



# Rényi's residual entropy: A quantile approach



Asok K. Nanda<sup>a</sup>, P.G. Sankaran<sup>b</sup>, S.M. Sunoj<sup>b,\*</sup>

<sup>a</sup> Department of Mathematics and Statistics, Indian Institute of Science Education and Research, Kolkata, India

<sup>b</sup> Department of Statistics, Cochin University of Science and Technology, Cochin, India

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## ABSTRACT

In the present paper, we introduce a quantile based Rényi's entropy function and its residual version. We study certain properties and applications of the measure. Unlike the residual Rényi's entropy function, the quantile version uniquely determines the distribution.

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## 1. Introduction

The notion of entropy, later extended to information theory and statistical mechanics, was originally developed by physicists in the context of equilibrium thermodynamics. In 1865, Rudolf Julius Emanuel Clausius, one of the founders of thermodynamics coined the term *entropy* derived from the Greek word *en-trepein* which means *energy turned to waste*, although the concept was introduced by him in the year 1850 in the context of classical thermodynamics. Later, a statistical basis to entropy was given by Ludwig Boltzmann, Willard Gibbs and James Clerk Maxwell (see, Nanda and Das (2006)). A general concept of entropy to quantify the statistical nature of *lost information* in phone-line signals mathematically was developed by Shannon (1948), an electrical engineer from Bell Telephone Laboratory. In the last few years, the literature on information theory has grown quite voluminous. Apart from communication theory, information theory has found lot of applications in many social, physical and biological sciences, viz., economics, statistics, accounting, language, psychology, ecology, pattern recognition, computer sciences, fuzzy sets etc. (see, Taneja (2001)).

Let  $X$  be an absolutely continuous nonnegative random variable (rv) representing the lifetime of a component with cumulative distribution function (CDF)  $F(t) = P(X \leq t)$  and survival function (SF)  $F(t) = P(X > t) = 1 - F(t)$ . The measure of uncertainty (Shannon, 1948) is defined by

$$H(X) = H(f) = - \int_0^{\infty} (\ln f(x)) f(x) dx = -E(\ln f(X)), \tag{1.1}$$

\* Corresponding author.

E-mail address: [smsunoj@gmail.com](mailto:smsunoj@gmail.com) (S.M. Sunoj).

where  $f(t)$  is the probability density function (PDF) of  $X$ . Eq. (1.1) gives the expected uncertainty contained in  $f(t)$  about the predictability of an outcome of  $X$ , which is known as Shannon information measure. Since the measure (1.1) may not be appropriate for used items, Ebrahimi (1996) modified (1.1) using the residual rv, given by

$$H(X; t) = H(f; t) = - \int_t^\infty \left( \frac{f(x)}{\bar{F}(t)} \right) \ln \left( \frac{f(x)}{\bar{F}(t)} \right) dx. \quad (1.2)$$

$H(X; t)$  is known as the Shannon residual entropy function. Note that  $H(X; t) = H(X_t)$ , where  $X_t = (X - t | X > t)$  is the residual time associated to  $X$ . For more properties of (1.1) and (1.2), we refer to Ebrahimi and Pellery (1995), Asadi and Ebrahimi (2000) and Borzadaran et al. (2007).

There are several generalizations of (1.1) available in the literature. One generalization is due to Rényi (1961). For a nonnegative rv  $X$  the Rényi's entropy is defined as

$$H_\beta(X) = \frac{1}{1 - \beta} \ln \int_0^\infty (f(x))^\beta dx, \quad \beta > 0, \beta \neq 1, \quad (1.3)$$

which, as  $\beta \rightarrow 0$ , reduces to Shannon's entropy. For used items, a residual version of Rényi's entropy is due to Abraham and Sankaran (2005), given by

$$H_\beta(X; t) = \frac{1}{1 - \beta} \ln \int_t^\infty \left( \frac{f(x)}{\bar{F}(t)} \right)^\beta dx, \quad \beta > 0, \beta \neq 1. \quad (1.4)$$

