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The role of lower partial moments in stochastic modeling

Summary - Lower partial moments plays an important role in the analysis of risks and in income/poverty studies. In the present paper, we further investigate its importance in stochastic modeling and prove some characterization theorems arising out of it. We also identify its relationships with other important applied models such as weighted and equilibrium models. Finally, some applications of lower partial moments in poverty studies are also examined.

Key Words - Lower partial moments; Weighted distributions; Equilibrium distributions; Income gap ratio.

1. Introduction

The concept of risk and its measurement plays an important role in many problems of economics, business and industry. There are several important risk measures available in literature. Many experimental studies (Adams and Montesi (1995), Unser (2002)), however, shows that corporate heads are mostly concerned with one sided risk, namely, the 'downside risk', a measure of distance between a risky situation and the corresponding risk free situation (see Dhaene *et al.*, 2003)).

Popular in literature, an important downside risk measure is the Lower Partial Moment (LPM) (see Bawa (1975), Fishburn (1977)). Consider a portfolio with a random return X and assume individual has a target return t. An outcome larger than t is nonrisky and desirable, then individual faces only a one-sided risk called the downside risk that occurs when X falls short of t. Therefore LPM provides a measure that a specified minimum return (target return) may not be earned by a financial investment. Clearly, lower partial moments provide a summary statistics for the downside risk. In the case of a continuous distribution, for a positive integer, the r^{th} order LPM of X is defined as a random variable (rv) having an absolutely continuous cumulative distribu-

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tion function (cdf) and a probability density function (pdf) $f(x)$, a subset of the real line (a, b), where a and b can be finite or infinite. If $E(X^r) < \infty$, then the r^{th} order LPM about a point t is defined as

$$l_r(t) = E\left[(X-t)^-\right]^r;\ r = 0, 1, 2, \ldots;\ t > 0 \tag{1.1}$$

where

$$(X-t)^- = \begin{cases} (t-X); & X < t \\ 0; & X \geq t. \end{cases}$$

When the cdf associated with X is $F(t)$ and t is the target rate of returns, then (1.1) can be written as

$$l_r(t) = \int_{-\infty}^{t} (t-x)^r dF(x). \tag{1.2}$$

Clearly, LPM is a function of the underlying distribution function and it is an increasing function of the target return; as t increases, $l_r(t)$ also increases. Some of the most frequently used risk measures are special cases of LPM's. For example, when the weighing coefficient $r = 0$, the probability of loss equals the 0^{th} order LPM $l_0(t)$ and for $r = 1$, it is the expected loss $l_1(t)$. In all these cases, the target value t is considered as a threshold point separating gains and losses.

Setting $r = 2$ yields a measure known as Target Semi-Variance (TSV). One reason for the interest in LPM measures of risk such as TSV is that they reflect investors' preferences better than the traditional measure, variance. Here, risk is measured by squared deviations below the target t and therefore it penalizes any extremely low returns in the same way that variance penalizes extreme values in either direction. In these cases, TSV fits investors risk preferences better than variance. Thus LPM's have several advantages over traditional measures of risk and for a survey of literature on LPM and its related topics we refer to Bawa (1975), Price *et al.* (1982), Harlow (1991), Eftekhari (1998), Lien and Tse (2000, 2001) and Brogan and Stidham (2005).

The standard practice in modeling statistical data is either to derive an appropriate model based on the physical properties of the system or to choose a flexible family of distributions and then find a member of the family that is appropriate to the data. In both the situations it would be helpful if we find characterization theorems that explain the distribution using important measures of indices. A plethora of work is being done in connection with the characterization of various probability distributions under different measures such as moments, hazard function, residual life function, coefficient of variation etc. For example, the characterization of log normal distribution came out by Krumbein and Pettijohn (1938) with the fact that this distribution provides a good fit to the observed particle sizes and along with the wide applicability of it in income studies, it has also been found applied recently in the context of mean lower partial moments (see Lee and Rao (1988)). Similarly, for various properties and characterizations of probability distributions which are useful in income studies and related areas we refer to Dimaki and Xekalaki (1990) and Reed (2006) and the references therein. Accordingly in the present paper, we focus attention on identifying various characterization relationships between LPM's to model some important probability distributions and families of distributions.

The measurement of downside risk is formally related to the measurement of poverty (see Breitmeyer *et. al.* (2004)). The target value could be interpreted as poverty line and the shortfall of an outcome as poverty gap. Thus in the present note, an attempt is also made to prove new characterizations to distributions such as Pareto, exponential and beta densities using the poverty measures such as average income below poverty line and income gap ratio.

The paper is organized as follows: Section 2 obtains some fundamental relationships and proves some characterization theorems for certain probability distributions. Sections 3 describe the usefulness of families of distributions in modeling problems and derive some new recurrence relationships pertaining to it using various LPM's. In Section 4 we introduce the concept of weighted distributions and obtain relationships for $l_r(t)$ and prove characterization theorems in the context of length biased and equilibrium models. Finally in Section 5, applications of LPM's in poverty studies are studied and obtained characterization theorems arising out of it.

