T-POLICY IN RELIABILITY AND INVENTORY

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Rekha.A under my guidance in the Department of Mathematics, Cochin University of Science & Technology and has not been included in any other thesis submitted previously for the award of any degree.

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Chapter 1

Introduction

Many real world phenomena require the analysis of systems in a probabilistic rather than deterministic setting. Stochastic models are becoming increasingly important for understanding and for assessing performance evaluation of complex systems in broad spectrum of fields such as Operations Research, Computer Science, Telecommunication and Engineering. In this thesis T-policy is implemented to the (s, S) inventory system with random lead time and also repair in the reliability of k-out-of-n system.

In this thesis we analyze an (s, S) inventory system with random lead time under T-policy and also a repairable k-out-of-n system with control policy governed by T-policy.

Inventory may be defined as a physical stock of goods kept in a system for the smooth and efficient business transactions. Inventory system may be considered as the system of keeping records of the amounts of commodities in stock. In an inventory problem lead time is defined as the time between the placement of order and the actual time at which units reach system. Several policies may be used to control an inventory system, of these, the most widely used is the (s, S) policy. Under this policy, whenever the position inventory reaches a level less than or equal to s for the first time measured from the previous replenishment epoch, a procurement is made to bring its level to S. Under a continuous review system, the (s, S) policy will usually imply the procurement of a fixed quantity $M = S - s$ of the commodity, while in periodic review systems the procurement quantity will vary. The (s, S) policy incorporates two decision variables s and S. The variable s is called the reorder level, which identifies when to order, while $S - s$ identifies how much to order.

During the lead time it may happen that there is no backlog, finite backlog (which will be met immediately on replenishment) or a large number of lost sales. In the latter two cases there is every chance of loss of customer goodwill and consequent loss to the system. Inorder to overcome this, T-policy is introduced during lead time.
We define the T-policy as follows: a replenishment does not occur within T units (a r.v) after the placement of an order, a local purchase is made either (i) bring the inventory level to S cancelling the replenishment order placed or (ii) to bring the inventory level to s or (iii) to bring the inventory level to 0 without cancelling the order (that is to meet all the backlogs, if any, without cancelling the order).

Local purchases by shop keepers are very common. This will ensure goodwill of customer to a great extent. Situations of this sort arise in practice. In shops when certain goods run out of stock and reaches a threshold (a negative level) due to backlogging the owner goes for local purchase. The local purchase involve higher cost to the system. The introduction of T-policy ensures the minimum number of loss of demands by taking decision at the right moment.

Inventory system of (s, S) type had been extensively studied in the past. A systematic account of such inventory system was first provided by Arrow, Karlin and Scarf [1958]. Further details of work carried out in this field can be found in Hadley and Whittin [1963], Veinott [1966], Sivazlian [1974]. (s, S) inventory policy with renewal demands and general lead time distribution was first considered by Srinivasan [1979]. Sahin [1979] deals with (s, S) policy where demand quantity is a continuous random variable and lead time is a constant. Sahin [1983] compute the binomial moments of the inventory level in an (s, S) inventory with compound renewal demand and arbitrarily distributed lead time. Manoharan, Krishnamoorthy and Madhusoodan [1987] investigate (s, S) inventory policy with unit demand and non-identically distributed inter-arrival times of demands having arbitrary lead time distribution.

Several models for perishable inventory systems can be found in the review article by Nahmias [1982]. N-policy in the queueing setup has been discussed by several authors (see Artalejo [1992], Gakis et.al [1995], Teghem Jr.[1986], Heyman [1967], Balachandran [1973].

(s, S) inventory system with N-policy during lead time have been introduced and investigated through a series of paper by Krishnamoorthy and Raju [1998, 1999] and Raju [1998]. In N-policy a local purchase is made when the number of backlogs reaches N.

We can make a note on some control policies in queueing system. Consider a steady state $M/G/1$ queueing system. Server remains in the system till all waiting customers are served. When the number of customers in the system reaches N, where $N \geq 1$, for the first time after the server is removed, it returns immediately and provides service until there are no customers in the system. This operating policy is called the N-policy in queueing context. In the T-policy the removed server returns to the system and provides service, on the elapse of T time units from the epoch of server removal, if there is at least one customer present in the waiting line. He continues to serve until there are no customers in the system, at which time the server is removed again to return after T time units. This process continues. Finally, if the workload or backlog, which is equal to the sum of the service time of waiting customers, exceeds D (where $D > 0$) for the first time after removal of the server, it returns to the system and provides service to all customers when the system is empty. Together with these, six different dyadic policies which are different combinations of the T-policy, N-policy and the D-policy are also studied in queueing literature. They are (i) the T^M/N-policy (ii) the T^M/D-policy (iii) the min(N, D)-policy (iv) min(T, N)-policy (v) the min(T, D)-policy (vi) the max(N, D)-policy. In the T^M/N-policy, a T-policy is first used once the server becomes idle. If following an idle period no customer appears in the first MT time units, where $M = 1, 2, \ldots$, is a given quantity, then the server switches to an N-policy. Thus, an N-policy is used if the server remains idle for MT time units, the N-policy is initiated at the end of MT time units. In the T^M/D policy a T-policy is again used first once the server becomes idle. If no customer appears during the first MT time units, where $M = 1, 2, \ldots$, is a given quantity, the server switches to a D-policy. Thus a D-policy is used if the server remains idle for MT time units.

Reliability of k-out-of-n system under D-policy has been studied by A Krishnamoorthy and P.V. Ushakumari [2000]. In the min(N, D) policy, following the start of an idle period or on completion of an idle period the server restarts serving and hence initiates a busy period, if either N customers have accumulated in the system ($N \geq 1$) or the total accumulated backlog of customers service time exceed D, whichever occurs first. Similar interpretations can be given to other policies also. For further details one may refer to Yadin and Naor [1963], Heyman [1977], Levy and Yechiali [1975], Balachandran and Tijms [1975], Bell [1971,73,80], Tegham [1986] and Gakis, Rhee and Sivazlian [1995].

We have also introduced the repair of a k-out-of-n system under T-policy. Several models are analysed under this set up.

Reliability is generally characterized or measured by the probability that an entity can perform one or several required functions under given conditions for a given time interval. The term 'entity' is used here to denote any component, subsystem, system or equipment that can be individually considered and tested separately. According to the entities, the notion of time interval should be replaced by the notion of number of cycles, distance travelled etc.

Reliability is defined as the ability of an entity to perform a required function under given conditions for a given time interval. It is measured by the probability that an entity E can perform a required function under given conditions for the time interval $[0, t]$. Thus $R(t) = P(E$ does not fail during $[0, t])$. The reverse of this ability is called unreliability. A system is a deterministic entity comprising an interconnected or interacting collection of discrete elements.

Suppose that a system has finite number n of independent components labelled $1, 2 \ldots,$ n and that the system is capable of just two modes of performance. Represent the mode of performance of the system by the Bernoulli r.v. X. Suppose that, given the structure of a system, the knowledge of its performance can be determined from that of its components. The system structures generally considered are described below.

i) *Series system* The system functions iff all the n components functions. We have $X = \min(X_1, X_2, \ldots X_n)$;, the reliability of n components is given by $P = P(X = 1) = P(\min(X_1, X_2, \ldots, X_n) = 1) = P(X_1 = 1, X_2 = 1, \ldots, X_n = 1) = \prod_{i=1}^{n} P_i$ (Here $X_i = 1$ indicates that ith component is operational and $P_i = P(X_i = 1)$, $i = 1, 2, \ldots, n$)

ii) *Parallel system* : The system functions iff at least one of the n components functions.

We have $X = \max(X_1, X_2, \ldots, X_n)$. The system reliability is given by

$$P = P(X = 1) \quad = P(\max(X_1, X_2, \ldots, X_n) = 1)$$
$$= 1 - P(X_1 = 0, X_2 = 0, \ldots, X_n = 0)$$
$$= 1 - \prod_{i=1}^{n}(1 - P_i)$$

iii) *k-out-of-n system* : The system function iff at least $k(1 \leq k \leq n)$ of the n components functions. As particular cases, we get the series system for $k = n$ and the parallel system for $k = 1$.

k-out-of-n system have been studied extensively (see, for example Angus [1988], Godbole, Potter and Sklar [1998], Pham and Upadhyaya [1988]. Madhujain and Ghimira [1997] discuss the reliability of k-r-out-of-n system. k-out-of-n system in discrete time with multiple repair facilities has been discussed in kapur, Garg, Sehgal and Jha [1997].

k-out-of-n system with the N-policy for repair of failed units has been discussed in detail by Krishnamoorthy, Ushakumari and Lakshmi [1998] under the assumption of exponential life times for components. Under this policy a server is called for repair as soon as the number of failed units reach $N(\leq n - k)$. Further Ushakumari and Krishnamoorthy[1998] examine the control problem of obtaining the optimal N value when the service times of units have arbitrary distribution. They analyze the semi-Markov process and the embedded Markov chain arising in this setup.

The optimal number of repairs in the context of analysing systems subject to shocks have been considered by Shen and Griffith [1996]. This can be regarded as the optimal N-policy for replacement. Rangan and Sarada [1992 a,b] discuss the optimal strategies of replacement for deteriorating system with changing failure distributions. Lam Yeh [1990] analyses a single repairable replacement model and in [1991] he obtains the optimal number of repairs before replacement.Rangan and Grace [1989] provide the optimal replacement policies for deteriorating systems with imperfect maintenance. Ushakumari [1998] has analyses a k-out-of-n system with repair of failed units under (N, T)-policy. Here the amount of time for which the server is not available in the system is a random variable which is the minimum of an exponentially distributed time duration T and the sum of N

independent exponentially distributed random variables that are not neccessarily identically distributed (ie. a generalized Erlang variate).

For some of the combination policies it is impossible to get analytical solution (eg. probability distribution of the system state). In such cases one can resort to numerical studies and also analyse certain performance characteristics.

In this thesis, we have considered a k-out-of-n system with repair under T-policy. Server is activated after the elapse of T time units where T is exponentially distributed with parameter α from the epoch at which it was inactivated after completion of repair of all failed units in the previous cycle, or the moment $n-k$ failed units accumulate, whichever occurs first. Thus server is activated at the moment which is $\min\{T, E_{n-k,\lambda}\}$ after his previous departure where $E_{n-k,\lambda}$ is an Erlang distributed r.v. with parameters $n-k$ and λ. He continues to remain active until all the failed units are repaired and then inactivated. The process continues in this fashion. The repaired units are assumed to be as good as new. Life time of components and service time (repair times) are assumed to be exponentially distributed with rates λ and μ, respectively. We consider three different situations: (a) cold system (b) warm system (c) hot system. A k-out-of-n system is called cold, warm or hot according as the functional units do not fail, fail at a lower rate or fail at the same rate when system is shown as that when it is up.

k-out-of-n system with repair and two modes of service under N-policy has been introduced by A. Krishnamoorthy and P.V. Ushakumari [1999]. In this thesis, we consider k-out-of-n system with repair and two modes of service under T-policy. In this case first server is available always and second server is activated on elapse of T time units. Reliability of a k-out-of-n system with repair and retrial of failed units has been introduced by A. Krishnamoorthy and P.V. Ushakumari [1999]. Retrial queues have been extensively studied by many researchers, an excellent account of which can be found in Falin and Templeton[1997]

Basic concepts

1.1 Definition : Renewal Process

Consider a specific phenomenon that occurs randomly in time. Let $w_1, w_2, \dots$ be the times between its successive occurrences. Write $S_0 = 0$; $S_{n+1} = S_n + W_{n+1}, n \in \mathbb{N}$. This sequence defines the times of occurrence of the event assuming that the time origin is taken to be an instant of such an occurrence. The sequence $S = \{S_n, n \in \mathbb{N}\}$ is called a renewal process provided that $w_1, w_2, \dots$ are independent and identically distributed non-negative random variables. Then the S_n, $n \in \mathbb{N}$ is called the nth renewal epoch.

Consider the number of renewals N_t in the interval $[0, t]$; this is $N_t(w) = \sum_{n=0}^{\infty} I_{[0,t]}$ $(S_n(w))$, $t \geq 0$, $w \in \Omega$, where $I_A(x) = 1$ or 0 according as $x \in A$ or $x \notin A$. Note that $N_0(w) \geq 1$ always, and that $N_t(w) = \inf\{n \in N \quad S_n(w) > t\}$. Thus the event $\{N_t = k\}$ is equal to the event $\{S_{k-1} \leq t; S_k > t\} = \{S_{k-1} \leq t\} \cap \{S_k \leq t\}^c$, and $\{S_k \leq t\} \subset \{S_{k-1} \leq t\}$. Since $S_k > S_{k-1}$. Thus, for any $k = 1, 2, \dots$ $P(N_t = k) = P(S_{k-1} \leq t) - P(S_k \leq t) = F^{(k-1)}(t) - F^{(k)}(t)$ where $F^{(l)}(.)$ is the l-fold convolution of F with itself. One can compute the expected member of renewals in $[0, t]$ by using this distribution :

$$R(t) = E[N_t] = \sum_{n=0}^{\infty} E[I_{[0,t]}(S_n)] = \sum_{n=0}^{\infty} P(S_n \leq t) = \sum_{n=0}^{\infty} F^n(t)$$

The function $R = 1 + F + F^2 + \dots$ is called the renewal function corresponding to the distribution F.

1.2 Definition : Regenerative Process

Consider a stochastic process $Z = \{Z_t, t \geq 0\}$ with state space E. Suppose that every time a specified event occurs, the future of the process Z after that time becomes a probabilistic replica of the past. Such times (usually random) are called regeneration times of Z, and the process Z is then said to be regenerative.

Let Z be a regenerative process with a discrete state space, and consider the probability $f(t)$ that $Z_t = i$ for some fixed state i. We condition the event $\{Z_t = i\}$ on the time S_1

of first regeneration, and argue as follows. The process Z regenerates itself at S_1 and the future process $\hat{Z}$ defined by $\hat{Z}_n = Z_{S_1+u}$ has the same probability law as Z itself. Given S_1, if $S_1 = S \leq t$, then $Z_t = \hat{Z}_{t-s}$, and therefore

$$P(Z_t = i/S_1) = P(\hat{Z}_{t-s} = i) = f(t-s) \text{ on } \{S_1 = s \leq t\}$$

Hence, if we define $g(t) = P\{Z_t = i, S_1 > t\}$ then we have $f(t) = g(t) + \int_{[0,t]} F(ds)f(t-s)$. This equation is called a renewal equation. Renewal theory is the study of the renewal equation $f = g + F * f$ where F is a distribution function on $R+$ and f and g are function which are bounded over finite intervals. The renewal equation has one and only one solution $f = R * g$ where $R = \sum F^n$ is the renewal function corresponding to F.

It is wellknown that with probability 1, $\frac{N_t}{t} \to \frac{1}{\mu}$ as $t \to \infty$ where $\mu = \int_0^{\infty} xdF(x)$ (see for example Ross [1970])

1.3 Markov Renewal Process

Definition : The Stochastic process $(X,T) = \{X_n, T_n, n \in N\}$ is said to be a Markov renewal process with state space E provided that

$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t/X_0, X_1, \ldots X_n, T_0, \ldots T_n\}$$
$$P\{X_{n+1} = j, T_{n+1} - T_n \leq t/X_n\} = Q(i,j,t) \quad \text{for all } n \in N, j \in E$$

and $t \in R+$ (Cinlar [1975])

Markov renewal theory combines renewal theory with the theory of Markov chains to create tools that are more powerful than those which either could provide. Consider a process which moves from one state to another with random sojourn times in between such that the successive states visited form a Markov chain and the sojourn time has distribution which depends on the state being visited as well as the next state to be entered.

The family of probabilities $\mathbb{Q} = \{Q(i,j,t) : i,j \in E, t \in R+\}$ is called a semi-Markov kernel over E. For each pair (i,j) the function $t \to Q(i,j,t)$ has all the properties of a distribution function except that $P(i,j) = lim_{t\to\infty} Q(i,j,t)$ is not necessarily equal to one, we can see that $P(i,j) \geq o, \sum_{j \in E} P(i,j) = 1$ that is, the $P(i,j)$ are the transition probabilities for some Markov chain with state space E.

1.4 Proposition

$X = \{X_n, n \in N\}$ is a Markov chain with state space E and transition matrix P. Another convenient picture in describing a Markov renewal process is provided by the process $Y = \{Y_t : t > 0\}$ defined by putting for each $t \geq 0$ and $w \in \Omega$

$$Y_t(w) = \begin{cases} X_n(w) & \text{if } T_n(w) \leq t < T_{n+1}(w) \\ \triangle & \text{if } t \geq \sup_n T_n(w) \text{ where } \triangle \text{ is a point not in } E \end{cases}$$

The stochastic process $Y = \{Y_t : t \geq 0\}$ defined as above is called the minimal semi-Markov process associated with (X, T)

1.5 Markov Renewal Function

Let $(X, T) = \{(X_n, T_n); \ n \in N\}$ be a Markov renewal process with semi-Markov kernel Q over a countable state space E. Define

$$Q^n(i,j,t) = P\{X_n = j, T_n \leq t / X_0 = i\}, \quad i, j \in E, t \in R+$$

for all $n \in N$, with

$$Q^0(i,j,t) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Then for $n \geq 0$ we have the succesive relation $Q^{n+1}(i,k,t) = \sum_{j \in E} \int_0^t Q(i,j,ds)$ $Q^n(j,k,t-s)$. Where the integration is on $[0, t]$.

Consider the function $R(i,j,t) = \sum_{n=0}^{\infty} P_i(X_n = j, T_n \leq t) = \sum_{n=0}^{\infty} Q^n(i,j,t)$. The functions $t \to R(i,j,t)$ are called Markov renewal functions and the collection $R = \{R(i,j,.); i,j \in E\}$ of these function is called a Markov renewal kernel.

Let $j \in E$ be fixed, and define $S_0^j, S_1^j, \ldots$ as the successive T_n for which $X_n = j$. Then $S_j = \{S_n^j \ ; \ n \in N\}$ is a (possibly delayed) renewal process.

Let $F(i,j,t)$ be the distribution of the first passage time from state i to state j, that is, let $F(i,j,t) = P_i(S_0^j \leq t)$, $i \neq j$ and let $F(j,j,t)$ be the distribution of time between successive ocuurence of j, that is, let $F(j,j,t) = P_j\{S_1^j \leq t\}$ $(P_j(S_0^j = 0) = 1)$

$$R(j,j,t) = \sum_{n=0}^{\infty} F^n(j,j,t)$$

and $R(i,j,t) = \int_0^t F(i,j,ds)R(j,j,t-s), i \neq j$ where $F^n(j,j,.)$ is the n-fold convolution of the distribution $F(j,j,.)$ with itself.

1.6 Scope of the Work

The thesis comprises five chapters. In Chapter 1 a brief summary of the topics relevant to the thesis, including the contributions of the author is given.

In Chapter 2, we introduce T-policy during lead time in (s,S) inventory system. In T-policy whenever a replenishment doesn't occur after the placement of an order within T units of time (a r v) a local purchase is made either to bring the inventory level to S cancelling the replenishment order placed or to bring the inventory level to s or to 0 without cancelling the order (the last policy serves to meet all the backlogs if any, without cancelling the order). The demand process is assumed to be Poisson with rate λ. As and when the inventory level drops to s, on order is placed for $M = S - s$ units. The lead time is exponentially distributed with parameter μ and T is exponentially distributed with parameter α. We denote by $I(t)$ the inventory level at time $t, t \geq 0$. $\{I(t), t \geq 0\}$ is a finite state space Markov chain with state space $A = \{-k, -k+1, \ldots, s, \ldots S\}$ when k is the maximum number of backlogs, allowed. We choose k such that $M - k > s$ to avoid perpe tual order placement. The probability of transition to i at time t starting from S at time 0 is denoted by $P_i(t), i \in A$. $P_{Si}(t) = P(I(t) = i/I(0) = S)$. The time dependent and steady system probabilities are computed. Also the optimal value of k is found out in the three cases by fixing s and S. The situation where T follows a general distribution is also considered. As above, demands are assumed to be Poisson with rate λ and lead time exponential with rate μ. The replenishment epochs $T_1, T_2, \ldots$ follow a regenerative process. Here, we consider only the first case. Time dependent probabilities are found out. Cost function is found out by examining the embedded Markov renewal process.

In Chapter 3 the reliability of a k-out-of-n system with repair under T policy is studied. T-policy in the queueing set up has been extensively studied (see Artalejo [1992]) However, this has not been brought to the investigation of the reliability of k-out-of-n system with repair inorder to minimize the system reliability. The repair is according to T-policy,server is called to the system after the elapse of T time units, where T is exponentially distibuted with parameter α, since his departure after completion of repair of failed unit or the moment $n-k$ failed units accumulate whichever occur first. He continue to remain in the system until all the failed units are repaired, once he arrives. We consider three different situations (a) cold system (b) warm system (c) hot system. The k-out-of-n system is called a cold system, warm or hot if once the system is down, the functioning components do not fail, fail at a lower rate or fail at the same rate.

Life times of units are assumed to have independent exponential distribution with parameter λ_i when i units are functioning. Repair time is also assumed to be exponentially distributed with rate μ. We have obtained the profit function and numerically, we have found optimal α which maximize the profit.

Next, the distribution of sevice time is taken as general. In this case we examine the system state at repair completion epochs. These epochs form a regenerative process provided failure time of components are exponentially distributed with rate λ and random variable T is assumed to follow exponential distribution with parameter value α. Here also, we consider three different states of components (i) cold (ii) warm and (iii) hot. In all these cases it is established that the cost function is convex and hence global minimum exists.

Chpater 4 deals with T-policy for k-out-of-n system with two modes of service. k-out-of-n system with repair time distribution of the I server exponential with rate μ_1 and that of the II server with rate μ_2. Here, we consider only cold system. Here, the II server is activated after the elapse of T time units since becoming idle from the time of completion of most repair of all failed units. Since, we are considering cold system, functional components do not fail after the system is down. We have obtained system rate probabilities and some performance measures. Some numerical illustrationsare provided.

In Chapter 5 we discuss some special models in reliability of a k-out-of-n system with repar under T-policy. In the first model, the repair is provided by an unreliable server. Here, T is assumed to be exponentially distributed with parameter value α. Repair time is exponentially distributed with rate μ. Server is subject to breakdown. The failure rate is assumed to be exponential with rate β and repair of server is also exponential with parameter value γ. $X(t)$ denotes the number of failed units

$$Y(t) = \begin{cases} 0 & \text{if Sever is inactive} \\ 1 & \text{if Server is activated} \\ 2 & \text{if Server is activated but down} \end{cases}$$

system state probabilities and some characteristics are obtained.

In the second model, though, the server is switched on after the elapse of T time units, he gets activated only after a random length of time. Let U be the activation time and is assumed to be exponentially distributed with rate θ. T is exponentially distributed with rate α and repair time exponentially distributed with rate μ. Hence the time elapsed until activation starting from all units operational, has generalized Erlang distribution. Here, $X(t)$ represent the number of failed units. $Y(t)$ equals 2, if server is active at time t, 1, if server is only switched on but not activated and 0 otherwise. Steady state probabilities and some performance measures are found out.

In the third model, we consider the time for the server to get inactivated. The system does not go directly to state $(0, 0)$ from $(1, 1)$, it goes to a state $(0, 2)$ and then to $(0, 0)$. Here $X(t)$ represents the number of failed units at t.

$$Y(t) = \begin{cases} 0 & \text{of server is inactive} \\ 1 & \text{if server is active} \\ 2 & \text{if server is switched off, but not inactivated} \end{cases}$$

Here the inactivation time is assumed to be exponentially distributed with rate η. All other assumption are as in the above models. Here also system state probabilities and some characteristics are obtained. In all the above three models, some numerical illustrations are provided.

Chapter 2

(s, S) Inventory system with lead time the T-policy

2.1 Introduction

In this chapter we consider an (s, S) inventory system with T-policy during lead time. During the lead time it may happen that there is no backlog, finite backlog (which will be met immediately on replenishment) or a large number of lost sales. In the latter two cases there is every chance of loss of customer goodwill and the consequent loss to the system. In order to over come this, we introduce T-policy during lead time to the (s, S) inventory system. (s, S) inventory system with N-policy during lead time have been introduced and investigated though a series of papers by Krishnamoorthy and Raju (1998, 1999). In N-policy a local purchase is made when the number of backlogs reaches N.

T-policy in the queueing set up has been discussed by several authors (See Artalejo (1992)). In T-policy whenever a replenishment doesn't occur after the placement of an order within a time of T units from the order placement epoch a local purchase is made either to bring the inventory level to S cancelling the replenishment order placed or to bring the inventory level to s or to clear all the backlogs without cancelling the order (that is to meet all the backlogs), if any without cancelling the order.

In Section 2.2 we analyse the three models and in Section 2.3 various characteristics of the models are established. Section 2.4 is concerned with the cost analysis and numerical illustration are given in Section 2.5. In Section 2.6. the general case of T following an arbitrary distribution is considered.

