ... $- u_{n_1-1}v_{n_2} - u_{n_1}v_{n_2} - u_{n_1}v_{n_2-1}$ where $d(u_1v_1, u_1v_{n_2}) = n_2 - 1$ and $d(u_1v_{n_2}, u_{n_1}v_{n_2-1}) = n_1$.
Also, $d(u_{n_1}v_{n_2-1}, u_{n_1-1}v_{n_2-1}) = 3$, $d(u_{n_1}v_{n_2-1}, u_{n_1}v_{n_2-2}) = 5$. Thus, the deletion of two edges increases the diam(G) by one.
Hence, $g'(G) = \text{diam}(G) + 1$. □

We shall now consider the case of cylinders.

Theorem 3.2.4. *Let* $G \cong P_{n_1} \square C_{n_2}$ *be a cylinder where* $n_1, n_2 \geqslant 4$. *Then* $f'(G) = diam(G) + 1$ *and* $g'(G) = diam(G) + 2$.

Proof. Let $G \cong P_{n_1} \square C_{n_2}$. Then $\kappa'(G) = 3$ and diam(G) $= n_1 - 1 + \lfloor n_2/2 \rfloor$. Let $d(v_w, v_z) = \text{diam}(C_{n_2})$.

If any two edges are deleted arbitrarily, then the diam(G) remains the same, since there is a path of length three between them in G.

Consider the vertices u_jv_w, u_jv_z in G. Let the edges $u_iv_p - u_iv_q$, $u_jv_p - u_jv_q$ where u_j is adjacent to u_i in P_{n_1} and $i, j \neq 1, n_1$, be deleted. Then, $d(u_iv_p, u_iv_q) = 3$ and $d(u_jv_p, u_jv_q) = 3$. If u_j is adjacent to u_i in P_{n_1} and $j = 1$ or n_1, then the diam(G) remains the same, since $d(u_iv_p, u_iv_q) = 3$, $d(u_iv_p, u_jv_q) = d(u_iv_q, u_jv_p) = 4$ and $d(u_jv_p, u_jv_q) = 5$ by a path $u_jv_p - u_iv_p - u_xv_p - u_xv_q - u_iv_q - u_jv_q$. Also, $d(u_jv_w, u_jv_z) \leqslant \lfloor n_2/2 \rfloor + 4$ by a path $u_jv_w - u_iv_w - u_xv_w - u_xv_{w+1}$ ... $u_xv_z - u_iv_z - u_jv_z$ where $d(u_jv_w, u_xv_w) = 2$, $d(u_xv_w, u_jv_z) = \lfloor n_2/2 \rfloor + 2$. Thus, the diam(G) remains the same.

Consider a pair of vertices u_1v_1, $u_{n_1}v_{\lfloor n_2/2 \rfloor + 1}$ in G. Let the edges $u_{n_1}v_{\lfloor n_2/2 \rfloor} - u_{n_1}v_{\lfloor n_2/2 \rfloor - 1}$, $u_{n_1}v_{\lfloor n_2/2 \rfloor} - u_{n_1-1}v_{\lfloor n_2/2 \rfloor}$, be deleted. Then, $d(u_1v_1, u_{n_1}v_{\lfloor n_2/2 \rfloor}) = \text{diam}(G) + 1$ by a path $u_1v_1 - u_1v_2 -$ ... $- u_1v_{\lfloor n_2/2 \rfloor + 1} - u_2v_{\lfloor n_2/2 \rfloor + 1} -$ ... $- u_{n_1}v_{\lfloor n_2/2 \rfloor + 1} - u_{n_1}v_{\lfloor n_2/2 \rfloor}$ where $d(u_1v_1, u_1v_{\lfloor n_2/2 \rfloor + 1}) = \lfloor n_2/2 \rfloor$ and $d(u_1v_{\lfloor n_2/2 \rfloor + 1}, u_{n_1}v_{\lfloor n_2/2 \rfloor}) = n_1$. Also, $d(u_{n_1}v_{\lfloor n_2/2 \rfloor}, u_{n_1-1}v_{\lfloor n_2/2 \rfloor}) = 3$, $d(u_{n_1}v_{\lfloor n_2/2 \rfloor}, u_{n_1}v_{\lfloor n_2/2 \rfloor + 1}) = 5$. Thus, the deletion of two edges increases the diam(G) by one [see Fig 3.7].
Hence, $f'(G) = \text{diam}(G) + 1$.

Now, we shall prove that $g'(G) = \text{diam}(G) + 2$. Consider a pair of vertices u_1v_1, $u_{n_1}v_{\lfloor n_2/2 \rfloor + 1}$ in G. Let the three edges $u_{n_1}v_{\lfloor n_2/2 \rfloor - 1} - u_{n_1}v_{\lfloor n_2/2 \rfloor - 2}$, $u_{n_1}v_{\lfloor n_2/2 \rfloor - 1} - u_{n_1-1}v_{\lfloor n_2/2 \rfloor - 1}$,

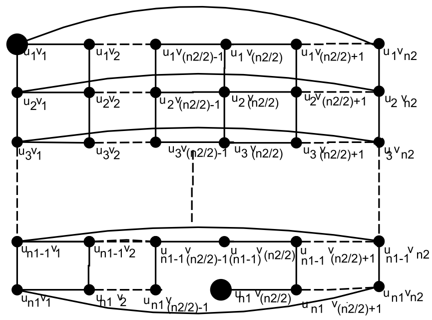

Fig 3.7: A graph $G \cong P_{n_1} \square C_{n_2}$ with $f'(G) = \text{diam}(G) + 1$.

and $u_{n_1}v_{\lfloor n_2/2 \rfloor} - u_{n_1-1}v_{\lfloor n_2/2 \rfloor}$, be deleted. Then, $d(u_1v_1, u_{n_1}v_{\lfloor n_2/2 \rfloor -1})$ $=$ daim$(G)+$ 2 by a path $u_1v_1 - u_1v_2 -$ ... $- u_1v_{\lfloor n_2/2 \rfloor +1} -$ $u_2v_{\lfloor n_2/2 \rfloor +1} -$ ... $- u_{n_1}v_{\lfloor n_2/2 \rfloor +1} - u_{n_1}v_{\lfloor (n_2/2) \rfloor} - u_{n_1}v_{\lfloor (n_2/2) \rfloor -1}$. Also, $d(u_{n_1}v_{\lfloor n_2/2 \rfloor}, u_{n_1-1}v_{\lfloor n_2/2 \rfloor}) = 3$, $d(u_{n_1}v_{\lfloor n_2/2 \rfloor -1}, u_{n_1}v_{\lfloor n_2/2 \rfloor -2}) = 7$, $d(u_{n_1}v_{\lfloor n_2/2 \rfloor -1}, u_{n_1-1}v_{\lfloor n_2/2 \rfloor -1}) = 5$. Thus, the deletion of three edges increases the diam(G) by two [see Fig 3.8].

Hence, $g'(G) = \text{diam}(G) + 2$. □

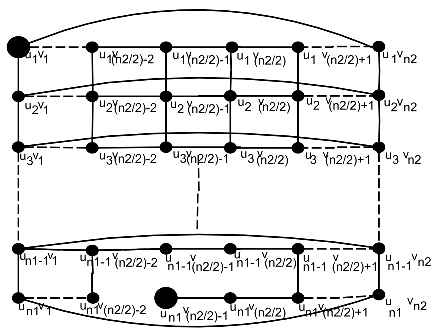

Fig 3.8: A graph $G \cong P_{n_1} \square C_{n_2}$ with $g'(G) = \text{diam}(G) + 2$.

Finally, we consider the case of tori.

Let $G \cong C_{n_1} \Box C_{n_2}$. For $n_1, n_2 \leqslant 5$, we observe that $f'(G)$ and $g'(G)$ are either diam(G)+ 1 or diam(G)+2. Hence, we consider $n_1, n_2 \geqslant 6$ and show that $f'(G)$ = diam(G)+1 and

$g'(G)$ = diam(G)+2.

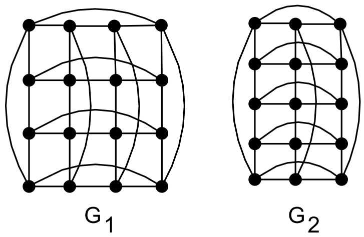

Fig 3.9: Graphs G_1 with $f'(G_1) = \text{diam}(G_1) + 1$ and G_2 with $f'(G_2) = \text{diam}(G_2) + 2$.

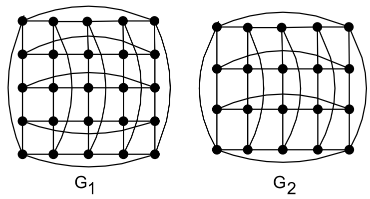

Fig 3.10: Graphs G_1 with $g'(G_1) = \text{diam}(G_1) + 1$ and G_2 with $g'(G_2) = \text{diam}(G_2) + 2$.

Theorem 3.2.5. *Let $G \cong C_{n_1} \Box C_{n_2}$ be a tori, where $n_1, n_2 \geqslant 6$, then $f'(G) = diam(G)+1$. Further, $g'(G) = diam(G)+2$ where n_1 and n_2 are odd with $n_1, n_2 \geqslant 6$.*

Proof. Let $G \cong C_{n_1} \Box C_{n_2}$. Then $\kappa'(G) = 4$ and diam(G)=$\lfloor n_1/2 \rfloor + \lfloor n_2/2 \rfloor$.

