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Quantile based stop-loss transform and its applications

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Abstract Partial moments are extensively used in actuarial science for the analysis of risks. Since the first order partial moments provide the expected loss in a stop-loss treaty with infinite cover as a function of priority, it is referred as the stop-loss transform. In the present work, we discuss distributional and geometric properties of the first and second order partial moments defined in terms of quantile function. Relationships of the scaled stop-loss transform curve with the Lorenz, Gini, Bonferroni and Leinkuhler curves are developed.

Keywords Partial moments · Stop-loss transform · Lorenz curve · Gini index

1 Introduction

Let X be a random variable with distribution function $F(x)$ and finite moment of order r. Then the rth partial moment about x is defined as

$$\alpha_r(x) = E(X-x)_+^r = \int_x^\infty (t-x)^r dF(t) \tag{1.1}$$

where

$$(X-x)_+ = \begin{cases} X-x, & \text{if } X \geq x \\ 0 & \text{if } X < x. \end{cases}$$

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The random variable $(X-x)_+$ is interpreted as the residual age in the context of life-length studies (Lin 2003) and the first two moments of (1.1) are extensively employed in actuarial studies for the analysis of risks (Denuit 2002). When X represents the income of an individual and x is the tax exemption level $(X-x)_+$ represents the taxable income, (1.1) is quite useful in the assessment of income tax.

Chong (1977) has developed a characterization result using (1.1) for exponential and geometric distributions. Gupta and Gupta (1983) discussed general properties of (1.1). They proved that (1.1) determines the underlying distribution uniquely for any positive real r. The survival function $\bar{F}(x)$ of X can be written in terms of $\alpha_r(x)$ as (Navarro et al. 1998; Sunoj 2004)

$$\bar{F}(x) = \frac{(-1)^r}{r!} \frac{d^r \alpha_r(x)}{dx^r}.$$

Sunoj (2004), Gupta (2007) have discussed properties of (1.1) with respect to length biased and equilibrium distributions.

The properties and applications of (1.1) discussed in the above papers are studied using the distribution function. An alternative approach for modelling and analysis of statistical data is to use quantile function defined by

$$Q(u) = \inf_x \{x : F(x) \geq u\}, \quad 0 \leq u \leq 1$$

Many of the quantile functions used in applied work do not have tractable distribution function, see Ramberg and Schmeiser (1974), Freimer et al. (1998), van Staden and Loots (2009), Hankin and Lee (2006), Nair et al. (2011). In such cases, the distribution function has to be evaluated through numerical methods to find $u = F(x)$, for chosen values of $x = Q(u)$. This renders the analysis of the analytical properties of the distribution using definitions based on the distribution function difficult. Moreover, many of the concepts used in the present work are in terms of quantile functions. This motivates us to formulate (1.1) in terms of quantile functions. Such a formulation will provide alternative tools for the analysis of statistical data. In view of this, we discuss properties of (1.1) based on quantile functions in the context of risk and income analysis.

The text is organized as follows. In Sect. 2, we give basic properties of quantile version of partial moments. Distributional and geometric properties of the measures are discussed in Sect. 3. The quantile partial mean is studied as a measure of income inequality in Sect. 4. In Sect. 5, the measure is related to Bonferroni curve and Leinkuhler curve. Finally Sect. 6 provides various other applications of the measure.

2 Basic results

We assume that X is a non-negative random variable with absolutely continuous distribution function $F(x)$ and probability density function $f(x)$. When $F(x)$ is strictly increasing, the quantile function $Q(u)$ is the solution of $F(x) = u$ as $x = Q(u)$. We take $Q(0) = 0$ generally, and an adjustment has to be made in the results when $Q(0) > 0$. The mean of the distribution assumed to be finite, is

$$\mu = \int_0^1 Q(p)dp \tag{2.1}$$

which is same as $\int_0^1 (1-p)q(p)dp$, where $q(u) = \frac{dQ(u)}{du}$ is the quantile density function.