When the system has the age  $t$ ,  $H_\beta(X; t)$  provides the spectrum of Rényi's information on the remaining life of the system for different values of  $\beta$ . Obviously,  $H_\beta(X; 0) = H_\beta(X)$ . For more properties and applications of (1.3) and (1.4), one could refer to Song (2001), Asadi et al. (2005, 2006), Baratpour et al. (2008), Zarezadeh and Asadi (2010), Li and Zhang (2011), Fashandi and Ahmadi (2012) and the references therein.

All these theoretical results and applications thereof are based on the distribution function. A probability distribution can also be specified in terms of the quantile functions (QFs). Recently, it has been showed by many authors that the QF defined by

$$Q(u) = F^{-1}(u) = \inf\{t | F(t) \geq u\}, \quad 0 \leq u \leq 1 \quad (1.5)$$

is an efficient and equivalent alternative to the distribution function in modelling and analysis of statistical data (see Gilchrist (2000), Nair and Sankaran (2009)). In many cases, QF is more convenient as it is less influenced by extreme observations, and thus provides a straightforward analysis with a limited amount of information. For detailed and recent studies on QF, its properties and usefulness in the identification of models we refer to Nair et al. (2008, 2011), Nair and Sankaran (2009), Sankaran and Nair (2009), Sankaran et al. (2010) and the references therein.

The quantile functions used in applied work like various forms of lambda distributions (Ramberg and Schmeiser, 1974; Freimer et al., 1988; van Staden and Loots, 2009; Gilchrist, 2000), the power-Pareto distribution (Gilchrist, 2000; Hankin and Lee, 2006), Govindarajulu distribution (Nair et al., 2011) etc. do not have tractable distribution functions. This makes the analytical study of the properties of these distributions by means of (1.1) or (1.2) difficult. Accordingly, Sunoj and Sankaran (2012) introduced quantile versions of the Shannon entropy (1.1) and its residual form (1.2). The quantile based residual entropy is defined by

$$\xi(u) = \xi(X; Q(u)) = \ln(1 - u) + (1 - u)^{-1} \int_u^1 \ln q(p) dp. \quad (1.6)$$

They have shown that unlike the measure (1.2), the quantile based residual entropy function determines the QF uniquely. The definition and properties of the quantile based entropy function of past lifetime are available in Sunoj et al. (2013).

In the present paper, we introduce a quantile based Rényi's residual entropy function and study its important properties. The proposed measure has several advantages. First, unlike (1.4), the proposed measure uniquely determines the quantile distribution function. Second, we derive entropy functions for certain quantile functions which do not have an explicit form for distribution functions. Finally we provide new characterizations for some families of distributions that are useful in lifetime data analysis.

The rest of the article is organized as follows. In Section 2, we give Rényi's entropy in terms of the quantile function. It is shown that the proposed measure determines the distribution uniquely. Various properties of the measure are discussed. We present certain characterization of distributions. In Section 3, we present ageing and ordering properties of residual Rényi's entropy function.

## 2. Quantile based Rényi's residual entropy

Suppose that  $X$  is a non-negative rv as described in Section 1. When the distribution function  $F$  is continuous, we have from (1.5),  $FQ(u) = u$ , where  $FQ(u)$  represents the composite function  $F(Q(u))$ . Denote the density quantile function by