If any two edges are deleted arbitrarily, then the diam(G) remains the same, since there is a path of length three between them in G.

Let the three edges $u_iv_p - u_iv_q$, $u_jv_p - u_jv_q$ and $u_xv_p - u_xv_q$ where u_j is adjacent to u_i and u_x in C_{n_1}, be deleted. Then $d(u_iv_p, u_iv_q) = 3 = d(u_xv_p, u_xv_q)$, $d(u_jv_p, u_jv_q) = 5$. Also, $d(u_jv_w, u_jv_z) = \lfloor n_2/2 \rfloor + 4$, $d(u_iv_w, u_iv_z) = \lfloor n_2/2 \rfloor + 2$ and $d(u_xv_w, u_xv_z) = \lfloor n_2/2 \rfloor + 2$. Thus, the diam(G) remains the same.

Consider a pair of diametral vertices u_1v_1, $u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor+1}$ in G. Let the three edges $u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor} - u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor-1}$, $u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor} - u_{\lfloor n_1/2 \rfloor}v_{\lfloor n_2/2 \rfloor}$ and $u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor} - u_{\lfloor n_1/2 \rfloor+2}v_{\lfloor n_2/2 \rfloor}$, be deleted. Then, the distance between these pairs of vertices is three. Now, $d(u_1v_1, u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor}) = \text{diam}(G) + 1$ by a path $u_1v_1 - u_1v_2 - \ \ldots \ - u_1v_{\lfloor n_2/2 \rfloor+1} - u_2v_{\lfloor n_2/2 \rfloor+1} - \ \ldots \ - u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor+1} - u_{\lfloor n_1/2 \rfloor+1}v_{\lfloor n_2/2 \rfloor}$. Thus, the deletion of three edges increases the diam(G) by one.
Hence, $f'(G) = \text{daim}(G) + 1$.

Now we shall prove that $g'(G) = \text{daim}(G) + 2$.

Consider a pair of vertices u_1v_1, $u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor+1}$ in G. Let the edges $u_1v_1 - u_1v_2$, $u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor} - u_{\lfloor n_1/2\rfloor}v_{\lfloor n_2/2\rfloor}$, $u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor} - u_{\lfloor n_1/2\rfloor+2}v_{\lfloor n_2/2\rfloor}$, $u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor} - u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor-1}$, be deleted. Then, $d(u_1v_1, u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}) = \text{diam}(G) + 2$ by a path $u_1v_1 - u_{n_1}v_1 - \dots - u_{n_1}v_{\lfloor n_2/2\rfloor+1} - u_{n_1-1}v_{\lfloor n_2/2\rfloor+1} - \dots - u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor+1} - u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor+1} - u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}$ where $d(u_1v_1, u_{n_1}v_{\lfloor n_2/2\rfloor+1}) = \lfloor n_2/2\rfloor+1$ and $d(u_{n_1}v_{\lfloor n_2/2\rfloor+1}, u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}) = \lfloor n_1/2\rfloor + 1$. Also, $d(u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}, u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor-1}) = 5$, $d(u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}, u_{\lfloor n_1/2\rfloor}v_{\lfloor n_2/2\rfloor}) = 3$, $d(u_1v_1, u_1v_2) = 3$ and $d(u_{\lfloor n_1/2\rfloor+1}v_{\lfloor n_2/2\rfloor}, u_{\lfloor n_1/2\rfloor+2}v_{\lfloor n_2/2\rfloor}) = 3$. Thus, the deletion of four edges increases the diam(G) by two.

Hence, $g'(G) = \text{daim}(G) + 2$. □

3.3 Wide Diameter of the lexicographic product of graphs

Lemma 3.3.1. *Let $G \cong H_1 \circ H_2$. If there exists a container of width w in H_1 which is of length l then there exists a container of width $\kappa(H_1) \times |V(H_2)|$ which will be of the same length l in*

G.

Proof. **Case 1:** Consider the vertices of the form $u_i v_j$ and $u_k v_j$ where $i \neq k$ and $i, k \in \{1, 2, ..., n_1\}$.

Since there exists a container of length l in H_1, there exists a container of length at most l between u_i and u_k. If $P_1 = u_i - u_{i+1} - u_{i+2} - \ ... \ - u_{k-1} - u_k$ is a path in the container $C_w(u_i, u_k)$ of H_1, then $u_i v_j - u_{i+1} v_j - u_{i+2} v_j - \ ... \ - u_{k-1} v_j - u_k v_j$ is the corresponding path connecting $u_i v_j$ and $u_k v_j$ in G. Also, by the structure of the lexicographic product, $u_i v_j - u_{i+1} v_a - u_{i+2} v_a - \ ... \ - u_{k-1} v_a - u_k v_j$ are also paths connecting $u_i v_j$ and $u_k v_j$ where $a \neq j$ and $a \in \{1, 2, ..., n_2\}$ in G. Thus, corresponding to the w internally vertex disjoint paths in $C_w(u_i, u_k)$ of H_1, we have shown the existence of $w\,|V(H_2)|$ internally disjoint paths between $u_i v_j$ and $u_k v_j$ in G which are of length at most l. Since the length of the container in H_1 is l, there exists a pair of vertices u_x and u_y in H_1 such that the path joining u_x and u_y is of length l. As proved above we can show that $C_w(u_x v_j, u_y v_j)$ in G is of length l.

Case 2: Consider the vertices of the form $u_i v_j$ and $u_i v_k$

where $j \neq k$ and $j, k \in \{1, 2, ..., n_2\}$.

By the structure of the lexicographic product, if u_i is adjacent to u_a in H_1, then both u_iv_j and u_iv_k will be adjacent to $u_av_1, u_av_2, \ldots, u_av_m$ in G. Thus there exists at least $d_G(u_i)\,|V(H_2)|$ internally vertex disjoint paths between u_iv_j and u_iv_k which are of length two. So we can say that for any vertex u_i in H_1, there exists $C_{\delta(G)|V(H_2)|}(u_iv_j, u_iv_k)$ of length two in G.

Case 3: Consider the vertices of the form u_iv_j and u_av_b where $i \neq a$ and $j \neq b$.

Consider the vertices u_i and u_a in H_1. By the assumption there exists a container of length at most l in between u_i and u_a in H_1. If $P_1 = u_i - u_{i+1} - u_{i+2} - \ \ldots \ - u_{a-1} - u_a$ is a path in the container $C_w(u_i, u_k)$, then $u_iv_j - u_{i+1}v_j - u_{i+2}v_j - \ \ldots \ - u_{a-1}v_j - u_av_b$ is a path connecting u_iv_j and u_av_b in G which is of length same as that of P_1. Again, by the structure of the lexicographic product, we can find $w\,|V(H_2)|$ internally vertex disjoint paths between u_iv_j and u_av_b and is of length at most l. Since the length of the container in H_1 is l, there exists a pair of vertices u_x and u_y in H_1 such that the path joining u_x and u_y is of length l. So $C_w(u_xv_j, u_yv_b)$ in G is of length exactly l.

Finally, since $1 \leqslant w \leqslant \kappa(G)$ and $\kappa(G) \leqslant \delta(G)$, the result follows.

$\square$

Theorem 3.3.2. *For any two connected graphs H_1 and H_2, Wide diameter $(H_1 \circ H_2)$ = Wide diameter (H_1).*

Proof. Suppose that $G \cong H_1 \circ H_2$.

Let $D_{\kappa(H_1)}(H_1) = k$. Then there exists a container of width $\kappa(H_1)$ in between any two vertices of H_1 which is of length at most k. Then, by Lemma 3.3.1, there exists a container of width $\kappa(H_1) \times |V(H_2)|$ in between any two vertices of G which is of length at most k.

Hence, $D_{\kappa(H_1)\times|V(H_2)|}(H_1 \circ H_2) \leqslant D_{\kappa(H_1)}(H_1)$.

Let $D_{\kappa(H_1)\times|V(H_2)|}(H_1 \circ H_2) = k$.

Consider any two vertices u_i and u_j in H_1. Clearly there exists $\kappa(H_1)$ internally disjoint paths joining u_i and u_j in H_1. Since $D_{\kappa(H_1)\times|V(H_2)|}(H_1 \circ H_2) = k$, there exist a container of length at most k joining u_iv_1 and u_jv_1. Thus there exist a container of width $\kappa(H_1)$ which is of length at most k joining u_i and u_j in G.

Hence, $D_{\kappa(H_1)}(H_1) \leqslant D_{\kappa(H_1)\times|V(H_2)|}(H_1 \circ H_2)$. $\square$

Chapter 4

Component factors of the product graphs

In this chapter we study the component factors of the product graphs. We show that if $G \cong H_1 * H_2$ where $* \in \{\Box, \boxtimes, \circ\}$ and H_1, H_2 are connected graphs then G has a $\{K_{1,n}, C_4\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2. In this chapter, we denote K_2 by $K_{1,1}$ and P_3 by $K_{1,2}$ for uniformity in notations.

Some results of this chapter are included in the following paper.
1. Chithra M.R., A. Vijayakumar, Component factors of the Cartesian product of graphs (Communicated).