2.2 Mathematical Formulation and Analysis of the Model

The demand process is assumed to be Poisson with rate λ. As and when the inventory level drops to s, an order is placed for $M = S - s$ units. The lead time is exponentially distributed with rate μ and T is exponentially distributed with parameter α. As mentioned above we consider three cases. It is to be noted that for the second and third cases several local purchases may take place before a replenishment takes place from the time of order placement. That is several T units may elapse before a replenishment takes place however, a local purchase may not be required at every such instant.

2.2.1 Analysis of the Model

Let $\{I(t), t \geq 0\}$ be the inventory level at time t. Then $\{I(t), t \geq 0\}$ is a finite state space Markov chain with state space $E = \{-k, -k+1, \dots, s, \dots, S\}$ where k is the maximum number of backlogs. We choose k such that $M - k > s$ to avoid perpetual order placement. At time 0 the system is assumed to be full, that is $I(0) = S$. We denote the transition probability of moving from S to i at time t by $P_i(t) = P_{Si}(t) = \{P(I(t) = i/I(0) = S)\}$

2.2.2 Model 1

If no replenishment takes place within a time of T units after the placement of order, a local purchase is made to bring the level to S. Then $P_i(t)$ satisfies the system of equations

$$\begin{aligned}
P_S'(t) &= -\lambda P_S(t) + \mu P_s(t) + \alpha \textstyle\sum_{i \leq s} P_i(t) \\
P_n'(t) &= -\lambda P_n(t) + \lambda P_{n+1}(t) + \mu P_{n-M}(t), M - k \leq n \leq S - 1 \\
P_n'(t) &= -\lambda P_n(t) + \lambda P_{n+1}(t), s + 1 \leq n \leq M - k - 1 \\
P_n'(t) &= -(\lambda(1 - \delta_{n,-k}) + \mu + \alpha) P_n(t) + \lambda P_{n+1}(t), -k \leq n \leq s
\end{aligned} \tag{2.1}$$

where $\delta_{i,j}$ represents Kronecker delta. Taking limit as $t \to \infty$, on both sides of the above system of equations and solving the resulting balance equations, we get the steady state distribution.

2.2.3 Steady State Distribution

Theorem 2.2.1. *Let $q_n = \lim_{t\to\infty} P_n(t)$, $n \in E$. These are given by*

$$q_n = \begin{cases} (\frac{\lambda+\mu+\alpha}{\lambda})^{k+n-1}(\frac{\mu+\alpha}{\lambda})q_{-k} & -k+1 \le n \le s+1 \\ (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}(\frac{\mu+\alpha}{\lambda})q_{-k} & s+2 \le n \le M-k \\ [(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}(\frac{\mu+\alpha}{\lambda}) - (\frac{\mu}{\lambda})(\frac{\lambda+\mu+\alpha}{\lambda})^{n-(M-k+1)}]q_{-k} & M-k+1 \le n \le S-1 \\ [(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}(\frac{\mu}{\lambda})(\frac{\mu+\alpha}{\lambda}) + (\frac{\alpha}{\lambda})(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}]q_{-k} & n = S \end{cases} \tag{2.2}$$

Using the normalising condition $\sum_{n=-k}^{S} q_n = 1$, we obtain the value of q_{-k}.

$$\begin{aligned} q_{-k} = [(\frac{\lambda+\alpha+\mu}{\lambda})^{k+s+1} + (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}[(S-s-2)(\frac{\mu+\alpha}{\lambda} + (\frac{\alpha}{\lambda})] \\ +(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}((\frac{\mu}{\lambda})(\frac{\mu+\alpha}{\lambda}) - \frac{\mu}{\mu+\alpha}) + \frac{\mu}{\mu+\alpha}]^{-1} \end{aligned}$$

2.2.4 Time Dependent Solution

The system of equations (2.1) can be written in the form $\mathbb{P}'(t) = A\mathbb{P}(t)$ where $\mathbb{P}'(t) = (P'_{-k}(t), P'_{-k+1}(t), \dots, P'_S(t))^T$, the column vector of first derivative of $P_i(t)$'s and A is the coefficient matrix and $P(0) = (0, 0, \dots, 1)^T$. The solution is given by $\mathbb{P}(t) = e^{AT}P(0)$. Since A is finite, the series converges and the solution is unique. We have the Jordan canonical form of A given by

$$\Lambda = C^{-1}AC = \begin{bmatrix} -(\mu+\alpha) & 0 & 0 \\ 0 & \begin{bmatrix} -(\lambda+\mu+\alpha) & 0 \\ 0 & 0 \\ 0 & -(\lambda+\mu+\alpha) \end{bmatrix} & \\ 0 & & \begin{bmatrix} -\lambda & \cdots & 0 \\ 0 & & 0 \\ 0 & & -\lambda \end{bmatrix} \end{bmatrix}$$

On solving, we get $\mathbb{P}(t) = Ce^{\Lambda t}C^{-1}\mathbb{P}(0)$.

2.2.5 Model 2

If no replenishment takes place upto time T measured from order placement point, a local purchase is made to bring the level to s without cancelling the order. Here we get the following system of equations which is satisfied by $P_i(t)$:

$$\begin{aligned} P_n'(t) &= -\lambda P_n(t) + \lambda(1-\delta_{ns})P_{n+1}(t) + \mu P_{n-M}(t), M-k \le n \le S \\ P_n'(t) &= -\lambda P_n(t) + \lambda P_{n+1}(t), s+1 \le n \le M-k-1 \\ P_s'(t) &= -(\lambda+\mu)P_s(t) + \lambda P_{s+1}(t) + \textstyle\sum_{l\le s} \alpha P_l(t) \\ P_n'(t) &= -(\lambda(1-\delta_{n,-k}) + \mu + \alpha)P_n(t) + \lambda P_{n+1}(t), -k \le n \le s-1 \end{aligned} \tag{2.3}$$

2.2.6 Steady State Distribution

Let $q_n = \lim_{t\to\infty} P_n(t)$, $n \in E$. Then the balance equation are given by

$$q_n = \begin{cases} (\frac{\lambda+\mu+\alpha}{\lambda})^{k+n-1}(\frac{\mu+\alpha}{\lambda})q_{-k}, -k+1 \le n \le s+1 \\ (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}(\frac{\mu}{\lambda})q_{-k}, s+1 \le n \le M-k \\ [(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}(\frac{\mu}{\lambda}) - (\frac{\mu}{\lambda})(\frac{\lambda+\mu+\alpha}{\lambda})^{n-(M-k+1)}]q_{-k}, \quad M-k+1 \le n \le S-1 \\ [(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}(\frac{\mu}{\lambda})(\frac{\mu+\alpha}{\lambda})]q_{-k}, n = S \end{cases} \tag{2.4}$$

Using the normalising condition $\sum_{n=-k}^{S} q_n = 1$, we obtain the value of q_{-k}

$$q_{-k} = [(\frac{\lambda+\alpha+\mu}{\lambda})^{k+s} + (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}[(S-s-1)\frac{\mu}{\lambda}+1] + (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}$$
$$((\frac{\mu}{\lambda})(\frac{\mu+\alpha}{\lambda}) - (\frac{\mu}{\mu+\alpha}) + \frac{\mu}{\mu+\alpha}]^{-1}$$

2.2.7 Time dependent solution

As in model 1, we obtain time dependent solution.

2.2.8 Model 3

If no replenishment takes place upto time T (r.v) measured from order placement point, a local purchase is made to meet all the backlogs, if any, without cancelling the replenishment order. There may be several instances of local purchase. We get the following system of difference differential equations.

$$\begin{aligned}
P_n'(t) &= -\lambda P_n(t) + \lambda(1-\delta_{n,S})P_{n+1}(t) + \mu P_{n-M}(t), \quad M-k \le n \le S \\
P_n'(t) &= -\lambda P_n(t) + \lambda P_{n+1}(t), \quad s+1 \le n \le M-k-1 \\
P_n'(t) &= -(\lambda+\mu)P_n(t) + \lambda P_{n+1}(t), \quad 1 \le n \le s \\
P_0'(t) &= -(\lambda+\mu)P_0(t) + \lambda P_1(t) + \alpha \textstyle\sum_{l \le 0} P_l(t), \\
P_n'(t) &= -(\lambda(1-\delta_{n,-k}) + \mu + \alpha)P_n(t) + \lambda P_{n+1}(t), \quad -k \le n \le s-1
\end{aligned} \tag{2.5}$$

As in the above two models solving the above equations after taking limit as $t \to \infty$ on both sides, we get the steady state distribution.

2.2.9 Steady State Distribution

$$q_n = \begin{cases}
(\frac{\lambda+\mu+\alpha}{\lambda})^{k+n-1}(\frac{\mu+\alpha}{\lambda})q_{-k} & -k+1 \le n \le 0 \\
(\frac{\lambda+\mu}{\lambda})^{n-1}(\frac{\lambda+\mu+\alpha}{\lambda})^{k}(\frac{\mu}{\lambda})q_{-k} & 1 \le n \le s+1 \\
(\frac{\lambda+\mu}{\lambda})^{s}(\frac{\mu}{\lambda})(\frac{\lambda+\mu+\alpha}{\lambda})^{k}q_{-k} & s+2 \le n \le M-k \\
[(\frac{\lambda+\mu+\alpha}{\lambda})^{k}(\frac{\mu}{\lambda})(\frac{\lambda+\mu}{\lambda})^{s} - (\frac{\mu}{\lambda})(\frac{\lambda+\mu+\alpha}{\lambda})^{n-(M-k+1)}]q_{-k} & M-k+1 \le n \le M+1 \\
(\frac{\lambda+\mu+\alpha}{\lambda})^{k}(\frac{\mu}{\lambda})(\frac{\mu+\lambda}{\lambda}) - (\frac{\mu}{\lambda})(\frac{\lambda+\mu}{\lambda})^{n-(M+1)}(\frac{\lambda+\mu+\alpha}{\lambda})^{k}]q_{-k} & M+2 \le n \le S
\end{cases} \tag{2.6}$$

Using the normalising condition $\sum_{n=-k}^{S} q_n = 1$, we obtain the value of q_{-k},

$$q_{-k} = [(\frac{\lambda+\alpha+\mu}{\lambda})^k(\frac{\lambda+\mu}{\lambda})^{s+1} + (\frac{\lambda+\mu+\alpha}{\lambda})^k(\frac{\lambda+\mu}{\lambda})^s[(S-s-1)(\frac{\mu}{\lambda})-1]$$
$$+(\frac{\lambda+\mu+\alpha}{\lambda})^k((\frac{\mu+\lambda}{\lambda}) - (\frac{\mu}{\lambda})(\lambda+\mu+\alpha)) + \frac{\mu}{\mu+\alpha}]^{-1}$$

Here also, we obtain time dependent solution as in model 1.

2.3 Characteristics of the Models

Model 1

2.3.1 Expected inventory level and expected backlog in the steady state

Expected inventory level in the steady state in given $E(I) = \sum_{i=1}^{S} iq_i$

Thus,

$$E(I) = \{(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s+1}[(s+1) - \frac{\lambda}{\mu+\alpha}] + (\frac{\lambda+\mu+\alpha}{\lambda})^k(\frac{\lambda}{\mu+\alpha})$$
$$+(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}[(\frac{(S-s-2)(S+s+1)}{2})(\frac{\mu+\alpha}{\lambda}) + \frac{S\alpha}{\lambda}]$$
$$+\frac{\mu}{(\mu+\alpha)^2}[(M-k+1)(\mu+\alpha) - (\lambda+\mu+\alpha)]$$
$$+(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}(\frac{\lambda\mu}{(\mu+\alpha)^2} + S(\frac{\mu(\mu+\alpha)}{\lambda^2} - \frac{\mu}{\mu+\alpha}) + \frac{\mu}{\mu+\alpha})\}q_{-k}$$

and the expected backlog $E(B) = \sum_{i=-k}^{-1} \mid i \mid q_i$

$$E(B) = \{(\frac{\lambda+\mu+\alpha}{\lambda})^{k-1}(\frac{\lambda}{\mu+\alpha}+1) - \frac{\lambda}{\mu+\alpha}\}q_{-k} \tag{2.7}$$

Theorem 2.3.1. *The distribution of time between two order placement epochs is given by*

$$G_1(x) = \int_{u=0}^{x}\int_{v=u}^{x}\sum_{l=0}^{k+s}\frac{e^{-\lambda u}(\lambda u)^l}{l!}\mu e^{-\mu u}\frac{e^{-\alpha u}e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-l-1}\lambda}{(S-s-l-1)!}dv\,du$$

if replenishment takes place before the elapse of T units of time. Otherwise,

$$G_2(x) = \int_{u=0}^{x}\int_{v=u}^{x}\sum_{l=0}^{k+s}\frac{e^{-\lambda u}(\lambda u)^l}{l!}e^{-\mu u}\alpha e^{-\alpha u}\frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-1}\lambda}{(S-s-1)!}dv\,du$$

and expected time elapsed between two order placement epochs is given by

$$E(\tau) = \frac{\mu}{\alpha+\mu}\left(\frac{1}{\mu}+\frac{1}{\lambda}((S-s)-\min([\frac{\lambda}{\mu}],s+k))\right)+\frac{\alpha}{\alpha+\mu}\left(\frac{1}{\alpha}+\frac{1}{\lambda}(S-s)\right)$$

Expected number of demand lost per unit time

$$E(D) = \frac{1}{E(\tau)}\left(\frac{\mu}{\alpha+\mu}(\max(0,\frac{\lambda}{\mu}-(s+k)))+\frac{\alpha}{\alpha+\mu}(\max(0,\frac{\lambda}{\alpha}-(s+k)))\right)$$

To arrive at the above expressions we proceed as follows.

Let τ be the time between successive order placement epochs. Consider an order placement epoch which is identified as time origin and in time u, measured from this epoch l demands occur. Suppose replenishment takes place before the elapse of T units of time. The maximum number l of demands met during the period is $\leq s+k$ where k is the maximum number of backlogs permitted. After replenishment, the level becomes $S-l$. To reach the level s, $(S-s-l)$ demands have to occur. Then the distribution $G_1(.)$ between two consecutive order placement epochs is given by

$$G_1(x) = \int_{u=0}^{x}\int_{v=u}^{x}\sum_{l=0}^{k+s}\frac{e^{-\lambda u}(\lambda u)^l}{l!}\mu e^{-\mu u}\frac{e^{-\alpha u}e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-l-1}\lambda}{(S-s-l-1)!}dv\,du$$

On the other hand if T occurs before replenishment then a local purchase is made to bring the level to S. To reach the level s, an additional $(S-s)$ demands should occur. Then the distribution function $G_2(.)$ is given by

$$G_2(x) = \int_{u=0}^{x}\int_{v=u}^{x}\sum_{l=0}^{k+s}\frac{e^{-\lambda u}(\lambda u)^l}{l!}e^{-\mu u}\alpha e^{-\alpha u}e^{-\lambda(v-u)}\frac{(\lambda(v-u))^{S-s-1}\lambda}{(S-s-1)!}dv\,du$$

Expected time elapsed between two order placement epochs is given by

$$E(\tau) = \frac{\mu}{\alpha+\mu}\left(\frac{1}{\mu}+\frac{1}{\lambda}((S-s)-\min([\frac{\lambda}{\mu}],\ s+k))\right)+\frac{\alpha}{\alpha+\mu}\left(\frac{1}{\alpha}+\frac{1}{\lambda}(S-s)\right)$$

and expected number of demands lost per unit time is

$$E(D) = \frac{1}{E(\tau)}\left(\frac{\mu}{\alpha+\mu}(\max(0,\frac{\lambda}{\mu}-(s+k)))+\frac{\alpha}{\alpha+\mu}(\max(0,\frac{\lambda}{\alpha}-(s+k))\right)$$

2.3.2 Model 2

Expected inventory level in the steady state is given by $E(I) = \sum_{i=1}^{S} i\, q_i$. Thus,

$$E(I) = \{(\frac{\lambda+\mu+\alpha}{\lambda})^{k+s}[s - \frac{\lambda}{\mu+\alpha} + \frac{\mu}{\lambda}(\frac{S-s-1}{2}(S+s))] + (\frac{\lambda+\mu+\alpha}{\lambda})^{k}$$
$$(\frac{\lambda}{\mu+\alpha}) + \frac{\mu}{(\mu+\alpha)}[(M-k+1) - \frac{\lambda+\mu+\alpha}{\mu+\alpha}] + (\frac{\lambda+\mu+\alpha}{\lambda})^{k+s-1}(\frac{\lambda\mu}{(\mu+\alpha)^2}$$
$$+S(\frac{\mu(\mu+\alpha)}{\lambda^2} - \frac{\mu}{\mu+\alpha}) + \frac{\mu}{\mu+\alpha})\}q_{-k}$$

and the expected backlog $E(B) = \sum_{i=-k}^{-1} |i| q_i$

$$E(B) = \{(\frac{\lambda+\mu+\alpha}{\lambda})^{k-1}(\frac{\lambda}{\mu+\alpha} + 1) - \frac{\lambda}{\mu+\alpha}\}q_{-k}$$

Lemma 2.3.1. *The distribution of time between two order placement epochs is given by*

$$F(x) = \int_{y=0}^{x}\int_{u=y}^{x}\sum_{m=1}^{\infty}\frac{e^{-\alpha y}(\alpha y)^{m-1}\alpha}{(m-1)!}\mu e^{-\mu u}e^{-\alpha(u-y)}\sum_{l}e^{-\lambda(u-y)}\frac{(\lambda(u-y))^{l}}{l!}$$
$$\frac{e^{-\lambda(x-u)}(\lambda(x-u))^{S-s-r}}{(S-s-r)!}dydu + \int_{u=0}^{x}\int_{v=u}^{x}\sum_{l=0}^{\infty}\frac{e^{-\lambda u}(\lambda u)^{l}}{l!}\mu e^{-\mu u}e^{\alpha u}$$
$$\frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-r-1}\lambda}{(S-s-r-1)!}dvdu$$

Proof: In this model, several local purchases may take place during a lead time. Suppose there are m instances of elapsed $T-$ times before a replenishment. But, a local purchase may not be required at each of these instants. The inventory level at an order placement epoch is s. Identifying this as the time origin, assume m local purchases take place within a time of y units. Immediately after the last local purchase the level is s. During the interval $(y, u]$, l demands occur and a natural replenishment takes place in $(u, u+du]$ with density function $\mu e^{-\mu u}$. No local purchase takes place is $(y, u]$. This has p.d.f $e^{-\alpha(u-y)}$.
In the remaining time the inventory level becomes s due to the arrival of $S-s-r$ demands where r is the number of demands met in $(y, u]$ that is $r = \min(l, s+k)$.

Expected elapsed time between two order placement epochs.

Let τ be the time elapsed between two order placement epochs. Then,

$$E(\tau) = \frac{\mu}{\alpha+\mu}(\frac{1}{\mu} + \frac{1}{\lambda}((S-s) - \min([\frac{\lambda}{\mu}], s+k))) + \frac{\alpha}{\alpha+\mu}(\frac{1}{\alpha}[\frac{\alpha}{\mu}] + \frac{1}{\mu}$$
$$+\frac{1}{\lambda}((S-s) - \min([\frac{\lambda}{\mu}], s+k)))$$

Expected number of demands lost per unit time is given by

$$E(D) = \frac{1}{E(\tau)}\{(\frac{\mu}{\alpha+\mu}(\max(0,[\frac{\lambda}{\mu} - (s+k)]) + \frac{\alpha}{\alpha+\mu}[\frac{\alpha}{\mu}]\lambda q_{-k}\}$$

2.3.3 Model 3

Expected inventory level in the steady state is given by $E(I) = \sum_{i=1}^{S} iq_i$

$$\begin{aligned} E(I) = \{ & (\frac{\lambda+\mu+\alpha}{\lambda})^k(\frac{\lambda+\mu}{\lambda})^{s+1}[(s+1) - \frac{\lambda}{\mu}] + (\frac{\lambda+\mu+\alpha}{\lambda})^k(\frac{\lambda}{\mu}) \\ & + (\frac{\lambda+\mu+\alpha)}{\lambda})^k(\frac{\lambda+\mu}{\lambda})^s((S-s-1)(\frac{S+s+2}{2})(\frac{\mu}{\lambda}) - (M+2)) \\ & + (\frac{\lambda+\mu+\alpha}{\lambda})^k(\frac{\lambda+\mu}{\lambda})^{s+1}(\frac{\lambda}{\mu} - (s-2)) + (\frac{\lambda+\mu+\alpha}{\lambda})^k \\ & (\frac{\lambda+\mu}{\lambda})((M+2) - \frac{\lambda}{\mu}) + (\frac{\lambda+\mu+\alpha}{\lambda})^{k+1}(\frac{\mu\lambda}{(\mu+\alpha)^2} \\ & - \frac{\mu}{(\mu+\alpha)}(M+1)) + \frac{\mu}{(\mu+\alpha)}((M-k+1) - (\frac{\lambda+\mu+\alpha}{\mu+\alpha}))\}q_{-k} \end{aligned}$$

and the expected backlog $E(B) = \sum_{i=-k}^{-1} \mid i \mid q_i$

$$E(B) = \{(\frac{\lambda+\mu+\alpha}{\lambda})^{k-1}(\frac{\lambda}{\mu+\alpha} + 1) - \frac{\lambda}{\mu+\alpha}\}q_{-k}$$

Theorem 2.3.2. *The distribution of time between two order placement epochs is given by* $H_1(.) + H_2(.)$ *where*

$$\begin{aligned} H_1(x) = & \int_{u=o}^{x}\int_{v=u}^{x} e^{-\alpha u}\mu e^{-\mu u}\sum_{l}\frac{e^{-\lambda u}(\lambda u)^l}{l!}\frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-\min(l,s+k)-1}}{(S-s-\min(l,s+k)-1)!}dvdu \\ & + \int_{y=0}^{x}\int_{u=y}^{x}\int_{v=u}^{x}\sum_{m=1}^{\infty}\frac{\alpha^m y^{m-1}e^{-\alpha y}}{(m-1)!}\mu e^{-\mu u}e^{-\alpha(u-y)}\sum_{l=0}^{s}\frac{e^{-\lambda y}(\lambda y)^l}{l!} \\ & \sum_{r=0}^{\infty}\frac{e^{-\lambda(u-y)}(\lambda(u-y))^r}{r!}\frac{e^{-\lambda(v-u)}\lambda(\lambda(v-u))^{S-s-(l+\min(l,s+k-l)-1)}}{(S-s-(l+\min(l,s+k-l)-1)!}dv\,du \end{aligned}$$

and

$$H_2(x) = \int_{y=0}^{x} \int_{u=y}^{x} \int_{y_{b_p}=y_{b_{p-1}}}^{y} \cdots \int_{y_{b_1}=0}^{y} \sum_{m} \alpha^{b_1-1} \frac{y_{b_1-1}^{b_1-2}}{(b_1-2)!} e^{-\alpha y_{b_1-1}}$$

$$\sum_{l_1=0}^{s} e^{-\lambda y_{b_1-1}} \frac{(\lambda y_{b_1-1})^{l_1}}{l_1!} \alpha e^{-\alpha y_{b_1}} \frac{y_{b_2-b_1-1}^{b_2-b_1-2} \alpha^{b_2-b_1-1} e^{-\alpha(y_{b_2-1}-y_{b_1})}}{(b_2-b_1-2)!}$$

$$e^{-\lambda(y_{b_2-1}-y_{b_1})} \alpha e^{-\alpha(y_{b_2}-y_{b_2-1})} \cdots \alpha^{m-b_p} y_{m-b_p}^{m-b_p-1} e^{-\alpha(y_m-y_{b_p})}$$

$$e^{-\lambda(y_m-y_{b_p})} \sum_l \frac{e^{\lambda y}(\lambda y)^l}{l!} \sum_r \frac{e^{-\lambda(u-y)}(\lambda(u-y))^r}{r!}$$

$$\frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-s-l-r-1}}{(S-s-l-r-1)!} dv\,du\,dy_{b_1} \cdots d_{y_{b_p}} : r = \min(l,k)$$

Proof: Consider the time interval $(0, y]$. The inventory level at an order placement epoch is s which is Identified as the time origin. Let there be m instances of elapse of T time units from the order placment epoch to the subsequent local purchase epochs $y_1, y_2, \ldots, y_{m-1}, y_m$. In the time interval $(u, u+du]$ natural replenishment takes place. Suppose no local purchase takes place in $(0, y]$. The inventory level at time y can be at most s. Assume l demands occur is $(y_m, y]$. In $(y, u]$ no local purchase is made and the number of demands met is r, where $r = \min(l, s+k-l)$.prior to time u, the inventory level will be $s - (l + \min(l, s+k-l))$. At time u after replenishment the level will be $S - (l + \min(l, s+k-l))$. In $(u, v]$, in order to reach the level $s, S - s - (l + \min(l, s+k-l))$ demand should occur.