**Table 1**  
Quantile functions and quantile based Rényi's entropy.

Distribution	Quantile functions	$H_\beta(X; Q(u))$
Exponential	$-\frac{1}{\lambda} \ln(1-u)$	$-\frac{\ln \beta}{(1-\beta)} - \ln \lambda$
Uniform	$a + (b-a)u$	$\ln[(b-a)(1-u)]$
Pareto II	$\alpha \left[ (1-u)^{-\frac{1}{c}} - 1 \right]$	$\ln\left(\frac{\alpha}{c}\right) + \frac{1}{(1-\beta)} \ln\left(\frac{c}{\beta(c+1)-1}\right) - \frac{1}{c} \ln(1-u)$
Rescaled beta	$\alpha \left[ 1 - (1-u)^{\frac{1}{c}} \right]$	$\ln\left(\frac{\alpha}{c}\right) + \frac{1}{(1-\beta)} \ln\left(\frac{c}{\beta(c-1)+1}\right) + \frac{1}{c} \ln(1-u)$
Pareto I	$\alpha(1-u)^{-\frac{1}{c}}$	$\ln\left(\frac{\alpha}{c}\right) + \frac{1}{(1-\beta)} \ln\left(\frac{c}{\beta(c+1)-1}\right) - \frac{1}{c} \ln(1-u)$
Power function	$\alpha u^{\frac{1}{c}}$	$\ln\left(\frac{\alpha}{c}\right) + \frac{1}{(1-\beta)} \ln\left(\frac{c}{\beta(c-1)+1}\right) + \frac{1}{(1-\beta)} \ln(1-u) \frac{\beta(c-1)+1}{c} - \frac{\beta}{(1-\beta)} \ln(1-u)$
Generalized Pareto	$\frac{b}{a} \left[ (1-u)^{-\frac{a}{(a+1)}} - 1 \right]$	$\log\left(\frac{b}{a+1}\right) + \frac{1}{(1-\beta)} \log\left(\frac{a+1}{\beta(2a+1)-a}\right) - \frac{a}{(a+1)} \log(1-u)$

$fQ(u) = f(Q(u))$  (see, Parzen (1979)) and the quantile density function by  $q(u) = Q'(u)$ , where the prime denotes the differentiation. Using (1.5), differentiation of  $F(Q(u)) = u$  provides

$$q(u)fQ(u) = 1. \tag{2.1}$$

From (1.4) and (2.1), Rényi's residual entropy denoted by  $H_\beta(X; Q(u))$  is defined as

$$\begin{aligned} (1-\beta)H_\beta(X; Q(u)) &= \ln \int_{Q(u)}^\infty \frac{f^\beta(x)}{(1-u)^\beta} dx \\ &= \ln \int_u^1 \frac{(fQ(p))^\beta}{(1-u)^\beta} q(p) dp \\ &= \ln \int_u^1 \left( \frac{1}{(1-u)q(p)} \right)^\beta q(p) dp \\ &= \ln \int_u^1 \frac{1}{(1-u)^\beta} (q(p))^{1-\beta} dp. \end{aligned} \tag{2.2}$$

The above expression may be referred as Rényi's residual quantile entropy. For different values of  $\beta$ ,  $H_\beta(X; Q(u))$  in (2.2) provides the spectrum of Rényi's information contained in the conditional density about the predictability of an outcome of  $X$  until  $100(1-u)\%$  point of distribution. As  $u \rightarrow 0$ , the expression (2.2) gives the Rényi's quantile entropy, which is the quantile version of (1.3). Eq. (2.2) can also be written as

$$(1-\beta)H_\beta(X; Q(u)) = \ln \int_u^1 (q(p))^{1-\beta} dp - \beta \ln(1-u)$$

or

$$\int_u^1 (q(p))^{1-\beta} dp = (1-u)^\beta \exp[(1-\beta)H_\beta(X; Q(u))]. \tag{2.3}$$

Differentiating (2.3) with respect to  $u$ , we get

$$(q(u))^{1-\beta} = (1-u)^{\beta-1} \exp[(1-\beta)H_\beta(X; Q(u))] [\beta - (1-\beta)(1-u)H'_\beta(X; Q(u))],$$

which leads to

$$q(u) = \frac{\exp[H_\beta(X; Q(u))]}{(1-u)} [\beta - (1-\beta)(1-u)H'_\beta(X; Q(u))]^{\frac{1}{(1-\beta)}}. \tag{2.4}$$