When $F(x)$ is strictly increasing, $f(x) > 0$ so that the quantile density function exists by virtue of the relations

$$f(Q(u))q(u) = 1.$$

Since the first two partial moments are generally in use, we confine the discussions to the cases $r = 1$ and $r = 2$ in (1.1). Setting $F(x) = u$ in (1.1)

$$P_r(u) = \alpha_r(Q(u)) = \int_u^1 (Q(p) - Q(u))^r dp. \tag{2.2}$$

When $r = 1$,

$$\begin{aligned} P_1(u) &= \int_u^1 (Q(p) - Q(u))dp \\ &= \int_u^1 (1-p)q(p)dp. \end{aligned} \tag{2.3}$$

Also

$$P_1(u) = \int_u^1 Q(p)dp - (1-u)Q(u) \tag{2.4}$$

As $P_1(u)$ provide the expected loss in a stop-loss treaty with infinite cover as a function of priority, it is called the stop-loss transform in risk analysis. We also have

$$P_2(u) = \alpha_2(Q(u)) = \int_u^1 (Q(p) - Q(u))^2 dp. \tag{2.5}$$

The variance of $(x - X)_+$ becomes

$$V_+(u) = \int_u^1 (Q(p) - Q(u))^2 dp - P_1^2(u) \tag{2.6}$$

which on simplification gives

$$V_+(u) = \int_u^1 Q^2(p)dp - (Q(u) + P_1(u))^2 + uQ^2(u). \tag{2.7}$$

From (2.4),

$$P_1'(u) = -(1-u)q(u). \tag{2.8}$$

Differentiating (2.7) and simplifying with the help of (2.8), we obtain

$$V_+'(u) = \frac{2uP_1(u)P_1'(u)}{1-u}. \tag{2.9}$$

Thus $P_1(u)$ determines $V_+(u)$ as

$$\begin{aligned} V_+(u) &= -\int_u^1 \frac{2pP_1(p)P_1'(p)dp}{1-p} \\ &= \frac{2}{1-u}P_1^2(u) + \int_u^1 \frac{P_1^2(p)dp}{(1-p)^2}. \end{aligned} \tag{2.10}$$

Conversely, from (2.9), we get

$$P_1^2(u) = -\int_u^1 \frac{(1-p)V_+'(p)}{p}dp$$

showing that $V_+(u)$ determines $P_1(u)$ also. Expression of $P_1(u)$ for various distributions that appear in the sequel are presented in Table 1.

Theorem 2.1 *The expression $Q_1(u) = \mu - P_1(u)$ is the quantile function of a distribution on* $[0, \mu]$.

Proof Let $Q_1(u) = \mu - P_1(u) = \int_0^u (1-p)q(p)dp$. Then $Q_1(u)$ is continuous, strictly increasing with $Q_1(0) = 0$ ans $Q_1(1) = \mu$ and hence it is the quantile function of a random variable Z on $[0, \mu]$.

Example 2.1 When X is exponential with parameters λ

$$P_1(u) = \frac{1-u}{\lambda}, \quad \text{where } \mu = \frac{1}{\lambda}.$$

Thus $Q_1(u)$ is the quantile function of the uniform distribution on $[0, \frac{1}{\lambda}]$.

Table 1 Quantile stop-loss transforms of distributions

Distribution	Quantile function	$P_1(u)$
Exponential	$-\lambda^{-1}\log(1-u)$	$\lambda^{-1}(1-u)$
Generalized lambda	$\lambda_1+\lambda_2^{-1}(u^{\lambda_3}-(1-u)^{\lambda_4})$	$\frac{1}{\lambda_2}\left[\frac{\lambda_4}{1+\lambda_4}(1-u)^{1+\lambda_4} + \frac{1-u^{\lambda_3+1}}{1+\lambda_3}-(1-u)u^{\lambda_3}\right]$
generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$b(1-u)^{\frac{1}{a+1}}$
van Staden-Loots	$\lambda_1+\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}(u^{\lambda_4}-1) -\frac{\lambda_3}{\lambda_4}((1-u)^{\lambda_4}-1)\right]$	$\lambda_2\left[\frac{1-\lambda_3}{\lambda_4}\left(\frac{1-u^{1+\lambda_4}}{1+\lambda_4} -(1-u)u^{\lambda_4}\right)+\frac{\lambda_3}{1+\lambda_4}(1-u)^{1+\lambda_4}\right]$
Govindarajulu	$\sigma\left[(\beta+1)u^{\beta}-\beta u^{\beta+1}\right]$	$\frac{\sigma}{\beta+2}\left[2-(\beta+1)(\beta+2)u^{\beta} +2\beta(\beta+2)u^{\beta+1}-\beta(\beta+1)u^{\beta+2}\right]$
Half logistic	$\sigma\log\frac{1+u}{1-u}$	$2\sigma\log\frac{2}{1+u}$
Exponential geometric	$\frac{1}{\lambda}\log\left(\frac{1-pu}{1-u}\right)$	$\frac{1-p}{\lambda p}\log\left(\frac{1-pu}{1-p}\right)$
Linear hazard quantile	$(a+b)^{-1}\log\left(\frac{a+bu}{a(1+u)}\right)$	$\frac{1}{b}\log\left(\frac{a+b}{a+bu}\right)$
Power	$\alpha u^{\frac{1}{\beta}}$	$\alpha\left[1-u^{\frac{1}{\beta}}-(\beta+1)^{-1}\left(1-u^{1+\frac{1}{\beta}}\right)\right]$

