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Note

Equal opportunity networks, distance-balanced graphs, and Wiener game

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HIGHLIGHTS

- A new characterization of distance balanced graph of even order under the name "Equal Opportunity Graphs".
- Construction of a new infinite family of distance balanced partial cubes.
- Introduction of a new game played on the vertices of a graphs named as "Wiener game".

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ABSTRACT

Given a graph G and a set $X \subseteq V(G)$, the relative Wiener index of X in G is defined as $W_X(G) = \sum_{\{u,v\}\in\binom{X}{2}} d_G(u,v)$. The graphs G (of even order) in which for every partition $V(G) = V_1 + V_2$ of the vertex set $V(G)$ such that $|V_1| = |V_2|$ we have $W_{V_1}(G) = W_{V_2}(G)$ are called equal opportunity graphs. In this note we prove that a graph G of even order is an equal opportunity graph if and only if it is a distance-balanced graph. The latter graphs are known by several characteristic properties, for instance, they are precisely the graphs G in which all vertices $u \in V(G)$ have the same total distance $D_G(u) = \sum_{v\in V(G)} d_G(u,v)$. Some related problems are posed along the way, and the so-called Wiener game is introduced.

1. Introduction

The Wiener index $W(G)$ of a (connected) graph G is defined as the sum of the distances between all pairs of vertices of G. It was introduced in 1947 in the seminal chemical paper [1]. The index was extensively studied in the last decades; see, for instance, the surveys [2,3] on the Wiener index of trees and hexagonal graphs. Although its main application

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is in mathematical chemistry, this natural concept appears in other areas of graph theory and applications. Note that investigations of the Wiener index are (essentially) equivalent to the studies of the average distance in graphs; cf. [4].

A related concept was recently introduced under the name terminal Wiener index, which is defined as the sum of the distances between all leaves in a graph [5,6]. In connection to the terminal Wiener index, the so-called terminal distance matrices were used in the mathematical modeling of proteins and genetic codes [7,8].

In this paper we are interested in another variation of the above concept(s), the network opportunity index. When partitioning a network topology into two equal pieces of nodes, the halves may have a very different structure, in particular their metric properties can be very different. If we have an option to design a network in advance (say, in the situation when two parties are competing in a common market with an objective to minimize the cost of transport between all its nodes), it seems fair to design a network in such a way that neither of the involved parties has an advantage to the other. Focusing on a simple model of an undirected graph, the opportunity index is the largest possible difference between the relative Wiener indices of two halves, over all partitions of its vertex set into two equal parts. We are especially interested in the so-called equal opportunity graphs that are defined as the graphs having the opportunity index equal to 0. In graph theory equal opportunity property yields another (metric) measure of symmetry.

Our main result asserts that equal opportunity graphs are precisely distance-balanced graphs (of even order), a class of graphs first studied by Handa [9] in the case of partial cubes. The concept was later generalized to all graphs in [10], where these graphs were also named distance-balanced. In particular it was observed that all vertex-transitive graphs are distance-balanced. Symmetry properties of distance-balanced graphs were studied in depth in [11]; see also [12]. In [13] distance-balanced graphs were characterized as the graphs in which all vertices have the same total distance. In the bipartite case, distance-balanced graphs can be characterized as the extremal graphs with respect to the so-called Szeged index, a result independently proved in [14,15]. Distance-balanced graphs with respect to different graph operations were studied in [10,13,16]. Cabello and Lukšič [17] considered the problem, which is the minimum number of edges to be added to a given graph to obtain a distance-balanced graph. They proved that the problem is NP-hard for graphs of diameter 3, but can be solved in polynomial time for graphs of diameter 2. Finally, Miklavič and Šparl [18] extended connectivity studies of Handa by constructing a bipartite distance-balanced graph that is neither a cycle nor 3-connected and classifying non-3-connected bipartite distance-balanced graphs for which the minimum distance between the vertices of a 2-cut equals 3.

In the next section we formally introduce the relevant concepts and give some preliminary results. Then, in Section 3, we prove the main result of this paper: equal opportunity graphs are precisely distance-balanced graphs of even order. We also construct a new infinite family of such graphs. In the concluding section, we propose a new game on graphs, called the Wiener game, which arises from a practical electricity distribution problem and is closely related to the opportunity index.

2. Preliminaries

The graphs considered are simple and connected. The *distance* $d_G(u, v)$ between vertices u and v in a graph G is the shortest path distance. For a vertex u of G the *total distance* $D_G(u)$ *of* u is $D_G(u) = \sum_{v \in V(G)} d_G(u, v)$. Whenever G will be clear from the context we will write $d(u, v)$ and $D(u)$ instead of $d_G(u, v)$ and $D_G(u)$, respectively.