2. Characterizations of distributions

By virtue of the relationship (1.2), we have

$$l_r(t) = r \int_{-\infty}^{t} (t-x)^{r-1} F(x)dx. \tag{2.1}$$

Differentiating (1.2) with respect to t, successively we get

$$\frac{d}{dt} l_r(t) = r \int_{-\infty}^{t} (t-x)^{r-1} f(x)dx,$$

and

$$\frac{d^2}{dt^2} l_r(t) = r(r-1) \int_{-\infty}^{t} (t-x)^{r-2} f(x)dx.$$

Proceeding similarly r times, we obtain

$$\frac{d^r}{dt^r} l_r(t) = r! F(t), \quad \text{where} \quad F(t) = \int_{-\infty}^{t} f(x)dx$$

or

$$F(t) = \frac{l_r^{(r)}(t)}{r!}, \tag{2.2}$$

where $l_r^{(r)}(t)$ is the r^{th} derivative of $l_r(t)$ with respect to t. Thus from (2.2), $l_r(t)$ determines the cdf uniquely.

It has been shown by Dimaki and Xekalaki (1990) about the importance of power distribution in income study. Now we give the following characterization theorem for power distribution based on a relationship between two consecutive LPM's.

Theorem 2.1. *Let X be a non-negative rv defined as in* (1.1) *and such that* $\lim_{t \to 0} t f(t) = 0$. *Then the ratio of consecutive lower partial moments, is of the form*

$$\frac{l_r(t)}{l_{r-1}(t)} = Ct, \tag{2.3}$$

where $0 < C < 1$ *is a constant characterizes power distribution with cdf*

$$F(t) = \left(\frac{t}{b}\right)^c, \quad 0 < t < b, \quad b, c > 0. \tag{2.4}$$

Proof. Suppose that the relation (2.3) holds, by using (1.2), we have

$$\int_0^t (t-x)^r f(x)dx = Ct \int_0^t (t-x)^{r-1} f(x)dx. \tag{2.5}$$

Use $t = (t - x + x)$ in (2.5) and on simplification, we get

$$(1-C) \int_0^t (t-x)^r f(x)dx = C \int_0^t x(t-x)^{r-1} f(x)dx. \tag{2.6}$$

Differentiating (2.6) r times and using (2.2), we obtain

$$(1-C)F(t) = Ctf(t)$$

or

$$\lambda(t) = \frac{(1 - C)}{Ct}. \tag{2.7}$$

which is the inverse Mill's ratio (see Tobin (1958)). Now from the uniqueness property of $\lambda(t)$ using the relationship $F(t) = \exp\left(-\int_t^\infty \lambda(x)dx\right)$, equation (2.7) corresponds to the power distribution with cdf (2.4).

Conversely, assuming (2.4) holds. Substituting for $f(t)$ in (1.2) and on simplification we get (2.3) and it is obtained from Table 2. □

Next to the normal distribution, the exponential distribution is possibly the most widely referenced continuous probability law. It appears as a text book or an in-class example in introductory probability and statistics course. A traditional characterization of exponential distribution is by the memory-less property. Therefore, in the following theorem a characterization of the exponential distribution using the LPM is proved.

Theorem 2.2. *Let X be a non-negative rv defined as in* (1.1)*, then the* r^{th} *order LPM satisfies a relationship of the form*

$$l_r(t) + Cl_{r-1}(t) = t^r, \tag{2.8}$$

where $C > 0$ *for all* $t > 0$ *if and only if X follows exponential distribution with cdf*

$$F(t) = 1 - e^{-\lambda t}; \quad t > 0, \lambda > 0. \tag{2.9}$$

Proof. Assume that the relation (2.8) holds, using (1.2), we get

$$\int_0^t (t - x)^r f(x)dx + C \int_0^t (t - x)^{r-1} f(x)dx = t^r. \tag{2.10}$$

Differentiating (2.10) r times with respect to t and using (2.2), we get

$$\frac{f(t)}{1 - F(t)} = \frac{r}{C}, \tag{2.11}$$

which is the reciprocal of the Mills ratio (hazard rate) (see Boyd (1959)). Now from the uniqueness property of hazard rate through the relationship$F(t) = 1 - \exp\left(-\int_0^t h(x)dx\right)$, equation (2.11) corresponds to the exponential distribution (2.9). The converse part is obtained by direct calculation and is given in Table 2. □

In the next theorem, a functional relationship for $l_r(t)$ is identified to characterize uniform distribution.

Theorem 2.3. *Let X be a non-negative rv defined as in* (1.1)*, a relationship connecting the* r^{th} *order LPM*

$$l_r(t) = C(t-a)^{r+1}, \tag{2.12}$$

where $C > 0$ *is a constant is satisfied for all* $t > 0$ *if and only if X follows uniform distribution with pdf*

$$f(t) = \frac{1}{(b-a)}; \quad a < t < b. \tag{2.13}$$

Proof. Assume (2.12) holds, using the similar steps as in Theorem 2.2, we get (2.13). The converse part is directly obtained from Table 2. □

The applicability of generalized Pareto distribution in economics, reliability etc. is well known (see Arnold and Laguna (1977), Nadarajah and Kotz (2003)). Therefore in the following theorem, we obtain a recurrence relationship between LPM's that characterize generalized Pareto distribution.