4.1 Component factors of the Cartesian product of graphs

Theorem 4.1.1. *Let $G \cong H_1 \square H_2$ be a connected graph where $|H_1| = n_1$ and $|H_2| = n_2$. Then G has a C_4-factor if and only if G is any one of the following graphs where,*

(I) H_1 or H_2 has a C_4-factor.

(II) both H_1 and H_2 have no C_4-factor and,

(a) both H_1 and H_2 are complete graphs with n_1, n_2 even and $n_1, n_2 \not\equiv 0$ mod 4.

(b) H_1 is a complete graph with n_1 even and H_2 is a not complete graph with n_2 even, has at least one vertex with at most one pendant vertex attached to it and has a $\{K_{1,1}\}$-factor.

(c) H_1 and H_2 are not complete graphs with n_1, n_2 even, both have at least one vertex with at most one pendant vertex attached to it and have a $\{K_{1,1}\}$-factor.

Proof. Let $G \cong H_1 \square H_2$ where $|H_1| = n_1$ and $|H_2| = n_2$.

(I) H_1 or H_2 has a C_4-factor.

Suppose that H_1 has a C_4-factor.

Consider the H_1 - layer at v_1 in G. Now, the vertices u_iv_1 where $i \in \{1, 2, 3, ..., n_1\}$ form a subgraph whose components are C_4, since H_1 has a C_4-factor. Similarly, the vertices u_iv_p where $i \in \{1, 2, 3, ..., n_1\}$ and $p \in \{2, 3, ..., n_2\}$ form a subgraph whose components are C_4. Hence, G has a C_4-factor.

(II) Both H_1 and H_2 have no C_4-factor.

(a) Suppose that both H_1 and H_2 are complete graphs with n_1, n_2 even and $n_1, n_2 \not\equiv 0 \ mod \ 4$.

Since both H_1 and H_2 are complete graphs with n_1,n_2 even, we can find a spanning subgraph of H_1 and H_2 whose components are $K_{1,1}$. Now, a $K_{1,1}$ from H_1 and a $K_{1,1}$ from H_2 form a C_4 in G. Thus, G has a spanning subgraph H whose components are C_4.

(b) Suppose that H_1 is a complete graph with n_1 even and H_2 is a not complete graph with n_2 even, has vertices with at most one pendant vertex attached to it and has a $\{K_{1,1}\}$-factor

There is a spanning subgraph of H_2 whose components are $K_{1,1}$. Also, H_1 has a $K_{1,1}$-factor. Hence, G has a C_4-factor.

(c) From II(b), it follows that G has a C_4-factor.

Conversely suppose that G has a C_4-factor. Let $G \cong H_1 \Box H_2$. Suppose that both n_1, n_2 are odd. Then H_1 and H_2 cannot have a spanning subgraph whose components are $K_{1,1}$. Hence, G has no C_4-factor. Thus, at least one graph should be of even order.

(i) n_1 even and n_2 odd.

If H_1 has no C_4-factor, then G has no C_4-factor, since n_2 is odd. Hence, H_1 has a C_4-factor. This proves (I).
Similar is the case when n_1 is odd and n_2 is even.

(iii) Both n_1, n_2 are even.

(a) $G \cong K_{n_1} \Box K_{n_2}$.
Clearly G has a C_4-factor. This proves II(a).

(b) $G \cong K_{n_1} \Box H_2$ where H_2 is a not complete graph.

If H_2 has a vertex v_x with at least two pendant vertices v_i, v_j attached to it, then in G $\langle u_pv_x, u_pv_i, u_qv_i, u_qv_x \rangle$ form a C_4 and $\langle u_pv_j, u_qv_j \rangle$ form an edge. Thus, G has no C_4-factor. Hence, H_2 is a not complete graph with n_2 even and has vertices with at most one pendant vertex attached to it.

Now, we know that H_1 has a spanning subgraph whose components are $K_{1,1}$. Hence, G has a C_4-factor only if H_2 has a $\{K_{1,1}\}$-factor This proves II(b).

(c) $G \cong H_1 \Box H_2$ where both H_1 and H_2 are not complete graphs.

If H_1 has a vertex u_p with at least two pendant vertices u_a, u_b attached to it and H_2 has a vertex v_x with at least two pendant vertices v_i, v_j attached to it, then in G $\langle u_pv_r, u_pv_s, u_av_s, u_av_r\rangle$ and $\langle u_pv_x, u_pv_i, u_qv_i, u_qv_x\rangle$ form C_4s and $\langle u_pv_r, u_bv_r\rangle$, $\langle u_pv_j, u_qv_j\rangle$ form $K_{1,1}$s. Thus, G has no C_4-factor. Hence, both have at least one vertex with at most one pendant vertex attached to it.

Now, G has a C_4-factor only if both H_1 and H_2 has a $\{K_{1,1}\}$-factor.

This proves II(c). $\square$

Theorem 4.1.2. *Let $G \cong K_{n_1} \Box K_{n_2}$ where $n_1, n_2 \geqslant 2$. Then G has a $\{K_{1,2}, C_4\}$-factor.*

Proof. We shall prove the theorem by considering the following three cases.

(I) Both n_1, n_2 are even.

From Theorem 4.1.1, it follows that G has a C_4-factor.

(II) Both n_1, n_2 are odd.

(a) Suppose that $n_1 \equiv 0 \ mod \ 3$.

Since $n_1 \equiv 0 \ mod \ 3$, we can find a spanning subgraph of K_{n_1} whose components are $K_{1,2}$. Now, G has a spanning subgraph H whose components are $K_{1,2}$, since K_{n_1} has a $K_{1,2}$-factor.

(b) Suppose that $n_1, n_2 \equiv 1 \ mod \ 3$.

Since $n_1 \equiv 1 \ mod \ 3$, we can find a spanning subgraph of K_{n_1} whose components are $K_{1,2}$ and $K_{1,1}$ where $u_1, u_2, ..., u_{n_1-4}$ are the vertices in the components of $K_{1,2}$ and $u_{n_1-3}, u_{n_1-2}, u_{n_1-1}, u_{n_1}$ are the vertices in the components of $K_{1,1}$. Now, consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-4}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$ in G, they form a subgraph whose components are $K_{1,2}$. The remaining vertices of G, $u_{n_1-3}v_p, u_{n_1-2}v_p, u_{n_1-1}v_p, u_{n_1}v_p$ where
$p \in \{1, 2, 3, ..., n_2 - 4\}$ form a subgraph whose components are $K_{1,2}$ and $\langle u_{n_1-3}v_{n_2-3}, u_{n_1-3}v_{n_2-2}, u_{n_1-2}v_{n_2-2}, u_{n_1-2}v_{n_2-3}\rangle$,
$\langle u_{n_1-3}v_{n_2-1}, u_{n_1-3}v_{n_2}, u_{n_1-2}v_{n_2}, u_{n_1-2}v_{n_2-1}\rangle$,

$\langle u_{n_1-1}v_{n_2-3}, u_{n_1-1}v_{n_2-2}, u_{n_1}v_{n_2-2}, u_{n_1}v_{n_2-3}\rangle$ and $\langle u_{n_1-1}v_{n_2-1}, u_{n_1-1}v_{n_2}, u_{n_1}v_{n_2}, u_{n_1}v_{n_2-1}\rangle$ form C_4s.

(c) Suppose that $n_1, n_2 \equiv 2\ mod\ 3$.

Since $n_1 \equiv 2\ mod\ 3$, we can find a spanning subgraph of K_{n_1} whose components are $K_{1,2}$ and $K_{1,1}$ where $u_1, u_2, ..., u_{n_1-2}$ are the vertices in the components of $K_{1,2}$ and u_{n_1-1}, u_{n_1} are the vertices in the components of $K_{1,1}$. Now, consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-2}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$ in G, they form a subgraph whose components are $K_{1,2}$. The remaining vertices of G, $u_{n_1-1}v_p$ where $p \in \{1, 2, 3, ..., n_2 - 2\}$ and $u_{n_1}v_p$ where $p \in \{1, 2, 3, ..., n_2 - 2\}$ form a subgraph whose components are $K_{1,2}$ and $\langle u_{n_1-1}v_{n_2-1}, u_{n_1-1}v_{n_2}, u_{n_1}v_{n_2}, u_{n_1}v_{n_2-1}\rangle$ form a C_4.

(d) Suppose that $n_1 \equiv 2\ mod\ 3$ and $n_2 \equiv 1\ mod\ 3$.

Clearly K_{n_1} and K_{n_2} has a $\{K_{1,1}, K_{1,2}\}$-factor. Consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-2}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$ in G, they form a subgraph whose components are $K_{1,2}$. The remaining vertices of G, $u_{n_1-1}v_p$ where $p \in \{1, 2, 3, ..., n_2 - 4\}$ and $u_{n_1}v_p$ where $p \in \{1, 2, 3, ..., n_2 - 4\}$ form a subgraph whose components are $K_{1,2}$ and $\langle u_{n_1-1}v_{n_2-3}, u_{n_1-1}v_{n_2-2}, u_{n_1}v_{n_2-2}, u_{n_1}v_{n_2-3}\rangle$,

$\langle u_{n_1-1}v_{n_2-1}, u_{n_1-1}v_{n_2}, u_{n_1}v_{n_2}, u_{n_1}v_{n_2-1}\rangle$ form C_4s.