The r-cube Q_r is the graph with $V(Q_r) = \{0, 1\}^r$ and two vertices r-tuples u and v are adjacent if and only if they differ in exactly one coordinate. A graph G is called a *partial cube* if it can be embedded as an induced subgraph into Q_r such that for each pair of vertices $u, v \in V(G)$, $d_G(u, v) = d_{Q_r}(u, v)$. (Note that this distance coincides with the number of coordinates in which the r-tuples u and v differ.) We also say that G is an *isometric subgraph* of Q_r.

The *Wiener index* $W(G)$ of a graph G is defined with

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u \in V(G)} D_G(u).$$

If $X \subseteq V(G)$ then the *relative Wiener index* of X in G is

$$W_X(G) = \sum_{\{u,v\} \in \binom{X}{2}} d_G(u, v),$$

where $\binom{X}{2}$ is the set of all 2-element subsets of X.

For an edge uv of a graph G, let W_{uv} be the set of vertices closer to u than to v, that is, $W_{uv} = \{x \in G : d(x, u) < d(x, v)\}$. If G is bipartite, then W_{uv} and W_{vu} form a partition of the vertex set of G. A graph G is *distance-balanced*, if $|W_{uv}| = |W_{vu}|$ holds for any edge uv of G. We recall the following result (see also [19, Theorem 1] for an alternative proof).

Theorem 2.1 *([13, Theorem 3.1]). Let G be a connected graph. Then G is distance-balanced if and only if* $|\{D(x) : x \in V(G)\}| = 1$.

In other words, distance-balanced graphs are precisely the graphs in which all the vertices have the same total distance.

Let G be a graph on $2n$ vertices, and let $V_1, V_2 \subset V(G)$ be n-sets of vertices of G such that $V_1 \cup V_2 = V(G)$ (note that this implies $V_1 \cap V_2 = \emptyset$). Then we say that $\{V_1, V_2\}$ is a *half-partition* of G. The *opportunity index* of a graph G is defined as

$$\mathrm{opp}(G) = \max\{|W_{V_1}(G) - W_{V_2}(G)| : \{V_1, V_2\} \text{ is a half-partition of } G\}.$$

There exist graphs G with $\mathrm{opp}(G)$ arbitrarily large. For instance, if G is the corona on K_n (the graph obtained from the complete graph on n vertices by attaching a leaf to each vertex), then $\mathrm{opp}(G) = n(n-1)$.

We conclude the section with a key definition: a graph G (of even order) is an *equal opportunity graph* if $\mathrm{opp}(G) = 0$.

3. Characterization of equal opportunity graphs

Here is our main result.

Theorem 3.1. *A graph G is an equal opportunity graph if and only if G is a distance-balanced graph of even order.*

Proof. Suppose first that G is a distance-balanced graph G of order $2n$. Let V_1 and V_2 be any sets of size n such that $V_1 \cup V_2 = V(G)$. For any $x \in V_1$ we can write $D_G(x) = \sum_{u\in V_1} d(x,u) + \sum_{v\in V_2} d(x,v)$. Summing up for all vertices $x \in V_1$ we get

$$W(G) = \sum_{x\in V_1}\left(\sum_{u\in V_1} d(x,u) + \sum_{v\in V_2} d(x,v)\right) = 2W_{V_1}(G) + 2W_{V_1,V_2}(G),$$

where $W_{V_1,V_2}(G)$ denotes the sum of all distances $d(x,y)$ where $x \in V_1, y \in V_2$. By applying the same reasoning for V_2 we find that

$$2W_{V_1}(G) + 2W_{V_1,V_2}(G) = 2W_{V_2}(G) + 2W_{V_1,V_2}(G)$$

which implies that $W_{V_1}(G) = W_{V_2}(G)$. We conclude that G is an equal opportunity graph.

Assume now that G is an equal opportunity graph. Then by definition, G is of even order $2n$. We are going to show that $D_G(x) = W(G)/n$ holds for any vertex x. This will imply, using Theorem 2.1, that G is a distance-balanced graph (of even order).