Suppose in $(0, y]$ there are p instances at which local purchase take place, that is, let $y_{b_1} y_{b_2}, \ldots y_{b_p}$ be a subsequence of $y_1, y_2, \ldots, y_{m-1}, y_m$ at which local purchase takes place. At y_{b_p} the level will be zero. In $(y_m, y]$. Suppose l demands occur and the number of demands met in $(y, u]$ is r where $r = \min(l, k)$. At time u, natural replenishment takes place and the levels is $S - l - r$. To reach the levels in the remaining time, $S - s - l - r$ demands should occur. The distribution $H(.)$ of the time between two order placement epochs is the sum of $H_1(.)$ and $H_2(.)$ where $H_1(.)$ is the distribution of time when no local purchase is made and $H_2(.)$ that when at least one local purchase is made. In the first case the number of instances of T-time units preceeding lead time is zero. Hence the distribution $H(.)$ is given by $H_1(.) + H_2(.)$

Expected time elapsed between two order placement epochs and expected demands lost are same given by the expressions as in model 2.

2.4 Cost Analysis

Let the various costs under the steady state be:
f-fixed cost of order placement, v-variable cost, h-holding cost per unit P- shortage cost per unit, L-cost associated with local purchase per unit, C-Cancellation cost per unit, R-unit cost for lost sales.

2.4.1 Total expected cost per unit time (TEC)

Model 1

$$(TEC)_1 = hE(I) + PE(B) + L(M + \sum_{i=-k}^{s} |\, i \,|\, q_{s-|i|})\frac{\alpha}{\alpha+\mu}\frac{1}{E(\tau)} + C\frac{\alpha}{\alpha+\mu}\frac{1}{E(\tau)}$$
$$+(f + vM)\frac{\mu}{\mu+\alpha}\frac{1}{E(\tau)} + RE(D)$$

2.4.2 Model 2

Here also the various costs are assumed to be as given for model 1. However there is no cancellation cost. Thus the total expected cost per unit time

$$(TEC)_2 = hE(I) + PE(B) + L(s + \sum_{i=-k}^{s} |\, i \,|\, q_{s-|i|})\frac{\alpha}{\alpha+\mu}\frac{1}{E(\tau)}$$
$$+(f + vM)\frac{\mu}{\mu+\alpha}\frac{1}{E(\tau)} + RE(D)$$

2.4.3 Model 3

The various costs are as in the above two models with no cancellation cost. The objective function is the total expected cost. We have to find an optimal value of k which minimises the total expected cost.

$$(TEC)_3 = hE(I) + PE(B) + L\left(\sum_{i=-k}^{-1} \mid i \mid q_{s-|i|}\right)\frac{\alpha}{\alpha+\mu}\frac{1}{E(\tau)}$$
$$+(f + vM)\frac{\mu}{\mu+\alpha}\frac{1}{E(\tau)} + RE(D)$$

The objective in all the above models is to find an optimal value of k which minimises the total expected cost. It's very difficult to obtain an analytic solution. It may be possible, but extremely hard due to unwieldy nature of the cost function, to prove that it is convex in k for given values of s, S and parameter values. However one can find numerically optimal value of k for fixed values of S, s and other parameters. Numerically we found that, with increasing values of k total expected cost function is convex in model 1 and monotonically decreasing in model 2 and 3.

2.5 Numerical Illustrations

The optimal value of k for the three models are found out with $\lambda = 23, \alpha = 20, \mu = 16$, $s = 2, h = 17, P = 20, L = 15, C = 8, R = 11, f = 15, v = 12$

$(TEC)_1$ FOR MODEL 1

k\S	19	20	21	22	23
1	493.145	501.667	510.188	518.707	527.225
2	493.070	501.667	510.116	518.635	527.153
3	493.076	501.597	510.116	518.633	527.149
4	493.082	501.601	510.119	518.635	527.150
5	493.086	501.604	510.122	518.637	527.152
6	493.088	501.606	510.123	518.638	527.153
7	493.089	501.607	510.124	518.639	527.153
8	493.089	501.607	510.124	518.639	527.153
9	493.089	501.608	510.124	518.640	527.153
10	493.089	501.608	510.124	518.640	527.154
11	493.090	501.608	510.124	518.640	527.154
12	493.090	501.608	510.124	518.640	527.154
13	493.090	501.608	510.124	518.640	527.154
14	493.090	501.608	510.124	518.640	527.154
15	—	501.608	510.124	518.640	527.154

k\ S	19	20	21	22	23
16	—	—	510.124	518.640	527.154
17	—	—	—	518.640	527.154
18	—	—	—	—	527.154

$(TEC)_2$ FOR MODEL 2

k\ S	19	20	21	22	23
1	342.311	349.314	356.472	363.762	371.165
2	328.166	335.184	342.356	349.658	357.071
3	323.021	330.022	337.178	344.466	351.868
4	321.139	328.127	335.271	342.548	349.940
5	320.455	327.434	334.571	341.842	349.228
6	320.207	327.182	334.315	341.583	348.966
7	320.118	327.091	334.222	341.488	348.869
8	320.086	327.058	334.189	341.454	348.834
9	320.074	327.047	334.177	341.441	348.822
10	320.070	327.042	334.172	341.437	348.817
11	320.069	327.041	334.171	341.435	348.815
12	320.069	327.040	334.170	341.435	348.815
13	320.068	327.040	334.170	341.434	348.815
14	320.068	327.040	334.170	341.434	348.814
15	—	327.040	334.170	341.434	348.814
16	—	—	334.170	341.434	348.814
17	—	—	—	341.434	348.814
18	—	—	—	—	348.814

$(TEC)_3$ FOR MODEL 3

k\ S	19	20	21	22	23
1	303.173	501.667	510.188	518.707	527.225
2	302.442	310.580	318.762	326.981	335.229
3	301.857	310.050	318.280	326.541	334.827
4	301.378	309.602	317.859	326.245	334.453
5	300.912	309.160	317.439	325.744	334.071
6	300.431	308.702	317.003	325.327	333.672
7	299.929	308.225	317.003	324.894	333.257
8	299.410	307.733	316.549	324.447	332.830
9	298.877	307.228	316.080	323.990	332.394
10	298.335	306.714	315.600	323.525	331.951
11	297.786	306.195	314.620	323.056	331.503
12	297.234	305.672	314.123	322.584	331.053

k\ S	19	20	21	22	23
13	296.678	305.146	313.625	322.110	330.601
14	296.120	304.619	312.125	321.635	330.148
15	—	304.091	312.624	321.158	329.694
16	—	—	312.122	320.681	329.239
17	—	—	—	320.204	328.784
18	—	—	—	—	328.329

Conclusion

The objective function is optimal for $k = 3$ in Model 1 for the other two Models optimal k is the value corresponding to minimum TEC. In the latter models the function is monotonically decreasing with increasing values of k.

2.6 General case

T arbitrarily distributed in the case of model 1.

Here we assume that T follows a general distribution $F(.)$. Also, one more condition is imposed on model 1, namely that if no replenishment occurs when the inventory level reaches k, a local purchase is made to bring the level to S, cancelling the order for replenishment.

Let $0 = T_0 < T_1 < T_2 < \cdots < T_n < \ldots$ be the replenishment epochs and $X_n = X(T_n+)$be the inventory level immediately after the n^{th} replacement. Then $\{X_n = X(T_n+), n = 0, 1, 2, \ldots\}$ is a Markov chain and $\{(X_n T_n), n = 0, 1, 2, \ldots\}$ is a Markov renewal process with states space $E_1 = \{M - k + 1, \ldots S\}$ embedded at replenishment epochs of the semi-Markov process $\{X(t), t \in R_+\}$.

Demands form a Poisson process with rate λ. The lead time is exponentially distributed with rate μ. Assume that $X(T_0+) = S$

2.6.1 The Semi-Markov Kernel

$\{Q(i, j, t), i, j \in E_1, t \geq 0\}$ where $Q(i, j, t) = P(X(T_{n+1}) = j, T_{n+1} - T_n \leq t / X(T_n) = i)$ is given by

$$Q(i,S,t) = \int_{u=0}^{t}\int_{v=u}^{t} E_{i-s,\lambda}(u)e^{-\lambda(v-u)}\mu e^{-\mu(v-u)}(1-F(v-u))dv\,du$$
$$+\int_0^t\int_{u=0}^{v} E_{i-s,\lambda}(u)\sum_{r=0}^{s+k-1}\frac{e^{-\lambda(v-u)}(\lambda(v-u))^r}{r!}f(v-u)e^{-\mu(v-u)}dv\,du$$
$$+\int_0^t\int_{u=0}^{v} E_{i-s,\lambda}(u)E_{s+k,\lambda}(v-u)e^{-\mu(v-u)}(1-F(v-u))dvdu, \quad i\in E_1$$

and

$$Q(i,S-l,t) = \int_{u=0}^{t}\int_{v=u}^{t} E_{i-s,\lambda}(u)e^{-\lambda(v-u)}\frac{(\lambda(v-u))^l}{l!}\mu e^{-\mu(v-u)}(1-F(v-u))dv\,du,$$
$$l = 1,2,3,\dots s+k-1$$

The steady state probabilities at the embedded points are obtained as the solution to $\Pi\mathbb{P} = \Pi$ with $\sum_i \pi_{i=1}$ where $\pi = (\pi_{M-k+1},\dots\pi_S)$ and $\mathbb{P} = (\lim_{t\to\infty} Q(i,j,t), i,j\in E_1)$.

2.6.2 Time Dependent system State Distribution

$\{X(t), t \in R+\}$ is a semi-Markov process on the set $E = \{-k,-k+1,\dots,S\}$. The embedded Markov Renewal process is $\{(X_n,T_n), n = 0,1,2,\dots\}$. Let $P_{Si}(t) = P(X(t) = i/X(0+) = S), i\in E$. $Q(.,.,t)$ defined by $Q(i,j,t) = P(X_{n+1} = j, T_{n+1} - T_n \le t/X_n = i)\ i,j\in E_1$ is given above.
Define $R(S,i,t) = \sum_{m=0}^{\infty} Q^{*m}(S,i,t)$ where $Q^{*m}(\dots,t)$ is the m-fold convolution of $Q(.,.,t)$ with itself and $Q^{*0}(i,j,t) = 1$ if $i=j$ and 0 otherwise.
Then $P_i(t) = K(S,i,t) + \int_0^t R(S,i,du)P_i(t-u)$ for $i\in E$, where $K(S,i,t) = P(X(t) = i, T_1 > t|X(0_+) = S)$, whose solution is given by

$$P_i(t) = \int_0^t \sum_{m\geq 0,k\in E,k\geq i,k\neq S}^{\infty} Q^{*m}(S,k,du)e^{-\lambda(t-u)}\frac{(\lambda(t-u))^{k-i}}{(k-i)!} \tag{2.8}$$

and

$P_S(t) = \int_0^t \sum_{m=0}^{\infty} Q^{*m}(S,S,du)e^{-\lambda(t-u)}$.

2.6.3 Limit Distribution

We shall compute q_r, $r \in E$, which represent the limiting probabilities of the system state at arbitrary epoch. First we compute the limiting probabilities immediately after a replenishment epoch. For this, consider the transition probability matrix $\mathbb{P} = (\lim_{t\to\infty} Q(i,j,t), i,j \in E_1)$ of the Markov chain $\{X_n\}$. The limiting probabilities are solution for $\Pi\mathbb{P} = \Pi$, with $\sum_i \pi_i = 1$. First consider the expected sojourn time m_i.
$m_i = \int_0^\infty (1 - \sum_k Q(i,k,t))dt$
Limiting probabilities of the system state at arbitrary epoch is given by

$$q_r = \lim_{t\to\infty} P(X(t) = r | X(0+) = S) \quad r \in E_1$$

Then,

$$q_S = \frac{\pi_s}{\sum_{i\in E_1} \pi_i m_i} \int_0^\infty e^{-\lambda t} dt$$

$$q_r = \frac{\sum_{j\in E_1} \sum_{j\geq r} \pi_j}{\sum_{i\in E_1} \pi_i m_i} \int_0^\infty \frac{e^{-\lambda u}(\lambda u)^{j-r}}{(j-r)!} du \quad r = S - s - k, \ldots, S - 1$$
$$= \frac{\sum_{j\in E_1} \pi_j}{\sum_{i\in E_1} p_i m_i} \int_0^\infty \frac{e^{-\lambda u}(\lambda u)^{j-r}}{(j-r)!} du \quad r = M - k - 1, \ldots s + 1$$

2.6.4 Distribution of time between consecutive replenishment epochs

Consider the levels at two replenishment epochs T_n and T_{n+1}, i, j be the levels at T_n and T_{n+1} respectively. Consider the order placement epoch u in $(T_n, T_{n+1}]$. The level at u is s. We have to find the distribution of time between replenishment epochs resulting in the following types of transitions:
They are (i) $S \to S$, (ii) $i < S \to S$, (iii) $S \to j < S$, (iv) $i < S \to j < S$

i) $S \to S$

$$G_1(t) = \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{S-s-1}\lambda}{(S-s-1)!}(1-F(v-u))e^{-\lambda(v-u)}\mu e^{-\mu(v-u)}dvdu$$
$$+ \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{S-s-1}\lambda}{(S-s-1)!} f(v-u)e^{-\mu(v-u)} \sum_{l=0}^{s+k-1} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^l}{l!} dvdu$$
$$+ \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{S-s-1}\lambda}{(S-s-1)!} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{s+k}}{(s+k)!} e^{-\mu(v-u)}(1-F(v-u))dvdu$$

ii) $i < S \to S$

$$G_2(t) = \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{i-s-1}\lambda}{(i-s-1)!} e^{-\lambda(v-u)}\mu e^{-\mu(v-u)}(1-F(v-u)dvdu$$
$$+ \int_0^t \int_{u=0}^v \frac{e^{-\lambda u}(\lambda u)^{i-s-1}.\lambda}{(i-s-1)!} \sum_{l=0}^{s+k-1} \frac{e^{-\lambda u}(\lambda u)^l}{l!} f(v-u)e^{-\mu(v-u)}dvdu, \quad i \in E_1$$

iii) $S \to j < S$

$$G_3(t) = \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{S-s-1}.\lambda}{(S-s-1)!} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-j}}{(S-j)!} \mu e^{-\mu(v-u)}$$
$$(1-F(v-u))dvdu \qquad S-s-k+1 \le j \le S-1, \ j \in E_1$$

iv) $i < S \to j < S$

$$G_4(t) = \int_0^t \int_{v=u}^t \frac{e^{-\lambda u}(\lambda u)^{i-s-1}\lambda}{(i-s-1)!} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{S-j}}{(S-j)!} \mu e^{-\mu(v-u)}$$
$$(1-F(v-u))dvdu, \quad s < j < S$$

Expected elapsed time between two order placement epochs

Let τ be the elapsed time between two order placement epochs. Then

$$E(\tau) = (\frac{S-s}{\lambda}) + (\frac{1}{\mu}\int_0^\infty \mu \bar{e}^{\mu t}(1-F(t))e^{-\lambda t}dt) + \frac{(S-s-\min([\frac{\lambda}{\mu}], s+k-1))}{\lambda}$$
$$+(\frac{1}{\mu}\int_0^\infty \sum_{l=1}^{s+k-1} \frac{e^{-\lambda t}(\lambda t)^l}{l!}\mu e^{-\mu t}(1-F(t))dt + (\frac{S-s}{\lambda}) + (\alpha \int_0^\infty e^{-\mu t} f(t) e^{-\lambda t}dt$$
$$+\int_0^\infty \sum_{l=1}^{s+k-1} \frac{e^{-\lambda t}(\lambda t)^l}{l!} e^{-\mu t} f(t)dt)$$

where $\alpha = \int_0^\infty (1-F(t))dt$

2.6.5 Cost Analysis

Let C_1 holding cost per unit time, C_2 Backlogged cost per unit per unit time, C_3 Cost due to natural purchase per unit, C_4 Cost due to local purchase per unit, C_5 Cancellation cost per unit.

Expected inventory level at an arbitrary epoch is $E(I) = \sum_{i=1}^{S} iP_i$ expected backlog $E(B) = \sum_{i=-k+1}^{-1} \mid i \mid P_i$.

Therefore,

$$TEC = C_1 \sum_{i=1}^{S} iP_i + C_2 \sum_{i=-k+1}^{-1} \mid i \mid P_i + \frac{C_3}{E(\tau)}(f + v(S-s)) \int_0^\infty (1-F(u))\mu e^{-\mu u}du$$
$$+\frac{C_4}{E(\tau)}((S-s) + ([\frac{\lambda}{\alpha}]_+(1-\delta_{([\frac{\lambda}{\alpha}] \geq s+k})) \int_0^\infty e^{-\mu u}(1-F(u))\frac{e^{-\lambda u}(\lambda u)^{s+k}}{(s+k)!}du))$$
$$+\frac{C_5}{E(\tau)}\int_0^\infty f(u)e^{-\mu u}du$$

where $[\frac{\lambda}{\alpha}]_+ = \{[\frac{\lambda}{\alpha}]$ if $\frac{\lambda}{\alpha} \leq s+k-1, s+k-1$ if $\frac{\lambda}{\alpha} > s+k-1$

The particular case of exponentially distributed was discussed in the previous sections. Some numerical illustrations were also provided there.

Chapter 3

k-out-of-n system with repair: T-policy

3.1 Introduction

In this chapter, we consider a k-out-of-n system. In k-out-of-n system, the system functions *iff* at least $k(1 \leq k \leq n)$ of the n components function. Server is activated on the elapse of T time units where T is exponentially distributed with parameter α from the epoch of it being inactivated previously. The activation time after switching on, is negligible. Thus server is brought to the system at the moment which is $\min\{T$, epoch of failure of $n-k$ units $\}$ after his previous departure. He continues to remain in the system until all the failed units are repaired, once he arrives. The process continues in this fashion. Both the continuous time Markovian case and the embedded Markov chain case are considered. Embedded case is discussed in section 3.3. We consider three different situations (a) cold system (b) warm system and (c) hot system. These are defined in section 3.2.1. We aim at finding out optimal T to maximize the profit, that is, to minimize the running cost and maximize the system reliability.

N-policy for repair of the k-out-of-n system has been studied extensively in Krishnamoorthy, Ushakumari and Lakshmi (1998). k-out-of-n system with general repair under N-policy has been studied by Ushakumari and Krishnamoorthy (1998). In these, the authors obtain the optimal number of components to fail before repair facility is activated inorder to minimize the running cost and maximize the system reliability.

Waiting until a large number of units (very close to $n-k$) fail inorder for the server to be called may lead to the system being down for longer duration thereby decreasing its up time and hence the reliability. Activating the server frequently results in high fixed cost. Hence we go for T-policy.

The chapter is presented as follows. Section 3.2 deals with the analysis of the model and it gives some preliminaries, notations, modelling and analysis of the problem under investigation. We outline the system state distribution in the finite time and in the long run for all the three models. Section 3.3 is devoted to the study of some measures of performance and section 3.4 discusses a control problem. It also provides some numerical illustration. Section 3.5 gives the general case where T is assumed to be arbitrarily distributed.

3.2 Analysis of the Model

Life times of units are assumed to have independent exponential distributions with parameter λ_i, when i units are functioning. T is exponentially distributed with parameter α. Repair time is also assumed to be exponentially distributed with rate μ.

Definition 3.2.1. *The k-out-of-n system is called a cold system if once the system is down (that is exactly $k-1$ functional units) there is no further failure of units that are not in failed state, until system starts functioning.*

Definition 3.2.2. *The system is called a warm system if functional units continue to deteriorate and so fail even when the system is down, but now at a lesser rate.*

Definition 3.2.3. *A hot system is one in which components deteriorate at the same rate during the system down state as they deteriorate when the system is up.*

We discuss these three situations separately. First, we introduce some notations.
$X(t)$: number of functional components at time t.
$Y(t)$: server state at time t,
Write

$$Y(t) = \begin{cases} 1 & \text{if the server is available at time } t \\ 0 & \text{otherwise} \end{cases}$$

Under assumptions made on the distribution of repair time, life time of components and on T, we see that $\{(X(t), Y(t)), t \in R_+\}$ is a Markov chain on $E_1 = \{(i,0)|k+1 \leq i \leq n\} \cup \{(i,1)|i = k-1, \ldots n\}$ for model a. (Definition 3.2.1) and $E_2 = \{(i,0)|k+1 \leq i \leq n\} \cup \{(i,1)|0 \leq i \leq n\}$ for models b and c (Definition 3.2.2 and 3.2.3 respectively). Denote by $P_{ij}(t)$ the system state probability at time t given $X(0) = n$, $Y(0) = 0$ that is $P_{ij}(t) = P((X(t), Y(t) = (i,j)|(X(0), Y(0)) = (n,0))$ for $(i,j) \in E_1(E_2))$

3.2.1 Transient Solution

Model a

Here the functioning units do not deteriorate while the system is down. The Kolmogrov forward differential difference equations satisfied by $P_{ij}(t)$ are

$$\begin{aligned}
P'_{m1}(t) &= -(m\lambda_m + \mu(1-\delta_{mn})P_{m1}(t) + (m+1)\lambda_{m+1}(1-\delta_{mn})P_{m+1,1}(t) \\
&\quad +\alpha(1-\delta_{mk})P_{m0}(t) + (m+1)\lambda_{m+1}\delta_{mk}P_{m+1,0}(t) + \mu P_{m-1,1}(t), \quad k \leq m \leq n \\
P'_{m0}(t) &= -(m\lambda_m + \alpha)P_{m0}(t) + (m+1)\lambda_{m+1}(1-\delta_{mn})P_{m+1,0}(t) \\
&\quad +\mu\delta_{mn}P_{m-1,1}(t), \quad k \leq m \leq n \\
P'_{k-1,1}(t) &= k\lambda_k P_{k1}(t) - \mu P_{k-1,1}(t)
\end{aligned} \tag{3.1}$$

where δ_{ij} is the Kronecker delta. The solution of equations 3.1 is given by $\mathbb{P}(t) = e^{tA}\mathbb{P}(0)$ where $\mathbb{P}(0)$ is the initial probability vector which has 1 corresponding to state $(n,0)$ and rest zeros. A is the matrix of coefficients on the right side of the system of equations.

3.2.2 Steady State Probabilities

From the above equations, by setting $q_{ij} = \lim_{t\to\infty} P_{ij}(t)$, $(i,j) \in E_1$, we get steady state probabilities

$$q_{n1} = \frac{\alpha}{n\lambda_n}q_{n0} \quad q_{n-1,1} = \frac{(n\lambda_n + \alpha)}{\mu}q_{n0} \quad q_{r1} = \prod_{l=r}^{n-1}\frac{(l+1)\lambda_{l+1}}{l\lambda_1}q_{n0}, \quad k+1 \leq r \leq n-1$$

$$q_{n-r,1} = \frac{(n-r+1)\lambda_{n-r+1}+\mu}{\mu} q_{n-r+1,1} - \frac{\alpha}{\mu} q_{n-r+1,0} - \frac{(n-r+2)\lambda_{n-r+2}}{\mu} q_{n-r+2,1}, \quad 2 \le r \le n-k$$

$q_{n-l,l}$ where $l = 1,2,\ldots n-k$ and $q_{n-l,0}$ for $l = 1,2,\ldots n-k-1$ can be expressed in terms of q_{n0}, $q_{k-1,1} = \frac{k\lambda_k}{\mu} q_{k1}$ q_{n0} can be determined from the relation $\sum_{i=k+1}^{n} q_{i0} + \sum_{i=k-1}^{n} q_{i1} = 1$. However the expressions for $q_{n-i,1}$ for different i values is unwieldy and so we consider the particular case of $\lambda_i = \frac{\lambda}{i}$ in further development. It is reasonable to make the assumption since deterioration rate increases with decreasing number of operational units.

3.2.3 Model b

In this model, when the number of functional components reduce to $k-1$, the units that have not failed start deteriorating at a rate $\delta < \lambda$. Then life time of functioning components are exponential with parameter δ. The Kolmogorov forward differential equations are

$$\begin{aligned} P'_{m1}(t) &= -(m\lambda_m + \mu(1-\delta_{mn}))P_{m1}(t) + \alpha(1-\delta_{mk})P_{m0}(t) \\ &+ (m+1)\lambda_{m+1}(1-\delta_{mn})P_{m+1,1}(t) + (m+1)\lambda_{m+1}\delta_{mk}P_{k+1,0}(t) \qquad (3.2) \\ &+ \mu(1-\delta_{mn})P_{m-1,1}(t), \quad k \le m \le n \end{aligned}$$

$$\begin{aligned} P'_{m0}(t) &= -(m\lambda_m+\alpha)P_{m0}(t) + (m+1)\lambda_{m+1}(1-\delta_{mn})P_{m+1,0}(t) + \mu\delta_{mn}P_{m-1,1}(t), \\ &\qquad k+1 \le m \le n \\ P'_{m1}(t) &= -(m\delta_m+\mu)P_{m1}(t) + (m+1)\delta_{m+1}P_{m+1,1}(t) + \mu P_{m-1,1}(t) \\ &\quad +(m+1)\lambda_{m+1}\delta_{m,k-1}P_{m+1,1}(t), \quad 0 < m \le k-1 \\ P'_{01}(t) &= -\mu P_{01}(t) + \delta P_{11}(t) \end{aligned}$$

These lead to the system state probabilities in the steady state with evolution of time

$$q_{k-l,1} = \frac{(k-l+1)\delta_{k-l+1}+\mu}{\mu} q_{k-l+1,1} - \frac{(k-l+2)\delta_{k-l+2}}{\mu} q_{k-l+2,1} 2 \le l \le k$$

The rest of the system state probabilities are as in model a $q_{k-l+1,1}$ and $q_{k-l+2,1}$, $l = 2,3,\ldots k$ are available in terms of q_{n0} which in turn can be obtained from the relation $\sum_{i=k+1}^{n} q_{i0} + \sum_{i=k-1}^{n} q_{i1} = 1$

3.2.4 Model C

Here the functional components deteriorate at the same rate during down state of the system as that when the system is up. The time dependent system state distribution can be obtained as in model a. The long run system state probabilities are given by

$$q_{k-l,1} = \frac{(k-l+1)\lambda_{k-l+1}+\mu}{\mu} q_{k-l+1,1} - \frac{(k-l+2)\lambda_{k-l+2}}{\mu} q_{k-l+2,1}, \quad 2 \le l \le k$$

and the rest of the system state probabilities are as in model a. The normalizing condition is $\sum_{(i,j)\in E_2} q_{ij} = 1$.