(III) n_1 odd and n_2 even.

(a) Suppose that $n_1 \equiv 0$ *mod* 3.

From II(a), it follows that G has a $K_{1,2}$-factor.

(b) Suppose that $n_1 \equiv 1$ *mod* 3.

Clearly K_{n_1} has a $\{K_{1,1}, K_{1,2}\}$-factor and K_{n_2} has a $K_{1,1}$-factor. Consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-4}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$ in G, they form a subgraph whose components are $K_{1,2}$. Now, $\langle u_{n_1-3}v_p, u_{n_1-2}v_p, u_{n_1-2}v_{p+1}, u_{n_1-3}v_{p+1}\rangle$, $\langle u_{n_1-1}v_p, u_{n_1}v_p, u_{n_1}v_{p+1}, u_{n_1-1}v_{p+1}\rangle$ where $p \in \{1, 2, 3, ..., n_2 - 1\}$ form C_4s.

(c) Suppose that $n_1 \equiv 2$ *mod* 3.

Clearly K_{n_1} has a $\{K_{1,1}, K_{1,2}\}$-factor and K_{n_2} has a $K_{1,1}$-factor. Consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-2}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$ in G, they form a subgraph whose components are $K_{1,2}$. Now, $\langle u_{n_1-1}v_p, u_{n_1}v_p, u_{n_1}v_{p+1}, u_{n_1-1}v_{p+1}\rangle$ where $p \in \{1, 2, 3, ..., n_2 - 1\}$ form C_4s.

Hence, G has a $\{K_{1,2}, C_4\}$-factor. $\square$

Lemma 4.1.3. *Let $G \cong K_{n_1} \square H_2$ be a connected graph where $n_1 \geqslant 2$ and H_2 is any not complete graph. Then G has a $\{K_{1,2}, C_4\}$-factor if G is any one of the following graphs where,*
(a) either K_{n_1} or H_2 has a $K_{1,2}$-factor or a C_4- factor.
(b) both n_1, n_2 are even and H_2 has at least one vertex with at most one pendant vertex attached to it and has a $\{K_{1,1}\}$-factor.
(c) n_1 even, n_2 odd and H_2 has at least one vertex with at most one pendant vertex attached to it and has a $\{K_{1,2}, K_{1,1}\}$-factor.
(d) n_1 odd, n_2 even and H_2 has at least one vertex with at most one pendant vertex attached to it and has a $\{K_{1,1}\}$-factor.

Proof. (a) Suppose that K_{n_1} or H_2 has a C_4-factor, then from Theorem 4.1.1 G has a C_4-factor.
Suppose that K_{n_1} or H_2 has a $K_{1,2}$-factor, then clearly G has a C_4-factor.
Suppose that (b) holds, then from Theorem 4.1.1 G has a C_4-factor.
Suppose that (c) holds.

Clearly, H_1 has a $K_{1,1}$-factor. We can find a spanning subgraph of H_2 whose components are $K_{1,2}$ and $K_{1,1}$ where $v_1, v_2, ..., v_p$ are the vertices in the components of $K_{1,2}$ and $v_{p+1}, v_{p+2}, ..., v_{n_2}$

are the vertices in the components of $K_{1,1}$. Consider the vertices $u_xv_1, u_xv_2, ..., u_xv_p$ where $x \in \{1, 2, 3, ..., n_1\}$ in G, they form a subgraph whose components are $K_{1,2}$. Now,
$\langle u_xv_{p+1}, u_xv_{p+2}, u_{x+1}v_{p+1}, u_{x+1}v_{p+2}\rangle$,...
$\langle u_xv_{n_2-1}, u_xv_{n_2}, u_{x+1}v_{n_2-1}, u_{x+1}v_{n_2}\rangle$ where $x \in \{1, 2, ..., n_1 - 1\}$ form C_4s.

Suppose that (d) holds.

If $n_1 \equiv 0$ *mod* 3, then K_{n_1} has a $K_{1,2}$-factor and hence the proof follows from Lemma 4.1.3(b).

If $n_1 \equiv 1$ *mod* 3, then we can find a spanning subgraph of K_{n_1} whose components are $K_{1,2}$ and $K_{1,1}$ where $u_1, u_2, ..., u_{n_1-4}$ are the vertices in the components of $K_{1,2}$ and $u_{n_1-3}, u_{n_1-2}, u_{n_1-1}, u_{n_1}$ are the vertices in the components of $K_{1,1}$. Consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-4}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$, they form a subgraph whose components are $K_{1,2}$. Now,
$\langle u_{n_1-3}v_y, u_{n_1-2}v_y, u_{n_1-2}v_{y+1}, u_{n_1-3}v_{y+1}\rangle$,
$\langle u_{n_1-1}v_y, u_{n_1}v_y, u_{n_1}v_{y+1}, u_{n_1-1}v_{y+1}\rangle$ where $y \in \{1, 2, ..., n_2 - 1\}$ form C_4s.

If $n_1 \equiv 2$ *mod* 3, then we can find a spanning subgraph of

K_{n_1} whose components are $K_{1,2}$ and $K_{1,1}$ where $u_1, u_2, ..., u_{n_1-2}$ are the vertices in the components of $K_{1,2}$ and u_{n_1-1}, u_{n_1} are the vertices in the components of $K_{1,1}$. Consider the vertices $u_1v_p, u_2v_p, ..., u_{n_1-2}v_p$ where $p \in \{1, 2, 3, ..., n_2\}$, they form a subgraph whose components are $K_{1,2}$. Now,
$\langle u_{n_1-1}v_y, u_{n_1}v_y, u_nv_{y+1}, u_{n_1-1}v_{y+1}\rangle$ where $y \in \{1, 2, ..., n_2 - 1\}$ form C_4s.
Hence, G has a $\{K_{1,2}, C_4\}$-factor. □

Lemma 4.1.4. *Let $G \cong K_{n_1}\square H_2$ be a connected graph where $n_1 \geqslant 2$ and H_2 is a not complete graph. Then G has a $\{K_{1,1}, K_{1,2}\}$-factor if G is any one of the following graphs where,*
(I) both n_1, n_2 are even and H_2 has either no pendant vertex or at least one vertex with at least one pendant vertex attached to it.
(II) n_1 even, n_2 odd and H_2 has either no pendant vertex or at least one vertex with at least one pendant vertex attached to it.
(III) n_1 odd, n_2 even and H_2 has either no pendant vertex or at least one vertex with at least one pendant vertex attached to it.
(IV) both n_1, n_2 are odd.

Proof. If $n_1 \equiv 0$ *mod* 3, then K_{n_1} has a $K_{1,2}$-factor. If

$n_1 \equiv 1$ *mod* 3 or $n_1 \equiv 2$ *mod* 3, then K_{n_1} has a $\{K_{1,1}, K_{1,2}\}$-factor. Hence, in all these cases G has a $\{K_{1,1}, K_{1,2}\}$-factor. $\square$

Theorem 4.1.5. *Let $G \cong K_{n_1} \square H_2$ be a connected graph where H_2 is a not complete graph. Then G has a $\{K_{1,1}, K_{1,2}, C_4\}$-factor.*

Proof. Follows from Lemma 4.1.3 and Lemma 4.1.4. $\square$

Lemma 4.1.6. *Let $G \cong H_1 \square H_2$ be a connected graph where H_1 and H_2 are not complete graphs. Then G has a $\{K_{1,1}, K_{1,2}, C_4\}$-factor if G is any one of the following graphs where,*
(I) either H_1 or H_2 has a a $K_{1,1}$-factor or a $K_{1,2}$-factor or a C_4-factor.
(II) H_1 and H_2 have no $K_{1,2}$-factor and C_4-factor and
(a) both n_1, n_2 are even and H_1, H_2 have at least one vertex with at most one pendant vertex attached to it and have a $K_{1,1}$-factor.
(b) H_1 has at least one vertex with at least two pendant vertices attached to it and H_2 has at least one vertex with at most one pendant vertex attached to it with n_2 even and has a $K_{1,1}$-factor.

Proof. Suppose that (I) holds, then clearly G has a $K_{1,2}$-

factor or a C_4-factor.

Suppose that II(a) holds, then from Theorem 4.1.1 G has a C_4-factor.

Suppose that II(b) holds.

