Some Problems in Graph Theory

STUDIES ON FUZZY GRAPHS

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By

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Certificate

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Chapter 1

INTRODUCTION

1.1 Fuzzy Set Theory – A Mathematical Model for Uncertainty

Most of our traditional tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. Precision assumes that parameters of a model represent exactly either our perception of the phenomenon modeled or the features of the real system that has been modeled. Now, as the complexity of a system increases our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance becomes almost mutually exclusive characteristics. Moreover in constructing a model, we always attempt to maximize its usefulness. This aim is closely connected with the relationship among three key characteristics of every system model: complexity, credibility and uncertainty. Uncertainty has a pivotal role in any efforts to maximize the usefulness of system models. All traditional logic habitually assumes that precise symbols are being employed.

One of the meanings attributed to the term 'uncertainty' is "vagueness". That is, the difficulty of making sharp or precise distinction. This applies even to many terms used in our day to day life, such as X is 'tall', Y is 'beautiful', the sky is 'cloudy' etc. It is important to realize that this imprecision or vagueness that are characteristic of natural language does not necessarily imply a loss of accuracy or meaningfulness.

A mathematical frame work to describe this phenomena was suggested by Lotfi. A. Zadeh in his seminal paper entitled "Fuzzy Sets" [45]. The crisp set is defined in such a way as to dichotomize the individuals in some universe of discourse in to two groups: members and non members, whose logic relies entirely on the classical Aristotlian one, "A or not A". A sharp, unambiguous distinction exists between the members and non members of the class represented by the crisp set. But, many of the terms that we commonly use , such as 'tall' , 'beauty' etc. which are called 'linguistic variables', do not exhibit this characteristic. Kosko [23] in his book calls this as Mismatch problem: The world is gray but science is black and white. Infact, the fuzzy principle is that "Everything is a matter of degree". Thus, the membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of degree. Consequently, the underlying logic is the fuzzy logic: A and Not A.

A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set. Formally, a fuzzy subset of a set S is a map $\sigma : S \to [0, 1]$, called the membership function where the transition from membership to non membership is gradual rather than abrupt. Therefore it is natural to treat fuzzy set as a kind of continuously valued logic.

From the very appearance of Zadeh's significant paper, the following question was in the air. Is not fuzzy set theory, probability theory in disguise? The answer has always been an emphatic 'no'. The swamp water example mentioned in [2] clearly indicates the spirit. Another immediately apparent difference is that sum of probabilities on a finite universal set must be equal to 1, while there is no such requirement for membership grades. Aristotle's law always hold in probability theory. Though a probability density function can be used to design a membership function, the converse situation may not hold. Thus, probability theory and fuzzy set theory put together can lead to a 'generalized information theory'.

The capability of fuzzy sets to express gradual transition from membership to non membership and vice versa has a broad utility. It provides us not only with a meaningful and powerful representation of measurement of uncertainties, but also with a meaningful representation of vague concepts expressed in natural language. Because every crisp set is fuzzy but not conversely, the mathematical embedding of conventional set theory into fuzzy sets is as natural as the idea of embedding the real numbers into the complex plane. Thus, the idea of fuzziness is one of enrichment, not of replacement.

Since it is not easily acceptable to define a concept on the basis of subjective feelings, the degree of membership $\sigma(x)$ of x is some times interpreted as the fraction of a sufficiently large number of referees agreeing with the statement that x belongs to S.

Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic, topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks, planning etc. Consequently, fuzzy set theory has emerged as a potential area of interdisciplinary research.

Some of the books discussing these various themes are Bezdek and Pal [2], Lootsma [25], Morderson and Malik[28], Cornelius . T. Leondes[24] and Klir and Bo Yuan[21].

We shall now list below some basic definitions and results from[33],[35].

Let σ be a fuzzy subset of S. Then the sets $\sigma^t = \{x \in S : \sigma(x) \geq t\}$ $\forall\, t \in [0, 1]$, are called the t level sets and the set $\sigma^* = \{x \in S : \sigma(x) > 0\}$ is called the support of σ. Note that σ^t and σ^* are crisp sets.