Example 2.2 Suppose X has power distribution specified by

$$Q(u)=\alpha u^{\frac{1}{\beta}}$$

on $[0,\alpha]$. Then $\mu=\frac{\alpha\beta}{\beta+1}$,

$$P_1(u)=\frac{\alpha}{1+\beta}\left[\beta-(1+\beta)u^{\frac{1}{\beta}}+u^{\frac{1}{\beta}+1}\right]$$

$$\text{and}\quad Q_1(u)=\frac{\alpha\beta}{1+\beta}\left[\left(1+\frac{1}{\beta}\right)u^{\frac{1}{\beta}}-\frac{1}{\beta}u^{\frac{1}{\beta}+1}\right] \tag{2.11}$$

However, (2.11) is the quantile function of the Govindarajulu distribution with parameters $(\mu,\frac{1}{\beta})$ and support $[0,\mu]$. Note that in the first example $Q_1(u)$ has a tractable distribution function, while in the second it is not the case.

3 Properties

The stop-loss transform $P_1(u)$ can be used to describe the basic properties of the distribution like measure of location, dispersion, skewness and kurtosis. This is achieved by evaluating the first four L-moments of the distribution in terms of $P_1(u)$. The first four L-moments of X given in Hosking (1996) are

$$L_1 = \int_0^1 Q(u)du = \mu = P_1(0)$$

$$L_2 = \int_0^1 (2u-1)Q(u)du = \int_0^1 P_1(u)du$$

$$L_3 = \int_0^1 (6u^2-6u+1)Q(u)du = \int_0^1 (4u-1)P_1(u)du$$

and

$$L_4 = \int_0^1 \left(20u^3 - 30u^2 + 12u - 1\right) Q(u)du$$
$$= \int_0^1 \left(1 - 10u + 15u^2\right) P_1(u)du$$

Note that the above formulae for L_2, L_3 and L_4 are obtained in terms of $P_1(u)$ after integrating by parts the first expressions in each case then and substituting the identity (2.8). Of these L_2 is twice the mean difference

$$\Delta = 2\int_0^\infty F(x)(1-F(x))dx.$$

Thus L_2 is a measure of spread. The L-skewness and L-kurtosis are respectively given by

$$\tau_3 = \frac{L_3}{L_2} \text{ and } \tau_4 = \frac{L_4}{L_2}.$$

When $P_1(u)$ is known, the above relationships help in computing the distributional characteristics without using the quantile function or the distribution function.

In order to study the geometrical properties as well as for comparison purposes it is convenient to consider the scaled version of the stop-loss transform defined by

$$S(u) = \frac{P_1(u)}{\mu}. \tag{3.1}$$

From (2.8), the derivative of (3.1) becomes

$$S'(u) = -\frac{(1-u)q(u)}{\mu}$$

showing that $S(u)$ is decreasing. Since $P_1(0) = \mu$ and $P_1(1) = 0$, we see that $S(0) = 1$ and $S(1) = 0$. Hence the curve $S(u)$ lies in the unit square. In the exponential case $S(u) = 1 - u$ is the diagonal of the unit square joining the points (0, 1) and (1, 0). The curve can be concave or convex or partly concave and partly convex. For example when X is rescaled beta