3.3 Some Performance Measures

We compute the optimal α for the three models. To do this, we need to compute the distribution of time during which the server is continuously available we assume $\lambda_i = \frac{\lambda}{i}$ for $i = k, \ldots, n$ for model a, $\lambda_i = \frac{\lambda}{i}$ for $i = k, \ldots n$ and $\delta_i = \frac{\delta}{i}$ for $i = 1, 2 \ldots, k-1$ for model b, $\lambda_i = \frac{\lambda}{i}$, $i = 1, 2, \ldots, n$ for model c. This assumption states that failure rate decreases with increasing number of functioning units, which is quite reasonable.

3.3.1 Model a

Theorem 3.3.1. *The system state probabilities in the long run are given by*

$$q_{n1} = \frac{\alpha}{\lambda} q_{n0}, \quad q_{n-1,1} = \frac{\lambda+\alpha}{\mu} q_{no}$$

$$q_{n-r,1} = \big[\lambda^{r-1}(\lambda+\alpha)^{n-r} + \lambda^r\mu((\lambda+\alpha)^{r-2} + \mu^{r-3}(\lambda+\alpha+\mu) + \ldots\ldots$$
$$+\mu(\lambda+\alpha)^{r-3} + \ldots + \mu^{r-4}(\lambda+\alpha)^2)\big] q_{n0} \backslash \mu^r(\lambda+\alpha)^{r-1}$$
$$2 \le r \le n-k$$

$$q_{k-1,1} = \frac{\lambda}{\mu} q_{k1}, \quad q_{r0} = \left(\frac{\lambda}{\lambda+\alpha}\right)^{n-r} q_{n0}, \quad k+1 \le r \le n-1$$

Proof. Consider the equation (3.1) from the equation

$$P'_{n1}(t) = -n\lambda_n P_{n1}(t) + \alpha P_{n0}(t)$$

we can write the steady state equation as

$$0 = -n\lambda_n q_{n1} + \alpha q_{n0}$$

Hence, $q_{n1} = \frac{\alpha}{n\lambda_n} q_{n0}$. Rest of the steady state equations are

$$\begin{aligned}
0 &= -(m\lambda_m + \mu(1-\delta_{mn}))q_{m1} + (m+1)\lambda_{m+1}(1-\delta_{mn})q_{m+1,1} + \alpha(1-\delta_{mk})q_{m0} \\
&\quad + (m+1)\lambda_{m+1}\delta_{mk}q_{m+1,0} + \mu q_{m-1,1}, \quad k \le m \le n \\
0 &= -(m\lambda_m + \alpha)q_{m0} + (m+1)\lambda_{m+1}q_{m+1,0}, \quad k+1 \le m < n. \\
0 &= k\lambda_k q_{k1} - \mu q_{k-1,1}
\end{aligned}$$

Solving the above equations, we get steady state probabilities in terms of q_{n0}. □

Note : The system availability at any epoch is given by $1 - q_{k-1,1}$. Hence the fraction of time the system is not available is $q_{k-1,1}$. Under the normalizing condition, we get q_{n0}.

3.3.2 Distribution of the time server is continuously available

Consider the Markov chain on the state space $\{(k-1,1), \ldots, (n,0)\}$ with state $(n,0)$ absorbing and the rest all transient. We have to compute the distribution of the time until reaching $(n,0)$ starting from one of the transient states (corresponding to server arrival). The infinitesimal generator of this chain is

$$\begin{array}{cc}
 & \begin{array}{ccccccc} (k-1,1) & (k,1) & & & (n-1,1) & (n,1) & (n,0) \end{array} \\
\begin{array}{c} (k-1,1) \\ (k,1) \\ \\ \\ (n-1,1) \\ (n,1) \\ (n,0) \end{array} &
\left(\begin{array}{ccccccc}
-\mu & \mu & 0 & & & & 0 \\
\lambda & -(\lambda+\mu) & \mu & 0 & & & 0 \\
0 & \lambda & & & & & \\
\vdots & & & & & & \\
0 & 0 & 0 & \lambda & -(\lambda+\mu) & 0 & \mu \\
0 & & 0 & & \lambda & -\lambda & 0 \\
0 & & & \cdots & & & 0
\end{array}\right)
\end{array}
= \begin{bmatrix} M_1 & \underline{e_\mu} \\ \underline{0} & 0 \end{bmatrix}$$

where M_1 is the matrix obtained by deleting the last row and last column of the generator and $\underline{e_\mu}$ is the column vector with last but one entry μ and all other zero. $\underline{0}$ is a row vector of zeros. The distribution of time till absorption is of phase type given by $F_1(x) = 1 - \underline{\alpha_1}\exp(T_1 x)\underline{e_1}$ for $x \geq 0$, where $\underline{\alpha_1}$ is the row vector of initial probability with entries $\alpha_{k-1}, \alpha_k, \ldots, \alpha_n$ where $\alpha_{k-1} = 0$, $\alpha_k = 1 - (\alpha_{k+1} + \ldots + \alpha_n)$ with $\alpha_i = P(S_{n-i} < T < S_{n-i+1})$ for $i = k+1, \ldots, n$ where the random variable S_i is the time till i failures take place starting from the instant at which all units function write $S_0 = 0$, then we have $S_0 < S_1 < \ldots < S_{n-k}$ and $\underline{e_1} = (1, 1, \ldots, 1)^T$.

3.3.3 Expected duration of time the server is busy in a cycle is given by

$$\sum_{i=k}^{n-2} \frac{1}{(\mu-\lambda)}((n-i) - (\frac{\lambda}{\mu})^{i-k+2}\frac{\mu(1-(\frac{\lambda}{\mu})^{n-i})}{(\mu-\lambda)}P(S_{n-i-1} < T < S_{n-i})$$
$$+\frac{1}{(\mu-\lambda)}(1-(\frac{\lambda}{\mu})^{i-k+1})\frac{\alpha}{\alpha+\lambda}$$

where

$$P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}, \quad k \leq i \leq n-1.$$

Let T_i denote the time to reach $(i+1, 1)$ starting from $(i, 1)$, $i \geq k-1$. We can recursively compute $E(T_i)$, $i \geq k-1$ from the relation $E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i-1})$ starting from $E(T_{k-1}) = \frac{1}{\mu}$

From the state $(i, 1)$ both $(i+1, 1)$ and $(i-1, 1)$ can be reached

$$\begin{aligned} (i,1) &\to (i+1,1) \\ (i,1) &\to (i-1,1) \to (i,1) \to (i+1,1) \end{aligned} \tag{3.3}$$

T_i denote the time to reach $(i+1, 1)$ from $(i, 1)$. Hence

$$\begin{aligned} E(T_i) &= \tfrac{1}{\lambda+\mu}\tfrac{\mu}{\lambda+\mu} + \tfrac{\lambda}{\lambda+\mu}(\tfrac{1}{\lambda+\mu} + E(T_{i-1}) + E(T_i)) \\ \text{ie,} \quad E(T_i)\tfrac{\mu}{\lambda+\mu} &= \tfrac{1}{\lambda+\mu} + \tfrac{\lambda}{\lambda+\mu}E(T_{i-1}) \end{aligned}$$

Thus we get the relation

$$\begin{aligned}
E(T_i) &= \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i-1}), i \geq k-1 \\
&= \frac{1}{\mu} + \frac{\lambda}{\mu}(\frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i-2})) \\
&= \frac{1}{\mu} + \frac{\lambda}{\mu^2} + (\frac{\lambda}{\mu})^2 E(T_{i-2}) \\
&\ldots \\
&= \frac{1}{\mu}\frac{(1-(\frac{\lambda}{\mu})^{i-k+2})}{(1-\frac{\lambda}{\mu})} = \frac{(1-(\frac{\lambda}{\mu})^{i-k+2})}{(\mu-\lambda)}
\end{aligned}$$

The expected time to reach $(n, 0)$ conditional on server getting activated between $(n-i)$th and $(n-i+1)$th component failures is $\sum_{j=i}^{n-1} E(T_j|S_i < T < S_{i+1})P(S_i < T < S_{i+1})$, where the random variable S_i is the time till i failures take place starting from the instant at which all units functions write $S_0 = 0$, then we have $S_0 < S_1 < \ldots < S_{n-k}$. With this the expected time to reach $(n, 0)$ is

$$\sum_{i=k}^{n-2} \frac{1}{(\mu-\lambda)}((n-i) - (\frac{\lambda}{\mu})^{i-k+2}\frac{\mu(1-(\frac{\lambda}{\mu})^{n-i})}{(\mu-\lambda)})P(S_{n-i-1} < T < S_{n-i})$$
$$+\frac{1}{(\mu-\lambda)}(1-(\frac{\lambda}{\mu})^{n-k+1})\frac{\alpha}{\alpha+\lambda}$$

where $P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \leq i \leq n-1$ and is obtained as follows.

$$\begin{aligned}
P(S_{n-i-1} < T < S_{n-i}) &= \int_0^\infty \int_0^x \frac{e^{-\lambda u}(\lambda u)^{n-i-1}.\lambda}{(n-i-1)!}\alpha e^{-\alpha x}e^{-\lambda(x-u)}dudx \\
&= \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}, \quad k \leq i \leq n-1
\end{aligned}$$

3.3.4 Expected time the server is not in the system in a cycle is given by

$$\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})$$

From the state $(n, 0)$ the system can move either to $(n, 1)$ or $(n-1, 0)$. If it goes to $(n-1, 0)$, then from this the system further moves to $(n-2, 0)$ or $(n-1, 1)$. This processing go on till the state $(k+1, 0)$ is reached. From $(k+1, 0)$ it can either go to $(k+1, 1)$ or $(k, 1)$. At $(k+1, 1)$ on failure of one unit the system goes to $(k, 1)$ by an assumption. Thus the expected amount of time the server is not in the system in a cycle is

$$\begin{aligned}
&\tfrac{1}{\alpha}P(T < S_1) + (\tfrac{1}{\alpha} + \tfrac{1}{\lambda}) \quad P(S_1 < T < S_2) + \cdots + (\tfrac{1}{\alpha} + \tfrac{n-k-1}{\lambda})P(S_{n-k-1} < T < S_{n-k}) \\
&+P(T > S_{n-k})\tfrac{n-k}{\lambda} \qquad = \tfrac{1}{\alpha}\tfrac{\alpha}{\lambda+\alpha} + (\tfrac{1}{\alpha} + \tfrac{1}{\lambda})\tfrac{\alpha\lambda}{(\lambda+\alpha)^2} + (\tfrac{1}{\alpha} + \tfrac{2}{\lambda})\tfrac{\alpha\lambda^2}{(\lambda+\alpha)^3} + \dots \\
&\qquad +(\tfrac{1}{\alpha} + \tfrac{n-k-1}{\lambda})\tfrac{\alpha\lambda^{n-k-1}}{(\lambda+\alpha)^{n-k}} + \tfrac{n-k}{\lambda}(\tfrac{\lambda}{\lambda+\alpha})^{n-k} = \tfrac{2}{\alpha}(1 - (\tfrac{\lambda}{\lambda+\alpha})^{n-k})
\end{aligned}$$

3.3.5 Expected duration of time the system is down in a cycle is

$$((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha) + (\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}} + (\frac{\lambda}{\mu})^{n-k+1}\frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{(\lambda+\alpha-\mu)})$$

It is well known that $\frac{q_{k-1,1}}{q_{n0}}$ gives the expected number of visits to $(k-1,1)$ before first return to $(n,0)$ (starting from $(n,0)$) (see Tijms (1994)). Further $\frac{1}{\mu}$ is the expected amount of time system remains in $(k-1,1)$ during each visit to that state. Hence expected duration of time system is down is $\frac{q_{k-1,1}}{\mu q_{n0}}$ which is equal to

$$\begin{aligned}
((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha) &+ (\frac{\lambda}{\mu})^{n-k+1}\frac{\mu^{n-k-3}(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}} \\
&+(\frac{\lambda}{\mu})^{n-k+1}\frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{(\lambda+\alpha-\mu)})
\end{aligned}$$

3.3.6 Model b

System state probabilities in the long run are the same as in model a for states $\{(k-1,1),\dots,(n-1,1),(k+1,0),\dots,(n,0)\}$. Further since the functional units deteriorate even when the system is down, we have for $l = 2,\dots,k, q_{k-l,1} = (\frac{\delta}{\mu})^{l-1}q_{k-1,1}, 2 \leq l \leq k$. The system is down for the fraction of time $\sum_{i=0}^{k-1} q_{i1}$. So the system reliability is $1-\sum_{i=0}^{k} q_{i1} = 1 - \frac{(1-(\delta/\mu)^k)}{(1-\delta/\mu)}q_{k-1,1}$

3.3.7 Distribution of server availability

Consider the Markov chain on the state space $\{(0,1),(1,1),\dots,(k,1),\dots(n-1,1),(n,0)\}$ with state $(n,0)$ absorbing. This distribution of phase type given by $F_2(x) = 1 - \underline{\alpha_2}\exp(M_2x)\underline{e_2}$, where M_2 is the matrix

$$
\begin{array}{c|c}
 & \begin{matrix} (0,1) & (1,1) & & & & (n-1,1) \end{matrix} \\
\begin{matrix} (0,1) \\ (1,1) \\ \vdots \\ (n-1,1) \end{matrix} &
\begin{pmatrix}
-\mu & \mu & 0 & & & 0 \\
0 & -(\mu+\delta) & \mu & 0 & & 0 \\
 & & \vdots & & & \\
0 & 0 & 0 & 0 & \lambda & -(\lambda+\mu)
\end{pmatrix}
\end{array}
$$

$\underline{\alpha_2}$ is the row vector of initial probabilities with first k entries zero, the rest of the entries are $\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n, \alpha_{n+1}$ where $\alpha_{k+1} = 1-(\alpha_{k+2}+\ldots+\alpha_{n+1})$ and $\alpha_i = P(S_{n-i+1} < T < S_{n-i+2}), i = k+2, \ldots n.$ $\alpha_{n+1} = P(T < S_1)$, $\underline{e_2} = (1, 1, \ldots, 1)^T$.

3.3.8 Expected duration of time the server is continuously busy

As in the earlier model, T_i denote the time to enter state $(i+1, 1)$ starting from $(i, 1)$. Here $E(T_0) = \frac{1}{\mu}$.

$$E(T_1) = \frac{1}{\mu}(1+\frac{\lambda}{\mu}) \quad E(T_{k-1}) = \frac{1}{\mu}\frac{(1-(\frac{\delta}{\mu})^k)}{(1-\frac{\delta}{\mu})}$$

$$E(T_k) = \frac{1}{\mu} + \frac{\lambda}{\mu}\frac{1}{\mu}\frac{(1-(\frac{\delta}{\mu})^k)}{(1-\frac{\delta}{\mu})}$$

We can recursively compute $E(T_i), i \geq 0$ from the relation $E(T_i) = \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i-1})$ starting from $E(T_{k-1}) = \frac{1}{\mu}\frac{(1-(\delta/\mu))^k}{(1-\delta/\mu)}$. Thus

$$E(T_j) = \frac{1}{\mu}\left[\frac{1-(\lambda/\mu)^{j+1-k}}{1-(\lambda/\mu)} + (\lambda/\mu)^{j+1-k}\frac{(1-(\delta/\mu)^k)}{(1-\delta/\mu)}\right]$$

The expected time to reach $(n, 0)$ conditional on server reaches between $(n-i)$th and $(n-i+1)$th component failures is $\sum_{j=i}^{n-1} E(T_j \mid S_i < T < S_{i+1})P(S_i < T < S_{i+1})$ which is equal to

$$\sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\frac{1}{\mu}\left[\frac{1-(\lambda/\mu)^{j+1-k}}{1-(\lambda/\mu)} + (\frac{\lambda}{\mu})^{j+1-k}\frac{1-(\delta/\mu)^k}{(1-\delta/\mu)}\right] P(S_{n-i-1} < T < S_{n-i}),$$

where $P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \leq i \leq n-1$

$$P(T < S_1) = \frac{\alpha}{\lambda+\alpha}$$

Expected time the server remains inactive during a cycle is same as in model a.

3.3.9 Expected duration of time the system is down in a cycle

To this end note that :

$\frac{q_{k-1,1}}{q_{n0}}$ gives the expected number of visits to $(k-1,1)$ before first return to $(n,0)$. Consider the class $\{(0,1),(1,1)\ldots(k-1,1)\}$. The process spends on the average $\frac{1}{\mu}\frac{(1-(\frac{\delta}{\mu})^k)}{(1-\frac{\delta}{\mu})}$ amount of time in this class during each visit before returning to state $(k,1)$. Hence expected duration of time the system is down in a cycle is

$$\frac{1}{\mu}\frac{(1-(\frac{\delta}{\mu})^k)}{(1-\frac{\delta}{\mu})}\frac{q_{k-1,1}}{q_{n0}} = \frac{1}{\mu}\frac{(1-(\frac{\delta}{\mu})^k)}{(1-(\frac{\delta}{\mu}))}((\frac{\lambda}{\mu})^{n-k}\frac{(\lambda+\alpha)}{\mu^2}+(\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}$$
$$\frac{(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}}+(\frac{\lambda}{\mu})^{n-k+1}\frac{1-(\frac{\mu}{\lambda+\alpha})^{n-k-3}}{\lambda+\alpha-\mu})$$

3.3.10 Model c

System state probabilities in the long run are the same as in model a for states $(k-1,1),\ldots,(n-1.1),(k+1,0),\ldots,(n,0)$. Further since the functional units deteriorate at the same rate even when the system is down as when it is up $q_{k-l,1} = (\frac{\lambda}{\mu})^{l-1}q_{k-1,1}$ for $l=2,3,\ldots,k$ can be expressed in terms of q_{n0}. System reliability is computed as earlier. q_{n0} can be obtained using the normalizing condition $\sum_{ij\in E_2} q_{ij} = 1$.

The distribution of the duration of time the server continuously remains in the system is given by

$$F_3(x) = 1 - \underline{\alpha_3}\exp(M_3x)\underline{e_3}$$

where $\underline{\alpha_3}$ is a $(n+1)$ component row vector with first k entries zero the rest of the entries are $\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_n,\alpha_{n+1}$ where $\alpha_{k+1} = 1-(\alpha_{k+2}+\ldots+\alpha_n)$ and $\alpha_i = P(S_{n-i+1} < T < S_{n-i+2}), i = k+2,\ldots,n, \alpha_{n+1} = P(T < S_1)$. $\underline{e_3}$ is also of the same dimension with all entries 1. M_3 is a non-singular matrix of order n gives by first n rows and n columns of the matrix $I-P$ where I is of order $(n+1)$ and P is the transition probability matrix of the chain on the set $\{(0,1),(1,1),\ldots,(n-1.1),(n,1).(n,0)\}$.

3.3.11 Expected amount of time the server is continuously busy

In this case $E(T_j) = \frac{1}{\mu}\frac{(1-(\lambda/\mu)^{j+1})}{1-(\frac{\lambda}{\mu})}$, $j \geq 0$ starting with $E(T_0) = \frac{1}{\mu}$. As in model b, we get $E(T_i) = \sum_{j=i}^{n-1} E(T_j \backslash S_i < T < S_{i+1})$,

$$P(S_i < T < S_{i+1}) = \sum_{i=0}^{n-1}\sum_{j=i}^{n-1}\frac{1}{\mu}[\frac{1-(\lambda/\mu)^{j+1}}{1-\lambda/\mu}]P(S_{n-i-1} < T < S_{n-i}),$$

where $P(S_{n-i-1} < T < S_{n-i}) = \frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$, $k \leq i \leq n-1$ Thus

$$E(T_i) = \sum_{i=0}^{n-2}(n-i) - (\frac{\lambda}{\mu})^{i+1}\frac{(1-(\frac{\lambda}{\mu})^{n-i})}{(1-\frac{\lambda}{\mu})}\frac{\alpha\lambda^{n-i-1}}{(\lambda+\alpha)^{n-i}}$$
$$+(1-(\frac{\lambda}{\mu})^n)\frac{\alpha}{\lambda+\alpha}$$

Here also the expected time the server is not in the system is the same as in the above two models.

3.3.12 Expected amount of time the system is nonfunctional

The process spends on the average $\frac{1}{\mu}\frac{(1-(\frac{\lambda}{\mu})^k)}{(1-\frac{\lambda}{\mu})}$ amount of time in the class $\{(0,1),(1,1),\ldots,(k-1,1)\}$. Expected amount of time the system is non-functional in a cycle is

$$\frac{1}{\mu}\frac{(1-(\frac{\lambda}{\mu})^k)}{1-\frac{\lambda}{\mu}}\frac{q_{k-1,1}}{q_{n0}}$$

3.4 A Control Problem

Here we attempt to find the optimal value of α by maximizing the profit and the system reliability. The following costs are considered.

1. Cost C per unit time due to the machine remaining non-functional
2. Profit per unit time when the server is not activated in the system.

Let C denote the cost per unit time due to the machine remaining non-functional and w denote the wages given to the server.

3.4.1 Model a

Profit per unit time when the server is not activated $= w(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}))$.
Expected cost per unit time due to the system remaining non functional $= C(\frac{1}{\mu})\frac{q_{k-1,1}}{q_{n0}} =$

$$C(\frac{\lambda}{\mu})^{n-k+1}((\frac{\mu}{\lambda})\frac{1}{\mu^2}(\lambda+\alpha)+\frac{\mu^{n-k-3}(\lambda+\alpha+\mu)}{(\lambda+\alpha)^{n-k-1}}+\frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{(\lambda+\alpha-\mu)})$$

Therefore the total expected profit per unit time $(TEP)_a$ is

$$w(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}))-C(\frac{\lambda}{\mu})^{n-k+1}((\frac{\mu}{\lambda})\frac{1}{\mu^2}(\lambda+\alpha)+\frac{\mu^{n-k-3}(\lambda+\alpha-\mu)}{(\lambda+\alpha)^{n-k-1}}$$
$$+\frac{1-(\frac{\mu}{\lambda+\alpha})^{n-k-3}}{\lambda+\alpha-\mu})$$

The above function is concave in α as it can be seen by differentiating it twice with respect to α. However it is difficult to find optimal α value from the first derivative equated to zero.

3.4.2 Model b

In model b, the total expected profit per unit time $(TEP)_b$ is

$$(TEP)_b = w(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}))-C((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha)$$
$$+(\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{\lambda+\alpha+\mu}{(\lambda+\alpha)^{n-k-1}}+(\frac{\lambda}{\mu})^{n-k+1}\frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{\lambda+\alpha-\mu})\frac{1-(\frac{\delta}{\mu})^k}{\mu-\delta}$$

Here again $(TEP)_b$ is a concave function in α as can be seen by differentiating the profit function with respect to α.

3.4.3 Model c

In this case the total expected profit per unit time $(TEP)_c$ is

$$(TEP)_c = w(\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}))-C((\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu^2}(\lambda+\alpha)$$
$$+(\frac{\lambda}{\mu})^{n-k+1}\mu^{n-k-3}\frac{\lambda+\alpha+\mu}{(\lambda+\alpha)^{n-k-1}}+(\frac{\lambda}{\mu})^{n-k+1}\frac{(1-(\frac{\mu}{\lambda+\alpha})^{n-k-3})}{(\lambda+\alpha-\mu)})\frac{1-(\frac{\lambda}{\mu})^k}{\mu-\lambda}$$

which is concave in α and hence has global maximum

3.4.4 Numerical illustration

For illustration, we calculate the total expected profit per unit time for given parameters for the three models and for various value of α. On comparing the three models for different set of parameters, we can see that total expected profit is maximum for model b.