Theorem 2.4. *Let X be a non-negative rv defined as in* (1.1) *with* $\lim_{t\to 0} tf(t) = 0$*, a relationship of the form*

$$l_r(t) + r(C_1 t + C_2) l_{r-1}(t) = (C_1+1)t^r, \tag{2.14}$$

where $C_i > 0$*;* $i = 1, 2$ *are constants holds for all* $t > 0$ *if and only if X follows generalized Pareto distribution with cdf*

$$F(t) = 1 - \left(\frac{q}{pt+q}\right)^{\frac{1}{p}+1}; \quad t > 0, p > -1, q > 0. \tag{2.15}$$

Proof. The 'if' part of the theorem can be obtained from Table 2. To prove the 'only if' part, assume (2.14) holds. Using (1.2) and on simplification, we obtain

$$\int_0^t (t-x)^r f(x)dx + r(C_1 t + C_2) \int_0^t (t-x)^{r-1} f(x)dx = (1+C_1)t^r. \tag{2.16}$$

Putting $t = (t - x + x)$ in (2.16) and on simplification, we get

$$(1+C_1) \int_0^t (t-x)^r f(x)dx + \int_0^t (C_1 x + C_2)(t-x)^{r-1} f(x)dx = (1+C_1)t^r. \tag{2.17}$$

Differentiating (2.17) r times with respect to t using (2.2), and on further simplification

$$r(1+C_1)F(t) + (C_1 t + C_2) f(t) = r(1+C_1)$$

which implies that

$$\frac{f(t)}{1-F(t)} = \frac{r(1+C_1)}{(C_1 t + C_2)}. \tag{2.18}$$

Now from the uniqueness property of hazard rate, (2.18) yields the required result. □

3. Recurrence relationships for families of distributions

In modeling problems, a common approach adopted is that the investigator initially chooses a family of distributions consisting of a wide variety of members with differing characteristics and then a member of the family that agrees with the data and/or the physical properties of the system are chosen as the final model. When using families of distributions as the starting point, often the general properties of the family will be of considerable use in identifying the appropriate member. Accordingly, there are several investigations available in literature for identifying some common characteristics pertaining to various systems of distributions and therefore any attempt at unearthing new properties is a worthwhile exercise. It also helps to unify the results obtained in the case of individual distributions that are obtained in separate studies. In terms of versatility, richness of members, the general family of distributions (Ruiz and Navarro (1994)), Pearson family, generalized Pearson family of distributions (see Ord (1972)) and exponential family appears to stand out as the best families of distributions in model building. Therefore in the present section, we prove some new recurrence relationships pertaining to these families of distributions using various LPM's .

Theorem 3.1. *Assume* $l_{g(r-1)}(t) = \int_{-\infty}^{t} (t-x)^{r-1} f(x)g(x)dx$*, then the pdf of a random variable X belongs to general family of distributions* (*see Ruiz and Navarro* (1994)) *given by*

$$\frac{f'(t)}{f(t)} = \frac{\mu - t - g'(t)}{g(t)}, \tag{3.1}$$

where μ *is a constant and* $g(t)$ *is a real function in* $(-\infty, \infty)$*, if and only if its* r^{th}*order LPM satisfies a recurrence relationship of the form*

$$K(t-a)^r + r l_{g(r-1)}(t) = (\mu - t) l_r(t) + l_{r+1}(t), \tag{3.2}$$

where $K = -f(a)g(a)$.

Proof. Assume that the relation (3.2) holds. By using (1.2) and on simplification, (3.2) becomes

$$\begin{aligned} K(t-a)^r + r \int_{-\infty}^{t} (t-x)^r f(x)g(x)dx \\ = \mu \int_{-\infty}^{t} (t-x)^r f(x)dx - \int_{-\infty}^{t} x(t-x)^r f(x)dx. \end{aligned} \tag{3.3}$$

Differentiating (3.3) $(r+1)$ times using (2.2) and on simplification, we get (3.1).

Conversely assuming (3.1) and multiplying both sides of (3.1) by $(t-x)^r$ and on integration using the assumption given in the theorem, we get (3.2). □

Table 1 provides some of the important members of the family (3.1) and identifies each of its recurrence relationships using (3.2).

Corollary 3.1. *When $g(t) = b_0 + b_1 t + b_2 t^2$ with $b_2 \neq -\frac{1}{2}$, the general family* (3.1) *reduces to the Pearson family of distributions whose pdf $f(t)$ satisfies a differential equation of the form*

$$\frac{d}{dt}\log f(t) = -\frac{(t+d)}{\left(b_0 + b_1 t + b_2 t^2\right)}, \tag{3.4}$$

where $f(t)$ is differentiable, b_0, b_1, b_2 and d are real constants. Substitute for $g(t)$ in (3.2) *and on integration we get a recurrence relationship for Pearson family given by*

$$\begin{aligned} &K(t-a)^r + r\left[b_0 + b_1 t + b_2 t^2\right] l_{r-1}(t) \\ &\qquad - \left[r(b_1 + 2b_2 t) + (\mu - t)\right] l_r(t) + \left[r b_2 - 1\right] l_{r+1}(t) = 0\,, \end{aligned} \tag{3.5}$$

where $K = -(b_0 + b_1 t + b_2 t^2) f(a)$.

TABLE 1: Recurrence relationships connecting some members of general family.