$$S(u) = (1-u)^{\frac{1}{\alpha}+1}$$

is convex. On the otherhand, the Pareto distribution has

$$S(u) = (1-u)^{-\frac{1}{c}+d}$$

which is concave. The Govindarajulu distribution given in Table 1 with parameters σ and β ($\sigma, \beta > 0$) has mean $\mu = 2\sigma(\beta+2)^{-1}$ and hence

$$S(u) = \frac{\mu}{2}\left[2 - (\beta+1)(\beta+2)u^{\beta} + 2\beta(\beta+2)u^{\beta+1} - \beta(\beta+1)u^{\beta+2}\right].$$

Then

$$S''(u) = \frac{\mu\beta(\beta+1)^2(\beta+2)}{2}(1-u)\left(u - \frac{\beta-1}{\beta+1}\right)u^{\beta-2}.$$

It is easy to see that $S(u)$ is convex for $0 < \beta < 1$.

For $\beta > 1$, $S'(u)$ attains a minimum at $u = \frac{\beta-1}{\beta+1}$ so that $S(u)$ is concave in $(0, \frac{\beta-1}{\beta+1})$ and convex in $(\frac{\beta-1}{\beta+1}, 1)$.

Remark 3.1 The above result is in contrast with the behaviour of $\alpha_1(x)$. In fact from

$$\alpha_1(x) = \int_0^x \bar{F}(t)dt$$

we get

$$\alpha_1''(x) = f(x) > 0$$

indicating that $\alpha_1(x)$ is both decreasing and convex.

4 Measures of income inequality

In the context of income analysis $\alpha_1(x)(P_1(u))$ has an interpretation. Although poverty is studied mostly with aid of income distributions there is equal interest in knowing the level of affluence in a population. For instance, Sen (1988), Belzunce et al. (1998) have developed the methodology to analyze the the inequality incomes among the rich individuals and proposed indices for their measurement. When X represents the income of an individual and x, the level of income above which the individual is considered affluent., $\alpha_1(x)(P_1(u))$ represents the average residual income beyond the affluence level (beyond the 100 $(1-u)$% of the distribution of X.).

One popular measure of income inequality is Gini index given by

$$\begin{aligned} G &= \frac{2\Delta}{\mu}. \\ &= \frac{4}{\mu}\int_0^\infty F(x)(1-F(x))dx \end{aligned} \tag{4.1}$$

From the definition of second L-moment, we can write G as

$$G = \frac{L_2}{\mu}. \tag{4.2}$$

Since first L-moment being the mean, G is the L-coefficient of variation.

The area under the $S(u)$ curve is given by

$$\begin{aligned} \int_0^1 S(p)dp &= \frac{1}{\mu}\int_0^1\int_u^1 (1-p)q(p)dp \\ &= \frac{1}{\mu}\int_0^1 u(1-u)q(u)du \\ &= \frac{L_2}{\mu} = G. \end{aligned} \tag{4.3}$$

The above formula provides an alternative expression to evaluate the Gini index and to represent it as the area of the $S(u)$ curve especially in terms of quantile functions. As examples Tarsitano (2004), Haritha et al. (2008) proposed respectively the generalized lambda and the generalized Tukey lambda distributions as flexible and adaptive models of income. In the latter case,

$$S(u) = \frac{(1-u)\lambda_2}{\mu}\left[\frac{(1-u)^{\lambda_4}}{1+\lambda_4} + \frac{1-u^{\lambda_3+1}}{\lambda_3(1-\lambda_3)(1-u)} - \frac{\lambda^{\lambda_3}}{\lambda_3}\right].$$

Employing (4.3), the Gini index is

$$G = (\mu\lambda_2)^{-1}\left[\frac{1}{(\lambda_2+1)(\lambda_2+2)} + \frac{1}{(\lambda_2+1)\lambda_4}\right]$$

where

$$\mu = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4+1} - \frac{1}{\lambda_3+1}\right).$$