Definition 1.1. Let σ and τ be two fuzzy subsets of S. Then

1. $\sigma \subseteq \tau$ *if* $\sigma(x) \leq \tau(x)$ $\forall x \in S$
2. $\sigma \subset \tau$ *if* $\sigma(x) \leq \tau(x)$ $\forall x \in S$ and $\exists$ atleast one $x \in S$ such that $\sigma(x) < \tau(x)$
3. $\sigma = \tau$ *if* $\sigma(x) = \tau(x)$ $\forall x \in S$

the restriction $\mu(x, y) \leq \sigma(x) \wedge \tau(y), \forall x \in S$ and $y \in T$ allows μ^t to be a relation from σ^t to τ^t, $\forall t \in [0,1]$ and μ^* to be a relation from σ^* to τ^*.

Definition 1.5. Let $\mu : S \times T \rightarrow [0,1]$ be a fuzzy relation from a fuzzy subset σ of S into a fuzzy subset τ of T and $\nu : T \times W \rightarrow [0,1]$ be a fuzzy relation from a fuzzy subset τ of T into a fuzzy subset ξ of W. Then $\mu \circ \nu : S \times W \rightarrow [0,1]$ defined by $\mu \circ \nu(x, z) = \vee\{\mu(x, y) \wedge \nu(y, z) : y \in T\}$ for all $x \in S, z \in W$, is called the composition of μ with ν.

Note. μ, σ, τ, ν and ξ be defined as in Definition 1.5. Then $\mu \circ \nu$ is a fuzzy relation from σ into ξ.

In the rest of the discussion we consider μ and ν to be fuzzy relations on a fuzzy subset σ of S. It is quite natural to represent a fuzzy relation in the form of a matrix. The composition operation reveals that $\mu \circ \nu$ can be computed similar to matrix multiplication, where the addition is replaced by $\vee$ and multiplication by $\wedge$. Since composition is associative, we use the notation μ^2 to denote $\mu \circ \mu$ and μ^k to denote $\mu^{k-1} \circ \mu$, k > 1.

Definition1.6. $\mu^{\infty}(x, y) = \vee\{\mu^k(x, y) : k = 1,2,3........\}, \forall x, y \in S.$

Now it is convenient to define $\mu^0(x, y) = 0$ if $x \neq y$ and $\mu^0(x, x) = \sigma(x)$ $\forall x, y \in S$.

Definition1.7. Let μ be a fuzzy relation on σ. Then μ is called reflexive if

$\mu(x, x) = \sigma(x)\ \forall\ x \in S$.

If μ is reflexive, then it follows that $\mu(x, y) \leq \mu(x, x)$ and $\mu(y, x) \leq \mu(x, x)\ \forall x,y \in S$.

Theorem 1.1. Let μ and ν be fuzzy relations on a fuzzy subset σ of S. Then the following properties hold .

1. If μ is reflexive, $\nu \subseteq \nu \circ \mu$ and $\nu \subseteq \mu \circ \nu$.
2. If μ is reflexive, $\mu \subseteq \mu^2$.
3. If μ is reflexive, $\mu^0 \subseteq \mu \subseteq \mu^2 \subseteq \mu^3 \subseteq \dots\dots\dots\dots \subseteq \mu^\infty$.
4. If μ is reflexive, $\mu^0(x, x) = \mu(x, x) = \mu^2(x, x) = \dots\dots\dots\dots\dots\dots = \mu^\infty(x, x) = \sigma(x)$ $\forall\ x \in S$.
5. If μ and ν are reflexive, so is $\mu \circ \nu$ and $\nu \circ \mu$.
6. If μ is reflexive, then μ^t is a reflexive relation on $\sigma^t\ \forall\ t \in [0, 1]$.

Definition 1.8. A fuzzy relation is called symmetric if $\mu(x, y) = \mu(y, x),\ \forall x, y \in S$.

Theorem 1.2. Let μ and ν be fuzzy relations on a fuzzy subset σ of S. Then the following properties hold.

1. If μ and ν are symmetric, then $\mu \circ \nu$ is symmetric if and only if $\mu \circ \nu = \nu \circ \mu$.
2. If μ is symmetric, then so is every power of μ .

3. If μ is symmetric, then μ^t is a symmetric relation on σ^t $\forall\, t \in [0, 1]$.

Definition 1.9. A fuzzy relation μ is transitive if $\mu^2 \subseteq \mu$.

It follows that μ^∞ is transitive for any fuzzy relation μ . The following are some of the properties of transitive fuzzy relation.

Theorem 1.3. Let μ , ν and π be fuzzy relations on a fuzzy subset σ of S. Then the following properties hold.