Comparison of three models

$n = 12, \lambda = 7.5, \mu = 13, k = 6, w = 70, C = 80, \delta = 5$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3	38.986	40.284	40.209
3.1	38.052	39.315	39.242
3.2	37.156	38.386	38.315
3.3	36.296	37.495	37.426
3.4	35.471	36.64	36.573
3.5	34.679	35.819	35.753
3.6	33.917	35.03	34.966
3.7	33.184	34.271	34.209
3.8	32.479	33.541	33.48
3.9	31.8	32.839	32.779
4	31.146	32.162	32.104
4.1	30.516	31.51	31.453
4.2	29.908	30.882	30.826
4.3	29.322	30.276	30.221
4.4	28.756	29.691	29.637
4.5	28.21	29.126	29.073
4.6	27.682	28.581	28.529
4.7	27.171	28.054	28.003
4.8	26.678	27.544	27.494
4.9	26.2	27.051	27.002
5	25.738	26.575	26.526

$n = 18, k = 7, \lambda = 9.5, \mu = 14, w = 80, C = 110, \delta = 4$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
2	49.157	50.57	50.402
2.1	47.805	49.153	48.992
2.2	46.518	47.805	47.652
2.3	45.292	46.523	46.376

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
2.4	44.124	45.302	45.161
2.5	43.01	44.138	44.003
2.6	41.946	43.028	42.898
2.7	40.929	41.968	41.844
2.8	39.957	40.955	40.835
2.9	39.027	39.986	39.871
3	38.136	39.059	38.949
3.1	37.282	38.172	38.065
3.2	36.463	37.321	37.218
3.3	35.677	36.505	36.406
3.4	34.922	35.722	35.627
3.5	34.796	35.03	35.002
3.6	33.499	34.248	34.158
3.7	32.828	33.553	33.466
3.8	32.182	32.885	32.806
3.9	31.56	32.241	32.216
4	30.96	31.621	31.542

$n = 10, K = 5, \lambda = 5.5, \mu = 10, w = 50, C = 100, \delta = 3$

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
3	27.435	29.251	29.106
3.1	26.737	28.512	28.37
3.2	26.069	27.806	27.667
3.3	25.43	27.131	26.995
3.4	24.819	26.484	26.351
3.5	24.232	24.865	25.734
3.6	23.67	25.271	25.143
3.7	23.13	24.702	24.576
3.8	22.612	24.156	24.032
3.9	22.114	23.631	23.509
4	21.635	23.126	23.007
4.1	21.174	22.641	22.524
4.2	20.73	22.174	22.059
4.3	20.303	21.725	21.611
4.4	19.891	21.292	21.18

Total expected profit/unit time			
α	$(TEP)_a$	$(TEP)_b$	$(TEP)_c$
4.5	20.047	20.953	20.88
4.6	19.11	20.472	20.363
4.7	18.739	20.083	19.976
4.8	18.382	19.708	19.602
4.9	18.036	19.346	19.241
5	17.701	18.996	18.893

3.5 General case

Here we assume that the repair time is arbitrarily distributed with distribution function $G(.)$ having density $g(.)$. The server is activated after the elapse of T time units since last inactivation after completion of most recent repair of all failed units or when the number of failed units accumulate to $n - k$ units, whichever occurs first. Life times of components are $i.i.d$ random variables. T is exponentially distributed with parameter α

3.5.1 Formulation and Analysis of the problem

Assume that at time $T_0 = 0$ the last of the failed units completed repair. That is we start the system at time zero with all units operational. $X(t)$ be the number of working components and $Y(t)$, the state of the repair man at time t. Write $X_n = X(T_n+)$ and $Y_n = Y(T_n+)$ for $n \in Z$.

We consider three cases (i) cold system (ii) warm system (iii) hot system where we designate the system as cold, warm or hot according as the functional components do not fail, fail at a slower rate or at the same rate during system down state as when the system functions, respectively.

We observe that $\{(X(t), Y(t)), t \in R_+\}$ is a semi-Markov process on $E_1 = \{(k-1,1), (k,1), \ldots, (n-1,1), (k+1,0) \ldots (n,0), (n,1)\}$ in model 1 and on $E_2 = \{(0,1), \ldots, (k-1,1), (k,1) \ldots (n,1), (k+1,0), \ldots, (n,0)\}$ in models 2 and 3 (the warm and hot systems).

Let time $T_0 = 0$, the system starts with all components operational. Thus $X(0+) = X_0 = n$ and $Y(0+) = Y_0 = 0$. Let $T_1, T_2, \ldots T_n, \ldots$ be the successive repair completion epochs of failed units.

A server is activated after the elapse of T time units since inactivation after completion of the most recent repair of all failed units or when the number of failed units accumulate to $n-k$, whichever occur first. Then, we have

Theorem 3.5.1. *$\{(X_n, T_n),\ n \in Z_+\}$ is a Markov renewal process with state space $E_3 = \{(k,1), \ldots, (n-1,1), (n,0)\}$ for model 1 and $E_4 = \{(1,1), \ldots (n-1,1), (n,0)\}$ for model 2 and 3 with semi-Markov kernel $Q(i,j,t)$ define as $Q(i,j,t) = P((X_{n+1}, Y_{n+1}) = (j,l);\ T_{n+1} - T_n \leq t \mid (X_n, Y_n) = (i,l)),\ t \in R_+$*

Proof. For model 1. there are given by

$$\begin{aligned}
Q((i,1)(j,1),t) &= \int_0^t \frac{e^{-\lambda u}(\lambda u)^{i-j+1}}{(i-j+1)!} g(u)du, \quad j \leq i+1, i \neq n-1 \\
Q((n-1,1),(n,0),t) &= \int_0^t e^{-\lambda u} g(u)du \\
Q((n-1,1),(j,1)t) &= \int_0^t \frac{e^{-\lambda u}(\lambda u)^{n-j-1}}{(n-j-1)!}(1-G(u))du, \quad j \leq n-1 \\
Q((n,0),(n,0),t) &= \int_{u=0}^t \int_{v=u}^t \int_{w=u}^t \alpha e^{-\alpha u} \lambda e^{-\lambda v} g(w-v) e^{-\lambda(w-v)} dw dv du \\
&+ \int_{u=0}^t \int_{v=u}^t \int_{w=v}^t \lambda e^{-\lambda u} \alpha e^{-\alpha v} g(w-v) e^{-\lambda(w-u)} dw dv du \\
Q((n,0),(i,1),t) &= \int_{u=0}^t \int_{v=u}^t \sum_{j=1}^{n-k-1} \frac{e^{-\lambda u}(\lambda u)^j}{j!} \frac{\alpha e^{-\alpha u} e^{-\lambda(v-u)}(\lambda(v-u)^{n-i-j+1})}{(n-i-j+1)!} \\
& g(v-u) dv du + \int_{u=0}^t \int_{v=u}^t \int_{w=v}^t \alpha e^{-\alpha u} \lambda e^{-\lambda v} g(w-v) e^{-\lambda(w-v)} \\
& \frac{(\lambda(w-v))^{n-i}}{(n-i)!} dw dv du
\end{aligned}$$

For model 2, we have

$$\begin{aligned}
Q((i,1),(j,1),t) &= \int_0^t \frac{e^{-\lambda u}(\lambda u)^{i-j+1}}{(i-j+1)!} g(u)du \quad i = k, \ldots, n-1, k \leq j \leq i+1 \\
Q((i,1),(j,1),t) &= \int_0^t \frac{e^{-\delta u}(\delta u)^{i-j+1}}{(i-j+1)!} g(u)du \quad i = 1, 2, \ldots, k-1; j \leq i+1
\end{aligned}$$

$$Q((i,1),(j,1),t) = \int_{u=0}^{t}\int_{v=u}^{t} \frac{e^{-\lambda u}(\lambda u)^{i-k}\lambda}{(i-k)!} \frac{e^{-\delta(v-u)}(\delta(v-u))^{k-1-(j-1)}}{(k-j)!} g(v)dvdu$$

$$i = k, k+1, \dots n-1;\; j = 1, 2, \dots k$$

$$Q((n-1,1),(n,0),t) = \int_0^t e^{-\lambda u} g(u)du$$

$$Q((n,0),(i,1),t) = \int_{u=0}^{t}\int_{w=u}^{t}\int_{v=w}^{t} \frac{\lambda e^{-\lambda u}(\lambda u)^{n-k-1}}{(n-k-1)!} e^{-\alpha u} g(v-u)\lambda e^{-\lambda(w-u)} e^{-\delta(v-w)}$$
$$\frac{(\delta(v-w))^{k-i}}{(k-i)!} dvdwdu + \int_{x_1=0}^{t}\int_{u=x_1}^{t}\int_{x_2=u}^{t}\int_{v=x_2}^{t} \sum_{j=1}^{n-k-1}$$
$$\frac{e^{-\lambda x_1}(\lambda x_1)^{j-1}\lambda}{(j-1)!} \alpha e^{-\alpha u} e^{-\lambda(x_2-x_1)} \frac{(\lambda(x_2-x_1))^{n-k-j} g(v-u)}{(n-k-j)! e^{-\lambda(u-x_1)}}$$
$$\frac{e^{-\delta(v-x_2)}(\delta(v-x_2))^{k-i}}{(k-i)!} dvdx_2dudx_1$$
$$+ \int_{u=0}^{t}\int_{w=u}^{t}\int_{x=w}^{t}\int_{v=x}^{t} \alpha e^{-\alpha u}\lambda e^{-\lambda w} g(v-w) e^{-\lambda(x-w)}$$
$$\frac{(\lambda(x-w))^{n-k}}{(n-k)!} \frac{e^{-\delta(v-x)}(\delta(v-x))^{k-i}}{(k-i)!} dvdxdwdu.\quad i = 1, 2, \dots, k$$

for model 3, we get $Q(i,j,t)$ as,

$$Q((i,1),(j,1),t) = \int_{u=0}^{t} \frac{e^{-\lambda u}(\lambda u)^{i-j+1}}{(i-j+1)!} g(u)du \quad i = 1, 2, \dots, k-1;\; j \le i+1$$

$$Q((i,1),(j,1),t) = \int_{u=0}^{t} \frac{e^{-\lambda u}(\lambda u)^{i-j}}{(i.j)!} g(u)du. i = k, k+1, \dots, n-1,\;\; j = 1, 2, \dots, k$$

$$Q((n,0),(i,1),t) = \int_{u=0}^{t}\int_{v=u}^{t} \frac{\lambda e^{-\lambda u}(\lambda u)^{n-k-1}}{(n-k-1)!} e^{-\alpha u} g(v-u) \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{k-i+1}}{(k-i+1)!} dvdu$$
$$+ \int_{u=0}^{t}\int_{v=u}^{t} \sum_{j=1}^{n-k-1} \frac{e^{-\lambda u}(\lambda u)^{j}}{j!} \alpha e^{-\alpha u} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{n-j-i+1}}{(n-j-i+1)!} g(v-u)dvdu$$
$$+ \int_{u=0}^{t}\int_{v=u}^{t}\int_{w=v}^{t} \alpha e^{-\alpha u}\lambda e^{-\lambda v} g(w-v) \frac{e^{-\lambda(w-v)}(\lambda(w-v))^{n-i}}{(n-i)!} dwdvdu$$

□

3.5.2 Time dependent solution

Model 1

Let $P_{(i_1j_1)(i_2j_2)}(t) = P((X(t) = i_2, Y(t) = j_2 \mid X(0) = i_1, Y(0) = j_1))$ and $P_{(i_1,j_1)}(t) = P((X(t) = i_1, Y(t) = j_1 \mid X(0) = n, Y(0) = 0))$. These probabilities are obtained from the definition of $Q(i,j,t)$, $i, j \in E_3$. Define

$$U((i,1),(j_1,1),t) = P((X(t),Y(t)) = (j_1,1), T_1 > t \mid (X(0),Y(0)) = (i_1,1))$$

Then $U((i_1,1),(j_1,1),t) = (1 - G(t))\frac{e^{-\lambda t}(\lambda t)^{i_1-j_1}}{(i_1-j_1)!}$ $k-1 \leq i, j_1 \leq n-1$ and $j_1 \leq i_1$. Further define

$$\begin{aligned} H((n,0),(n,0),t) &= \int_{u=0}^{t}\int_{v=u}^{t} \alpha e^{-\alpha u}\lambda e^{-\lambda v}\sum_{l\geq 1} Q^{*l}((n-1,1),(n,0),t-v)dvdu \\ &+ \int_0^t \sum_{m=1}^{n-k-1} \frac{e^{-\lambda u}(\lambda u)^m}{m!}\alpha e^{-\alpha u}\sum_{l\geq m} Q^{*l}((n-m,1),(n,0),t-u)du \\ &+ \int_0^t \frac{e^{-\lambda u}(\lambda u)^{n-k-1}.\lambda}{(n-k-1)!}e^{-\alpha u}\sum_{l\geq n-k} Q^{*l}((k,1),(n,0),t-u)du \end{aligned}$$

where $Q^{*l}((i,1),(j,1),t)$ is the l-fold convolution of Q with itself and

$Q^{*(0)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ This represents the distribution of time of first return to $(n,0)$ starting from $(n,0)$

$$\begin{aligned} P_{(i,0)}(t) &= \int_0^t \sum_{l=0}^{\infty} H^{*l}((n,0),(n,0),du)e^{-\lambda(t-u)}\frac{(\lambda(t-u))^{n-i}}{(n-i)!}du, \quad i = k+1,\ldots n \\ P_{(i,1)}(t) &= \int_0^t \sum_{j=k}^{n-1} Q((n,0),(j,1),du)P_{(j,1),(i,1)}(t-u)du \end{aligned}$$

3.5.3 Limiting distribution

Let $\mathbb{Q} = (\lim_{t\to\infty} Q((i_1,j_1),(i_2,j_2),t))$, $(i_1,j_1),(i_2,j_2) \in E_3$ and $\Pi = (\pi(k,1),\pi(k+1,1),\ldots,\pi(n-1,1),\pi(n,0))$ is the stationary vector where $\pi(i,j) = \lim_{n\to\infty} P(X_n = i, Y_n = j \mid X_0 = n, Y_0 = 0)$ where $j = 0$ if $i = n$ and $j = 1$ when $i = k, k+1, \ldots, n-1$. These probabilities can be computed from $\Pi\mathbb{Q} = \Pi$ and $\sum_{i=k}^{n-1}\pi(i,1) + \pi(n,0) = 1$. The long run system state distribution at arbitrary epoch can be derived as follows. Define

$$q_{ij} = \lim_{t\to\infty} P((X(t) = i, Y(t) = j \mid X(0) = n, Y(0) = 0))$$

for $i = k+1,\ldots,n-1,n; j = 0, i = k-1,\ldots,n-1; j = 1$.

Model 1

Let $\mu = \int_0^\infty (1 - G(t))dt$ which we assume to be finite. Then,

$$q_{n0} = \frac{\pi(n,0)}{\mu} \quad q_{n1} = \frac{\pi(n,0)\alpha}{(\lambda+\alpha)\mu} \quad q_{i0} = \frac{\pi(n,0)}{\mu}\frac{\lambda^{n-i}}{(\alpha+\lambda)^{n-i+1}}, i = k+1,\ldots,n-1$$

$$q_{i1} = \sum_{j \leq i} \frac{\pi(j,1)}{\mu} \int_0^\infty \frac{e^{-\lambda u}(\lambda u)^{j-i}}{(j-i)!}(1 - G(u))du$$

Next, we find $q_{(k-1,1)}$. State $(k-1,1)$ can be reached from the states $(j,1), j = k,\ldots,n-1$ and $(n,0)$. We have derived $\pi(j,1), j = k,\ldots,n-1$ and $\pi(n,0)$. From the state $(j,1)$ state $(k-1,1)$ can be reached by the failure of the $j-k+1$ units. From state $(n,0)$, state $(k-1,1)$ can be reached due to failure of $n-k+1$ units. Here we consider three cases (i) server arrives before any failure (ii) server arrives between l^{th} and $(l+1)$th failure (iii) server arrives only on the failure of $n-k$ units.

Therefore,

$$q_{(k-1,1)} = \sum_{j=k}^{n-1} \frac{\pi(j,1)}{\mu} \int_0^\infty \frac{e^{\lambda u}(\lambda u)^{j-k+1}}{(j-k+1)!}(1 - G(u))du + \frac{\pi(n,0)}{\mu}$$

$$\Big[\int_{u=0}^\infty \int_{v=u}^\infty \sum_{l=1}^{n-k+1} \frac{e^{-\lambda u}(\lambda u)^l}{l!}\alpha e^{-\alpha u}\frac{e^{-\lambda(v-u)}(\lambda(v-u))^{n-k-l+1}}{(n-k-l+1)!}(1 - G(v-u))dvdu$$

$$+ \int_{u=0}^\infty \int_{t=u}^\infty \frac{e^{-\lambda u}(\lambda u)^{n-k-1}\lambda}{(n-k-1)!}e^{-\alpha u}[1 - G(t-u)]e^{-\lambda(t-u)}(\lambda(t-u))dtdu$$

$$+ \int_{u=0}^\infty \int_{v=u}^\infty \int_{w=v}^\infty \alpha e^{-\alpha u}\lambda e^{-\lambda v}\frac{e^{-\lambda(w-v)}(\lambda(w-v))^{n-k}}{(n-k)!}(1 - G(w-v))dwdvdu\Big]$$

Model 2

We have $\mathbb{Q} = (\lim_{t\to\infty} Q((i_1,j_1),(i_2,j_2),t)), (i_1,j_1),(i_2,j_2) \in E_4$ with $\pi(i,j) = \lim_{n\to\infty} P(X_n = i, Y_n = j), (i,j) \in E_4$. Let $\Pi = \{\pi(1,1), \pi(2,1), \ldots, \pi(n-1,1), \pi(n,0)\}$. Then Π is given by $\Pi\mathbb{Q} = \Pi$ with $\sum_{(i,j)\in E_4} \pi(i,j) = 1$. Next, we find out q_{ij}, $(i,j) \in E_2$. $q_{ij} = \lim_{t\to\infty} P(X(t) = i, Y(t) = j)$. These have the same form as in model 1, except for q_{i1} for $i = 0,1,2,\ldots,k-2$.

$$q_{i1} = \sum_{m\geq k} \frac{\pi(m,1)}{\mu} \int_{u=0}^{\infty}\int_{v=u}^{\infty} \frac{e^{-\lambda u}(\lambda u)^{m-k}\lambda}{(m-k)!} e^{-\alpha u} \frac{e^{-\delta(v-u)}(\delta(v-u))^{k-1-i}}{(k-1-i)!}(1-G(u))dvdu$$
$$+\frac{\pi(n,0)}{\mu}[\int_{u=0}^{\infty}\int_{v=u}^{\infty} \frac{e^{-\lambda u}(\lambda u)^{n-k}\lambda}{(n-k)!} e^{-\alpha u} \frac{e^{-\delta(v-u)}(\delta(v-u))^{k-1-i}}{(k-1-i)!}(1-G(u))dvdu$$
$$+\int_{u=0}^{\infty}\int_{v=u}^{\infty}\int_{w=v}^{\infty} \sum_{j=0}^{n-k-1} \frac{e^{-\lambda u}(\lambda u)^{j}}{j!}\alpha e^{-\alpha u} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{n-k-j+1}}{(n-k-j+1)!}$$
$$\frac{e^{-\lambda(w-v)}(\lambda(w-v))^{k-i}}{(k-i)!} g(w-v)dwdvdu$$

Model 3

In the case of model 3, we have $\mathbb{Q} = (\lim_{t\to\infty} Q((i_1,j_1),(i_2,j_2),t)),\ (i_1,j_1),(i_2,j_2) \in E_4$. Here the failure rate is λ even when the system is down. The limiting system state probabilities can be obtained. The expressions for q_{ij} remains identical except for the states $(i,1), i = 0,1,2,\dots,k-2$.

$$q_{i1} = \sum_{m\geq k} \frac{\pi(m,1)}{\mu} \int_{u=0}^{\infty}\int_{v=u}^{\infty} \frac{e^{-\lambda u}(\lambda u)^{m-k}\lambda}{(m-k)!} e^{-\alpha u} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{k-1-i}}{(k-1-i)!}(1-G(u))du$$
$$+\frac{\pi(n,0)}{\mu}[\int_{u=0}^{\infty}\int_{v=u}^{\infty} \frac{e^{-\lambda u}(\lambda u)^{n-k}\lambda}{(n-k)!} e^{-\alpha u} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{k-1-i}}{(k-1-i)!}(1-G(u))dvdu$$
$$+\int_{u=0}^{\infty}\int_{v=u}^{\infty}\int_{w=v}^{\infty} \sum_{j=0}^{n-k-1} \frac{e^{-\lambda u}(\lambda u)^{j}}{j!}\alpha e^{-\alpha u} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{n-k-j+1}}{(n-k-j+1)!}$$
$$\frac{e^{-\lambda(w-v)}(\lambda(w-v))^{k-i}}{(k-i)!} g(w-v)dwdvdu]$$

3.6 Control problem

Here we derive optimal value of α for a suitable cost function associated with the problem. For that, first we compute the distribution of time between two successive $(n,0)$ to $(n,0)$ transition that is the distribution of the time of the first return. The distribution of the time duration since the server arrival till all the failed units are repaired can be derived as follows.

3.6.1 Model 1

Suppose $B(t)$ is the distribution of the random variable. Let u be the time at which i units are repaired. During this time there may be none, one or more failures. Suppose there are j failures. Thus busy period generated by j failures has distribution $B_j(.)$. Thus

$$B(t) = B_1(t)P(T < S_1) + \sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1})B_i(t) + P(T > S_{n-k})B_{n-k}(t) =$$

$$\frac{\alpha}{\alpha+\lambda}\int_0^t \sum_{j=0}^{n-k-1} g(u)\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du + \sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1})\int_0^t \sum_{j=0}^{n-k-1} g^{*i}(u)$$
$$\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du + P(T > S_{n-k})\int_0^t \sum_{j=0}^{n-k-1} g^{*n-k}(u)\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du$$

where $g(.)$ is the density of the service time of a single unit. When $n-k$ is large we have

$$\begin{aligned} B(t) &= \tfrac{\alpha}{\alpha+\lambda}\int_0^t \textstyle\sum_{j=0}^{\infty} g(u)\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du \\ &+ \textstyle\sum_{i=1}^{n-k-1} P(S_i < T < S_{i+1})\int_0^t \sum_{j=0}^{\infty} g^{*i}(u)\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du \\ &+ P(T > S_{n-k})\int_0^t \textstyle\sum_{j=0}^{\infty} g^{*n-k}(u)\frac{e^{-\lambda u}(\lambda u)^j}{j!}B_j(t-u)du \end{aligned} \tag{3.4}$$

$b(t)$ is the density function corresponding to $B(t)$. Differentiating (3.4) and taking Laplace transform, we get

$$\begin{aligned} L(b(t)) = \tfrac{\alpha}{\alpha+\lambda}\ & L(g(u)) \mid s' = s+\lambda-\lambda L(b(t)) \\ &+ \textstyle\sum_{i=1}^{n-k-1} \frac{\alpha\lambda^i}{(\lambda+\alpha)^{i+1}} L(g^{*i}(u)) \mid s' = s+\lambda-\lambda L(b(t)) \\ &+ (\tfrac{\lambda}{\lambda+\alpha})^{n-k} L(g^{*n-k}(u)) \mid s' = s+\lambda-\lambda L(b(t)) \end{aligned} \tag{3.5}$$

inverting this, we get

$$b(t) = \frac{\alpha}{\alpha+\lambda}\sum_{l=1}^{\infty}\frac{1}{l}\frac{(\lambda t)^{l-1}}{(l-1)!}e^{-\lambda t}g^{*l+1}(t) + \sum_{i=1}^{n-k-1}\alpha\frac{\lambda^i}{(\lambda+\alpha)^{i+1}}\sum_{l=1}^{\infty}\frac{1}{l}\frac{(\lambda t)^{l-1}}{(l-1)!}$$
$$e^{-\lambda t}g^{*l+1}(t) + (\frac{\lambda}{\lambda+\alpha})^{n-k}\sum_{l=1}^{\infty}\frac{1}{l}\frac{(\lambda t)^{l-1}}{(l-1)!}e^{-\lambda t}g^{*n-k+l}(t)$$

Let b be the expected length of the busy period. Differentiating (3.4) and using

$$\frac{-d}{ds}L(g(u))/_{s=0} = \mu^{-1} \text{ and } \frac{-d}{ds}L(b(t))/_{s=0} = b.$$

we get $b = \frac{1}{\mu}(1+\lambda b)$ which gives $b = \frac{1}{(\mu-\lambda)}$

3.6.2 Expected length of time the server continuously remains inactive is given by

$$\frac{1}{\alpha}P(T < S_1) + (\frac{1}{\alpha} + \frac{1}{\lambda})P(S_1 < T < S_2) + \dots$$
$$+ (\frac{1}{\alpha} + \frac{n-k-1}{\lambda})P(S_{n-k-1} < T < S_{n-k}) + \frac{n-k}{\lambda}P(T > S_{n-k})$$
$$= \frac{1}{\alpha} \quad \frac{\alpha}{\alpha+\lambda} + (\frac{1}{\alpha} + \frac{1}{\lambda})\frac{\alpha\lambda}{(\lambda+\alpha)^2} + (\frac{1}{\alpha} + \frac{2}{\lambda})\frac{\alpha\lambda^2}{(\lambda+\alpha)^3} + \dots$$
$$+ (\frac{1}{\alpha} + \frac{n-k-1}{\lambda})\frac{\alpha\lambda^{n-k-1}}{(\lambda+\alpha)^{n-k}} + (\frac{\lambda}{\lambda+\alpha})^{n-k}\frac{n-k}{\lambda} = \frac{2}{\alpha}(1 - (\frac{\lambda}{\lambda+\alpha})^{n-k})$$

3.6.3 Expected duration of a busy cycle (that is the length of the time of first return to $(n, 0)$ starting from $(n, 0)$) is

$$\frac{1}{(\mu-\lambda)} + \frac{2}{\alpha}(1 - (\frac{\lambda}{\lambda+\alpha})^{n-k})$$

The fraction of time the server remains continuously in the system is

$$\frac{\frac{1}{(\mu-\lambda)}}{\frac{1}{(\mu-\lambda)} + \frac{2}{\alpha}(1 - (\frac{\lambda}{\lambda+\alpha})^{n-k})}$$

3.6.4 Total expected cost per unit time

Let C_1 be the fixed cost of hiring the server and C_2 the wage of the server per unit time. The total expected cost per unit time

$$(TEC)_1 = C_1(\frac{1}{(\mu-\lambda)} + \frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}))^{-1} + C_2\frac{1}{(\mu-\lambda)}[\frac{1}{(\mu-\lambda)} + \frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]^{-1}$$

It is seen that $(TEC)_1$ is convex in α. Hence global minimum value α^* that minimizes $(TEC)_1$ exists.