Distribution	$f(t)$	$g(t)$	$l_{r+1}(t)$
Beta	$\frac{1}{B(a,b)} t^{(a-1)}(1-t)^{(b-1)}$; $0<t<1,\ a>0,\ b>0$	$\frac{t(1-t)}{(a+b)}$	$[r+a+b]^{-1}\Big[((2r+a+b)t-r-\mu(a+b))\, l_r(t) + rt(1-t)l_{r-1}(t)\Big]$
Gamma	$\frac{m^p}{)p}\exp(-mt)t^{(p-1)}$; $t>0,\quad m>0,\quad p>0$	mt	$rmtl_{r-1}(t) + (t - rm - \mu)l_r(t)$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp-\left(\frac{t-\mu}{\sigma}\right)^2$; $-\infty<t<\infty, \sigma>0,$ $-\infty<\mu<\infty$	σ^2	$r\sigma^2 l_{r-1}(t) + (t-\mu)l_r(t)$
Maxwell	$4\lambda^{-3/2}\pi^{-1/2}t^2\exp\left(-\frac{t^2}{\lambda^2}\right)$; $t>0, \lambda>0$	$\frac{\lambda^2}{2}\left(1+\frac{\lambda^2}{t^2}\right)$	$2tl_r(t) - \frac{1}{2}\left[\lambda^2(r+3)-2t^2\right] l_{r-1}(t) + \frac{r}{2}\lambda^2 t l_{r-2}(t)$

Theorem 3.2. *Assume $\lim\limits_{t\to-\infty} tf(t) = 0$, then the distribution of a rv X belongs to generalized Pearson family of distributions* (*see Ord* (1972)), *where the pdf of X satisfies a differential equation*

$$\frac{d}{dt}\log f(t) = -\frac{\left(a_0 + a_1 t + a_2 t^2\right)}{\left(b_0 + b_1 t + b_2 t^2\right)}, \tag{3.6}$$

where a_i, b_i; $i = 0, 1, 2$ *are real constants, if and only if its* r^{th}*order LPM satisfies a recurrence relationship of the form*

$$\begin{aligned} l_r(t) = C(t-a)^{r-2} &+ [(c_1 + 2c_2 t) + r d_2] l_{r-1}(t) \\ &- [(r-1)(d_1 + 2d_2 t) + (c_0 + c_1 t + c_2 t^2)] l_{r-2}(t) \\ &+ (r-2)(d_0 + d_1 t + d_2 t^2) l_{r-3}(t) \end{aligned}$$

where $C = \frac{-1}{a_2}\left((b_0 + b_1 a - b_2 a^2) f(a)\right)$, $c_i = \frac{a_i}{a_2}$ *and* $d_i = \frac{b_i}{a_2}$; $i = 0, 1, 2$ *are real constants provided* $a_2 \neq 0$*, and when* $a_2 = 0$*,*

$$\begin{aligned} l_r(t) = [r(d_1 + 2d_2 t) &+ (c_0 + c_1 t)]\, l_{r-1}(t) \\ &- (r-1)(d_0 + d_1 t + d_2 t^2)\, l_{r-2}(t) - C(t-a)^{r-1}, \end{aligned} \tag{3.7}$$

where

$$c_i = \frac{a_i}{(a_1 + (r+1)b_2)}; \quad i = 0, 1, d_j = \frac{b_j}{(a_1 + (r+1)b_2)}; \quad j = 1, 2$$

and

$$C = \frac{-1}{(a_1 + (r+1)b_2)}\left((b_0 + b_1 a - b_2 a^2) f(a)\right)$$

provided $(a_1 + (r+1)b_2) \neq 0$.

Proof. When $a_2 \neq 0$: Assume that the distribution of X belongs to the generalized Pearson family, multiply both sides of (3.6) by $(t-x)^{r-2}$ and on integration over the limits a to t we get (3.7).

Conversely assume (3.7), by using (1.2) and substituting for c_i and d_i; $i = 0, 1, 2$ we obtain

$$\begin{aligned} C(t-a)^{r-2} &- \int_{-\infty}^{t} (b_1 + 2b_2 x)(t-x)^{r-2} f(x)dx \\ &- \int_{-\infty}^{t} (a_0 + a_1 x + a_2 x^2)(t-x)^{r-2} f(x)dx \\ &+ (r-2) \int_{-\infty}^{t} (b_0 + b_1 x + b_2 x^2)(t-x)^{r-3} f(x)dx. \end{aligned} \tag{3.8}$$

Differentiating equation (3.8) r times and using (2.2) we get (3.6). □

Theorem 3.3. *Let X be a rv with support* $(0, \infty)$ *admitting an absolutely continuous distribution function* $F(x)$*, then the distribution of X belongs to exponential family with pdf*

$$f(t) = \exp\left[\theta t + C(t) + D(\theta)\right], \quad x \in (0, \infty), \theta > 0, \tag{3.9}$$

where $C(.)$ *and* $D(.)$ *are arbitrary functions, if and only if its* r^{th} *order LPM's satisfy a recurrence relationship*

$$l_{r+1}(t) = \left(t + D'(\theta)\right) l_r(t) - \frac{d}{d\theta} l_r(t), \tag{3.10}$$

where $D'(\theta)$ *is the derivative of* $D(\theta)$ *with respect to* θ*.*

Proof. From the definition of LPM's

$$l_r(t) = \int_0^t (t - x)^r \exp\left(\theta x + C(x) + D(\theta)\right) dx. \tag{3.11}$$

Differentiating (3.11) with respect to θ, we get (3.10).