1. If μ is transitive and $\pi \subseteq \mu$, $\nu \subseteq \mu$, then $\pi \circ \nu \subseteq \mu$.
2. If μ is transitive, then so is every power of μ.
3. If μ is transitive, ν is reflexive and $\nu \subseteq \mu$, then $\mu \circ \nu = \nu \circ \mu = \mu$.
4. If μ is reflexive and transitive, then $\mu^2 = \mu$.
5. If μ is reflexive and transitive, then $\mu^0 \subseteq \mu = \mu^2 = \mu^3 = \dots\dots\dots\dots = \mu^\infty$.
6. If μ and ν are transitive and $\mu \circ \nu = \nu \circ \mu$, then $\mu \circ \nu$ is transitive.
7. If μ is symmetric and transitive, then, $\mu(x, y) \leq \mu(x, x)$ and $\mu(y, x) \leq \mu(x, x)$ $\forall\, x, y \in S$.
8. If μ is transitive, then μ^t is a transitive relation on σ^t $\forall\, t \in [0, 1]$.

A fuzzy relation on S which is reflexive, symmetric and transitive is called a fuzzy equivalence relation on S.

More properties of fuzzy sets are in [20],[21], [47].

1.2 Theory of Fuzzy Graphs – Definitions and Basic Concepts

It is quite well known that graphs are simply models of relations. A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a 'Fuzzy Graph Model'.

Application of fuzzy relations are widespread and important; especially in the field of clustering analysis, neural networks, computer networks, pattern recognition, decision making and expert systems. In each of these, the basic mathematical structure is that of a fuzzy graph.

We know that a graph is a symmetric binary relation on a nonempty set V. Similarly, a fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. The first definition of a fuzzy graph was by Kaufmann[18] in 1973, based on Zadeh's fuzzy relations [46]. But it was Azriel Rosenfeld [35] who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975. During the same time R.T.Yeh and S.Y. Bang [44] have also introduced various connectedness concepts in fuzzy graphs.

Definition 1.10. Let V be a non empty set. A fuzzy graph is a pair of functions $G : (\sigma, \mu)$ where σ is a fuzzy subset of V and μ is a symmetric fuzzy relation on σ . i.e. $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ such that $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$ for all u, v *in* V.

We denote the underlying (crisp) graph of $G : (\sigma, \mu)$ by $G^* :(\sigma^*, \mu^*)$ where σ^* is referred to as the (nonempty) set V of nodes and $\mu^* = E \subseteq V \times V$. Note that the crisp graph (V, E) is a special case of a fuzzy graph with each vertex and edge of (V,E) having degree of membership 1. We need not consider loops and we assume that μ is reflexive. Also, the underlying set V is assumed to be finite and σ can be chosen in any manner so as to satisfy the definition of a fuzzy graph in all the examples.

The basic definitions and properties that follow are from [33], [35].

Definition 1.11. The fuzzy graph $H: (\tau, \nu)$ is called a partial fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau \subseteq \sigma$ and $\nu \subseteq \mu$. In particular, we call $H: (\tau, \nu)$ a fuzzy subgraph of $G : (\sigma, \mu)$ if $\tau(u) = \sigma(u)\ \forall u \in \tau^*$ *and* $\nu(u, v) = \mu(u, v)\ \forall (u, v) \in \nu^*$.

For any threshold t, $0 \le t \le 1$, $\sigma^t = \{ u \in V : \sigma(u) \ge t \}$ and $\mu^t = \{ (u,v) \in V \times V : \mu(u,v) \ge t \}$. Since $\mu(u,v) \le \sigma(u) \wedge \sigma(v)$ for all u,v in V we have $\mu^t \subseteq \sigma^t \times \sigma^t$, so that (σ^t, μ^t) is a graph with vertex set σ^t and edge set μ^t for $t \in [0,1]$.

Note. Let G : (σ, μ) be a fuzzy graph. If $0 \le t_1 \le t_2 \le 1$, then $(\sigma^{t_2}, \mu^{t_2})$ is a subgraph of $(\sigma^{t_1}, \mu^{t_1})$

Note. Let $H : (\tau, \nu)$ be a partial fuzzy subgraph of G: (σ, μ). For any threshold $0 \le t \le 1$, (τ^t, ν^t) is a subgraph of (σ^t, μ^t).

Definition 1.12. For any fuzzy subset τ of V such that $\tau \subseteq \sigma$, the partial fuzzy subgraph of (σ, μ) induced by τ is the maximal partial fuzzy subgraph of (σ, μ) that has fuzzy nodeset τ. This is the partial fuzzy subgraph (τ, ν) where

$\tau(u, v) = \tau(u) \wedge \tau(v) \wedge \mu(u\ v)$, *for all* $u, v \in V$.