3.6.5 Model 2

Inorder to compute the distribution of the time of first return to $(n, 0)$, note that, once the system is down, further failures take place at rate δ. The distribution of the time duration the system is in the set of states $\{(0,1),(1,1),\dots(k-2,1)\}$ continuously is obtained by considering a process that starts at $(k-2,1)$ and returns to $(k-1,1)$ for the first time. The distribution is given by

$$B(t)=\int_0^t g(u)\sum_{j=0}^{k-2}\frac{e^{-\delta u}(\delta u)^j}{j!}B_j(t-u)du$$

For large k, we get

$$B(t)=\int_0^t g(u)\sum_{j=0}^{\infty}\frac{e^{-\delta u}(\delta u)^j}{j!}B_j(t-u)du$$

This has mean $b=\frac{1}{(\mu-\delta)}$. Thus the expected time between two successive visits to $(n,0)$ is

$$\frac{1}{(\mu-\delta)}+\frac{1}{(\mu+\delta)}+\frac{1}{(\mu-\lambda)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}).$$

Therefore

$$(TEC)_2=C_1[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{(\mu^2-\delta^2)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]^{-1}+C_2[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{\mu^2-\delta^2}]$$
$$[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{(\mu^2-\delta^2)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]^{-1}$$

This is convex in α and so global minimum value α^* of α exists.

3.6.6 Model 3

In this case the first return to $(n,0)$ starting from $(n,0)$ has on the average the duration $[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{(\mu^2-\lambda^2)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]$. Hence the expected cost per unit time is

$$(TEC)_3=C_1[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{(\mu^2-\lambda^2)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]^{-1}+C_2[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{\mu^2-\lambda^2}]$$
$$[\frac{1}{(\mu-\lambda)}+\frac{2\mu}{(\mu^2-\lambda^2)}+\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})]^{-1}$$

This is also convex in α. Hence optimal value of α that minimize $(TEC)_3$ exists.

3.6.7 Numerical illustration

For illustration we calculate the total expected cost for given parameters for the three models and for values of α. On comparing the three models for different sets of parameters, we can see that total expected cost is minimum for model 3.

Table 1. $C_1 = 100$ $C_2 = 80$ $n = 10$ $k = 5$ $\lambda = 5.5$ $\mu = 10$ $\delta = 3$

Total expected cost per unit time			
α	$(TEC)_1$	$(TEC)_2$	$(TEC)_3$
2.0	116.944	110.063	108.564
2.1	119.240	112.307	110.670
2.2	122.669	114.520	112.736
2.3	125.555	116.700	114.772
2.4	128.424	118.846	116.771
2.5	131.273	120.962	118.735
2.6	134.103	123.043	120.665
2.7	136.783	125.092	122.562
2.8	139.698	127.108	124.425
2.9	142.464	129.091	126.255
3.0	145.205	131.043	128.051

Table 2. $C_1 = 200$ $C_2 = 110$ $n = 12$ $k = 6$ $\lambda = 8.5$ $\mu = 15$ $\delta = 5$

Total expected cost per unit time			
α	$(TEC)_1$	$(TEC)_2$	$(TEC)_3$
2.0	248.882	228.390	223.26
2.1	254.551	232.817	227.394

Total expected cost per unit time			
α	$(TEC)_1$	$(TEC)_2$	$(TEC)_3$
2.2	260.211	237.209	231.494
2.3	265.865	241.567	235.548
2.4	271.512	245.886	239.565
2.5	276.690	250.169	243.539
2.6	282.770	254.414	247.474
2.7	288.378	258.621	251.367
2.8	293.972	262.790	255.217
2.9	299.548	266.920	259.026
3.0	305.106	271.009	262.792

Chapter 4

k-out-of-n system with repair and two modes of service : the T-policy

4.1 Introduction

For a k-out-of-n system with repair, we assume that there are two types of servers; I server (S_1) and II server (S_2). S_1 is available always and attends repair one at a time. However S_2 is activated only after elapse of T units of time from the epoch since the most recent completion of repair of all failed units. S_1 repairs failed units with exponentially distributed service times with rate μ_1, where as S_2 provides service which is exponentially distributed with rate μ_2 (again with single repair at a time). When all units are back to working state S_2 is switched off to be activated again on elapse of a random duration T which is assumed to follow an exponential distribution with rate α. Life times of components are exponentially distributed with rate λ. Repaired units are assumed to be as good as new.

k-out-of-n system with repair and two modes of service under N-policy has been investigated by Krishnamoorthy and Ushakumari [1999]. In this case S_1 is always alert and is serving if any unit is waiting for repair. S_2 is activated only when the number of failed units accumulate to $N(> 1)$. S_2 is activated to increase the repair rate in order that the system will have failure free operation for a longer duration.

Section 4.2 deals with the modeling and analysis and provides the system state distributions in finite time and in the long run. Section 4.3 is devoted to the study of some measures of performance.

4.2 Modelling and Analysis

Life times of components are exponentially distributed with rate λ when the number of failed units reaches $n-k+1$, the system is down. Once the system is down, operational components do not deteriorate further until the system starts functioning.
Let $X(t)$ be the number of failed units at time t and $Y(t)$ be the state of S_2 at that epoch. Write

$$Y(t) = \begin{cases} 1 & \text{if } S_2 \text{ is active at } t \\ 0 & \text{otherwise} \end{cases}$$

Then the bivariate process $\{(X(t), Y(t)),\ t \in R_+\}$ is a Markov process on the state space $E = \{n-k+1, n-k, \ldots, 1\} \times \{0,1\} \cup \{0,0\}$. Define $P_{ij}(t) = P\{(X(t), Y(t)) = (i,j),\ (i,j) \in E\}$ and let $P((X(0), Y(0)) = (0,0)) = 1$

The Kolmogorov forward differential difference equations satisfied by $P_{ij}(t)$ are

$$\begin{aligned}
P'_{00}(t) &= -\lambda P_{00}(t) + \mu_1 P_{10}(t) + \mu_2 P_{11}(t) \\
P'_{i0}(t) &= -(\lambda + \alpha + \mu_1)P_{i0}(t) + \mu_1 P_{i+1,0}(t) + \lambda P_{i-1,0}(t), \quad 1 \le i \le n-k \\
P'_{i1}(t) &= -(\lambda(1-\delta_{in-k+1}) + \mu_1 + \mu_2)P_{i1}(t) + (\mu_1 + \mu_2)(1-\delta_{in-k+1}) \\
&\quad P_{i+1,1}(t) + \alpha P_{i0}(t) + \lambda(1-\delta_{i1})P_{i-1,1}(t), \quad 1 \le i \le n-k+1 \\
P'_{n-k+1,0}(t) &= -(\alpha + \mu_1) P_{n-k+1,0}(t) + \lambda P_{n-k,0}(t)
\end{aligned} \tag{4.1}$$

where δ_{ij} is the Kronecker delta.

4.2.1 Transient solution

The above system of differential equations has a unique solution given by $\mathbb{P}(t) = e^{At}\mathbb{P}(0)$ where $\mathbb{P}(0)$ is the initial probability vector which has 1 corresponding to state $(0,0)$and rest zeros. A is the matrix of coefficient on the right side of (4.1)

4.2.2 Steady state solution

Theorem 4.2.1. *The stationary probability vector* $\Pi = [\pi_0, \Pi_1 \dots, \Pi_{n-k+1}]$ *is given by* $\Pi_i = \pi_0 \underline{\beta} R^i \; 1 \le i \le n-k$.

$$\Pi_{n-k+1} = \pi_0 \underline{\beta} R^{n-k}(-\lambda S^{-1})$$

where $\pi_0 = \{\underline{\beta}[\sum_{i=0}^{n-k} R^i - \lambda R^{n-k} S^{-1}]\underline{e}\}^{-1}$ *and* $R = \lambda(\lambda I - \lambda B^{oo} - S)^{-1}$ *where* $B^{oo} = \underline{e}.\underline{\beta}$ *where* $\underline{e}$ *is the column vector of 1s' and* $\underline{\beta}$ *is the initial probability vector.*

The infinitesimal generator of the Markov chain is given [refer next page]. We write the states lexicographically as $(0,0), (1,0), (1,1), (2,0), (2,1) \dots, (n-k+1, 0), (n-k+1, 1)$

Stationary probability vectors of the system state are given by $\Pi_i = \pi_0 \underline{\beta} R^i$, $1 \le i \le n-k$ and $\Pi_{n-k+1} = \pi_0 \underline{\beta} R^{n-k}(-\lambda S^{-1})$ where $R = \lambda(\lambda I - \lambda B^{oo} - S)^{-1}$ and $B^{oo} = \underline{e}\underline{\beta}$. β is the initial probability vector, ie. $\underline{\beta} = (\beta_1, \beta_2)$ (see M. F. Neuts [1981])

$$B^{oo} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [\beta_1, \beta_2] = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_1 & \beta_2 \end{bmatrix}.$$

$$R = \lambda \begin{bmatrix} [\lambda(1-\beta_1) + \alpha + \mu_1 & -(\lambda\beta_2 + \alpha) \\ -\lambda\beta_1 & \lambda(1-\beta_2) + \mu_1 + \mu_2 \end{bmatrix}^{-1}$$

and $\pi_0 = \{\underline{\beta}(\sum_{i=0}^{n-k} R^i - \lambda R^{n-k} S^{-1})\underline{e}\}^{-1}$

4.2.3 Numerical illustration

For given values of parameters, we obtain the stationary probability vector as follows. $\lambda = 6, \mu_1 = 8, \mu_2 = 10, \alpha = 8, n = 15, k = 5, \beta_1 = \frac{1}{3}, \beta_2 = \frac{2}{3}$ we get

$$R = \begin{bmatrix} 0.319 & 0.191 \\ 0.032 & 0.319 \end{bmatrix}$$

$\pi_0 = 0.597$; $\Pi_1 = [0.076, 0.165]$; $\Pi_2 = [0.03, 0.067]$

$\Pi_3 = [0.012, 0.027]$; $\Pi_4 = [0.005, 0.011]$; $\Pi_5 = [0.002, 0.004]$

$\Pi_6 = [7.149X10^{-4}, 0.002]$ $\quad \Pi_7 = [2.835X10^{-4}, 6.902X16^{-4}]$

$\Pi_8 = [1.125X10^{-4}, 2.746X10^{-4}]$ $\quad \Pi_9 = [4.467X10^{-5}, 1.092X10^{-4}]$

$\Pi_{10} = [1.774X10^{-5}, 4.34X10^{-5}$ $\quad \Pi_{11} = [6.653X10^{-6}, 1.742X10^{-5}]$

The infinitesimal generator of the Markov chain

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccc} (0,0) & (1,0) & (1,1) & (2,0) & (2,1) & (3,0) & (3,1) & & & & (n-k,0) & (n-k,1) & (n-k+1,0) & (n-k+1,1) \end{array} \\
\begin{array}{c} (0,0) \\ (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (3,0) \\ (3,1) \\ \\ (n-k,0) \\ (n-k,1) \\ (n-k+1,0) \\ (n-k+1,1) \end{array} &
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 \\
\mu_1 & -(\lambda+\alpha+\mu_1) & \alpha & \lambda & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\
\mu_2 & 0 & -(\lambda+\mu_1+\mu_2) & 0 & \lambda & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & -(\lambda+\alpha+\mu_1) & \alpha & \lambda & 0 & 0 & & & 0 & 0 & 0 & 0 \\
0 & 0 & (\mu_1+\mu_2) & 0 & -(\lambda+\mu_1+\mu_2) & 0 & \lambda & 0 & & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_1 & 0 & -(\lambda+\alpha+\mu_1) & \alpha & \lambda & 0 & & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\mu_1+\mu_2) & 0 & -(\lambda+\mu_1+\mu_2) & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & -(\lambda+\alpha+\mu_1) & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & \lambda+\mu_1+\mu_2 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \mu_1 & 0 & -(\alpha+\mu_1) & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & (\mu_1+\mu_2) & 0 & -(\mu_1+\mu_2)
\end{pmatrix}
\end{array}
$$

4.3 Some performance measures

4.3.1 Distribution of the time duration of continuous activity of S_1

There are two possibilities, one of them is return to $(0,0)$ without S_2 activated and in the other, return to $(0,0)$ with S_2 activated in between. Consider the Markov chain on the state space $\{(0,0),(1,0),(2,0),\ldots,(n-k+1,0).(1,1),(2,1),\ldots(n-k+1,1)\}$. The distribution of the first part is of phase type given by $(1-\underline{\Psi}_1\exp(V_1x)\underline{e})\sum_{i=1}^{n-k+1}\frac{1-(\frac{\lambda}{\mu})^i}{1-(\frac{\lambda}{\mu})^{i+1}}$, where $\underline{\Psi}_1$ is the initial probability vector, V_1 is the matrix obtained by deleting the rows and columns corresponding to the states $(0,0),(1,1).(2,1),\ldots(n-k+1,1)$ $\frac{1-(\lambda/\mu)^i}{1-(\lambda/\mu)^{i+1}}$ is the probability of reaching $(0,0)$ before going to $(i,1)$ starting from $(1,0)$.

Distribution of the second part, ie., distribution of the time taken to return to $(0,0)$ with S_2 activated in between is given by $(1-\underline{\Psi}_2\exp(V_2x)\underline{e}_2)*E_{1,\alpha}$ where $\underline{\Psi}_2$ is the initial probability vector. V_2 is the matrix obtained by deleting the row and column corresponding to $(0,0)$. Hence distribution of the time duration of continuous activity of S_1 is

$$(1-\underline{\psi}_1\exp(V_1x)\underline{e}_1)\sum_{i=1}^{n-k+1}\frac{1-(\frac{\lambda}{\mu})^i}{1-(\frac{\lambda}{\mu})^{i+1}}+(1-\underline{\Psi}_2\exp(V_2x)\underline{e}_2)*E_{1,\alpha}$$

where $\underline{e}_1$ and $\underline{e}_2$ are column vectors of 1s' with appropriate orders.

4.3.2 Distribution of continuous activity of S_2

Distribution of the time till absorption into $(0,0)$ starting from $(i,1)$. Consider the Markov chain on the state space $\{(0,0),(1,1),(2,1),\ldots(n-k+1,1)\}$. Then the distribution of time till absorption into $(0,0)$ starting from $(i,1)$ is of phase type given by $[1-\underline{\Psi}_3\exp(V_3x)\underline{e}_3]$ where V_3 is the matrix obtained from the infinitesimal generator by deleting the row and column corresponding to the state $(0,0)$.

4.3.3 Distribution of the time required to reach $(i,1)$ starting from $(0,0)$

Consider the Markov chain on the state space $\{(0,0),(1,0),\ldots,(n-k+1,0),(i,1)\}$. The distribution of the time required to reach $(i,1)$ starting from $(0,0)$ is $1-\underline{\Psi}_4\exp(V_4x)\underline{e}_4$

where V_4 is the matrix obtained by deleting the row and column corresponding to the state $(i, 1)$

4.3.4 Distribution of cycle length

To find the distribution of the time required to reach $(0, 0)$ starting from $(0, 0)$.

$(0, 0)$ can be reached from $(i, 0)$. From $(i, 0)$, $(0, 0)$ can be reached with S_2 activated or without S_2 activated. The distribution of this time duration is in 5.2.1

$$ie(1 - \underline{\Psi_1} \exp(V_1 x)\underline{e_1}) \sum_{i=1}^{n-k+1} \frac{1 - (\frac{\lambda}{\mu})^i}{1 - (\frac{\lambda}{\mu})^{i+1}} + (1 - \underline{\Psi_2} \exp(V_2 x)\underline{e_2}) * E_{1,a}$$

Next we compute the distribution of time taken to reach $(i, 0)$ from (0.0). To this end consider the Markov chain in the state space $\{(0, 0), (1, 0), (2, 0) \ldots, (n-k+1, 0), (1, 1), (2, 1), \ldots (n - k + 1, 1)\}$. The distribution of time required to reach $(i, 0)$ starting from $(0, 0)$ is given by $1 - \underline{\Psi_5} \exp(V_5 x)\underline{e_5}$ where V_5 is the matrix obtained by deleting the rows and columns corresponding to $(2, 0), (3, 0), \ldots, (n - k + 1, 0)$.
Hence distribution of cycle length is equal to

$$(1 - \underline{\Psi_5} \exp(V_5 x)\underline{e_5}) * [(1 - \underline{\Psi_1} \exp(V_1 x)\underline{e_1}) \sum_{i=1}^{n-k+1} \frac{1 - (\frac{\lambda}{\mu})^i}{1 - (\frac{\lambda}{\mu})^{i+1}} + (1 - \underline{\Psi_2} \exp(V_2 x)\underline{e_2}) * E_{1,a}]$$

4.3.5 Expected length of time S_2 is continuously busy

For $i = 1, 2, \ldots, n - k + 1$ define T_{i1} as the time to reach $(i - 1, 1)$ starting from $(i, 1)$ and T_{11} be the time to reach $(0, 0)$ starting from $(1, 1)$. From the state $(i, 1)$, the system moves to the state $(i + 1, 1)$ or $(i - 1, 1)$.

$$E(T_{n-k+1}) = \frac{1}{\mu_1 + \mu_2}$$

Then

$$E(T_{i1}) = \frac{1}{\lambda + \mu_1 + \mu_2} \frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2} + \frac{\lambda}{\lambda + \mu_1 + \mu_2} [\frac{1}{\lambda + \mu_1 + \mu_2} + E(T_{i+1,1}) + E(T_{i1})]$$

Thus, we get the relation

$$E(T_{i1}) = \frac{1}{\mu_1 + \mu_2} + \frac{\lambda}{\mu_1 + \mu_2} E(T_{i+1,1}).$$

Recursively, we obtain

$$E(T_{i1}) = \frac{1-(\frac{\lambda}{\mu_1+\mu_2})^{n-k-i+2}}{(\mu_1+\mu_2)-\lambda}$$

for $i = 1, 2, \ldots, n-k+1$.

The expected time the server is busy is the expected time to reach $(0, 0)$ starting from some state $(i, 1)$. We can find out this from the above relation. Expected time server is busy is given by

$$\sum_{i=1}^{n-k+1} E(T_{i1}) = \sum_{i=1}^{n-k+1} \frac{1-(\frac{\lambda}{\mu_1+\mu_2})^{n-k-i+2}}{(\mu_1+\mu_2)-\lambda} = \frac{(n-k+1)}{((\mu_1+\mu_2)-\lambda)} - \frac{\lambda}{(\mu_1+\mu_2)^2}$$
$$(1-(\frac{\lambda}{\mu_1+\mu_2})^{n-k+1})$$

4.3.6 The expected time the first server alone is continuously active is given by

$$\sum_{i=1}^{n-k+1} \frac{1-(\frac{\lambda}{\mu_1+\alpha})^{n-k-i+2}}{((\mu_1+\alpha)-\lambda)}$$

for $i = 1, 2, \ldots, n-k+1$. Define T_{i0} as the time to reach $(i-1, 0)$ starting from $(i, 0)$ without S_2 activated. From state $(i, 0)$, the system moves to $(i+1, 0)$ or $(i-1, 0)$ or $(i, 1)$. Here, we have to find the expected time S_1 alone is active. So, we consider conditional expectation that is expected time given S_2 is not activated. Hence for $(i, 0)$, transitions to $(i+1, 0)$ or $(i-1, 0)$ are only considered.

$$E(T_{i0})(1-\frac{\alpha}{\lambda+\mu+\alpha}) = \frac{1}{\lambda+\mu_1+\alpha}\frac{\mu_1}{\lambda+\mu_1+\alpha} + \frac{\lambda}{\lambda+\mu_1+\alpha}\Big(\frac{1}{\lambda+\mu_1+\alpha}$$
$$+E(T_{i+1,0})(1-\frac{\alpha}{\lambda+\mu_1+\alpha}) + E(T_{i0})(1-\frac{\alpha}{\lambda+\mu_1+\alpha})\Big)$$

Thus, we get the relation

$$E(T_{i0}) = \frac{1}{\mu_1+\alpha} + \frac{\lambda}{\mu_1+\alpha}E(T_{i+1,0})$$

Finally we obtain the relation starting with $E(T_{n-k+1,0}) = \frac{1}{\mu_1+\alpha}$

$$E(T_{i0}) = \frac{1-(\frac{\lambda}{\mu_1+\alpha})^{n-k-i+2}}{((\mu_1+\alpha)-\lambda)}$$

for $i = 1, 2, \ldots n-k+1$. Thus the expected length of a cycle with first server alone is active is given by

$$\sum_{i=1}^{n-k+1} E(T_{i0}) = \frac{(n-k+1)}{((\mu_1+\alpha)-\lambda)} - \frac{\lambda}{((\mu_1+\alpha)-\lambda)^2}(1-(\frac{\lambda}{\mu_1+\alpha})^{n-k+1})$$

4.3.7 Expected time to reach $(i, 0)$ starting from $(0, 0)$ with S_2 not activated

For $i = 0, 1, 2, \ldots, n-k$ define T'_{i0} denote the time to reach $(i+1, 0)$ starting from $(i, 0)$. From $(i, 0)$, the system can go to $(i+1, 0)$ or $(i-1, 0)$ or $(i, 1)$. Here, since we have to find the expected time with S_2 not activated we have to assume that S_2 is not activated. So, we consider expected time to reach $(i, 0)$ start from $(0, 0)$ with S_2 not activated. Then,

$$\begin{aligned} E(T'_{i0})(1-\frac{\alpha}{\lambda+\mu_1+\alpha}) &= \frac{1}{\lambda+\mu_1+\alpha}\frac{\lambda}{\lambda+\mu_1+\alpha} + \frac{\mu_1}{\lambda+\mu_1+\alpha}(\frac{1}{\lambda+\mu_1+\alpha} \\ &+E(T'_{i-1,0})(1-\frac{\alpha}{\lambda+\mu_1+\alpha}) + E(T'_{i0})(1-\frac{\alpha}{\lambda+\mu_1+\alpha})) \end{aligned}$$

from above, we get the relation

$$E(T'_{i0}) = \frac{1}{\lambda+\alpha} + \frac{\mu_1}{\lambda+\alpha}E(T'_{i-1,0})$$

$E(T'_{00}) = \frac{1}{\lambda+\alpha}$, recursively we get the relation

$$E(T'_{i0}) = \frac{1-(\frac{\mu_1}{\lambda+\alpha})^{n+1}}{(\lambda+\alpha)-\mu_1}$$

for $i = 0, 1, \ldots, n-k$. Thus expected time to reach $(n-k+1, 0)$ starting from the state $(0, 0)$ is given by

$$\sum_{i=0}^{n-k} E(T'_{i0}) = \sum_{i=0}^{n-k} \frac{1-(\frac{\mu_1}{\lambda+\alpha})^{i+1}}{(\lambda+\alpha)-\mu_1} = \frac{(n-k)}{((\lambda+\alpha)-\mu_1)} - \frac{\mu_1}{(\lambda+\alpha-\mu_1)^2}(1-(\frac{\mu_1}{\lambda+\alpha})^{n-k+1})$$

Chapter 5

Some special models in Reliability of k-out-of-n system with repair under T-policy

5.1 Introduction

In this chapter, we consider a k-out-of-n system with (i) an unreliable server (ii) activation time for the server which is a positive random variable (iii) positive inactivation time of the server.

First we consider a repair facility which consists of a single server which is subject to failure. Here lifetimes of components are exponentially distributed with rate λ. Repair time of components are exponentially distributed with parameter μ. Failure of the server and its repair are exponentially distributed with rate β and γ. We call this model 1.