The residual Rényi's entropy  $H_\beta(X; t)$  given in (1.4) in general does not provide an explicit relationship between  $H_\beta(X; t)$  and  $f(t)$ . Asadi et al. (2005) have proved that  $H_\beta(X; t)$  uniquely determines the distribution when the density function is monotone. However, Eq. (2.4) provides a direct relationship between  $q(u)$  and  $H_\beta(X; Q(u))$  which shows that  $H_\beta(X; Q(u))$  uniquely determines the underlying distribution. For example, when  $H_\beta(X; Q(u)) = a + bu$ , a linear function in  $u$ , then from (2.4) the quantile density function  $q(u)$  is of the form  $q(u) = \frac{\exp(a+bu)}{(1-u)} [\beta - (1-\beta)b(1-u)]^{\frac{1}{1-\beta}}$ . On the other hand, for the residual entropy function  $H_\beta(X; t)$  given in (1.4), no direct relationship exists analogous to (2.4) that determines the distribution uniquely. Table 1 provides some important quantile lifetime models and its  $H_\beta(X; Q(u))$ .

Unlike the models in Table 1, there are some models that do not have any closed form expressions for CDF or PDF, but have simple QF's or quantile density functions. For instance, consider the quantile density function (see Nair et al. (2011)) given by

$$q(u) = cu^\alpha (1 - u)^{\frac{1}{1-\beta}}. \tag{2.5}$$

From (2.5), we get  $H_\beta(X; Q(u))$  as

$$H_\beta(X; Q(u)) = \ln c - \frac{1}{(1 - \beta)} \ln [\alpha(1 - \beta) + 1][\alpha(1 - \beta) + 2] - \frac{\beta}{(1 - \beta)(1 - u)} + \frac{1}{(1 - \beta)} \ln [(\alpha(1 - \beta) + 2)(1 - u^{\alpha(1-\beta)+1}) - (\alpha(1 - \beta) + 2)(1 - u^{\alpha(1-\beta)+1})].$$

When the quantile density function is of the form

$$q(u) = (1 - u)^{-A}(-\ln(1 - u))^{-M},$$

then  $H_\beta(X; Q(u))$  becomes

$$H_\beta(X; Q(u)) = \frac{1 - M(1 - \beta)}{(1 - \beta)} \ln(-\ln(1 - u)) - \frac{1}{(1 - \beta)} \ln(M(1 - \beta) - 1) - \frac{\beta}{(1 - \beta)} \ln(1 - u). \tag{2.6}$$

To study ageing behaviour of  $H_\beta(X; Q(u))$ , we differentiate (2.2) with respect to  $u$ , which gives

$$\begin{aligned} (1 - \beta) \frac{d}{du} H_\beta(X; Q(u)) &= \frac{d}{du} \left[ \ln \int_u^1 (q(p))^{1-\beta} dp - \beta \ln(1 - u) \right] \\ &= \frac{(q(u))^{1-\beta}}{\int_u^1 (q(p))^{1-\beta} dp} + \frac{\beta}{1 - u} \\ &= \frac{\beta \int_u^1 (q(p))^{1-\beta} dp - (1 - u)(q(u))^{1-\beta}}{(1 - u) \int_u^1 (q(p))^{1-\beta} dp}. \end{aligned}$$

**Case I:** Let  $0 < \beta < 1$ . If  $H_\beta(X; Q(u))$  is increasing in  $u$ , then we have

$$\beta \int_u^1 (q(p))^{1-\beta} dp \geq (1 - u)(q(u))^{1-\beta},$$

which leads to

$$\begin{aligned} e^{(1-\beta)H_\beta(X; Q(u))} &\geq \frac{1}{\beta} [q(u)(1 - u)]^{1-\beta} \\ &= \frac{(H(u))^{\beta-1}}{\beta}, \end{aligned}$$

where  $H(u)$  is the hazard quantile function defined by

$$H(u) = [(1 - u)q(u)]^{-1}.$$

$H(u)$  explains the conditional probability of failure in the next small interval of time given survival until  $100(1 - u)\%$  point of distribution (see Nair and Sankaran (2009)). The above inequality can be rewritten as

$$H_\beta(X; Q(u)) \geq -\ln H(u) - \frac{\ln \beta}{1 - \beta}. \tag{2.7}$$

It is to be noted that, as  $\beta \rightarrow 1-$ , the right hand side of (2.7) tends to  $1 - \ln H(u)$ .