Since G is an equal opportunity graph, $W(X) = W(X^c)$ holds for any $X \subset V(G)$ with $|X| = n$, where X^c denotes the complement of X. Fixing $x \in V(G)$ and summing over all half-sized subsets that contain x we thus have

$$\sum_{\substack{x\in X\subset V(G)\\ |X|=n}} \left(W(X) - W(X^c)\right) = 0.$$

Considering how many times a fixed pair of vertices appears in the above summation we then get:

$$\binom{2n-2}{n-2}\sum_{u\neq x} d(x,u) + \binom{2n-3}{n-3}\sum_{u,v\neq x} d(u,v) - \binom{2n-3}{n-2}\sum_{u,v\neq x} d(u,v) = 0,$$

which can be rewritten as

$$\binom{2n-2}{n-2} D(x) = \left[\binom{2n-3}{n-2} - \binom{2n-3}{n-3}\right]\sum_{u,v\neq x} d(u,v),$$

or, equivalently,

$$\binom{2n-2}{n-2} D(x) = \frac{1}{n-1}\binom{2n-2}{n-2}\sum_{u,v\neq x} d(u,v).$$

It follows that $(n-1)D(x) = \sum_{u,v\neq x} d(u,v)$. Adding $D(x)$ to both sides of this equality we get $nD(x) = \sum_{u,v} d(u,v) = W(G)$. Hence G is a distance-balanced graph. □

We have thus seen that equal opportunity graphs are precisely distance-balanced graphs of even order. Hence it is desirable to know many interesting (infinite) families of such graphs. There exist non-regular distance-balanced partial cubes, for instance, the Handa graph [9]; see also [15]. We next construct a new infinite family of (non-regular) distance-balanced partial cubes, and so, an infinite family of equal opportunity graphs.

For any $3 \le s \le r$, let $Q_{r,s}$ be the graph obtained from Q_r by removing the vertices which either have the first s coordinates equal to 1 or have the first s coordinates equal to 0. For instance, $Q_{3,3} = C_6$ because it is obtained from Q_3 by removing vertices 111 and 000. (A seemingly more general construction would be to first select some subset of s coordinates and then remove the corresponding vertices; however due to the symmetry of Q_r a graph isomorphic to $Q_{r,s}$ would be constructed.) Now we have the following proposition.

Proposition 3.2. *If $3 \le s \le r$, then $Q_{r,s}$ is an equal opportunity graph.*

Proof. We first claim that $Q_{r,s}$ is a partial cube. To this end let $u = u_1 \dots u_r$ and $v = v_1 \dots v_r$ be arbitrary vertices of $Q_{r,s}$ and suppose that u and v differ in b coordinates. To prove the claim it suffices to show that there exists a u, v-path in $Q_{r,s}$ of length b.

Let $i_1, \dots, i_a, i_{a+1}, \dots, i_b$ be the coordinates in which u and v differ, where $i_1, \dots, i_a \in \{1, \dots, s\}$ and $i_{a+1}, \dots, i_b \in \{s+1, \dots, r\}$. Let $u' = u_1 \dots u_s$ and $v' = v_1 \dots v_s$. If not all u_i, $i \in I_{u,s} := \{1, \dots, s\} \setminus \{i_1, \dots, i_a\}$, are equal, then it is straightforward to construct a desired u, v-path of length b in $Q_{r,s}$. Hence we may assume without loss of generality that $u_i = 0$ for any $i \in I_{u,s}$ (and so also $v_i = 0$ for any $i \in I_{u,s}$). Then there exists at least one index i' from $\{i_1, \dots, i_a\}$ such that $u_{i'} = 0$, for otherwise all the coordinates of v' would be 0, a contradiction. But now we can easily find a shortest u, v-path of length b in $Q_{r,s}$ by first changing $u_{i'}$ to 1.

We have thus proved that $Q_{r,s}$ is a partial cube. It follows that the distance between two vertices of $Q_{r,s}$ is the number of coordinates in which they differ. Since $Q_{r,s}$ is obtained from Q_r by removing $2 \cdot 2^{r-s}$ vertices, $|V(Q_{r,s})| = 2^r - 2^{r-s+1}$. Moreover, by the way $Q_{r,s}$ is constructed, if u is an arbitrary vertex of $Q_{r,s}$, then u differs in a fixed coordinate from precisely $(2^r - 2^{r-s+1})/2$ other vertices. It follows that for any vertex u we have $D_{Q_{r,s}}(u) = r(2^r - 2^{r-s+1})/2$. From Theorem 2.1 we thus infer that $Q_{r,s}$ is distance-balanced. Theorem 3.1 now implies that $Q_{r,s}$ is an equal opportunity graph. □

The *Fibonacci cube* Γ_r, $r \geq 1$, is obtained from Q_r by removing all vertices that contain two consecutive ones, cf. [20], while the *generalized Fibonacci cube* $\Gamma_r(f)$, where f is a given fixed binary string, is obtained from Q_r by removing all vertices that contain f as a substring [21]. Since the construction of the graphs $Q_{r,s}$ introduced above is of similar nature, it seems interesting to further study this class of partial cubes.