Consider the vertices u_is, u_js in H_1 where $i, j \in \{1, 2, ..., n_1\}$ and u_js are the pendant vertices in H_1. We can find a spanning subgraph of H_1 whose components are $K_{1,2}$, $K_{1,1}$ and K_1 where u_is are the vertices in the components of $K_{1,2}$, $K_{1,1}$ and u_js are the vertices in the K_1. Then in G, the vertices u_iv_p where $p \in \{1, 2, ..., n_2\}$ form a subgraph whose components are $K_{1,2}$ and $K_{1,1}$. Now, $\langle u_jv_p, u_jv_{p+1}\rangle$ where $p \in \{1, 2, ..., n_2 - 1\}$ form $K_{1,1}$s. Hence, G has a $\{K_{1,1}, K_{1,2}\}$-factor. □

Lemma 4.1.7. *Let $G \cong H_1 \square H_2$ be a connected graph where H_1 and H_2 are not complete graphs and have at least one vertex with at least one pendant vertex attached to it. Then G has a $\{K_{1,n}\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2.*

Proof. If H_1 and H_2 have at least one vertex with at least one pendant vertices attached to it, then H_1 has a $\{K_{1,p}\}$-factor and H_2 has a $\{K_{1,q}\}$-factor where $p, q \leqslant t$ and t is the maximum degree of an induced sub graph $K_{1,t}$ in H_1 or H_2. Hence, G has a $\{K_{1,n}\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2. $\square$

Theorem 4.1.8. *Let $G \cong H_1 \Box H_2$ be a connected graph where H_1 and H_2 are not complete graphs. Then G has a $\{K_{1,n}, C_4\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2.*

Proof. Follows from Lemma 4.1.6 and Lemma 4.1.7. $\square$

Theorem 4.1.9. *Let $G \cong H_1 * H_2$ where $* \in \{\Box, \boxtimes, \circ\}$ and H_1, H_2 are connected graphs. Then G has a $\{K_{1,n}, C_4\}$-factor where $n \leqslant t$ and t is the maximum degree of an induced subgraph $K_{1,t}$ in H_1 or H_2.*

Proof. Follows from Theorem 4.1.2, 4.1.5, 4.1.8 and the fact that the Cartesian product of two connected graphs is a spanning subgraph of the strong product and the lexicographic product of graphs. $\square$

4.2 Path factors of Hypercubes and Hamming graphs

Theorem 4.2.1. *The hypercube Q_n has a $\{P_4\}$-factor.*

Proof. We prove the theorem by induction.

Let $n = 2$.

Then Q_2 is C_4 and it has a path of length four.

Let $n = 3$.

Then Q_3 has a $\{P_4\}$-factor as shown in Fig 4.1.

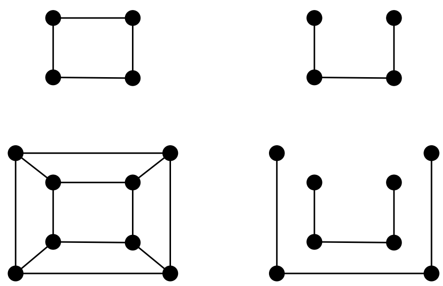

Fig 4.1: The graphs Q_2, Q_3 with a P_4-factor.

Assume that for $n = k$, Q_k has a $\{P_4\}$-factor.

Next, we have to prove that Q_{k+1} has a $\{P_4\}$-factor. We have $Q_{k+1} \cong Q_k \Box K_2$. Now, in G the vertices $u_1v_1, u_1v_2, ..., u_1v_{2^k}$ form P_4s, since Q_k has a $\{P_4\}$-factor. Similarly, $u_2v_1, u_2v_2, ..., u_2v_{2^k}$ form P_4s. Hence, Q_{k+1} has a P_4-factor. $\square$

Theorem 4.2.2. *A Hamming graph* $G \cong K_{n_1} \Box K_{n_2} \Box K_{n_3} \Box \ldots \Box K_{n_k}$ *has a* $\{P_3, P_4\}$*-factor.*

Proof. Let $G \cong K_{n_1} \Box K_{n_2} \Box K_{n_3} \Box \ \ldots \ \Box K_{n_k}$.

Suppose that there exist one n_i such that $n_i \leqslant n_j$ and $i \geqslant 3$. We know that the Cartesian product is associative. Hence, we consider G as $G \cong (K_{n_1} \Box K_{n_2} \Box K_{n_3} \ \ldots \ \Box K_{n_k}) \Box K_{n_i}$. Let $H \cong (K_{n_1} \Box K_{n_2} \Box K_{n_3} \Box \ \ldots \ \Box K_{n_k})$. Thus $G \cong H \Box K_{n_i}$. Let u_j and v_p where $j \in \{1, 2, 3, ..., |H|\}$ and $p \in \{1, 2, 3, ..., n_i\}$ be the vertices of H and K_{n_i} respectively.

If $n_i \equiv 0 \ mod \ 3$, then K_{n_1} has a P_3-factor. Consider $u_j v_p$-$u_j v_q$ where $j \in \{1, 2, 3, ..., |H|\}$ and $p, q \in \{1, 2, 3, ..., n_i\}$ in G. Since $n_i \equiv 0 \ mod \ 3$, K_{n_1} has a P_3-factor and hence G has a P_3-factor.

If $n_i \equiv 1 \ mod \ 3$, then we can find a spanning subgraph of K_{n_i} whose components are P_3 and P_4 where $v_1, v_2, ..., v_{n_i-4}$ are the vertices in the components of P_3 and $v_{n_i-3}, v_{n_i-2}, v_{n_i-1}, v_{n_i}$ are the vertices in the components of P_4. Now, in G the vertices $u_1 v_p, u_2 v_p, ..., u_{n_i-4} v_p$ where $p \in \{1, 2, 3, ..., |H|\}$ form a subgraph whose components are P_3. The $\langle u_j v_{n_i-3}, u_j v_{n_i-2}, u_j v_{n_i-1}, u_j v_{n_i} \rangle$

where $j \in \{1, 2, \ \dots \ , |H|\}$ form P_4s. Thus, G has a $\{P_3, P_4\}$-factor.

If $n_i \equiv 2 \ mod \ 3$, then we can find a spanning subgraph of K_{n_i} whose components are P_3 and P_4 where $v_1, v_2, ..., v_{n_i-8}$ are the vertices in the components of P_3 and $v_{n_i-7}, v_{n_i-6}, ..., v_{n_i-1}, v_{n_i}$ are the vertices in the components of P_4. Now, in G the vertices $u_jv_1, u_jv_2, ..., u_jv_{n_i-8}$ where $j \in \{1, 2, 3, ..., |H|\}$ form a subgraph whose components are P_3. The $\langle u_jv_{n_i-7}, u_jv_{n_i-6}, u_jv_{n_i-5}, u_jv_{n_i-4}\rangle$ and $\langle u_jv_{n_i-3}, u_jv_{n_i-2}, u_jv_{n_i-1}, u_jv_{n_i}\rangle$ where $j \in \{1, 2, 3, ..., |H|\}$ form P_4s. Thus, G has a $\{P_3, P_4\}$-factor. $\square$

Chapter 5

Domination criticality in the Cartesian product of graphs

A connected dominating set is an important notion and has many applications in routing and management of networks. In this chapter we study the Cartesian product of graphs G with connected domination number, $\gamma_c(G) = 2, 3$ and characterize such graphs. Also, we characterize the k - γ - vertex (edge) critical graphs and k - γ_c - vertex (edge) critical graphs for $k = 2$, 3 where γ denotes the domination number of G. We also

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Some results of this chapter are included in the following paper.
1. Chithra M.R., A. Vijayakumar, Domination criticality in product graphs (communicated).

discuss the vertex criticality in grids.

5.1 Domination critical graphs

Theorem 5.1.1. *Let $G \cong H_1 \square H_2$ be a connected graph. Then $\gamma(G) = 2$ if and only if $H_1 = K_2$ and H_2 is either a C_4 or has a universal vertex.*

Consider $G \cong K_2 \square C_4$, then a minimum dominating set of G is

$D = \{u_1v_1, u_2v_3\}$. If $G \cong K_2 \square H_2$, where H_2 has a universal vertex v_i, then a minimum dominating set of G is $D = \{u_1v_i, u_2v_i\}$. Hence, $\gamma(G) = 2$ in both the cases.

Conversely suppose that $\gamma(G) = 2$.

Suppose that both H_1 and H_2 are not complete graphs. Then, $\gamma(G) \geqslant min\{|H_1|, |H_2|\} \geqslant 3$.
Hence, at least one graph (say) H_1 should be complete.

Let $G \cong K_{n_1} \square H_2$.
Suppose that H_1 is a complete graph of order at least three. If H_2 has a universal vertex, then a minimum dominating set of G con-

tains vertices from each layer of G and $3 \leqslant \gamma(G) \leqslant min\{n_1, n_2\}$. If H_2 does not has a universal vertex, then $\gamma(H_2) \geqslant 2$ and a minimum dominating set of G contains vertices from each layer of G and $3 \leqslant \gamma(G) \leqslant n_1$. Thus, in both the cases $\gamma(G) \geqslant 3$. Hence, $n_1 = 2$. Thus, $G \cong K_2 \square H_2$.

Let $n_2 \geqslant 2$.
Then $\gamma(G) \leqslant min\{2\gamma(H_2), n_2\gamma(K_2)\} = min\{2\gamma(H_2), n_2\}$ (1).
From (1) we have $\gamma(G) = 2$ only when H_2 has a universal vertex, since $n_2 \geqslant 2$.

Next, we consider the case when $\gamma(H_2) \geqslant 2$.
Let $n_2 \geqslant 5$.
Suppose that H_2 contains a vertex v_i of degree $(n_2 - 2)$ and v_i is not adjacent to v_j, then $\gamma(H_2) = 2$. Now, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_1v_j\}$ and $\gamma(G) = 3$. Suppose that H_2 contains a vertex of degree at most $(n_2 - 3)$, then $\gamma(H_2) = 2$. Let v_p be a vertex of degree $(n_2 - 3)$ and is not adjacent to v_q and v_r in H_2. Then, in G the vertices u_1v_i and u_2v_i dominate $2n_2 - 4$ vertices and the remaining four vertices u_1v_q, u_1v_r, u_2v_q and u_2v_r cannot be dominated by a single vertex. Hence, in these cases $\gamma(G) \geqslant 3$. Thus, $n_2 \leqslant 4$.