Conversely, assume that (3.10) holds, then by using (1.2) and on simplification, we get

$$\frac{d}{d\theta} l_r(t) = \int_0^t (t - x)^r \left(x + D'(\theta)\right) f(x) dx. \tag{3.12}$$

Differentiating (3.11) $(r + 1)$ times with respect to t and on simplification we get (3.9). □

A list of various distributions with its pdf and the corresponding recurrence relationships using LPM are given in Table 2.

In the next section, we study various properties of LPM's in the context of weighted distributions.

4. Weighted models

The concept of weighted distributions was introduced in connection with modeling statistical data and in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. A survey of research in various fields of applications is available in Gupta and Kirmani (1990), Nair and Sunoj (2003), Sunoj and Maya (2006) and Di Crescenzo and Longobardi (2006). Let X be a non-negative and absolutely continuous rv with pdf $f(t)$, then the pdf $f^{w}(t)$ for the weighted rv X_w associated to X and to a positive real function $w(.)$ are defined by

$$f^{w}(t) = \frac{w(t)f(t)}{E\left(w(X)\right)} \tag{4.1}$$

where $E\left(w(X)\right) < \infty$. In this case, the r^{th}order LPM corresponding to the weighted distribution at a point t is denoted as $l_r^{w}(t)$ and is defined as

$$l_r^{w}(t) = E\left((X_w - t)^{-}\right)^{r}; r = 0, 1, 2, \ldots, t > 0. \tag{4.2}$$

By using (1.1), (4.2) can be written as

$$l_r^{w}(t) = \frac{1}{\mu_w}\int_0^t (t-x)^r w(x)f(x)dx. \tag{4.3}$$

In the next two subsections, we find some relationships connecting LPM's of original and length-biased and equilibrium models and examine its properties.

4.1. Length-biased models

When the weight function is proportional to lengths of units of interest $(i.e, w(t) = t)$, then the model (4.1) is known as a length-biased model. The r^{th}order LPM corresponding to the length-biased model (1.2) is denoted as $l_r^{L}(t)$ and is given by

$$l_r^{L}(t) = \frac{1}{\mu}\int_0^t (t-x)^r x f(x)dx. \tag{4.4}$$

Substituting$x = (t - (t - x))$, (4.4) becomes

$$l_r^{L}(t) = \frac{1}{\mu}\left(t l_r(t) - l_{r+1}(t)\right). \tag{4.5}$$

The following theorem characterizes power distribution using a relationship connecting the r^{th}order LPM's of original and length-biased models.

Theorem 4.1. *Let X be a non-negative rv as defined as in* (1.1). *Assume* $\lim_{t\to 0} tf(t) = 0$, *then the ratio of the* r^{th}*order LPM's of original and length-biased model satisfies a relationship of the form*

$$\frac{l_r^{L}(t)}{l_r(t)} = Ct, \tag{4.6}$$

where $C > 0$ *is a constant is satisfied for all* $t > 0$ *if and only if X follows power distribution with cdf* (2.4).

Proof. Suppose that the relation (4.6) holds. Using (4.5) and (1.2) and on simplification, we get

$$\left(\frac{1}{\mu}-C-1\right)\int_0^t (t-x)^{r+1}+\left(\frac{1}{\mu}-C\right)\int_0^t x(t-x)^r=0. \tag{4.7}$$

Differentiating (4.7) $(r+1)$ times and on simplification using (2.2), we obtain

$$\lambda(t)=\frac{\left(\frac{1}{\mu}-C-1\right)}{\left(C-\frac{1}{\mu}\right)t}. \tag{4.8}$$

Now using the uniqueness property of $\lambda(t)$, (4.8) implies the required result.

The converse part is obtained from Table 2. □

Theorem 4.2. *Let X be a non-negative rv as defined as in* (1.1) *and* $\lim_{t\to 0} tf(t)=0$*, then the r^{th}order LPM's of original and length-biased model satisfies a relationship of the form*

$$(C_1t+C_2)l_r(t)-l_r^L(t)=C_1t^{r+1}, \tag{4.9}$$

where $C_i(>0)$; $i=1,2$ are constants, holds for all $t>0$ if and only if X follows generalized Pareto distribution with cdf (2.15).

Proof. Assuming (4.9), from (1.2) and (4.4), we have

$$C_1\int_0^t (t-x)^{r+1}f(x)dx+\int_0^t (C_3x+C_2)(t-x)^r f(x)dx=C_1t^{r+1}, \tag{4.10}$$

where $C_3=\left(C_1-\frac{1}{\mu}\right)$, differentiating (4.10) $(r+1)$ times with respect to t and on simplification using (2.2), we get $\frac{f(t)}{1-F(t)}=\frac{C_1(r+1)}{(C_3t+C_2)}$, which is the hazard rate of generalized Pareto distribution. From the uniqueness property of hazard rate, the remaining part of the theorem can be proved.

Conversely assume that X is specified by generalized Pareto distribution. Substituting (2.15) in (4.5) and using (1.2) and on simplification, we get (4.9) with $C_1=\frac{(p+1)}{q(1-pr)}$ and $C_2=\frac{(r+1)}{(1-pr)}$. □

Table 2 provides LPM of some of the distributions in original and length biased case.

TABLE 2: $l_r(t)$ and $l_r^L(t)$ of certain probability distributions.