**Case II:** Let  $\beta > 1$ . If  $H_\beta(X; Q(u))$  is increasing in  $u$ , then we have

$$\beta \int_u^1 (q(p))^{1-\beta} dp \leq (1 - u)(q(u))^{1-\beta},$$

which can equivalently be written as

$$\begin{aligned} e^{(1-\beta)H_\beta(X; Q(u))} &\leq \frac{1}{\beta} [q(u)(1 - u)]^{1-\beta} \\ &= \frac{(H(u))^{\beta-1}}{\beta}, \end{aligned}$$

or

$$H_\beta(X; Q(u)) \geq -\ln H(u) - \frac{\ln \beta}{1 - \beta}. \tag{2.8}$$

As  $\beta \rightarrow 1+$ , the right hand side of (2.8) tends to  $1 - \ln H(u)$ . Thus, we see that, as  $\beta \rightarrow 1$ , we get the bounds for the residual quantile entropy function given in Sunoj and Sankaran (2012), as expected. We prove later that equality in (2.8) holds if and only if the underlying distribution is exponential.

When  $H_\beta(X; Q(u))$  is decreasing in  $u$ , the derivation is similar. Now, combining the above two cases, we get the bounds for  $H_\beta(X; Q(u))$  as given below. It is to be noted that the bounds do not depend on whether  $\beta$  is greater than or less than unity.

**Theorem 2.1.** *If  $H_\beta(X; Q(u))$  is increasing (decreasing) in  $u$ , then*

$$H_\beta(X; Q(u)) \geq (\leq) \frac{\ln \beta}{\beta - 1} - \ln H(u) \rightarrow 1 - \ln H(u), \text{ as } \beta \rightarrow 1.$$

Next we prove a characterization theorem for exponential, Pareto II and rescaled beta models using  $H_\beta(X; Q(u))$ .

**Theorem 2.2.** *The relationship*

$$H_\beta(X; Q(u)) = A + B \ln(1 - u) \tag{2.9}$$

where  $A > 0$  holds for  $u > 0$ , if and only if  $X$  follows exponential when  $B = 0$ , Pareto II when  $B < 0$  and rescaled beta when  $B > 0$ .

**Proof.** The necessary part follows from Table 1 and the converse part follows from (2.4).

**Remark 2.1.** When  $A = 0$  and  $B > 0$ , Eq. (2.9) is a characterization of Uniform distribution.

### 3. Ageing and ordering properties

We now study the ageing and ordering properties of  $H_\beta(X; Q(u))$ .

**Definition 3.1.**  $X$  is said to have increasing (decreasing) Rényi's quantile entropy (IRRQE) (DRRQE) if  $H_\beta(X; Q(u))$  is nondecreasing (nonincreasing) in  $u$ .