Next a k-out-of-n system with repair under T-policy with positive activation time of the server is considered. Here, though the server is switched on, it gets activated only after random length of time. Let U be the activation time of the server, ie., the amount of time required to get activated from the time it is switched on. The activation time is assumed to be exponentially distributed with rate θ. Lifetimes of components and their repair times are exponentially distributed with rate λ and μ respectively. T is assumed to be exponential with rate α. This is referred to as model 2.

Finally, we consider a k-out-n-system with repair with inactivation time of the server. Here, server gets activated on clapse of T time units or the moment $n - k$ units fail, whichever occurs first. In the models discussed earlier the system goes to $(0, 0)$ from $(1, 1)$ on repair of a failed unit. In this model since there is a positive inactivation time the system goes to $(0, 2)$. On the server being switched off, where the status 2 of the server is defined later. From $(0, 2)$ it may go to $(0, 0)$ or $(1, 1)$ depending on whether a failure does not or does occur before inactivation. That is, though the server is switched off, he does not get inactivated. Denote by W the time required for the server to get inactivated from the moment it is switched off. W is assumed to be exponentially distributed with rate η. Lifetimes of the components and repair times are assumed to be exponential with rate λ and μ respectively. These are the assumptions underlying model 3.

In all the three models, we obtain system state probabilities and some characteristics. Also we investigate the system reliability in the case of a cold system above.

Section 5.2 considers modelling and analysis of k-out-of-n system with unreliable server. Section 5.3 gives the stationary probability distribution and some numerical illustrations. Section 5.4 is devoted to some system state characteristics. Section 5.5 analyses the model of k-out-of-n system with an activation time for the server. Section 5.6 gives system state probabilities of this model and section 5.7 deals with some performance measures. Finally in Section 5.8 we analyse the last mentioned model.

5.2 Modelling and analysis

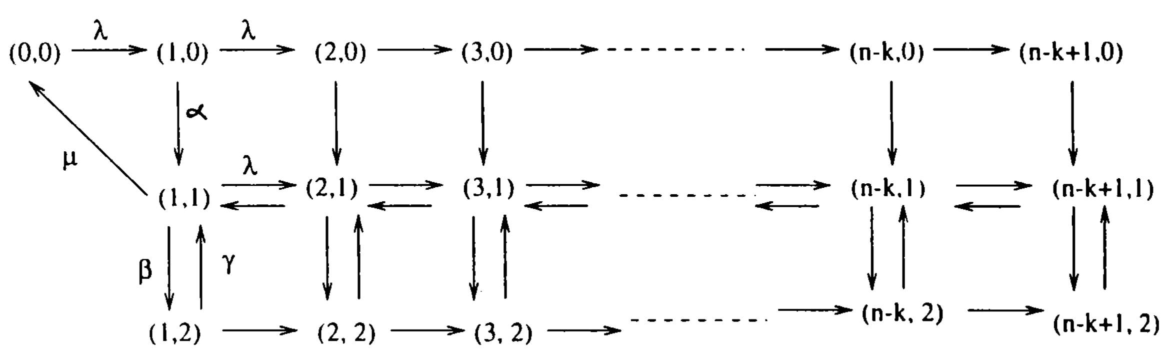
Model 1

Here we assume that the server is switched on only if there is at least one failed unit the system.

Let $X(t)$ = number of failed units at time t.

$$Y(t) = \begin{cases} 0 & \text{if server is inactive (but not in failed state)} \\ 1 & \text{if server is active at time } t \\ 2 & \text{if server is activated but down at } t \end{cases}$$

$\{(X(t), Y(t),\ t \in R_+)\}$ is a Markov chain on the state space.
$E = \{(i,0)/0 \le i \le n-k+1\} \cup \{(i,1)/1 \le i \le n-k+1\} \cup \{(i,2)/1 \le i \le n-k+1\}$. Let $P_{ij}(t) = P((X(t), Y(t)) = (i,j)/(X(0), Y(0)) = (0,0))$. State transition are as follows,



5.3 Stationary Probability Distribution

The infinitesimal generator of the Markov chain is given [refer next page]. We write the states lexicographically ie. $(0,0), (1,0), (1,1), (1,2), (2,0), \ldots\ldots\ldots, (n-k+1,1), (n-k+1,2)$

We now write the infinitesimal generator in the following form,

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ (n-k) \\ (n-k+1) \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & (n-k) & (n-k+1) \end{array} \\
\begin{pmatrix}
-\lambda & \lambda\beta & 0 & 0 & & 0 & 0 \\
S^o & S-\lambda I & \lambda I & 0 & & & 0 \\
0 & 0 & S^oB^o & S-\lambda I & \lambda I & & 0 \\
 & & \vdots & & & & \vdots \\
0 & 0 & 0 & & S^oB^o & S-\lambda I & \lambda I \\
0 & 0 & 0 & & 0 & S^oB^o & S
\end{pmatrix} \end{array}$$

where $S^o = \begin{bmatrix} 0 \\ \mu \\ 0 \end{bmatrix}$, $S^oB^o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

The infinitesimal generator of the Markov chain

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccc} (0,0) & (1,0) & (1,1) & (1,2) & (2,0) & (2,1) & (2,2) & (n-k,0) & (n-k,1) & (n-k,2) & (n-k+1,0) & (n-k+1,1) & (n-k+1,2) \end{array} \\
\begin{array}{c} (0,0) \\ (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ \\ (n-k,0) \\ (n-k,1) \\ (n-k,2) \\ (n-k+1,0) \\ (n-k+1,1) \\ (n-k+1,2) \end{array} &
\begin{pmatrix}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(\lambda+\alpha) & \alpha & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu & 0 & -(\lambda+\mu+\beta) & \beta & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \gamma & -(\lambda+\gamma) & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -(\lambda+\alpha) & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & -(\lambda+\mu+\beta) & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma & -(\lambda+\gamma) & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\alpha) & \alpha & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(\lambda+\mu+\beta) & \beta & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -(\lambda+\gamma) & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & -(\mu+\beta) & \beta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma & -\gamma
\end{pmatrix}
\end{array}
$$

$$S - \lambda I = \begin{bmatrix} -(\lambda+\alpha) & \alpha & 0 \\ 0 & -(\lambda+\mu+\beta) & \beta \\ 0 & \gamma & -(\lambda+\gamma) \end{bmatrix}$$

In this case, the stationary probability vectors are given by $\Pi_i = \pi_0 \underline{\beta} R^i$, $1 \leq i \leq n-k$ and $\Pi_{n-k+1} = \pi_0 \underline{\beta} R^{n-k}(-\lambda S^{-1})$ where $R = \lambda(\lambda I - \lambda B^{oo} - S)^{-1}$ and $B^{oo} = \underline{e}.\underline{\beta}$. $\underline{\beta}$ is the initial probability vector (See Neuts (1981)).

$$B^{oo} = \underline{e}.\underline{\beta} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [\beta_1 \beta_2 \beta_3] = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

$$R = \lambda \begin{bmatrix} \lambda(1-\beta_1)+\alpha & -(\lambda\beta_2+\alpha) & -\lambda\beta_3 \\ -\lambda\beta_1 & \lambda(1-\beta_2)+\mu+\beta & -(\lambda\beta_3+\beta) \\ -\lambda\beta_1 & -(\lambda\beta_2+\gamma) & \lambda(1-\beta)+\gamma \end{bmatrix}^{-1}$$

and $\pi_0 = \{\underline{\beta}(\sum_{i=0}^{n-k} R^i - \lambda R^{n-k} S^{-1})\underline{e}\}^{-1}$.

5.3.1 Numerical illustration

For given values of parameters, we obtain the stationary probability vectors as follows. $\lambda = 6$, $\mu = 10$, $\alpha = 8$, $\beta = 5$, $\gamma = 9$, $\beta_1 = 1/3$, $\beta_2 = 1/2$, $\beta = 1/6$, $n = 15$, $k = 5$.
we get

$$R = \begin{bmatrix} 0.545 & 0.164 & 0.109 \\ 0.121 & 0.503 & 0.224 \\ 0.182 & 0.455 & 0.636 \end{bmatrix}$$

Here $n - k = 10$

$\pi_0 = 0.098$; $\quad \Pi_1 = (0.036, 0.04, 0.027)$

$\Pi_2 = (0.029, 0.038, 0.03)$; $\quad \Pi_3 = (0.026, 0.037, 0.031)$

$\Pi_4 = (0.024, 0.037, 0.031)$; $\quad \Pi_5 = (0.023, 0.037, 0.031)$

$\Pi_6 = (0.023, 0.036, 0.03)$; $\quad \Pi_7 = (0.022, 0.036, 0.030)$

$\Pi_8 = (0.022, 0.035, 0.029)$; $\quad \Pi_9 = (0.022, 0.035, 0.029)$

$\Pi_{10} = (0.021, 0.034, 0.029)$; $\quad \Pi_{11} = (0.014, 0.045, 0.012)$

5.4 Some system state characteristics

5.4.1 Distribution of first passage time to break down state

Consider the Markov chain on the state space $\{(0,0), (1,0), \ldots, (n-k+1,0), (1,1), (2,1), \ldots, (n-k+1,1), (1,2), (2,2), \ldots (n-k+1,2)\}$. Consider the class $\{(1,2), (2,2), \ldots, (n-k+1,2)\}$. To find the distribution of the time required to reach $(i,2)$ for $1 \leq i \leq n-k+1$. Let (i,l), where $l = 0$ or 1 be any of the transient states. The infinitesimal generator of $(X(t), Y(t))$ be denoted by $\begin{bmatrix} Q & Q^o \\ \underline{0} & 0 \end{bmatrix}$. Here Q is the matrix obtained by deleting the rows and columns corresponding to the states in the class $\{(1,2), (2,2), \ldots, (n-k+1,2)\}$. The distribution of the time required to reach $(i,2)$ is of phase type given by $F_1(x) = 1 - \underline{\alpha}_1 \exp(Qx)\underline{e}$ with $\underline{\alpha}_1$ the initial probability vector.

5.4.2 Distribution of the time from server activation till all failed units are repaired

We have to find the distribution of time required to reach (0,0) starting from $(i,1)$ without visiting $(i,2)$ for $1 \leq i \leq n-k+1$. Let Q_1 be the matrix obtained by deleting the rows and columns corresponding to the states $(1,2), (2,2), \ldots, (n-k+1,2)$ and state $(0,0)$. the distribution of time required to reach $(0,0)$ starting from $(i,1)$ is of phase type given by $F_2(x) = 1 - \underline{\alpha}_2 \exp(Q_1 x)\underline{e}$ where $\underline{\alpha}_2$ is the initial probability vector.

Then, the distribution of the time required to reach (0,0) starting from $(i,1)$ before going to $(i,2)$ is given by

$$F_2(x) \sum_{i=1}^{n-k+1} \left(\frac{1-(\lambda/\mu)^i}{1-(\lambda/\mu)^{i+1}}\right)$$

5.4.3 Distribution of time server remains continuously in the system

We need to find the distribution of the time required to reach (0,0) starting from some $(i,1)$, at which an $(i,0)$ to $(i,1)$ transition took place, for $1 \leq i \leq n-k+1$. Let Q_2 be the matrix obtained by deleting the row and column corresponding to the state $(0,0)$.

Distribution of the time required to reach (0,0) is given by $F_3(x) = 1 - \underline{\alpha_3}\exp(Q_2 x)\underline{e}$. Distribution of time server remains continuously in the system given by $F_3(x)$.

5.4.4 Distribution of cycle length

To find the distribution of the time required to each (0,0) starting from (0,0).

Consider the infinitesimal generator of the transition probabilities. Regard (0,0) as an absorbing state. Then distribution of the time required to reach (0,0) starting from any of the transient states $(i, 1)$ is $F_3(x) = 1 - \underline{\alpha_3}\exp(Q_2 x)\underline{e}$. Distribution of the time required to reach (0,0) starting from (0,0) is given by $P(S_i < T < S_{i+1})\ E_{i,\lambda} * E_{1,\alpha} * F_3(x)$ where $E_{i,\lambda}$ is an Erlang distribution of order i, parameter λ. S_i is the time till i failures take place.

5.4.5 Expected time server remains continuously in the system

From the distribution of time server remains continuously in the system (Section 5.4.3), we can compute the expected time as $-\underline{\alpha}_3 Q_2^{-1}\underline{e}$. To find the inverse is a difficult task. So, we go for an iterative procedure.

Let T_{i1} denote the time to reach $(i-1, 1)$ starting from (i,1) and T_{i2} denote the time to reach $(i, 1)$ starting from $(i, 2)$. Th possible transitions are :

$$(i,1) \to (i+1,1) \to (i+1,2) \to (i,1) \to (i-1,1)$$
$$(i,1) \to (i,2) \to (i,1) \to (i-1,1)$$
$$(i,1) \to (i-1\,1)$$

Thus

$$E(T_{i1}) = \frac{1}{\lambda+\mu+\beta}\frac{\mu}{\lambda+\mu+\beta} + \frac{\beta}{\lambda+\mu+\beta}\Big(\frac{1}{\lambda+\mu+\beta} + E(T_{i2}) + E(T_{i1})\Big)$$
$$+\frac{\lambda}{\lambda+\mu+\beta}\Big(\frac{1}{\lambda+\mu+\beta} + \frac{1}{\lambda+\mu+\beta}\frac{\beta}{\lambda+\mu+\beta} + E(T_{i+1,2}) + E(T_{i+1,1}) + E(T_{i1})\Big)$$

where

$$E(T_{i2}) = \frac{1}{\lambda+\gamma}\frac{\gamma}{\lambda+\gamma} + \frac{\lambda}{\lambda+\gamma}\Big(\frac{1}{\lambda+\gamma} + E(T_{i+1,2}) + \frac{1}{\lambda+\mu+\beta}\frac{\mu}{\lambda+\mu+\beta}\Big)$$
$$= \frac{\lambda\mu}{(\lambda+\gamma)(\lambda+\mu+\beta)^2} + \frac{\lambda}{\lambda+\gamma}E(T_{i+1,2}) + \frac{1}{\lambda+\gamma}; \quad i = 1, 2, \ldots, n-k+1$$

Hence

$$E(T_{i1}) = \frac{(\lambda+\beta+\gamma)}{\mu(\lambda+\gamma)} + \frac{\beta\lambda(\lambda+\mu+\gamma)}{\mu(\lambda+\mu+\beta)^2(\lambda+\gamma)} + \frac{\lambda(\lambda+\beta+\gamma)}{\mu(\lambda+\gamma)}E(T_{i+1,2}) + \frac{\lambda}{\mu}E(T_{i+1,1})$$

which gives

$$E(T_{i1}) = \frac{(\lambda+\beta+\gamma)}{\mu\gamma} + \frac{\beta\lambda(\lambda+\mu+\gamma)}{\mu(\lambda+\gamma)(\lambda+\mu+\beta)^2} + \frac{\lambda^2(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\lambda+\mu+\beta)^2} - \frac{\lambda^2(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\lambda+\mu+\beta)^2}(\frac{\lambda}{\lambda+\gamma})^{n-k-i} + (\lambda/\mu)E(T_{i+1,1})$$

Recursively, we get

$$E(T_{i1}) = (\frac{\lambda+\beta+\gamma}{\mu\gamma})[\frac{1-(\lambda/\mu)^{n-k+1-i}}{1-\lambda/\mu}] + \frac{\beta\lambda(\lambda+\mu+\gamma)}{\mu(\lambda+\gamma)(\lambda+\mu+\beta)^2}(\frac{1-(\lambda/\mu)^{n-k+1-i}}{1-(\lambda/\mu)}) + \frac{\lambda^2(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\lambda+\mu+\beta)^2}\frac{(1-(\lambda/\mu)^{n-k+1-i})}{1-(\lambda/\mu)} - \frac{\lambda^2(\lambda+\beta+\gamma)}{\gamma(\lambda+\mu+\beta)^2}((\frac{\lambda}{\lambda+\gamma})^{n-k-i} + (\lambda/(\lambda+\gamma))^{n-k-i-1} + \ldots + (\lambda/\mu)^{n-k-i}) + (\lambda/\mu)^{n-k+1-i}\frac{1}{\mu}$$

ie.,

$$E(T_{i1}) = [\frac{\lambda+\beta+\gamma}{\gamma(\mu-\gamma)} + \frac{\beta\lambda(\lambda+\mu+\gamma)}{(\lambda+\gamma)(\lambda+\mu+\beta)^2(\mu-\lambda)} + \frac{\lambda^2\mu(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\lambda+\mu+\beta)^2(\mu-\lambda)}](1-(\lambda/\mu)^{n-k+1-i}) + (\lambda/\mu)^{n-k+1-i}(\frac{\lambda\mu(\lambda+\beta+\gamma)\mu}{\gamma(\lambda+\mu+\beta)^2(\mu-(\lambda+\gamma))} + \frac{1}{\mu}) - \frac{\lambda(\lambda+\beta+\gamma)\mu}{\gamma(\lambda+\mu+\beta)^2(\mu-(\lambda+\gamma))}(\lambda/(\lambda+\gamma))^{n-k+1-i} \quad \text{for } i = 1,2,\ldots,n-k+1$$

Now, $\sum_{i=1}^{n-k+1} E(T_{i1})$ gives the expected time server remains continuously in the system which is equal to

$$(n-k+1)[\frac{\lambda+\beta+\gamma}{\gamma(\mu-\lambda)} + \frac{\beta\lambda(\lambda+\mu+\gamma)}{(\lambda+\gamma)(\lambda+\mu+\beta)^2(\mu-\lambda)} + \frac{\lambda^2\mu(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\mu-\lambda)(\lambda+\mu+\beta)^2}] - \frac{\mu}{(\mu-\lambda)}(1-(\lambda/\mu)^{n-k+1})[\frac{\lambda+\beta+\gamma}{\gamma(\mu-\lambda)} + \frac{\beta\lambda(\lambda+\mu+\gamma)}{(\lambda+\gamma)(\lambda+\mu+\beta)^2(\mu-\lambda)} + \frac{\lambda^2\mu(\lambda+\beta+\gamma)}{\gamma(\lambda+\gamma)(\mu-\lambda)(\lambda+\mu+\beta)^2}] + \frac{\mu}{(\mu-\lambda)}(1-(\lambda/\mu)^{n-k+1})(\frac{1}{\mu} + \frac{\lambda\mu(\lambda+\beta+\gamma)}{\gamma(\lambda+\mu+\beta)^2(\mu-(\lambda+\gamma))}) - \frac{\lambda\mu(\lambda+\beta+\gamma)(\lambda+\gamma)}{\gamma^2(\lambda+\mu+\beta)^2(\mu-(\lambda+\gamma))}(1-(\frac{\lambda}{\lambda+\gamma})^{n-k+1})$$

5.4.6 Expected time server remains in down state

The expected number of visits to $(i, 2)$ before visiting the state (0,0) is $\frac{q_{i2}}{q_{00}}$. Expected time the system remains in the state $(i, 2)$ is $\frac{1}{\lambda+\gamma}$, when $i = 1, 2, \ldots, n-k$ and the expected time the system remains in the state $(n-k+1, 2)$ is $\frac{1}{\gamma}$. Hence, expected time the server is in breakdown state during a cycle is

$$\sum_{i=1}^{n-k} \frac{1}{\lambda+\gamma} \frac{q_{i2}}{q_{00}} + \frac{1}{\gamma} \frac{q_{n-k+1,2}}{q_{00}}$$

Thus, expected time server remains busy in a cycle

$$= \sum_{i=1}^{n-k} (E(T_{i1}) - \frac{q_{i2}}{q_{00}}.\frac{1}{\lambda+\gamma}) + E(T_{n-k+1,2}) - \frac{q_{n-k+1,2}}{q_{00}}.\frac{1}{\gamma}$$

where q_{i2}/q_{00} can be computed for given parameters of the process.

5.5 k-out-of-n system with activation time

Lifetimes of the components are assumed to be exponentially distributed with parameter λ. Server is switched on after the elapse of T time units since the epoch of its inactivation. (ie. completion of repair of all failed units in the previous cycle) or until accumulation of $n-k$ failed units, whichever occurs first. The server does not get activated the moment it is switched on. It takes a random length of time U which is assumed to be exponentially distributed with rate θ. T is exponentially distributed with rate α and repair time exponentially distributed with rate μ. Hence the time elapsed until activation starting from all units operational, has generalized Erlang distribution. In chapter 3, we considered the case when activation time is zero. We get the results there by taking $\lim_{\theta \to \infty}$ in this section.

5.5.1 Mathematical Formulation

Let $X(t)$ represent the number of failed units at time t.

$$Y(t) = \begin{cases} 2 & \text{if server is active at time } t \\ 1 & \text{if server is only switched on but not active at } t \\ 0 & \text{otherwise} \end{cases}$$

$\{(X(t), Y(t)), t \in R_+\}$ is a Markov chain with state space.

$A = \{(i,0)/0 \le i \le n-k-1\} \cup \{(i,1)/0 \le i \le n-k+1\} \cup \{(i,2)/0 \le i \le n-k+1\}$

The difference-differential equations satisfied by $P_{ij}(t)$ are

$$
\begin{aligned}
P'_{00}(t) &= -(\lambda+\alpha)P_{00}(t) + \mu P_{12}(t) \\
P'_{01}(t) &= -(\lambda+\theta)P_{01}(t) + \alpha P_{00}(t) \\
P'_{02}(t) &= -\lambda P_{02}(t) + \theta P_{01}(t) \\
P'_{i0}(t) &= -(\lambda+\alpha)P_{i0}(t) + \lambda P_{i-1,0}(t); \quad 1 \le i \le n-k-1 \\
P'_{i1}(t) &= -(\lambda+\theta)P_{i1}(t) + \lambda P_{i-1,1}(t) + \alpha P_{i0}(t)(1-\delta_{in-k}) + \lambda\delta_{in-k}P_{i-1,0}(t); \\
&\qquad 1 \le i \le n-k \\
P'_{i2}(t) &= -(\lambda(1-\delta_{in-k+1})+\mu)P_{i2}(t) + \lambda P_{i-1,2}(t) + \mu P_{i+1,2}(t)(1-\delta_{in-k+1}) \\
&\quad + \beta(1-\delta_{in-k+1})P_{i1}(t) + \lambda\delta_{in-k+1}P_{i-1,1}(t); \quad 1 \le i \le n-k+1
\end{aligned}
$$

5.6 Steady state distribution

Let $q_{ij} = \lim_{t\to\infty} P_{ij}(t)$. Then the steady state probabilities are given by $q_{01} = \frac{\alpha}{\lambda+\theta}q_{00}$; $q_{12} = (\frac{\lambda+\alpha}{\mu})q_{00}$; $q_{02} = \frac{\theta\alpha}{\lambda(\lambda+\theta)}q_{00}$; $q_{i0} = (\lambda/(\lambda+\alpha))^i q_{00}$, $1 \le i \le n-k-1$

$$
\begin{aligned}
q_{i1} &= \frac{\alpha\lambda^i}{(\theta+\lambda)}\left(\frac{1}{(\theta+\lambda)^i} + \frac{1}{(\theta+\lambda)^{i-1}(\alpha+\lambda)} + \cdots + \frac{1}{(\alpha+\lambda)^i}\right); \qquad 1 \le i \le n-k-1 \\
&= \frac{\alpha}{\theta-\alpha}\left(\frac{\lambda}{\alpha+\lambda}\right)^i\left(1-\left(\frac{\alpha+\lambda}{\theta+\lambda}\right)^{i+1}\right)q_{00} \text{ for } 1 \le i \le n-k-1 \\
q_{n-k,1} &= \left(\frac{\lambda}{\lambda+\theta}\right)\left(\frac{\alpha\lambda^{n-k-1}}{\theta+\lambda}\left(\frac{1}{(\theta+\lambda)^{n-k-1}} + \frac{1}{(\alpha+\lambda)}\frac{1}{\theta+\lambda}\right)^{n-k-2} + \cdots \right. \\
&\quad \left. + \frac{1}{(\alpha+\lambda)^{n-k-1}}\right) + \left(\frac{\lambda}{\alpha+\lambda}\right)^{n-k-1}\Big)q_{00} \\
&= \left(\frac{\lambda}{\lambda+\theta}\right)\left[\left(\frac{\lambda}{\lambda+\alpha}\right)^i\left(1-\left(\frac{\alpha+\lambda}{\theta+\lambda}\right)^{i+1}\right)\right]q_{00}
\end{aligned}
$$

$$q_{i2} = (\frac{\alpha+\lambda}{\mu})[\frac{1-(\lambda/\mu)^i}{1-(\lambda/\mu)}] - \frac{\alpha\theta\lambda}{\mu(\theta+\lambda)}(\frac{1}{\lambda} + \frac{\lambda}{\alpha+\lambda} + \frac{1}{\theta+\lambda})$$
$$-\sum_{j=2}^{i-1} \frac{\alpha\theta\lambda^j}{\mu^j(\theta+\lambda)}(\frac{\mu^{j-1}}{(\theta+\lambda)^j} + \frac{\mu^{j-1}}{(\theta+\lambda)^{j-1}(\alpha+\lambda)} + \cdots$$
$$+ \frac{\mu^{j-1}}{(\alpha+\lambda)^j} + \cdots + \frac{1}{\lambda} + \frac{1}{\alpha+\lambda} + \frac{1}{\theta+\lambda})$$
$$= (\frac{\alpha+\lambda}{\mu-\lambda}) - \frac{\alpha\theta\lambda}{\mu(\theta+\lambda)}(\frac{1}{\lambda} + \frac{1}{\alpha+\lambda} + \frac{1}{\theta+\lambda}) - \frac{\lambda^3}{(\theta-\alpha)(\alpha+\lambda)^2(\mu-\lambda)}$$
$$+ \frac{\alpha\theta\lambda^2}{(\alpha+\lambda)(\theta-\alpha)(\lambda+\alpha-\mu)}(\frac{(\lambda/\mu)^{i-2}}{(\mu-\lambda)} - \frac{(\lambda/(\lambda+\alpha))^{i-2}}{\alpha})$$
$$+ \frac{\alpha\theta\lambda^2(\theta+\lambda)}{(\theta-\alpha)(\alpha+\lambda)^2(\theta+\lambda-\mu)}(\frac{(\lambda/(\theta+\lambda))^{i-2}}{\theta} - \frac{(\lambda/\mu)^{i-2}}{(\mu-\lambda)})$$

q_{00} can be obtained using the normalizing condition

$$\sum_{i=0}^{n-k-1} q_{i0} + \sum_{i=1}^{n-k} q_{i1} + \sum_{i=1}^{n-k+1} q_{i2} = 1$$

The system reliability is given by $1 - q_{n-k+1,2}$. Fraction of time the system is down is $q_{n-k+1,2}$

5.7 Some performance measures

5.7.1 Distribution of duration of server availability

Consider the Markov chain on $\{(0,0),(1,2),(2,2),\ldots,(n-k+1,2)\}$. We have to compute the distribution of time until the system reaches (0,0) starting from one of the transient states $(i,2)$. Consider the infinitesimal generator $\begin{bmatrix} S & S^0 \\ \underline{0} & 0 \end{bmatrix}$, where S is the matrix obtained by deleting the row and column corresponding to the state (0,0). Then distribution of time to reach $(0,0)$ starting from $(i,2)$ is phase type given by $1 - \underline{\gamma_1}\exp(Sx)\underline{e}$, where $\underline{\gamma_1}$ is the initial probability vector.