Definition 1.13. The fuzzy graph $H : (\tau, \nu)$ is called a fuzzy subgraph of $G : (\sigma, \mu)$ induced by P if $P \subseteq V$, $\tau(u) = \sigma(u)\ \forall u \in P$ and $\nu(u, v) = \mu(u, v)\ \forall u, v \in P$.

Definition 1.14. A partial fuzzy subgraph (τ, ν) spans the fuzzy graph (σ, μ) if $\sigma = \tau$. In this case (τ, ν) is called a partial fuzzy spanning subgraph of (σ, μ).

Next we introduce the concept of a fuzzy spanning subgraph as a special case of partial fuzzy spanning subgraph.

Definition 1.15. A fuzzy subgraph (τ, ν) spans the fuzzy graph (σ, μ) if $\sigma = \tau$

and $\nu(u,v) = \begin{cases} \mu(u,v) \, if (u,v) \in \nu^* \\ 0 \, , otherwise. \end{cases}$

In this case we call (τ, ν), a fuzzy spanning subgraph of $G : (\sigma, \mu)$.

The following examples illustrate these basic concepts.

Example 1.1.

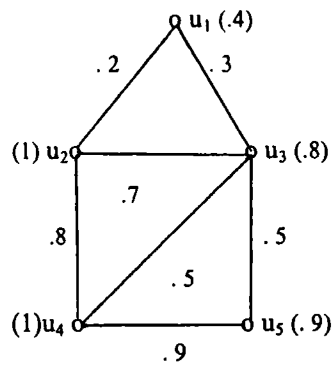


Fig.1.1a

A fuzzy graph G $:(\sigma, \mu)$

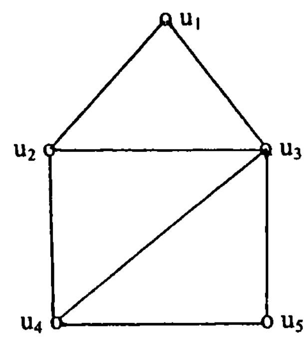


Fig1.1b

(crisp) graph G* $:(\sigma^*, \mu^*)$

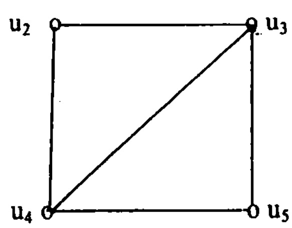


Fig.1.1c

$G' : (\sigma', \mu')$ where $t = 0.5$

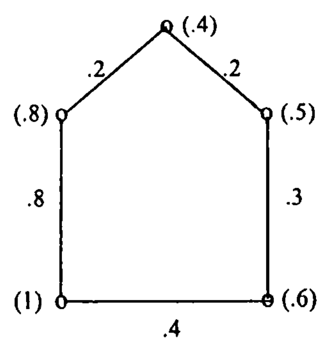


Fig.1.1d

A partial fuzzy subgraph of G

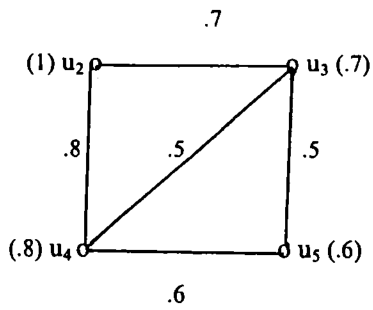


Fig.1.1e

A partial fuzzy subgraph induced by τ where $\tau(u_2) = 1$, $\tau(u_3) = 0.7$, $\tau(u_4) = 0.8$ and $\tau(u_5) = 0.6$

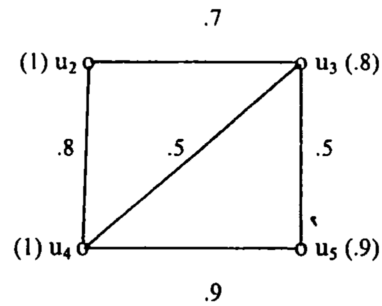
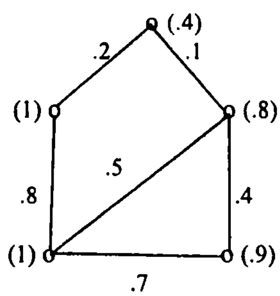