A second popular measure of income inequality is the Pietra index given by

$$T = \frac{E|X-\mu|}{2\mu}$$

which is

$$T = \mu^{-1} \int_0^{F(\mu)} (\mu - Q(p))dp. \tag{4.4}$$

We deduce from (2.4) that

$$P_1(u) = \mu - \int_0^u Q(p)dp - (1-u)Q(u)$$

and

$$\frac{1}{\mu} \int_0^u Q(p)dp = 1 - S(u) - \frac{(1-u)}{\mu} Q(u) \tag{4.5}$$

From (4.4)

$$T = F(\mu) - \frac{1}{\mu} \int_0^{F(\mu)} Q(p)dp. \tag{4.6}$$

From (4.5) and (4.6), we have

$$T = F(\mu) + S(F(\mu)) - 1 + 1 - F(\mu) = S(F(\mu)).$$

Notice that the mean deviation about the mean can also be written in terms of $P_1(u)$ as

$$E(|X-\mu|) = 2P_1(F(\mu)). \tag{4.7}$$

Another well known measure of income inequality is the Lorenz curve given by

$$L(u) = \frac{1}{\mu} \int_0^u Q(p)dp \tag{4.8}$$

which is predominantly used to study economic variables like income, wealth, land holdings etc. The curve $L(u)$ is increasing and convex with $L(0) = 0$, $L(1) = 1$ and

$L(u) \leq u$. The Gini index G is the area between the line of equality $L(u) = u$ and $L(u)$, while the Pietra index is the maximum vertical deviation between $L(u)$ and $L(u) = u$. Since $\mu L'(u) = Q(u)$, Eq. (4.5) gives relationship between $L(u)$ and $S(u)$ as

$$S(u) = 1 - L(u) - (1 - u)L'(u). \tag{4.9}$$

Taking (4.9) as a linear differential equation

$$L'(u) + \frac{L(u)}{1 - u} = \frac{1 - S(u)}{1 - u},$$

we have

$$L(u) = (1 - u) \int_0^u \frac{1 - S(p)}{(1 - p)^2} dp. \tag{4.10}$$

The Eqs. (4.9) and (4.10) reveal that $L(u)$ and $S(u)$ determine each other uniquely.

Even though (4.10) is a general identity connecting $L(u)$ and $S(u)$, we can have simple relationships between the two that characterize distributions. This is illustrated in the following theorem.

Theorem 4.1 *The relationship*

$$L(u) = A + Bu + CS(u) \tag{4.11}$$

holds for all u, if and only if X is distributed with

$$Q(u) = \mu B + \mu \frac{(C + A - BC)}{C} (1 - u)^{\frac{1}{C}}, \quad 0 \leq u \leq 1 \tag{4.12}$$

provided that $BC - A - C > 0$.

Proof Assume that (4.11) holds. Then

$$L'(u) + \frac{L(u)}{1 - u} = \frac{1}{1 - u} - \frac{L(u) - A - Bu}{C(1 - u)} \tag{4.13}$$

or

$$L'(u) + \frac{C + 1}{C(1 - u)} L(u) = \frac{C + A + Bu}{C}$$

Writing (4.13) as

$$\frac{d}{du} (1 - u)^{-\frac{C+1}{C}} L(u) = \frac{C + A + Bu}{C} (1 - u)^{-\frac{C+1}{C} - 1}$$

and integrating over $(0, u)$, we get

$$L(u) = \frac{C + A - BC}{C + 1} + Bu + \frac{BC - C - A}{C + 1}(1 - u)^{\frac{C+1}{C}}.$$

Since $Q(u) = \mu L'(u)$, we have

$$Q(u) = \mu B + \frac{(C + A - BC)}{C}\mu(1 - u)^{\frac{1}{C}}. \tag{4.14}$$

As $Q(u)$ has to be an increasing function

$$BC - A - C > 0.$$

The converse is obtained by direct calculation of $L(u)$ and $S(u)$ from (4.8) and (4.9).

Remark 4.1 Setting $B = 0$, we have

$$L(u) = A + CS(u)$$

and

$$\begin{aligned} Q(u) &= \frac{(C + A)\mu}{C}(1 - u)^{\frac{1}{C}} \\ &= \sigma(1 - u)^{-\alpha}, \quad C = -\frac{1}{\alpha}, \ \frac{(C + A)\mu}{C} = \sigma \end{aligned}$$

the quantile function of the Pareto I distribution which is basic in income analysis, where $\frac{C+A}{C} > 0$.