Distribution	$f(t)$	$l_r(t)$	$l_r^L(t)$
Exponen.	$\lambda \exp(-\lambda t)$; $\lambda > 0, t > 0$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$	$(r+1+\lambda t) l_r(t) - \lambda t^{r+1}$
Pareto I	$ck^c t^{-(c+1)}$; $t > k$, $k, c > 0$	$\frac{c}{(c-r)} \left[(t-k)^r - \frac{rt}{c} l_{r-1}(t)\right]$	$\frac{(c-1)}{k(c-r-1)} \left[t l_r(t) - (t-k)^{(r+1)}\right]$
Power function	$\frac{c}{b^c} t^{(c-1)}$; $t > 0, b, c > 0$	$\frac{rt}{(r+c)} l_{r-1}(t)$	$\frac{(c+1)t}{b(r+c+1)} l_r(t)$
Finite range	$pd(1-pt)^{d-1}$; $0 < t < \frac{1}{p}$, $p, d > 0$	$\frac{d}{(d+r)} \left[t^r - \frac{r}{pd}(1-pt) l_{r-1}(t)\right]$	$\frac{pd(d+1)}{(d+r+1)} \left[\frac{(r+1+pdt)}{pd} l_r(t) - t^{r+1}\right]$
Pareto II	$pq(1+pt)^{-q-1}$; $t > 0, p, q > 0$	$\frac{q}{(q-r)} \left[t^r - \frac{r}{pq}(1+pt) l_{r-1}(t)\right]$	$\frac{pq(q-1)}{(q-r-1)} \left[\frac{(r+1+pqt)}{pq} l_r(t) - t^{r+1}\right]$
Uniform	$\frac{1}{(b-a)}$; $a < t < b$	$\frac{(t-a)^{r+1}}{(b-a)(r+1)}$	$\frac{2[t+(r+1)a]}{(b^2-a^2)(r+1)(r+2)} (t-a)^{r+1}$
General. Pareto	$q^{\frac{1}{p}+1}(p+1)\left(\frac{1}{(pt+q)}\right)^{\frac{1}{p}+2}$; $t > 0, p > -1, q > 0$	$\frac{1}{[1+p(1-r)]} \left[(p+1)t^r - r(pt+q) l_{r-1}(t)\right]$	$\frac{(p+1)}{q(1-pr)} \left[t l_r(t) - t^{r+1}\right] + \frac{(r+1) l_r(t)}{(1-pr)}$

4.2. Equilibrium models

The equilibrium distribution arises naturally in renewal theory (see Cox (1962)). It is the distribution of the backward or forward recurrence time in the limiting case. A formal definition of the equilibrium distribution is as follows. Let X be a rv admitting an absolutely continuous cdf $F(t)$ with respect to Lebesgue measure in the support of the set of non-negative real numbers. Associated with X, the equilibrium rv X_E is defined with pdf

$$f_E(t) = \frac{R(t)}{\mu}; \quad t > 0 \tag{4.11}$$

where $R(t) = 1 - F(t)$ and $\mu = E(X) < \infty$. The form of the equilibrium distribution (4.11) can also be obtained as a particular case of weighted distribution (4.1), the case when weight function is equal to Mill's ratio, i.e., $w(t) = \frac{R(t)}{f(t)}$. Then the r^{th}order LPM corresponding to the equilibrium model is denoted as $l_r^E(t)$ and is defined as

$$l_r^E(t) = \frac{1}{\mu(r+1)} \left(t^{r+1} - l_{r+1}(t)\right); \quad t > 0. \tag{4.12}$$

Equilibrium distribution has also been found application in economics. For a recent discussion of it and its usefulness in economics, actuary and reliability studies we refer to Hesselager *et al.* (1997), Sunoj (2004), Willmot *et al.* (2005) and Gupta (2007).

Theorem 4.3. *Let X be a non-negative rv with an absolute continuous cdf $F(t)$, then a relationship $l_r(t) = l_r^E(t)$ is satisfied for all $t > 0$ if and only if X follows an exponential distribution with cdf* (2.9).

Proof. Assume $l_r(t) = l_r^E(t)$, by using (4.1) and (1.2) we get

$$\frac{1}{\mu(r+1)}\left(t^{r+1} - \int_0^t (t-x)^{r+1} f(x)dx\right) = \int_0^t (t-x)^r f(x)dx. \tag{4.13}$$

Differentiating (4.13) $(r+1)$ times with respect to t and on simplification using (2.2) we obtain

$$\frac{f(t)}{1-F(t)} = \frac{1}{\mu}. \tag{4.14}$$

From the uniqueness property of hazard rate, (4.14) corresponds to exponential distribution. The converse part is obtained from Table 3. □

Theorem 4.4. *For a non-negative rv X having an absolute continuous cdf $F(t)$ and assume $\lim_{t\to 0} tf(t) = 0$, then the r^{th} order LPM of original and equilibrium model satisfies a relationship of the form*

$$l_r^E(t) + Btl_r(t) = At^{r+1}, \tag{4.15}$$

where $A > 0$ and $B > 0$ are constants, holds for all $t > 0$ if and only if X follows power distribution with cdf (2.4).