5.7.2 Expected duration of time the system remains non-functional in a cycle

$\frac{q_{n-k+1,2}}{q_{00}}$ gives the expected number of visits to $(n-k+1,2)$ before first return to (0,0). Further $\frac{1}{\mu}$ is the expected amount of time system remains in that state during each visit to it. Hence expected duration of time system is down is $\frac{1}{\mu}\frac{q_{n-k-1,2}}{q_{00}}$ which is equal to

$$\frac{1}{\mu}\Big[\frac{(\alpha+\lambda)}{(\mu-\lambda)}-\frac{\alpha\theta\lambda}{\mu(\theta+\lambda)}(\frac{1}{\lambda}+\frac{1}{\alpha+\lambda}+\frac{1}{\theta+\lambda})-\frac{\lambda^3}{(\alpha+\lambda)^2(\theta-\alpha)(\mu-\lambda)}$$
$$+\frac{\alpha\theta\lambda^2}{(\theta-\alpha)(\alpha+\lambda)(\lambda+\alpha-\mu)}(\frac{(\lambda/\mu)^{n-k-1}}{(\mu-\lambda)}-\frac{(\lambda/(\lambda+\alpha))^{n-k-1}}{\alpha})$$
$$+\frac{\alpha\theta\lambda^2(\theta+\lambda)}{(\theta-\alpha)(\alpha+\lambda)^2(\theta+\lambda-\mu)}[\frac{(\lambda/(\theta+\lambda))^{n-k-1}}{\theta}-\frac{(\lambda/\mu)^{n-k-1}}{(\mu-\lambda)}]\Big]$$

5.7.3 Expected time server remains active during a cycle

Define T_{i2} as the time to reach $(i-1,2)$ starting from $(i,2)$. The following transitions are possible

$$(i,2)\rightarrow(i+1,2)\rightarrow(i,2)\rightarrow(i-1,2)$$
$$(i,2)\rightarrow(i-1,2)$$

$$E(T_{i2})=\frac{1}{\lambda+\mu}.\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}(\frac{1}{\lambda+\mu}+E(T_{i+1,2})+E(T_{i2}))$$

Hence $E(T_{i2})=\frac{1}{\mu}+\frac{\lambda}{\mu}E(T_{i+1,2})$ recursively we get

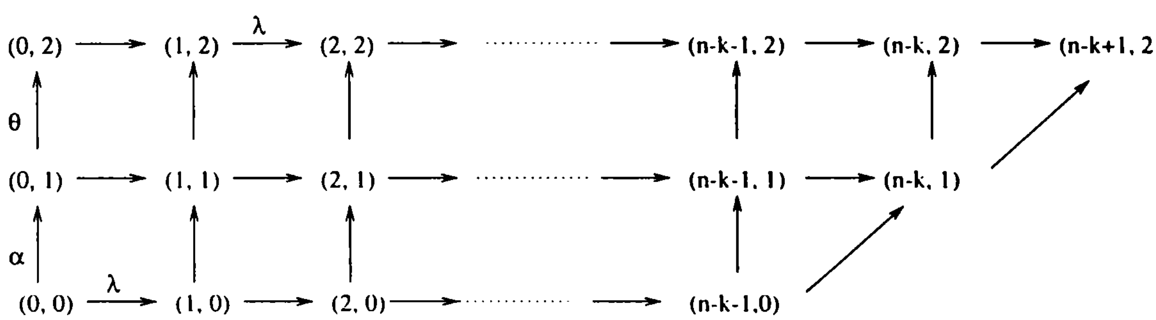
$$E(T_{i2})=\frac{1-(\lambda/\mu)^{n-k+2-i}}{\mu-\lambda}$$

for $i=1,2,\ldots,n-k+1$ starting with $E(T_{n-k+1,2})=\frac{1}{\mu}$. The expected time to reach (0,0) starting from $(i,2)$ is $\sum_{i=0}^{n-k+1}E(T_{i2})$, where T_{02} denote the time to reach $(0,0)$ starting from $(0,2)$.

5.7.4 Expected amount of time server is inactive in a cycle

From the state (0,0), the system goes to $(1,0)$ on failure of one unit or it goes to (0,1) on elapse of T time units.

From the state (0,1), system goes to (0,2) on elapse of activation time or it goes to (1,1) on failure of one unit. The process goes on in this fashion. a possible transition is indicated in the diagram.



The expected time server is not in the system

$$=(\frac{1}{\alpha}+\frac{1}{\theta})P(T+U<S_1)+(\frac{1}{\alpha}+\frac{1}{\theta}+\frac{1}{\lambda})P(S_1<T+U<S_2)$$
$$+\cdots+(\frac{1}{\alpha}+\frac{1}{\theta}+\frac{n-k-1}{\lambda})P(S_{n-k-1}<T+U<S_{n-k})+\frac{n-k}{\lambda}P(T+U>S_{n-k})$$

where

$$P(S_i<T+U<S_{i+1})=\int_{u=0}^{\infty}\int_{v=u}^{\infty}\frac{e^{-\lambda u}(\lambda u)^{i-1}\lambda}{(i-1)!}\frac{\alpha\theta(e^{-\alpha v}-e^{-\theta v})}{(\theta-\alpha)}e^{-\lambda(v-u)}dvdu$$
$$=\frac{\alpha\lambda^i\theta}{(\theta-\alpha)}(\frac{1}{(\lambda+\alpha)^{i+1}}-\frac{1}{(\lambda+\theta)^{i+1}})$$

for $i=0,1,\dots,n-k-1$ and $P(T+U>S_{n-k})=\frac{\alpha\theta\lambda^{n-k}}{(\theta-\alpha)}(\frac{1}{\alpha(\lambda+\alpha)^{n-k}}-\frac{1}{\theta(\lambda+\theta)^{n-k}})$. Thus we get expected time server is not in the system

$$=2(\frac{1}{\alpha}+\frac{1}{\theta})-(\frac{\lambda}{\lambda+\alpha})^{n-k}(\frac{(2\theta+\alpha)}{\alpha(\theta-\alpha)})-(\frac{\lambda}{\lambda+\theta})^{n-k}(\frac{2\alpha+\theta}{\theta(\theta-\alpha)})$$

5.7.5 Expected cycle length

$E(\tau)=E$(busy period) $+\,E$(time server remains inactive in the system)

From 5.7.3 and 5.7.4, we get

$$E(\tau)=\frac{(n-k)(\mu-\lambda)+2\mu-3\lambda}{(\mu-\lambda)^2}+\frac{\lambda}{(\mu-\lambda)^2}(\frac{\lambda}{\mu})^{n-k+2}+2(\frac{1}{\alpha}+\frac{1}{\theta})$$
$$-(\frac{\lambda}{\lambda+\alpha})^{n-k}\frac{(2\theta+\alpha)}{\alpha(\theta-\alpha)}-(\frac{\lambda}{\lambda+\theta})^{n-k}\frac{(2\alpha+\theta)}{\theta(\theta-\alpha)}$$

5.7.6 Cost Analysis

Let C_1 be the fixed cost of hiring the server, w be the wage of the server per unit time and C_2 be the cost per unit time due to the system remaining non-functional.

Then, total expected cost per unit time,

$$TEC = C_1/E(\tau) + w[\frac{(n-k+2)}{(\mu-\lambda)} - \frac{\lambda}{(\mu-\lambda)^2}(1-(\frac{\lambda}{\mu})^{n-k+2})]/E(\tau)$$
$$+ C_2\frac{1}{\mu}\frac{q_{n-k+1,2}}{q_{00}}$$

where $\frac{1}{\mu}\frac{q_{n-k+1,2}}{q_{00}}$ is the expected time the system remains non-functional is a cycle and is given in 5.7.2. TEC is convex in α. Hence, global minimum value α^* that minimizes TEC exits. Numerically, TEC is evaluated for given set of parameters and various values of α and is given below.

Total expected cost per unit time		
α	$c_1 = 100, w = 50, \lambda = 5, \mu = 10$ $n = 20, K = 10, \theta = 4, C_2 = 60$	$C_1 = 30, w = 20\ \lambda = 15\ \mu = 20$ $n = 30, K = 10, \theta = 1\ C_2 = 10$
2.0	64.403	18.679
2.1	64.993	18.842
2.2	65.554	18.989
2.3	66.084	19.120
2.4	66.583	19.241
2.5	67.054	19.349
2.6	67.494	19.450
2.7	67.903	19.542
2.8	68.281	19.628
2.9	68.624	19.709
3.0	68.926	19.782

5.8 k-out-of-n system with inactivation time

Model 3

5.8.1 Mathematical modelling and analysis

Lifetimes of components are assumed to be exponentially distributed with rate λ. Service time follows on exponential distribution with rate μ. T is exponentially distributed with rate α.

Let $X(t)$ denote the number of failed units at time t.

$$Y(t) = \begin{cases} 0 & \text{if server is inactive at } t \\ 1 & \text{if server is active at } t \\ 2 & \text{if server is switched off, but has not} \\ & \text{become inactive at } t \end{cases}$$

In models discussed earlier from the state $(1,1)$, the state (0,0) is reached on completion of repair of the failed units provided no units fail in between. Here the system goes to $(0,2)$ from $(1,1)$ before reaching (0,0) provided no failure takes place during this period.
$\{(X(t), Y(t)),\ t \in R_+\}$ from a Markov chain on the state space.
$B = \{(i,0)|0 \leq i \leq n-k-1\} \cup \{(i,1)|0 \leq i \leq n-k-1\} \cup \{(0,2)\}$

The difference-differential equations satisfied by $P_{ij}(t)$ are given by

$$\begin{aligned}
P'_{00}(t) &= -(\lambda+\alpha)P_{00}(t) + \eta P_{02}(t) \\
P'_{02}(t) &= -(\lambda+\eta)P_{02}(t) + \mu P_{11}(t) \\
P'_{n-k,1}(t) &= -(\lambda+\mu)P_{n-k,1}(t) + \lambda P_{n-k-1,0}(t) + \lambda P_{n-k-1,1}(t) \\
&\quad + \mu P_{n-k+1,1}(t) \\
P'_{m0}(t) &= -(\lambda+\alpha)P_{m0}(t) + \lambda P_{m-1,0}(t); \quad 0 < m \leq n-k+1 \\
P'_{m1}(t) &= -(\lambda+\mu)P_{m1}(t) + \mu P_{m+1,1}(t) + \lambda P_{m-1,1}(t) + \alpha P_{m0}(t) + \lambda\delta_{m1}P_{02}(t) \\
&\quad 0 < m \leq n-k-1 \\
P'_{01}(t) &= -\lambda P_{01}(t) + \alpha P_{00}(t) \\
P'_{n-k+1,1}(t) &= \lambda P_{n-k,1}(t) - \mu P_{n-k+1,1}
\end{aligned}$$

5.8.2 Steady state probabilities

Let $q_{ij} = \lim_{t\to\infty} P_{ij}(t)$, $i,j \in B$ On solving the equations, we get

$$q_{01} = \frac{\alpha}{\lambda}q_{00} \qquad q_{i0} = (\frac{\lambda}{\lambda+\alpha})^i q_{00};\ i = 1,2,\ldots,n-k-1$$

$$q_{02} = (\frac{\lambda+\alpha}{\eta})q_{00} \qquad q_{11} = (\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{\eta})q_{00}$$

$$q_{21} = [\frac{\lambda^2}{\mu(\lambda+\alpha)} + (\frac{\lambda}{\mu})(\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{\eta})]q_{00}$$

$$q_{31} = [\frac{\lambda^3}{\mu(\lambda+\alpha)^2} + \frac{\lambda^3}{\mu^2(\lambda+\alpha)} + (\frac{\lambda}{\mu})^2(\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{n})]q_{00}$$

$$q_{i1} = [\frac{\lambda^i(1-(\frac{\lambda+\alpha}{\mu})^{i-3})}{(\lambda+\alpha)^{i-1}(\mu-(\lambda+\alpha))} + \frac{\lambda^i(\lambda+\alpha+\mu)}{\mu^{n-k}(\lambda+\alpha)^2} + (\frac{\lambda}{\mu})^{i-1}(\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{\eta})]q_{00}$$

$$4 \leq i \leq n-k$$

$$q_{n-k+1,1} = \left[\frac{\lambda^{n-k+1}(1-(\frac{\lambda+\alpha}{\mu})^{n-k-3})}{\mu(\lambda+\alpha)^{n-k-1}(\mu-(\lambda+\alpha))} + \frac{\lambda^{n-k+1}(\lambda+\alpha+\mu)}{\mu^{n-k}(\lambda+\alpha)^2}\right.$$
$$\left. + (\frac{\lambda}{\mu})^{n-k}(\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{\eta})\right]q_{00}$$

System reliability is given by $1 - q_{n-k+1,1}$. Fraction of time system is down is $q_{n-k+1,1}$. q_{00} can be obtained using normalising condition

$$\sum_{i=0}^{n-k-1} q_{i0} + \sum_{i=0}^{n-k+1} q_{i1} + q_{02} = 1$$

$$q_{00} = \left[(\frac{\lambda+\alpha}{\alpha})(1-(\frac{\lambda}{\lambda+\alpha})^{n-k}) + (\frac{\alpha}{\lambda} + \frac{\lambda^2}{\mu(\lambda+\alpha)} + \frac{\lambda+\alpha}{\eta})\right.$$
$$+ \frac{1}{\mu(\mu-\lambda)}(1-(\lambda/\mu)^{n-k+1})(\frac{(\lambda+\eta)(\lambda+\alpha)\mu}{\eta} + \frac{\lambda^3(\lambda+\alpha+\mu)}{(\lambda+\alpha)^2})$$
$$+ \frac{\lambda^4}{\alpha(\lambda+\alpha)^2}(1-(\frac{\lambda}{\mu})^{n-k-3}) - \frac{\lambda^4}{\alpha\mu(\lambda+\alpha)^2}(\frac{\lambda}{\lambda+\alpha})^{n-k-3}$$
$$\left. - \frac{\lambda^{n-k+2}}{\mu\alpha(\lambda+\alpha)^{n-k-1}(\mu-(\lambda+\alpha))} + \frac{\lambda^{n-k+2}}{\alpha(\lambda+\alpha)^3\mu^{n-k-3}(\mu-(\lambda+\alpha))}\right]^{-1}$$

5.8.3 Distribution of time duration server remains continuously in the system

This is the distribution of time from activation till it becomes inactive. Consider the Markov chain on the state space $\{(0,0), (0,2), (1,1), \ldots, (n-k-1,1), (n-k,1), (n-k+1,1)\}$. The distribution of time taken to reach (0,0) starting from any of the transient states $(i,1)$ is given by $G_1(x) = 1 - \underline{\alpha}\exp(Dx)\underline{e}$ where D is the matrix obtained by deleting the row and column corresponding to state (0,0) and $\underline{\alpha}$ is the initial probability vector.

5.8.4 Expected time server remains busy in a cycle

First, we find expected time to reach cycle $(0,2)$ starting from some $(i,1)$. Define T_{i1}, for $i = 1, 2, \dots, n-k+1$ as the time required to reach $(0,2)$ from $(i,1)$. Following are the possible transitions

$$
\begin{aligned}
&(i,1) \to (i-1,1)\\
&(i,1) \to (i+1,1) \to (i,1) \to (i-1,1)
\end{aligned}
$$

Then,

$$
\begin{aligned}
E(T_{i1}) &= \frac{1}{\lambda+\mu}\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}(\frac{1}{\lambda+\mu} + E(T_{i+1,1})) + E(T_{i1})\\
\text{ie. } E(T_{i1}) &= \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{i+1,1})
\end{aligned}
$$

for $i = 1, 2, \dots, n-k+1$. From the above relation, recursively we get

$$
E(T_{i1}) = \frac{1-(\lambda/\mu)^{n-k+2-i}}{(\mu-\lambda)}
$$

starting from $E(T_{n-k+1,1}) = \frac{1}{\mu}\cdot \sum_{i=1}^{n-k+1} E(T_{i1})$ gives the expected time to reach $(0,2)$ starting from $(n-k+1,1)$. Next, we find the expected time to reach (0,0) starting from (0,2). The following transitions are possible

$$
\begin{aligned}
&(0,2) \to (0,0)\\
&(0,2) \to (1,1) \to (0,2) \to (0,0)
\end{aligned}
$$

Let T_{02} denote the time to reach (0,0) starting from $(0,2)$. Then,

$$
E(T_{02}) = \frac{1}{\lambda+\eta}\frac{\eta}{\lambda+\eta} + \frac{\lambda}{\lambda+\eta}(\frac{1}{\lambda+\eta} + E(T_{11}) + E(T_{02}))
$$

$$
\begin{aligned}
\text{ie. } E(T_{02})(1-\frac{\lambda}{\lambda+\eta}) &= \frac{\lambda+\eta}{(\lambda+\eta)^2} + \frac{\lambda}{\lambda+\eta}E(T_{11})\\
\text{ie. } E(T_{02}) &= \frac{1}{\eta} + \frac{\lambda}{\eta}\frac{(1-(\lambda/\mu)^{n-k+1})}{\mu-\lambda}
\end{aligned}
$$

Expected time server remains continuously in the system is the expected time to reach $(0,2)$ from (0,0) and to reach (0,0) from $(0,2)$. Thus, expected time server remains continuously in the system is given by

$$\sum_{i=1}^{n-k+1} E(T_{i1}) + E(T_{02})$$
$$= \frac{(n-k+2)}{(\mu-\lambda)} - \frac{\lambda}{(\mu-\lambda)^2}(1-(\frac{\lambda}{\mu})^{n-k+2}) + \frac{1}{\eta} + \frac{\lambda}{\eta}\frac{(1-(\lambda/\mu)^{n-k+1})}{\mu-\lambda}$$

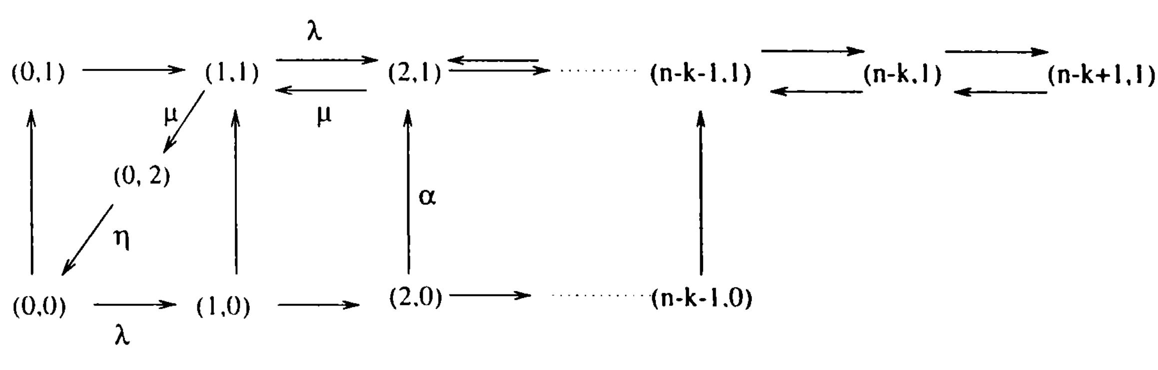
5.8.5 Expected time system remains non-functional

$\frac{q_{n-k-1,1}}{q_{00}}$ gives the expected number of visits to $(n-k+1,1)$ before first return to $(0,0)$. $\frac{1}{\mu}$ is the expected sojourn time in the state $(n-k+1,1)$. Thus, expected time system remains non-functional is given by

$$\frac{1}{\mu}\frac{q_{n-k+1}}{q_{00}} = \frac{\lambda^{n-k+1}(1-(\frac{\lambda+\alpha}{\mu})^{n-k-3})}{\mu^2(\lambda+\alpha)^{n-k-1}(\mu-(\lambda+\alpha))} + \frac{\lambda^{n-k+1}(\lambda+\alpha+\mu)}{\mu^{n-k+1}(\lambda+\alpha)^2} + (\frac{\lambda}{\mu})^{n-k}\frac{1}{\mu}(\frac{\lambda+\eta}{\mu})(\frac{\lambda+\alpha}{\eta})$$

5.8.6 Expected time server remains inactive during a cycle

From state $(0,0)$ system goes to (1,0) or (0,1) on failure of one unit or on elapse of T time units respectively. From state $(1,0)$, system goes to $(1,1)$ or $(2,0)$. The process goes on like this. The possible transition are given below.



Expected time server remains inactive
$= \frac{1}{\alpha}P(T < S_1) + (\frac{1}{\alpha} + \frac{1}{\lambda})P(S_1 < T < S_2) + \ldots + (\frac{1}{\alpha} + \frac{n-k-1}{\lambda})P(S_{n-k-1} < T < S_{n-k}) + \frac{n-k}{\lambda}P(T > S_{n-k})$ where S_i denote the time till i failures take place. Write $S_0 = 0$

and $S_0 < S_1 < S_2 \cdots < S_{n-k}$.

$$P(S_i < T < S_{i+1}) = \frac{\alpha\lambda^i}{(\lambda+\alpha)^{i+1}} \quad \text{for } 0 \leq i \leq n-k-1$$

This, expected time server remains inactive during a cycle $= \frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})$

5.8.7 Cost Analysis

Let C denote the cost per unit time due to the machine remaining non functional and w be the wage of the server per unit time.

Then, total expected profit per unit time,

$$TEP = w\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda})^{n-k}) - \frac{C}{\mu}\frac{q_{n-k+1,1}}{q_{00}} = w\frac{2}{\alpha}(1-(\frac{\lambda}{\lambda+\alpha})^{n-k})$$
$$-\frac{C}{\mu}\Big[\frac{(\lambda^{n-k+1}(1-(\frac{\lambda+\alpha}{\mu})^{n-k-3}))}{\mu(\lambda+\alpha)^{n-k-1}(\mu-(\mu+\alpha))}$$
$$+\frac{\lambda^{n-k+1}(\lambda+\alpha+\mu)}{\mu^{n-k}(\lambda+\alpha)^2} + (\frac{\lambda}{\mu})^{n-k}(\frac{\lambda+\eta}{\alpha})(\frac{\lambda+\alpha}{\mu})\Big]$$

It is seen that TEP is concave in α. The objective is to find an optimal α which maximizes the profit. Numerically TEP is evaluated for given parameters and for various values of α and is given below.

Total expected profit per unit time		
α	$C = 30, \lambda = 10, \mu = 12$ $k = 5, w = 50, n = 30\ \eta = 5$	$C = 50, w = 100\ \lambda = 5\ \mu = 12$ $n = 50, k = 5, \eta = 2$
2.0	49.236	100
2.1	46.551	95.238
2.2	44.529	90.909
2.3	42.667	86.957
2.4	40.448	83.333
2.5	39.358	80.000
2.6	37.882	76.923
2.7	36.511	74.074
2.8	35.232	71.429
2.9	34.038	68.966
3.0	32.92	66.662

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