Remark 4.2 When $A + C = 0$, the left endpoint of the support is zero. In this case, setting $B = -\frac{1}{a}$ and $C = -\frac{a+1}{a}$, satisfying $B > 1$

$$Q(u) = \frac{\mu}{a}\left[(1 - u)^{-\frac{C}{C+1}} - 1\right]$$

the quantile function of the generalized Pareto model with parameters (a, μ). When $-1 < a < 0$ we have rescaled beta and Pareto II for $a > 0$. The exponential distribution is not a special case of (4.11), but has to be worked out separately as

$$L(u) = u + (1 - u)\log(1 - u).$$

5 Relationship to other curves

Another concept used to assess income inequality is the Bonferroni curve. It is defined in the orthogonal plane $(F(x), B_1(x))$ where

$$B_1(x) = \frac{\int_0^x tf(t)dt}{\mu F(x)} \tag{5.1}$$

The curve lies with in the unit square. We refer to Giorgi and Crescenzi (2001), Pundir et al. (2005) for a detailed study of $B_1(x)$. In terms of quantiles (5.1) become

$$B(u) = B_1(Q(u)) = \frac{\int_0^u Q(p)dp}{\mu u}. \tag{5.2}$$

Bonferroni curve is strictly increasing, but need not be concave or convex. If u is the proportion of units whose income does not exceed x, then $B(u)$ is the ratio between the average income of this group and the population mean. A main difference between $L(u)$ and $B(u)$ is that while the former represent fractions of total income, latter indicate relative income level. Since

$$B(u) = u^{-1}L(u). \tag{5.3}$$

$B(u)$ always be below $L(u)$. Also $B(u)$ determines the distribution of X uniquely through (upto a scale factor)

$$Q(u) = \mu(B(u) + uB'(u)).$$

From (2.4),

$$\begin{aligned} P_1(u) &= \mu - \int_0^u Q(p)dp - (1-u)Q(u) \\ &= \mu[1 - \beta(u) - u(1-u)B'(u)]. \end{aligned}$$

By virtue of (5.3), Eq. (4.10) provides an inversion formula

$$B(u) = \frac{1-u}{u}\int_0^u \frac{1-S(p)}{(1-p)^2}dp. \tag{5.4}$$

Characterizations by relationship between $S(u)$ and $B(u)$ are easily derived from those between $S(u)$ and $L(u)$. Hence no separate treatment of such problems is attempted here.

Our next object of comparison is the Leinkuhler curve which is closely related to he other curves described above. It is defined as

$$\begin{aligned} K(u) &= \frac{1}{\mu}\int_{1-u}^1 Q(p)dp = \frac{1}{\mu}\left[\int_0^1 Q(p)dp - \int_0^{1-u} Q(p)dp\right] \\ &= 1 - \frac{1}{\mu}\int_0^{1-u} Q(p)dp \end{aligned} \tag{5.5}$$

A general definition of the curve is given in Sarabia (2008) and methods of generating it is discussed in Sarabia et al. (2010). The curve is mainly used in studying concentration of bibliometric distribution in information sciences. It also find applications in economics as the plot of cumulative proportion of productivity against cumulative proportion of sources. The following relationships between the various curves are evident,

$$
\begin{aligned}
K(u) &= 1 - L(1-u) \\
K(u) &= 1 - (1-u)B(1-u) \\
\text{and} \quad K(u) &= 1 - u\int_0^{1-u} \frac{1-S(p)}{(1-p)^2}dp.
\end{aligned}
$$

Differentiating the last equation, we have

$$
S(u) = K(1-u) - (1-u)K'(1-u).
$$

The $K(u)$ curve is continuous, concave and increasing with $K(0) = 0$ and $K(1) = 1$. It determines the distribution through

$$
Q(u) = \mu K'(1-u) \tag{5.6}
$$

An interesting result originating from (5.6) is that $[\mu K^1(u)]^{-1}$ is the quantile function of $\frac{1}{X}$. This follows from the fact that

$$
\frac{1}{Q(1-u)} = \frac{1}{\mu K'(u)}.
$$

The area under the Leinkuhler curve becomes

$$
\begin{aligned}
K = \int_0^1 K(p)dp &= 1 - \frac{1}{\mu}\int_0^1\int_0^{1-u} Q(p)dp \\
&= 1 - \frac{1}{\mu}\int_0^1 (1-u)Q(u)du \\
&= 1 - \frac{1}{2u}\int_0^1 (2u-1)Q(u)du - \frac{1}{2} \\
&= \frac{1}{2}(1+G).
\end{aligned}
$$