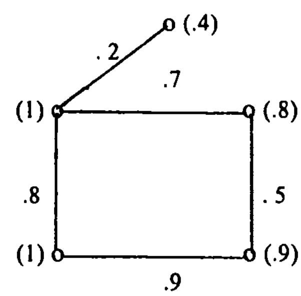
Fig.1.1f

The fuzzy subgraph induced by P where $P = \{ u_2, u_3, u_4, u_5 \}$



Figl.1g

A partial fuzzy spanning subgraph of G



Figl.1h

A fuzzy spanning subgraph of G

Yeh R.T. and S.Y. Bang [44]have extended the definition of degree of a node as follows. Let $G : (\sigma, \mu)$ be a fuzzy graph. Degree of a node v is defined to be $d(v) = \sum_{v \neq u} \mu(u,v)$. The minimum degree of G is $\delta(G) = \min_{v \in V} \{d(v)\}$ and the maximum degree of G is $\Delta(G) = \max_{v \in V} \{d(v)\}$.

From the above definition and from the symmetry of the fuzzy relation, we have,

$$\sum_{v \in V} \deg v = 2\sum_{v \neq u} \mu(u,v).$$

Definition1.16. A fuzzy graph G : (σ, μ) is strong if

$$\mu(u,v) = \sigma(u) \wedge \sigma(v) \forall (u,v) \in \mu^*$$

and is complete if $\mu(u,v) = \sigma(u) \wedge \sigma(v) \forall u,v \in \sigma^*$.

Note that every complete fuzzy graph is strong but not conversely. Also if $G : (\sigma, \mu)$ is a complete fuzzy graph then $G^* : (\sigma^*, \mu^*)$ is a complete graph.

Example 1.2.

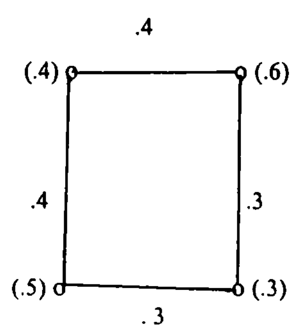


Fig.1.2a

A strong fuzzy graph

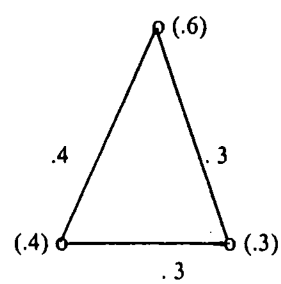


Fig.1.2b

A complete fuzzy graph

Definition 1.17. A path P in a fuzzy graph $G : (\sigma, \mu)$ is a sequence of distinct nodes u_0 ,u_1 ,..........., u_n such that $\mu(u_{i-1}, u_i) > 0,\ 1 \leq i \leq n$.

Here $n \geq 1$ is called the length of the path P. A single node u may also be considered as a path. In this case the path is of length 0. The consecutive pairs (u_{i-1}, u_i) are called arcs of the path. We call P a cycle if $u_0 = u_n$ and $n \geq 3$.

Definition 1.18. The strength of a path P is defined as $\overset{n}{\underset{i=1}{\wedge}} \mu(u_{i-1}, u_i)$.

In other words, the strength of a path is defined to be the degree of membership of a weakest arc of the path. If the path has length 0, it is convenient to define its strength to be $\sigma(u_0)$.

Next we have the concept of a strongest path in a fuzzy graph which plays an important role in the structure of fuzzy graphs.

Definition 1.19. A strongest path joining any two nodes u and v is that path which has strength $\mu^{\infty}(u, v)$ and $\mu^{\infty}(u, v)$ is called the strength of connectedness between u and v.

Example 1.3. In Example 1.1(a), a strongest path joining u_2 and u_5 is the path $P : u_2, u_4, u_5$ with $\mu^{\infty}(u_2, u_5) = 0.8$.

Also , $\mu^{\infty}(u_1, u_2) = \mu^{\infty}(u_1, u_3) = \mu^{\infty}(u_1, u_4) = \mu^{\infty}(u_1, u_5) = 0.3$,

$\mu^{\infty}(u_2, u_3) = 0.7$, $\mu^{\infty}(u_2, u_4) = 0.8$, $\mu^{\infty}(u_3, u_4) = 0.7 = \mu^{\infty}(u_3, u_5)$ and $\mu^{\infty}(u_4, u_5) = 0.9$.

Definition 1.20. A fuzzy graph $G : (\sigma, \mu)$ is connected if any two nodes are joined by a path. Maximal connected partial subgraphs are call ed components.