For exponential distribution  $H_\beta(X; Q(u))$  is constant so that it is the boundary class of the above two nonparametric classes. For the exponential distribution, the hazard quantile function  $H(u)$  is also constant. When  $X$  follows  $U(0, 1)$ ,  $Q(u) = u$ , but  $H_\beta(X; Q(u)) = \ln(1 - u)$ . Therefore  $H_\beta(X; Q(u))$  is nonincreasing in  $u$ , while its hazard quantile  $H(u) = \frac{1}{1-u}$  is nondecreasing. On the other hand, when  $X$  follows Pareto II,  $H_\beta(X; Q(u))$  is nondecreasing in  $u$ , while  $H(u)$  is nonincreasing in  $u$ . Also from (2.6), we have  $H'_\beta(X; Q(u)) = \frac{1-M(1-\beta)}{(1-\beta)(-\ln(1-u))(1-u)} + \frac{\beta}{(1-\beta)(1-u)}$  which is positive for  $0 < M < 1, 0 < \beta < 1$ , and negative for  $M < 0, \beta > 1$ . These observations establish the fact that there is no direct relationship between increasing (decreasing) hazard rate (IHR) (DHR) class and IRRQE (DRRQE) class.

Below we see how the monotonicity of  $H_\beta(X; Q(u))$  is affected by increasing transformation. The following lemma helps us to prove the results on monotonicity of  $H_\beta(X; Q(u))$ .

**Lemma 3.1.** *Let  $f(u, x) : \mathcal{R}_+^2 \rightarrow \mathcal{R}_+$  and  $g : \mathcal{R}_+ \rightarrow \mathcal{R}_+$  be any two functions. If  $\int_u^\infty f(u, x)dx$  is increasing and  $g(u)$  is increasing (decreasing) in  $u$ , then  $\int_u^\infty f(u, x)g(x)dx$  is increasing (decreasing) in  $u$ , provided the integrals exist.*

**Proof.** Given that  $\int_u^\infty f(u, x)dx$  is increasing in  $u$ . This means that

$$\int_u^\infty \frac{\partial}{\partial u} (f(u, x)g(u))dx \geq (\leq) f(u, u)g(u). \tag{3.1}$$

We have to show that  $\int_u^\infty f(u, x)g(x)dx$  is increasing (decreasing) in  $u$ , which means

$$\int_u^\infty \frac{\partial}{\partial u} f(u, x)g(x)dx \geq (\leq) f(u, u)g(u). \tag{3.2}$$

From (3.1) and (3.2), it is sufficient to show that

$$\int_u^\infty \frac{\partial}{\partial u} f(u, x)g(x)dx \geq (\leq) \int_u^\infty \frac{\partial}{\partial u} (f(u, x)g(u))dx.$$

Or, equivalently,

$$\int_u^\infty \frac{\partial}{\partial u} (f(u, x) (g(x) - g(u))) dx \geq (\leq) 0.$$

This holds if  $g(\cdot)$  is increasing (decreasing).

In the following theorem, we write  $H_\beta(z)$  to mean that it is the  $H_\beta$  function, as defined earlier, for the random variable  $Z$ .

**Theorem 3.1.** For a nonnegative random variable  $X$ , define  $Y = \phi(X)$ , where  $\phi(\cdot)$  is a nonnegative, real-valued, increasing and convex (concave) function. Denote  $Q_X(u)$  and  $Q_Y(u)$  as QF's of rv's  $X$  and  $Y$  respectively.

- (i) For  $0 < \beta < 1$ ,  $H_\beta^Y(Q_Y(u))$  is increasing (decreasing) in  $u$  whenever  $H_\beta^X(Q_X(u))$  is increasing (decreasing) in  $u$ .
- (ii) For  $\beta > 1$ ,  $H_\beta^Y(Q_Y(u))$  is decreasing (increasing) in  $u$  whenever  $H_\beta^X(Q_X(u))$  is increasing (decreasing) in  $u$ .