Proof. Assume that (4.15) holds. Using (4.12), (1.2) and on simplification, we get

$$\begin{aligned}&\left(\frac{1}{\mu(r+1)} - A\right)t^{(r+1)}\\ &-\left(\frac{1}{\mu(r+1)} - B\right)\int_0^t (t-x)^{r+1} f(x)dx + B\int_0^t x(t-x)^r f(x)dx = 0.\end{aligned} \tag{4.16}$$

Differentiating (4.16) $(r+2)$ times with respect to t and using (2.2) and the regularity condition, we obtain

$$\frac{f'(t)}{f(t)} = \frac{B_2}{t}, \tag{4.17}$$

where $B_2 = \left(\frac{1}{\mu B} - (r+2)\right)$. Integrating (4.17) with respect to t, yields the required result. Converse part of the theorem is obtained from Table 3. □

Theorem 4.5. *For a non-negative rv X as defined in* (1.1)*, the* r^{th} *order LPM of original and equilibrium model satisfies a relationship of the form*

$$(At+B)l_r(t) - l_r^E(t) = At^{(r+1)}, \tag{4.18}$$

where $A,\ B\quad (\geq 0)$ *are constants holds for all* $t > 0$ *if and only if X follows generalized Pareto distribution with cdf* (2.15).

Proof. Assume the relation (4.18) holds, from (1.2) and (4.12), we obtain

$$A_1 \int_0^t (t-x)^{r+1} f(x)dx + \int_0^t (Ax+B)(t-x)^r f(x)dx = A_1 t^{r+1} \tag{4.19}$$

where $A_1 = \left(A + \frac{1}{\mu(r+1)}\right)$. Assume $\lim_{t\to 0} tf(t) = 0$ and differentiating (4.19) $(r+1)$ times with respect to t and using (2.2), we get

$$\frac{f(t)}{1-F(t)} = \frac{A_1(r+1)}{(At+B)}. \tag{4.20}$$

From the uniqueness property of hazard rate, (4.20) provides (2.15). The proof of converse part is obtained from Table 3. □

The r^{th}order LPM's of some probability distributions of original and equilibrium models are listed in Table 3.

TABLE 3: $l_r(t)$ and $l_r^E(t)$ of certain probability distributions.

Distribution	$f(t)$	$l_r(t)$	$l_r^E(t)$
Exponen.	$\lambda\exp(-\lambda t)$; $\lambda > 0, t > 0$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$	$t^r - \frac{r}{\lambda} l_{r-1}(t)$
Pareto I	$ck^c t^{-(c+1)}$; $t > k,\ c, k > 0$	$\frac{c}{(c-r)}\left[(t-k)^r - \frac{rt}{c} l_{r-1}(t)\right]$	$\frac{(c-1)}{kc(r+1)}\left[t^{r+1} - \frac{c}{(c-r-1)}(t-k)^{r+1} - \frac{(r+1)t}{c} l_r(t)\right]$
Power function	$\frac{c}{b^c} t^{(c-1)}$; $t > 0;\quad b, c > 0$	$\frac{rt}{(r+c)} l_{r-1}(t)$	$\frac{(c+1)}{bc}\left[\frac{t^{r+1}}{(r+1)} - \frac{t l_r(t)}{(r+c+1)}\right]$
Finite range	$pd(1-pt)^{d-1}$; $0 < t < \frac{1}{p},\ p, d > 0$	$\frac{d}{(d+r)}\left[t^r - \frac{r}{pd}(1-pt) l_{r-1}(t)\right]$	$\frac{(d+1)}{(d+r+1)}\left[pt^{r+1} + (1+pt) l_r(t)\right]$
Pareto II	$pq(1+pt)^{-q-1}$; $t > 0,\ p, q > 0$	$\frac{q}{(q-r)}\left[t^r - \frac{r}{pq}(1+pt) l_{r-1}(t)\right]$	$\frac{(q-1)}{(q-r-1)}\left[(1+pt) l_r(t) - pt^{r+1}\right]$
Uniform	$\frac{1}{(b-a)}$; $a < t < b$	$\frac{(t-a)^{r+1}}{(b-a)(r+1)}$	$\frac{2}{(b+a)(r+1)}\left[t^{r+1} - \frac{(t-a)^{r+2}}{(b-a)(r+2)}\right]$
General. Pareto	$q^{\frac{1}{p}+1}(p+1)\left(\frac{1}{(pt+q)}\right)^{\frac{1}{p}+2}$; $t > 0, a > -1, b > 0$	$\frac{1}{[1+p(1-r)]}\left[(p+1)t^r - r(pt+q) l_{r-1}(t)\right]$	$\frac{1}{q(1-pr)}\left[(pt+q) l_r(t) - pt^{r+1}\right]$

5. Applications

One of the main applications of LPM is that it can be used to find some poverty indices in the income analysis. Poverty measures are generally a kind of inequality measure that confines attention to a specified bottom slice of the income distribution, *i.e.* they only care for poor people. In measuring the indices of poverty, an important statistic is the proportion of population that falls below the poverty line. The measures of poverty ignore most of the income distribution and often give substantial weight to an individual being or just below the poverty line whereas no weight is given to those slightly above the poverty line. These measurements involve two problems, the identification of the poor and the aggregation of information about the poor (see Sen (1976)). Most studies focus on income distribution as an indicator to identify the poor.