Thus the Gini index is $G = 2K - 1$. The area K is same as that above the Lorenz curve. The Bonferroni index is defined us

$$B = 1 - \int_0^1 B(u)du$$

which is shown in Pundir et al. (2005) to have upper bound $\frac{1+G}{2}$. This gives $B \leq K$.

Finally we have the total time on test transform defined in terms of quantile function as Nair et al. (2008)

$$T(u) = \int_0^u (1-p)q(p)dp.$$

Obviously

$$S(u) = \mu - T(u) \tag{5.7}$$

The relationship between $T(u)$ and $L(u)$ with several properties in this connection have been discussed in Chandra and Singpurwalla (1981), Pham and Turkkan (1994). Various characterization results of probability distributions using $T(u)$ and applications of $T(u)$ in different areas can easily be converted in terms of $S(u)$ using the identity (5.7).

6 Discussion

As revealed through our discussions, $\alpha_1(x)(P_1(u))$ can be used for the modelling and analysis of lifetime data. Since $P_1(u)$ determines the underlying distribution uniquely, various functional forms of $P_1(u)$ enable us to develop characterizations of lifetime distributions. The measure $P_1(u)$ can be related to well known basic reliability concepts such as hazard quantile function and mean residual quantile function introduced in Nair et al. (2008). Accordingly the proposed measure can provide alternative definitions for ageing criteria in reliability theory. The work in these directions will be reported in a separate paper. The relationship with other curves used in different contexts, makes the curve of $P_1(u)$ also as a useful alternative in such situations, with appropriate interpretations.

In this context some remarks about the stochastic ordering of stop-loss transform seems to be in order.

When X and Y are non negative random variables with finite expectations, X is said to be smaller than Y in increasing convex order. $X \leq_{\text{icx}} Y$ if and only if

$$\int_x^\infty \bar{F}_X(t)dt \leq \int_x^\infty \bar{F}_Y(t)dt, \quad \text{for all } x. \tag{6.1}$$

or equivalently

$$\int_{u}^{1} Q_X(p)dp \leq \int_{u}^{1} Q_Y(p)dp. \tag{6.2}$$

It may be easy to see if $\alpha_X(x)$ and $\alpha_Y(x)$ are stop-loss transforms of X and Y, the dominance of Y over X, denoted by $X \leq_{\text{SL}} Y$, is defined as $\alpha_X(x) \leq \alpha_Y(x)$ for all x. Hence

$$X \leq_{\text{SL}} Y \Leftrightarrow X \leq_{\text{icx}} Y.$$

In terms of quantile functions we can define X to be smaller than Y in stop-loss quantile function denoted by $X \leq_{\text{SLQ}} Y$ if and only if

$$\int_{u}^{1} (1-p)q_X(p)dp \leq \int_{u}^{1} (1-p)q_Y(p)dp. \tag{6.3}$$

Note that, by integration by parts (5.3) becomes identical to

$$\int_{u}^{1} Q_X(p)dp - (1-u)Q_X(u) \leq \int_{u}^{1} Q_Y(p)dp - (1-u)Q_Y(u)$$

which is same as X is smaller than Y in excess wealth order, $X \leq_{\text{ew}} Y$ studied in (Shaked and Shanthikumar, 2007, p.165). Thus $X \leq_{\text{SLQ}} Y \Leftrightarrow X \leq_{\text{ew}} Y$.

When the lower end of the support of X does not exceed that of Y, $X \leq_{\text{ew}} Y \Leftrightarrow X \leq_{\text{icx}} Y$. Thus the $\leq_{\text{SLQ}}$ order is stronger than the $\leq_{\text{SL}}$ order. Since the two stop-loss transform orders are equivalent to two well known orders, their properties and relationships with other orders can be obtained from Shaked and Shanthikumar (2007).

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