**Proof.** (i) Denote  $q_X(\cdot)$  as the quantile density function of  $X$ . From the given condition, we have

$$\frac{1}{1 - \beta} \ln \int_u^1 \frac{(q_X(p))^{1-\beta}}{(1 - u)^\beta} dp \text{ is increasing (decreasing) in } u,$$

which gives that

$$\ln \int_u^1 \frac{(q_X(p))^{1-\beta}}{(1 - u)^\beta} dp \text{ is increasing (decreasing) in } u.$$

Now, from the definition of  $H_\beta$ , we have

$$\begin{aligned} (1 - \beta)H_\beta^Y(Q_Y(u)) &= \ln \int_u^1 \frac{(q_Y(p))^{1-\beta}}{(1 - u)^\beta} dp \\ &= \ln \int_u^1 \frac{[q_X(p)\phi'(Q_X(p))]^{1-\beta}}{(1 - u)^\beta} dp. \end{aligned}$$

This can equivalently be written as

$$H_\beta^Y(Q_Y(u)) = \frac{1}{1 - \beta} \ln \int_u^1 \left[ \frac{(q_X(p))^{1-\beta}}{(1 - u)^\beta} \right] (\phi'(Q_X(p)))^{1-\beta} dp. \tag{3.3}$$

Since  $0 < \beta < 1$  and  $\phi$  is nonnegative, increasing and convex (concave), we have  $[\phi'(Q(p))]^{1-\beta}$  is increasing (decreasing) and is nonnegative. Hence, by Lemma 3.1, (3.3) is increasing (decreasing). This proves (i). When  $\beta > 1$ ,  $[\phi'(Q(p))]^{1-\beta} = \frac{1}{[\phi'(Q(p))]^{\beta-1}}$  is decreasing (increasing) in  $p$ , since  $\phi$  is increasing and convex (concave). Hence, we have

$$\frac{1}{1 - \beta} \ln \int_u^1 \left[ \frac{(q_X(p))^{1-\beta}}{(1 - u)^\beta} \right] (\phi'(Q_X(p)))^{1-\beta} dp$$

is decreasing (increasing)  $u$ .

The following example illustrates the utility of Theorem 3.1.

**Example 3.1.** Let  $X$  have the exponential distribution with failure rate  $\lambda$  and let  $Y = X^{1/\alpha}$ ,  $\alpha > 0$ . Then  $Y$  has Weibull distribution with  $Q(u) = \lambda^{-1/\alpha}(-\ln(1 - u))^{1/\alpha}$ . The function  $\phi(x) = x^{1/\alpha}$ ,  $x > 0$ ,  $\alpha > 0$  is convex (concave) if  $0 < \alpha < 1$  ( $\alpha > 1$ ) and is nonnegative. Thus due to Theorem 3.1, for  $0 < \beta < 1$  and  $0 < \alpha < 1$  the Weibull distribution is IRRQE. For  $\beta > 1$  and  $\alpha > 1$ , Weibull distribution is DRRQE.

Below we define a stochastic order based on the comparison of  $H_\beta$  functions corresponding to two nonnegative random variables  $X$  and  $Y$ .

**Definition 3.2.**  $X$  is said to be smaller than  $Y$  in the Rényi quantile entropy order (written as  $X \leq_{RQE} Y$ ), if  $H_\beta^X(Q_X(u)) \leq H_\beta^Y(Q_Y(u))$  for all  $u \in [0, 1]$ .

**Example 3.2.** Suppose  $X_i$  follows an exponential distribution with parameter  $\lambda_i$ ,  $i = 1, 2$ , we can easily see that  $\lambda_1 \geq \lambda_2$  implies that  $X_1 \leq_{RQE} X_2$ . Suppose that  $X_i$  follows Pareto I distribution with parameters  $\alpha_i = 1$  and  $b_i = \frac{1}{c_i}$ ,  $i = 1, 2$ . If  $b_1 < b_2$  and  $0 < \beta < 1$  ( $\beta > 1$ ), then  $X_1 \leq_{RQE} (\geq) X_2$ .

Below we see that the RQE order defined above is a partial order.

**Theorem 3.2.** The RQE order defined above is reflexive, associative and anti-symmetric.

**Proof.** From the definition of the RQE order the reflexive and associative properties are easy to verify. To prove the antisymmetric property, we note that  $X \leq_{RQE} Y$  and  $Y \leq_{RQE} X$  together give  $q_X(u) = q_Y(u)$  for all  $u \in [0, 1]$ . This gives  $Q_X(u) = Q_Y(u)$  for all  $u \in [0, 1]$ .