4. Opportunity index and Wiener game

Given a graph G it is interesting to know what its opportunity index is. Loosely speaking, the bigger the difference, the less metric-symmetric the graph is. When a graph is a model for a real-life problem (say in economy, location theory, or social choice phenomena) then the network opportunity measures the unfairness or social inequality of a given topology. Thus, in many situations the design of equal opportunity networks is highly desirable.

Let us present an example from distribution networks. Consider a network that consists of nodes connected by transmission lines. Each node is a source of items as well as a distribution center which supplies customers and the other nodes of the network with items. Some of the items transmitted between the nodes are lost: the loss rate is proportional to the number of transmission lines on a shortest path between two nodes. We assume that all sources and distribution centers have the same capacity and the same customer demand, respectively.

The customers are served by two distribution companies, say A and B, such that each company is allowed to control half of the nodes of the grid. The company A (resp. B) can use only the items obtained from the sources that belong to A (resp. B). In order to make a selection of the nodes fair, A and B alternate taking turns choosing a node. The goal of a company is to minimize the losses on transmission lines. If nodes are considered as vertices with nodes being adjacent if there is a transmission line between them, we have to minimize the sum of distances between vertices of A (or B) in the underlying graph of a network.

An example of the above concept would be an electrical grid that consists of high-voltage transmission lines that connect intermittent energy sources. An intermittent energy source is a source of energy that is not continuously available due to some factor outside direct control, e.g., wind turbines and solar power stations. An intermittent energy source supplies individual customers. Moreover, since the variability of production from a single source can be high, it exchanges electrical energy with the other intermittent energy sources of the grid.

These examples initiate an introduction of the game played on vertices of a graph, which we call the *Wiener game*. (For more on combinatorial games; see the survey [22].) This game is played on a connected graph G of even order. Vertices are chosen, one at a time, by two players—player A and player B. Player A starts the game and the players alternate by taking turns choosing a vertex from G until all the vertices have been selected. Let V_A and V_B be the sets of vertices selected by players A and B, respectively. Since the order of G is even, $|V_A| = |V_B|$. The goal of both players is to make

$$\sum_{\{u,v\} \in \binom{V_A}{2}} d_G(u, v) \quad \text{and} \quad \sum_{\{u,v\} \in \binom{V_B}{2}} d_G(u, v)$$

as small as possible, respectively. Assuming that both were playing optimally and that sets V_A were selected by the two players, we set $W_A(G) = \sum_{\{u,v\} \in \binom{V_A}{2}} d_G(u, v)$ and $W_B(G) = \sum_{\{u,v\} \in \binom{V_B}{2}} d_G(u, v)$. We say that player A (resp. B) *wins* the game if $W_A(G) < W_B(G)$ (resp. $W_B(G) < W_A(G)$), otherwise the game is a *draw*.

Note that $|W_A(G) - W_B(G)| \leq \mathrm{opp}(G)$, and in practical situations, the player who wins the game, often wants to maximize $|W_A(G) - W_B(G)|$. This yields another (difficult) problem of making the advantage as big as possible and determine its value. Note that this difference can be arbitrarily large. For instance, $W_A(K_{1,2n-1}) - W_B(K_{1,2n-1}) = \mathrm{opp}(K_{1,2n-1}) = n - 1$. Of course, $\mathrm{opp}(G)$ is in general only an upper bound for this difference. Anyway, studying $\mathrm{opp}(G)$ for arbitrary graphs G may be of independent interest. Also studying lower bounds for $|W_A(G) - W_B(G)|$ and finding classes of graphs on which this lower bound is always positive (graphs on which the game can never be a draw) will be interesting. From a complementary point of view, graphs G in which $|W_A(G) - W_B(G)| = 0$ for some half-partition $\{A, B\}$ of G are interesting in the sense that a fair half-partition of their nodes can be achieved. Again finding optimum strategies for each player in special classes of graphs is challenging.

The following observation is a direct consequence of Theorem 3.1.

Corollary 4.1. *If G is a distance-balanced graph of even order, then the Wiener game on G is a draw, regardless of the strategy used by either of the players.*

In fact, we can even allow the first player to first choose half of the vertices and the second player is left with the other half, and the game will still be a draw.

To conclude the paper we propose a further study of the Wiener game and the opportunity index.

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