Note. A fuzzy graph $G : (\sigma, \mu)$ is connected if and only if $\mu^{\infty}(u, v) > 0 \ \forall u, v \in V$. Also, in a (crisp) graph each path is a strongest path with strength 1.

Definition 1.21. A maximum spanning tree of a connected fuzzy graph $G : (\sigma, \mu)$ is a fuzzy spanning subgraph $T: (\sigma, \nu)$, such that T^* is a tree, and for which $\sum_{u \neq v} \nu(u,v)$ is maximum.

Analogous to minimum spanning tree algorithm for crisp graphs, an algorithm to obtain a maximum spanning tree of a connected fuzzy graph is given in [4]. Note that the strength of the unique u-v path in T gives the strength of connectedness between u and v for all u,v. Also if $G : (\sigma, \mu)$ is such that G^* is a tree, then T is G itself . In example 1.1a, a maximum spanning tree is given in Fig. 1.3.

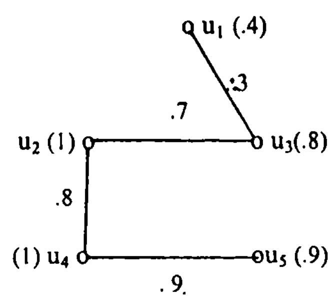


Fig.1.3

The notions of bridge , cutnode, tree, block and metric are extended to fuzzy graphs as follows.

Definition 1.22. An arc (u, v) is a fuzzy bridge of $G : (\sigma, \mu)$ if the deletion of (u, v) reduces the strength of connectedness between some pair of nodes.

Equivalently, (u, v) is a fuzzy bridge if and only if there are nodes x, y such that (u, v) is an arc of every strongest x - y path.

Definition 1.23. A node is a fuzzy cutnode of $G : (\sigma, \mu)$ if removal of it reduces the strength of connectedness between some other pair of nodes.

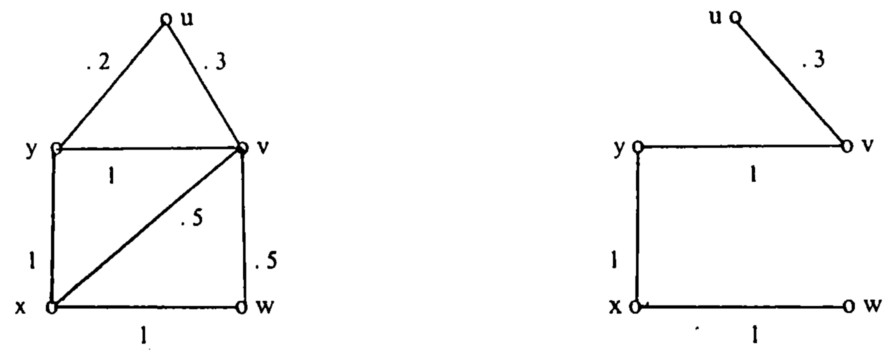
Equivalently, w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest $u - v$ path.

Example1.4. In Fig.1.1a, (u_1, u_3), (u_3, u_2), (u_2, u_4) and (u_4, u_5) are the fuzzy bridges and u_2, u_3, u_4 are the fuzzy cutnodes of $G : (\sigma, \mu)$.

Definition 1.24. A connected fuzzy graph $G : (\sigma, \mu)$ is a fuzzy tree if it has a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree, where for all arcs (u, v) not in F $\mu(u, v) < \nu^{\infty}(u, v)$.

Equivalently, there is a path in F between u and v whose strength exceeds $\mu(u, v)$ for all (u, v) not in F. Note that if G is such that G^* is a tree then F is G itself.

Example 1.5



Fuzzy tree $G : (\sigma, \mu)$ Fig.1.4 Spanning subgraph $F: (\sigma, \nu)$

Here $\mu(u, y) = .2 < .3 = \nu^{\infty}(u, y)$, $\mu(v, w) = .5 < 1 = \nu^{\infty}(v, w)$ *and*

$\mu(v, x) = .5 < 1 = \nu^{\infty}(v, x)$.

Definition 1.25. Let $G : (\sigma, \mu)$ be a fuzzy graph such that G^* is a cycle. Then G is called a fuzzy cycle if it has more than one weakest arc.

Definition 1.26. A connected fuzzy graph $G : (\sigma, \mu)$ with no fuzzy cutnodes is called a block.