We now show that the RQE order is closed under increasing convex transformation. We state below a lemma, which is used to prove the theorem.

**Lemma 3.2.** Let  $f(u, x) : [0, 1] \times \mathcal{R}_+ \rightarrow \mathcal{R}_+$  be such that  $\int_u^1 f(u, x) dx \geq 0$  for all  $u \in [0, 1]$ , and  $g(x)$  be any nonnegative increasing function in  $x$ . Then  $\int_u^1 f(u, x)g(x)dx \geq 0$ .

**Theorem 3.3.** Let  $X$  and  $Y$  be two random variables such that  $X \leq_{RQE} Y$ . Then, for any nonnegative increasing convex function  $\phi$ , we have  $\phi(X) \leq_{RQE} \phi(Y)$ .

**Proof.** We first consider the case when  $0 < \beta < 1$ . To show  $\phi(X) \leq_{RQE} \phi(Y)$ , it is enough to show that, for all  $u \in [0, 1]$ ,

$$\frac{1}{1-\beta} \ln \int_u^1 \left[ \frac{(q_Y(p))^{1-\beta}}{(1-u)^\beta} \right] \phi'(Q_Y(p))^{1-\beta} dp \geq \frac{1}{1-\beta} \ln \int_u^1 \left[ \frac{(q_X(p))^{1-\beta}}{(1-u)^\beta} \right] \phi'(Q_X(p))^{1-\beta} dp. \quad (3.4)$$

Since  $X \leq_{RQE} Y$ , we have, for all  $u \in [0, 1]$

$$\frac{1}{1-\beta} \ln \int_u^1 \frac{(q_Y(p))^{1-\beta}}{(1-u)^{1-\beta}} dp \geq \frac{1}{1-\beta} \ln \int_u^1 \frac{(q_X(p))^{1-\beta}}{(1-u)^{1-\beta}} dp, \quad (3.5)$$

which gives, for all  $u \in [0, 1]$

$$\int_u^1 (q_Y(p))^{1-\beta} dp \geq \int_u^1 (q_X(p))^{1-\beta} dp. \quad (3.6)$$

Thus from (3.6), it follows that for  $u \in [0, 1]$ ,  $q_Y(u) \leq q_X(u)$ , and so  $Q_Y(u) \geq Q_X(u)$ . Since  $\phi(\cdot)$  is convex,  $\phi'(\cdot)$  is increasing in  $u$  so that  $\phi'(Q_Y(u)) \geq \phi'(Q_X(u))$ . Thus from (3.5) and Lemma 2.2, it follows that the inequality (3.4) holds. When  $\beta > 1$ , (3.5) leads to

$$\int_u^1 (q_Y(p))^{1-\beta} dp \leq \int_u^1 (q_X(p))^{1-\beta} dp. \quad (3.7)$$

From (3.7), it again follows that  $Q_Y(u) \geq Q_X(u)$  and  $(Q_Y(u))^{1-\beta} \leq (Q_X(u))^{1-\beta}$ . Thus from Lemma 2.2, we obtain

$$\int_u^1 (q_Y(p))^{1-\beta} \phi'(Q_Y(p))^{1-\beta} dp \leq \int_u^1 (q_X(p))^{1-\beta} \phi'(Q_X(p))^{1-\beta} dp,$$

which leads to (3.4). The proof is complete.

**Definition 3.3.**  $X$  is said to be smaller than  $Y$  in quantile failure rate ordering ( $X \leq_{QFR} Y$ ) if  $H_X(u) \geq H_Y(u)$  for all  $u \geq 0$ .

**Theorem 3.4.** If  $X \leq_{QFR} Y$ , then  $X \leq_{RQE} Y$ .

**Proof.** The proof follows from (2.2).

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