Based on the properties of poverty measures, it has been known for some time that a close formal tie between risk and inequality exists. The income inequality arises in situations where not all people in a society earn the same income in a given period. Similarly, a distribution of returns is called risky if there are events where portfolio values are different. Besides, the resemblance of poverty and downside risk is also striking as both have their focus on the lower part of the distribution, concentrating on income of the poor and the bad outcomes respectively. The poverty line t in income studies divides the poor from the non poor corresponds to the critical line t that divides critical events with portfolio values less than or equal to t from the uncritical events with portfolio values greater than t. Thus the result that we obtained in poverty studies is also useful in downside risk studies where X represents the random return of a portfolio.

In the present context, suppose X represents the income of a community of individuals and define a minimum income requirement, the poverty line t, such that all individuals i who earn income $x_i < t$ are said to be poor and the rv takes the value $(t-x)$ and zero for those individuals whose income below or above poverty line respectively. Thus in income analysis, it is a useful measure for studying some poverty measures. In this case, the zero order LPM $l_0(t)$ gives the proportion of poor people and their income distribution is given as

$$ {}_tF_X(x) = \begin{cases} \frac{l_0(x)}{l_0(t)}; X \le t \\ 1; X > t \end{cases} \tag{5.1} $$

(see Belzunce *et al.* (1995)), and $l_0^L(t)$ measures the proportion of total income earned by income units having income less than or equal to t. In income studies, an index, which measures how poor the poor are, is the income gap ratio $\alpha(t)$, where the income gap of an individual is $(t-x)$. Another measure useful in income analysis is $\mu(t)$, the average income below the poverty line and they are defined as

$$\alpha(t) = \frac{l_1(t)}{t l_0(t)} \tag{5.2}$$

$$\mu(t) = t - \frac{l_1(t)}{l_0(t)}. \tag{5.3}$$

From (5.2) and (5.3), it is clear that

$$\mu(t) = t\,(1 - \alpha(t))\,. \tag{5.4}$$

Using the above relationships, the following theorems are immediate.

Theorem 5.1. *Let X be a non-negative rv representing the income of a community of individuals and have an absolute continuous cdf $F(t)$, then the average income below poverty line satisfies a relationship*

$$\mu(t) = \mu[1 - t(1 + Ct)\lambda(t)], \tag{5.5}$$

where C is a constant, if and only if X follows Pareto II with cdf

$$F(t) = 1 - (1 + pt)^{-q}; \quad t > 0, p, q > 0 \tag{5.6}$$

for, exponential distribution with cdf (2.9) for $C = 0$, or finite range distributions with cdf

$$F(t) = 1 - (1 - pt)^{d}; \quad 0 < t < \frac{1}{p}, \quad d > 0, p > 0 \tag{5.7}$$

for $C < 0$.

Proof. Assume that the relation (5.5) holds. Using (5.3), (1.2) and on simplification, we get

$$\int_0^t x f(x) dx = \mu\,[F(t) - t(1 + Ct) f(t)]\,. \tag{5.8}$$

On differentiating (5.8) with respect to t using the assumption that $\lim_{t\to 0} t f(t) = 0$, we obtain

$$\frac{f'(t)}{f(t)} = \frac{(1 + 2\mu C)}{\mu(1 + Ct)}. \tag{5.9}$$

Integrating (5.9) with respect to t yields the distributions Pareto II, exponential and finite range according to $C > 0$, $C = 0$ and $C < 0$ respectively.

The converse part is obtained by direct calculation. □

REMARK 5.1. Even if the distributions (5.6), (2.9) and (5.7) in Theorem 5.1 satisfies the relationship (5.5), the inequality $\mu(t) \le t$ will be true only for Pareto II and exponential distributions. For finite range distribution (5.7), the inequality fails as it is not useful for modeling poverty data.

Corollary 5.1. *The income gap ratio for the poor people satisfies a relationship*

$$\alpha(t) = 1 - \frac{\mu}{t}\left(1 - t(1+kt)\lambda(t)\right), \tag{5.10}$$

for all $t > 0$ *if and only if* X *follows Pareto II, exponential and finite range distributions respectively according as* $k > 0$, $k = 0$ *and* $k < 0$.

Theorem 5.2. *Let* X *be a non-negative rv representing the income of a community of individuals and have an absolute continuous cdf* $F(t)$ *and if* $\lim_{t\to 0} g(t)f(t) = 0$*, then the income gap ratio* $\alpha(t)$ *satisfies a relationship of the form*

$$\alpha(t) = 1 - \frac{1}{t}\left(\mu - g(t)\lambda(t)\right) \tag{5.11}$$

for all $t > 0$ *if and only if the distribution belongs to the general family* (3.1).

Proof. Assume the relation (5.11) holds, then using (5.2) and (1.2) and on simplification we get

$$\int_0^t (t-x)f(x)dx = tF(t) - \mu F(t) + g(t)f(t). \tag{5.12}$$

Differentiating (5.12) with respect to t and on simplification, we obtain the required result.

To prove the converse, assuming (3.1), we have

$$\frac{d}{dt}\left(f(t)g(t)\right) = (\mu - t)f(t). \tag{5.13}$$

Integrating (5.13) using the assumption and on simplification, we get

$$f(t)g(t) = (\mu - t)l_0(t) + l_1(t). \tag{5.14}$$

Dividing each term of (5.14) by $tl_0(t)$, we get (5.11). □

Corollary 5.2. *For the family given in* (3.1) *the average income below poverty line is*

$$\mu(t) = \mu - g(t)\lambda(t).$$

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