In [35] it was observed that blocks in fuzzy graphs may have fuzzy bridges. In the following Example, G_1 is a block without fuzzy bridge and G_2 is a block with fuzzy bridges, (u_1, u_2) and (u_3, u_4).

G_1 Fig.1.5. G_2

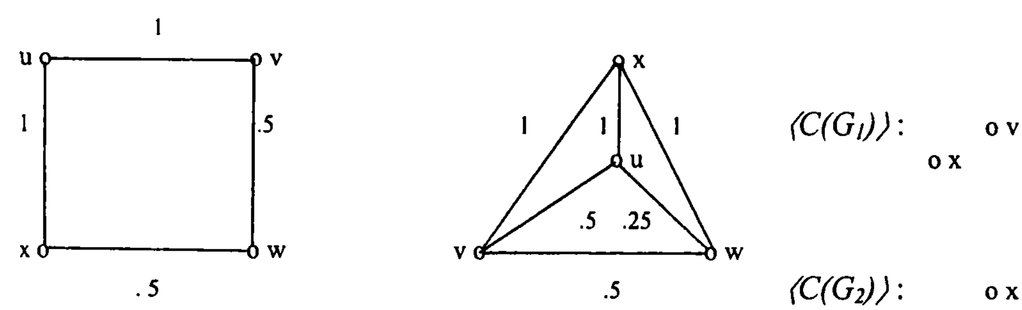
Definition 1.27. The μ - distance $\delta(u, v)$ is the smallest μ-length of any u-v path, where the μ - length of a path $P: u_0, u_1, \ldots\ldots\ldots, u_n$ is $\ell(P) = \sum_{i=1}^{n} \frac{1}{\mu(u_{i-1}, u_i)}$.

If $n = 0$, then define $\ell(P) = 0$.

Note. In a connected fuzzy graph G, $\delta(u, v)$ is a metric.

Based on this metric, Bhattacharya [3] has defined the concepts of eccentricity and center in fuzzy graphs. The eccentricity $e(v)$ of a node v in a connected fuzzy graph G is max $\delta(u, v)$ for all u in G. The radius $r(G)$ is the minimum eccentricity of the nodes, the diameter $d(G)$ is the maximum eccentricity. A node v is a central node if $e(v) = r(G)$. We call $\langle C(G) \rangle = H : (\tau, \nu)$, the fuzzy subgraph of $G : (\sigma, \mu)$ induced by the central nodes of G, the center of G.

Example : In the following fuzzy graph G_1 (Fig 1.6), $\delta(u, w) = 3$, $\delta(v, x) = 2$, $e(u) = e(w) = 3$, $e(x) = e(v) = 2$, $r(G_1) = 2$ and $d(G_1) = 3$. For G_2, $\delta(u, v) = 2$, $\delta(v, x) = 1$, $e(u) = e(v) = e(w) = 2$, $e(x) = 1$, $r(G_2) = 1$ and $d(G_2) = 2$.



G_1 Fig. 1. 6 G_2

The concept of isomorphism of two fuzzy graphs has been defined in [3,5]. Consider the fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ with $\sigma_1{}^* = V_1$ and $\sigma_2{}^* = V_2$.

Definition 1.28. An isomorphism between two fuzzy graphs G_1 and G_2 is a bijective map $h : V_1 \to V_2$ that satisfies

$$\sigma_1(u) = \sigma_2 h(u) \quad \forall u \in V_1 and$$
$$\mu_1(u,v) = \mu_2(h(u),h(v)) \forall u,v \in V_1 and \;\; we \;\; write G_1 \approx G_2.$$

An automorphism of G is an isomorphism of G with itself.

The operations on fuzzy graphs such as union, join, cartesian product and composition of graphs has been defined in [30] . In the following definitions an arc between two nodes u and v is denoted by uv rather than (u, v), because in the cartesian product of two graphs, a node of the graph itself is an ordered pair.