Now, by an exhaustive verification of all graphs with $n_2 \leqslant 4$ it follows that $G \cong K_2 \Box C_4$.

Illustration of Theorem 5.1.1

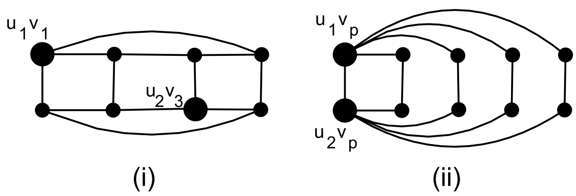

Fig 5.1: (i) $G = K_2 \Box C_4$, $\gamma(G) = 2$ (ii) $G = K_2 \Box K_{1,4}$, $\gamma(G) = 2$.

Corollary 5.1.2. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is 2 - γ - vertex critical if and only if $G = C_4$.*

Proof. In Theorem 5.1.1 we have characterized the Cartesian product of graphs with $\gamma(G) = 2$. Hence, we need to prove the theorem only for such Gs.

First, note that $G \cong K_2 \Box C_4$ is not 2 - γ - vertex critical.

Now, consider $G \cong K_2 \Box K_{n_2}$, where $n_2 \geqslant 3$. Then, a minimum dominating set $D = \{u_1v_x, u_2v_x\}$ of G contains a vertex from each layer of K_{n_2}. Now, let a vertex u_iv_p where $p \in \{1, 2, ..., n_2\}$, be deleted. If $p = x$, then we can find another minimum dominating set $D = \{u_1v_y, u_2v_y\}$. If $p \neq x$, then the minimum dominating set $D = \{u_1v_x, u_2v_x\}$ of G remains the same. Thus, in both the cases $\gamma(G - v) = \gamma(G)$ =2 $\forall v \in V(G)$. Hence, $H_2 = K_2$.

Consider $G \cong K_2\square H_2$ where H_2 is a not complete graph with a universal vertex v_p. Then, a minimum dominating set $D = \{u_1v_p, u_2v_p\}$ of G contains a vertex from each layer of H_2. Now, let a vertex u_1v_q where $q \in \{1, 2, ..., n_2\}$, be deleted. If $p \neq q$, then the minimum dominating set $D = \{u_1v_x, u_2v_x\}$ of G remains the same. If $p = q$, then in G the vertex u_2v_q dominate the n_2 vertices u_2v_i and the remaining n_2 vertices cannot be dominated by a single vertex, since we have deleted the universal vertex from the layer of H_2. Hence, $\gamma(G) \geqslant 3$. Thus, $G \cong K_2\square H_2$ is not 2 - γ - vertex critical. $\square$

Corollary 5.1.3. *Let $G \cong H_1\square H_2$ be a connected graph. Then G is* 2 - γ - *edge critical if and only if $G = C_4$.*

Proof. In Theorem 5.1.1 we have characterized the Cartesian product of graphs with $\gamma(G) = 2$. Hence, we need to prove the theorem only for such Gs.

First, note that $G \cong K_2 \square C_4$ is not 2 - γ - edge critical.

Consider $G \cong K_2 \square H_2$, where H_2 is a not complete graph with a universal vertex or a complete graph with $n_2 \geqslant 3$. Let an edge $u_1v_p - u_2v_i$ where $i \in \{1, 2, 3, \ \dots\ , n_2\}$, be added. Then, the addition of an edge does not make either G a complete graph or a graph with a universal vertex. Thus, $\gamma(G)$ remains the same. Hence, $H_2 = K_2$. □

Corollary 5.1.4. *Let $G \cong H_1 \square H_2$ be a connected graph. Then $\gamma_c(G) = \gamma(G) = 2$ if and only if $H_1 = K_2$ and H_2 has a universal vertex.*

Proof. It suffices to show that the dominating set of G in Theorem 5.1.1 is connected.

Consider $G \cong K_2 \square C_4$. Then a minimum dominating set of G is $D = \{u_1v_1, u_2v_3\}$ and $\gamma(G) = 2$. From Fig 5.1, it is clear that, $< D >$ is disconnected.

Consider $G \cong K_2 \square H_2$ where H_2 is a complete graph or a not complete graph with a universal vertex v_p.
Then a minimum dominating set of G is $D = \{u_1v_p, u_2v_p\}$ and $< D >$ is connected. Hence, $\gamma_c(G) = \gamma(G) = 2$. □

Corollary 5.1.5. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is 2 - γ_c - vertex critical if and only if $G = C_4$.*

Corollary 5.1.6. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is 2 - γ_c - edge critical if and only if $G = C_4$.*

Theorem 5.1.7. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then $\gamma(G) = 3$ if and only if G is the Cartesian product of any one of the following graphs where,*

(a) $H_1 = K_3$ or P_3 and H_2 has a universal vertex.

(b) $H_1 = K_2$ and H_2 has a vertex of degree $n_2 - 2$.

(c) $H_1 = K_2$ and H_2 has a vertex v_r of degree n_2-3 and is not adjacent to the vertices v_p and v_q with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$.

(d) $H_1 = K_3$ or P_3 and $H_2 = C_4$.

Proof. Let $G \cong H_1 \Box H_2$ where H_1 is a K_3 or P_3 and H_2 has a universal vertex v_i. Then, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_3v_i\}$ and $\gamma(G) = 3$.

If $G \cong K_2 \Box H_2$ where H_2 has a vertex v_j of degree $n_2 - 2$ and v_j is not adjacent to v_p in H_2, then a minimum dominating set of G is $D = \{u_1v_j, u_2v_j, u_1v_p\}$ and $\gamma(G) = 3$.

Further if $G \cong K_2 \Box H_2$ where H_2 has a vertex v_r of degree

$n_2 - 3$ and is not adjacent to the vertices v_p and v_q with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$, then a minimum dominating set of G is $D = \{u_1v_r, u_2v_p, u_2v_q\}$ and $\gamma(G) = 3$.
Now, $G \cong H_1 \Box C_4$ where H_1 is a K_3 or P_3, then a minimum dominating set of G is $D = \{u_1v_1, u_2v_3, u_3v_1\}$ and $\gamma(G) = 3$.

Conversely suppose that $\gamma(G) = 3$.

(I) Suppose that both H_1 and H_2 are complete graphs, where $n_1, n_2 \geqslant 4$.

Then $\gamma(G) \geqslant min\{4, 4\} = 4$. Thus, at least one graph (say) H_1 has order $n_1 \leqslant 3$. But $\gamma(G) = 3$ only when H_1 is a K_3. Hence, $G \cong K_3 \Box K_{n_2}$ where $n_2 \geqslant 3$.

(II) Suppose that H_1 is a complete graph and H_2 is not a complete graph.

If n_1, $n_2 \geqslant 4$, then $\gamma(G) \geqslant 4$. Thus, to prove the theorem we have to consider the following cases.

(1) Let $n_1 = 2$ and $n_2 = 3$, then $\gamma(G) = 2$.

(2) Let $n_1 = 2$ and $n_2 \geqslant 4$.

Consider $G \cong K_2 \Box H_2$. From (1) we have $\gamma(G) \leqslant min\{2\gamma(H_2), n_2\}$. Thus it is clear that we do not have to consider the case when $\gamma(H_2) = 1$, since $\gamma(G) = 3$. Hence, $\gamma(H_2) \geqslant 2$.

If $\gamma(H_2) \geqslant 3$, then $\gamma(G) \geqslant 4$. Hence, we need consider only the case when $\gamma(H_2) = 2$.

Now, suppose that H_2 is not a complete graph with $\gamma(H_2) = 2$.

Suppose that a minimum dominating set of H_2 is $D = \{v_p, v_q\}$.

Let a minimum dominating set of G be $D = \{u_1v_p, u_1v_q, u_2v_p\}$. The vertices u_1v_p and u_1v_q dominate $n_2 + 2$ vertices in G. Now, the remaining $2n_2 - (n_2 + 2) = n_2 - 2$ vertices will be dominated by a single vertex u_2v_p, only if $\deg(v_p) = n_2 - 2$. Hence, H_2 has a vertex of degree $n_2 - 2$. This, proves (b).

Let a minimum dominating set of G contain a vertex u_1v_r where v_r is not a neighbour of v_p and v_q in H_2. The vertex u_1v_r dominate the $n_2 - 1$ vertices u_1v_x and u_2v_r, where $x \neq r \in \{1, 2, ..., n_2\}$ in G. If the dominating set contain the vertex u_2v_r, then the vertices u_1v_r and u_2v_r dominate $2n_2 - 4$ vertices in G. The remaining four vertices u_1v_q, u_1v_q, u_2v_p and u_2v_q cannot be dominated by a single vertex and hence $\gamma(G) \geqslant 3$. Thus, the dominating set does not contain the vertex u_2v_r. Since, a minimum dominating set of G contain the vertex u_1v_r and v_r is not a neighbour of v_p and v_q in H_2, the dominating set of G should contain the vertices u_2v_p and u_2v_q. Now, the remaining $2n_2-(n_2-1)=n_2+1$ vertices will be dominated by the vertices u_2v_p and u_2v_q, only if $N[v_p]\cup N[v_q]=V(H_2)-v_r$. Hence, H_2 has a vertex v_r of degree n_2-3 and is not adjacent to the vertices v_p and v_q with $N[v_p]\cup N[v_q]\cup\{v_r\}=V(H_2)$. This, proves (c).

(3) By an exhaustive verification of all graphs with $n_2=4$ it follows that $G\cong K_3\Box C_4$.

(4) Let $n_1=3$ and $n_2\geqslant 4$.

Consider $G\cong K_3\Box H_2$. Let $\gamma(H_2)\geqslant 2$, then $\gamma(G)\geqslant 4$. Thus, H_2 has a universal vertex.

(5) Let $n_2=3$ and $n_1\geqslant 3$.

Consider $G\cong K_{n_1}\Box P_3$, then a minimum dominating set of G is

$D=\{u_1v_1,u_1v_2,u_1v_3\}$. The vertex u_1v_1 dominate the vertices u_iv_1 where $i\in\{1,2,\ ...\ ,n_1\}$, since H_1 is a complete graph. Similarly, the vertices u_1v_2 and u_1v_3 dominate the remaining vertices in G. Thus, $\gamma(G) = 3$.

(III) Suppose that both H_1 and H_2 are not complete graphs. If $n_1, n_2 \geqslant 4$, then $\gamma(G) \geqslant 4$. Hence, $n_1 = 3$ and $n_2 \geqslant 4$.
If $\gamma(H_2) \geqslant 3$, clearly $\gamma(G) \geqslant 4$. Hence, the domination number of H_2 is at most 2.

We know that $\gamma(G) \leqslant min\{3\gamma(H_2), n_2\}$.
If $\gamma(H_2) = 1$ where v_i is a universal vertex in H_2, then $\gamma(G) = 3$. Hence, $G \cong P_3 \Box H_2$ where H_2 has a universal vertex.

Now, suppose that $\gamma(H_2) = 2$ and $n_2 \geqslant 6$, then by a similar argument of II(2) it follows that $\gamma(G) \geqslant 4$. Hence, $n_2 \leqslant 5$.
By an exhaustive verification of all graphs with $n_2 = 3, 4, 5$ it follows that $G \cong P_3 \Box C_4$. $\Box$

Illustration of Theorem 5.1.7.

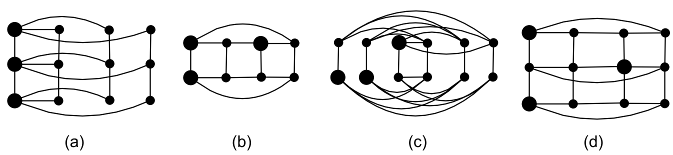

Fig 5.2: (a) $G = P_3 \Box K_{1,3}$, $\gamma(G) = 3$.
(b) $G = K_2 \Box C_4$, $\gamma(G) = 3$.
(c) $G = K_2 \Box K_{3,3}$, $\gamma(G) = 3$.
(d) $G = P_3 \Box C_4$, $\gamma(G) = 3$.

Corollary 5.1.8. *Let $G \cong H_1 \square H_2$ be a connected graph. Then G is a* 3 *-* γ *- vertex critical graph if and only if $H_1 = H_2 = K_3$.*

Proof. It suffices to prove that $\gamma(G - v) < \gamma(G)$ $\forall v \in G$ of Theorem 5.1.7.

Consider $G \cong K_3 \square K_{n_2}$ where $n_2 \geqslant 4$. Then, a minimum dominating set $D = \{u_1v_x, u_2v_x, u_3v_x\}$ of G contains a vertex from each layer of K_{n_2}. Now, let a vertex u_iv_p where $p \in \{1, 2, ..., n_2\}$, be deleted. If $p = x$, then we can find another minimum dominating set $D = \{u_1v_y, u_2v_y, u_3v_y\}$. If $p \neq x$, then the minimum dominating set $D = \{u_1v_x, u_2v_x, u_3v_x\}$ of G remains the same. Thus, in both cases $\gamma(G - v) = \gamma(G)$ =3 $\forall v \in V(G)$. Hence, $H_2 = K_3$.

Consider $G \cong K_3 \square H_2$ or $G \cong P_3 \square H_2$ where H_2 has a universal vertex v_i. Then, a minimum dominating set
$D = \{u_1v_i, u_2v_i, u_3v_i\}$ of G contains a vertex from each layer of H_2. Now, let a vertex u_1v_q where $q \in \{1, 2, ..., n_2\}$, be deleted. If $i \neq q$ then, the minimum dominating set $D = \{u_1v_i, u_2v_i, u_3v_i\}$ of G remains the same. If $p = i$, then in G, the vertices u_2v_i and u_3v_i dominate the $2n_2$ vertices and the remaining n_2 vertices

u_1v_x, where $q \in \{1, 2, ..., n_2\}$ cannot be dominated by a single vertex, since we have deleted the universal vertex from the layer of H_2. Hence, $\gamma(G) \geqslant 3$. Thus $\gamma(G - v) \geqslant \gamma(G)\ \forall v \in V(G)$. Hence, $G \cong K_3 \Box H_2$ or $G \cong P_3 \Box H_2$ is not 3 - γ - vertex critical.

Consider $G \cong K_2 \Box H_2$ where H_2 has a vertex v_i of degree $(n_2 - 2)$ and is not adjacent to the vertex v_j. Then, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_1v_j\}$ and $\gamma(G) = 3$. Now, let a vertex u_1v_q where $q \in \{1, 2, ..., n_2\}$, be deleted. If $i \neq q$, then the minimum dominating set $D = \{u_1v_i, u_2v_i, u_1v_j\}$ of G remains the same. If $q = i$, then in G, the vertices u_2v_i and u_1v_j dominate the $n_2 + 1$ vertices and the remaining $n_2 - 1$ vertices u_1v_x cannot be dominated by a single vertex, since we have deleted the vertex u_1v_i from the layer of H_2. Hence, $\gamma(G) \geqslant 3$ and $G \cong K_2 \Box H_2$, where H_2 has a vertex v_i of degree $(n_2 - 2)$, is not 3 - γ - vertex critical.

Consider $G \cong K_2 \Box H_2$, where H_2 has a vertex v_p of degree at most $(n_2 - 3)$ and v_p is not adjacent to v_q and v_r. Then then by a similar argument, as in the above case, it follows that $\gamma(G) \geqslant 3$. Hence, $G \cong K_2 \Box H_2$, where H_2 has a vertex of degree at most $(n_2 - 3)$, is not 3 - γ - vertex critical.

Consider $G \cong K_3 \Box C_4$, $G \cong P_3 \Box C_4$ and $G \cong K_2 \Box C_5$. In all these cases G is not 3 - γ - vertex critical. $\Box$

Corollary 5.1.9. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then G is a* 3 - γ - *edge critical graph if and only if $H_1 = H_2 = K_3$.*

Proof. It suffices to prove that $\gamma(G + e) = 2 \ \forall e \notin G$ of Theorem 5.1.7.

Consider $G \cong K_3 \Box H_2$ or $G \cong P_3 \Box H_2$, where H_2 has a universal vertex v_1 and $n_2 \geqslant 4$. Then, a minimum dominating set of G is $D = \{u_1v_1, u_2v_1, u_3v_1\}$. In G, the vertex u_1v_1 dominate the n_2 vertices u_1v_i, where $i \in \{1, 2, 3, \ ... \ , n_2\}$ and u_2v_1 dominate the n_2 vertices u_2v_i, where $i \in \{1, 2, 3, \ ... \ , n_2\}$. Let an edge $u_1v_1 - u_2v_p$, be added. Then, in G the vertex u_1v_1 dominate the $n_2 + 1$ vertices u_1v_i, u_2v_p, where $i \in \{1, 2, 3, \ ... \ , n_2\}$ and u_2v_1 dominate the $n_2 - 1$ vertices u_2v_i, where $i \neq p \in \{1, 2, 3, \ ... \ , n_2\}$ and u_3v_1 dominate the n_2 vertices u_3v_i, where $i \in \{1, 2, 3, \ ... \ , n_2\}$. Hence,the minimum dominating set $D = \{u_1v_1, u_2v_1, u_3v_1\}$ of G remains the same. Thus, $n_2 = 3$. By an exhaustive verification of all such graphs, it follows G is a 3 - γ - edge critical graph if and only if $H_1 = H_2 = K_3$.

Consider $G \cong K_2 \Box H_2$, where H_2 has a vertex v_i of degree n_2-2 and v_i is not adjacent to v_j. Then, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_1v_j\}$. In G, the vertex u_1v_i dominate the $n_2 - 1$ vertices u_1v_p, where $p \neq j \in \{1, 2, 3, \ \dots \ , n_2\}$ and u_2v_i dominate the $n_2 - 1$ vertices u_2v_p, where $p \neq j \in \{1, 2, 3, \ \dots \ , n_2\}$. Let an edge $u_1v_i - u_2v_j$, be added. Then, in G the vertex u_1v_i dominate the n_2 vertices u_1v_p, u_2v_j, where $p \neq j \in \{1, 2, 3, \ \dots \ , n_2\}$ and u_2v_1 dominate the $n_2 - 1$ vertices u_2v_p, where $p \neq j \in \{1, 2, 3, \ \dots \ , n_2\}$ and the remaining one vertex u_1v_j is not dominated by the vertices u_1v_i and u_2v_i. Hence, the minimum dominating set $D = \{u_1v_i, u_2v_i, u_1v_j\}$ of G remains the same. Thus, $G \cong K_2 \Box H_2$, where H_2 has a vertex v_i of degree $n_2 - 2$, is not 3 - γ - edge critical.

Consider $G \cong K_2 \Box H_2$, where H_2 has a vertex v_p of degree at most $(n_2 - 3)$ and v_p is not adjacent to v_q and v_r. Then, by a similar argument, as in the above case, it follows that $\gamma(G) = 3$. Hence, $G \cong K_2 \Box H_2$ where H_2 has a vertex of degree at most $(n_2 - 3)$, is not 3 - γ - edge critical.

In all other cases, G is not 3 - γ - edge critical. $\square$

Corollary 5.1.10. *Let $G \cong H_1 \Box H_2$ be a connected graph. Then*

$\gamma_c(G) = \gamma(G) = 3$ if and only if $H_1 = K_3$ or P_3 and H_2 has a universal vertex.

Proof. It suffices to prove that the dominating set of G in Theorem 5.1.7 is connected.

Consider $G \cong K_3 \Box H_2$ or $G \cong P_3 \Box H_2$, where H_2 has a universal vertex v_i.
Then, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_3v_i\}$ and $\gamma(G) = 3$. Also, $< D >$ is connected. Hence, $\gamma_c(G) = 3$.

Consider $G \cong K_2 \Box H_2$, where H_2 has a vertex v_j of degree $(n_2 - 2)$ and v_j is not adjacent to v_x.
Then, a minimum dominating set of G is $D = \{u_1v_j, u_2v_j, u_1v_x\}$ and $\gamma(G) = 3$. Also, $< D >$ is disconnected, since v_j is not adjacent to v_x in H_2. Hence, $\gamma_c(G) \geqslant 4$.

Consider $G \cong K_2 \Box H_2$, where H_2 has a vertex v_p of degree $(n_2 - 3)$ and v_p is not adjacent to v_q and v_r with $N[v_p] \cup N[v_q] \cup \{v_r\} = V(H_2)$.
Then, a minimum dominating set of G is $D = \{u_1v_p, u_2v_q, u_2v_r\}$ and $\gamma(G) = 3$. Also, $< D >$ is disconnected, since v_p is not adjacent to v_q and v_r in H_2. Hence, $\gamma_c(G) \geqslant 4$.

In all other cases, $\gamma(G) = 3$ and $< D >$ is disconnected. Hence, $\gamma_c(G) \geqslant 4$. □

Corollary 5.1.11. *Let $G \cong H_1 \square H_2$ be a connected graph. Then G is a 3 - γ_c - vertex critical graph if and only if $H_1 = H_2 = K_3$.*

Corollary 5.1.12. *Let $G \cong H_1 \square H_2$ be a connected graph. Then G is a 3 - γ_c - edge critical graph if and only if $H_1 = H_2 = K_3$.*

5.2 Vertex criticality in grids

Theorem 5.2.1. *Let $G \cong P_{n_1} \square P_{n_2}$. Then G is vertex critical if and only if $G \cong P_2 \square P_2$.*

Proof. It suffices to prove the converse.

Let $G \cong P_{n_1} \square P_{n_2}$, where $n_1, n_2 \geqslant 3$.
Let $u_i v_j \in D$, where $i \neq 1, n_1$. Since, each vertex in D will dominate at most five vertices, it will dominate two vertices from the P_{n_2} - layer at u_i and two vertices each from the P_{n_2} - layer at u_{i-1} and P_{n_2} - layer at u_{i+1}, where u_{i-1}, u_{i+1} are the neighbours of u_i in P_{n_1}. Let a vertex $u_i v_{j-1}$, be deleted.

Then, the minimum dominating set D of G remains the same. Hence, G is not a vertex critical graph. Thus, $n_1 = n_2 = 2$. $\square$

Theorem 5.2.2. *Let $G \cong P_{n_1} \square P_{n_2}$. If $n_1, n_2 \geqslant 4$, then a minimal dominating set of G is disconnected.*

Proof. Let $u_i v_j \in D$. Since, the maximum degree of a vertex in G is four, each vertex in D will dominate at most five vertices, it will dominate two vertices from the P_{n_2} - layer at u_i and two vertices each from the P_{n_2} - layer at u_{i-1} and P_{n_2} - layer at u_{i+1}, where u_{i-1}, u_{i+1} are the neighbours of u_i in P_{n_1}.

Now, suppose that the vertex $u_p v_{j+1} \in D$. If $p \neq i$, then $< D >$ is disconnected. If $p = i$, then D should contain a vertex either from $u_{i-1}v_x - u_{i-1}v_y$ or from $u_{i+1}v_x - u_{i+1}v_y$, since $n_1, n_2 \geqslant 4$. Then, $< D >$ is disconnected. $\square$

Corollary 5.2.3. *Let $G \cong P_{n_1} \square P_{n_2}$. Then $\gamma_c(G) = \gamma(G)$ if and only if G is any one of the following graphs where,*

(a) $G \cong P_2 \square P_2$.

(b) $G \cong P_2 \square P_3$.

(c) $G \cong P_3 \square P_3$.

(d) $G \cong P_3 \square P_4$.

Concluding Remarks

In this thesis we have discussed mainly two metric related notions - the diameter variability and the diameter vulnerability, in graph products. Also studied are the notions of the component factors and the domination criticality. This study is quite far from being complete. We list below some problem which we found are interesting, but could not be attempted for various reasons.

1. Obtain an upper bound for $D^k(G * H)$ and $D^{-k}(G * H)$ where $* \in \{\Box, \boxtimes, \circ\}$.

2. Characterize the graphs with $f'(G * H) = \text{diam}(G) + 1$, where $* \in \{\boxtimes, \circ\}$.

3. Characterize the graphs with $f'(G) = f(G)$.

4. Study some other component factors of the graph products

5. Characterize the graphs with $\gamma(G * H) = 2, 3$ where $* \in \{\boxtimes, \circ\}$.

6. Characterize the graphs with $\gamma_c(G * H) = 2, 3$ where $* \in \{\boxtimes, \circ\}$.

List of symbols

C_n	-	Cycle of length n
P_n	-	Path of length $n-1$
K_n	-	Complete graph on n vertices
$K_{1,n}$	-	Star graph of size n
$deg(v)$	-	Degree of v
$\Delta(G)$	-	Maximum degree of vertices in G
$\delta(G)$	-	Minimum degree of vertices in G
$\mathrm{diam}(G)$	-	Diameter of G
$d(u,v)$	-	Distance between u and v in G
G^c	-	Complement of G
$V(G)$	-	Vertex set of G
$\lvert V(G)\rvert$	-	Number of vertices of G
$E(G)$	-	Edge set of G
$\lvert E(G)\rvert$	-	Number of edges of G
$< V >$	-	Graph induced by V
$\lceil x \rceil$	-	Smallest integer $\geqslant x$
$\lfloor x \rfloor$	-	Greatest integer $\leqslant x$

$\kappa(G)$	-	Vertex connectivity of G
$\kappa'(G)$	-	Edge connectivity of G
$G \cong H$	-	G is isomorphic to H
$G\Box H$	-	Cartesian product of G and H
$G \boxtimes H$	-	Strong product of G and H
$G \circ H$	-	Lexicographic product of G and H
$D^{-k}(G)$, $D^k(G)$, $D^0(G)$	-	Diameter variability of G
Q_n	-	Hypercube on n vertices
$K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_k}$	-	Hamming graph
$f(G)$	-	Fault diameter of G
$f'(G)$	-	Diameter vulnerability of G
$C_w(u,v)$	-	w-container between u and v
$D_w(G)$	-	w- wide diameter of G
$\gamma(G)$	-	Domination number
$\gamma_c(G)$	-	Connected domination number

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Index

adjacent, 7

bipartite graph, 9

Cartesian product, 13, 38, 86, 89, 122, 140
clique, 9
closed neighborhood, 7
complement, 10
complete
  bipartite graph, 10
  graph, 9
component factor, 15, 121
connected, 9
  domination number, 11
cut vertex, 10
cycle, 9
cylinder, 13, 112

degree, 7
diameter, 9
  fault, 18, 98, 100, 105
  maximal, 12, 53, 66, 83
  minimal, 12, 47, 62, 80
  variability, 17, 37
disconnected graph, 9
distance, 9
dominating set, 10

eccentricity, 9
edge, 7
edge connectivity, 10
Edge critical graphs, 11, 143, 145
end vertex, 7

graph, 7
grid, 13, 107

Hamming graph, 15, 136
hypercube, 14, 135

incident, 7
isolated vertex, 8
isomorphic, 8

k-connected, 10
k-vertex cut, 10
k-regular, 8

layer, 14
length of the container, 16
lexicographic product, 14, 67, 95, 100

neighborhood, 7

order, 7

path, 9
pendant vertex, 8

radius, 9

size, 7
strong product, 13, 54, 92, 105
subgraph, 8
  induced, 8
  spanning, 8

torus, 13, 115
totally disconnected, 7, 10
trail, 9
tree, 9

u-v path, 9
universal vertex, 8

vertex, 7
  connectivity, 10
vertex critical graphs, 11, 142, 145, 155

w-container, 16
walk, 8
wide diameter, 16, 117
width, 16