Definition 1.29. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $G_1{}^* : (V_1, E_1)$ and $G_2{}^* : (V_2, E_2)$ with $V_1 \cap V_2 = \phi$ and let $G^* = G_1{}^* \cup G_2{}^* = (V_1 \cup V_2, E_1 \cup E_2)$ be the union of $G_1{}^*$ and $G_2{}^*$. Then the union of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ defined by

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u) \text{ if } u \in V_1 - V_2 \\ \sigma_2(u) \text{ if } u \in V_2 - V_1 \end{cases} \quad and$$

$$(\mu_1 \cup \mu_2)(uv) = \begin{cases} \mu_1(uv) \text{ if } uv \in X_1 - X_2 \\ \mu_2(uv) \text{ if } uv \in X_2 - X_1 \end{cases}$$

Definition 1.30. Consider the join $G^* = G_1^* + G_2^* = (V_1 \cup V_2, E_1 \cup E_2 \cup E')$ of graphs where E' is the set of all arcs joining the nodes of V_1 and V_2 where we assume that $V_1 \cap V_2 = \phi$. Then the join of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G = G_1 + G_2$: $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ defined by

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u) \;\; \forall \; u \in V_1 \cup V_2 \quad and$$

$$(\mu_1 + \mu_2)(uv) = \begin{cases} (\mu_1 \cup \mu_2)(uv) \text{ if } uv \in E_1 \cup E_2 \text{ and} \\ \sigma_1(u) \wedge \sigma_2(v) \text{ if } uv \in E'. \end{cases}$$

Definition 1.31. Let $G^* = G_1^* \times G_2^* = (V, E^{ll})$ be the cartesian product of two graphs where $V = V_1 \times V_2$ and $E^{ll} = \{ (u, u_2)(u, v_2) : u \in V_1, u_2v_2 \in E_2 \} \cup \{ (u_1, w)(v_1, w) : w \in V_2, u_1v_1 \in E_1 \}$. Then the cartesian product $G = G_1 \times G_2 : (\sigma_1 \times \sigma_2, \mu_1 \times \mu_2)$ is a fuzzy graph defined by

$$(\sigma_1 \times \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2) \quad \forall (u_1, u_2) \in V \text{ and}$$
$$\mu_1 \times \mu_2((u, u_2)(u, v_2)) = \sigma_1(u) \wedge \mu_2(u_2v_2) \quad \forall u \in V_1, \forall u_2v_2 \in E_2,$$
$$\mu_1 \times \mu_2((u_1, w)(v_1, w)) = \sigma_2(w) \wedge \mu_1(u_1v_1) \quad \forall w \in V_2, \forall u_1v_1 \in E_1.$$

Definition 1.32. Let $G^* = G_1^* \circ G_2^* = (V_1 \times V_2, E)$ be the composition of two graphs, where $E = \{ (u, u_2)(u, v_2) : u \in V_1, u_2v_2 \in E_2 \} \cup \{ (u_1, w)(v_1, w) : w \in V_2, u_1v_1 \in E_1 \} \cup \{ (u_1, u_2)(v_1, v_2) : u_1v_1 \in E_1, u_2 \neq v_2 \}$. Then the composition of fuzzy graphs

$G = G_1 \circ G_2 : (\sigma_1 \circ \sigma_2, \mu_1 \circ \mu_2)$ is a fuzzy graph defined by

$(\sigma_1 \circ \sigma_2)(u_1, u_2) = \sigma_1(u_1) \wedge \sigma_2(u_2), \forall (u_1, u_2) \in V_1 \times V_2$ *and*
$(\mu_1 \circ \mu_2)((u, u_2)(u, v_2)) = \sigma_1(u) \wedge \mu_2(u_2 v_2), \forall u \in V_1, \forall u_2 v_2 \in E_2;$
$(\mu_1 \circ \mu_2)((u_1, w)(v_1, w)) = \sigma_2(w) \wedge \mu_1(u_1 v_1), \forall w \in V_2, \forall u_1 v_1 \in E_1;$
$(\mu_1 \circ \mu_2)((u_1, u_2)(v_1, v_2)) = \sigma_2(u_2) \wedge \sigma_2(v_2) \wedge \mu_1(u_1 v_1), \forall (u_1, u_2)(v_1, v_2) \in E' - E''$
where $E'' = \{(u, u_2)(u, v_2) : u \in V_1, \forall u_2 v_2 \in E_2\} \cup \{(u_1, w)(v_1, w) : w \in V_2, u_1 v_1 \in E_1\}.$

1.3 Fuzzy Graph Theory – Survey of Results

After the pioneering work of A.Rosenfeld [35] and R.T.Yeh and S.Y. Bang [44] in 1975, where some basic fuzzy graph theoretic concepts and applications have been indicated, several authors have been finding deeper results, and fuzzy analogues of many other graph theoretic concepts. This include fuzzy trees [10], fuzzy line graphs[29], autmorphism of fuzzy graphs [5], fuzzy interval graphs [8], cycles and cocycles of fuzzy graphs [31] etc.

We shall list below some of the known results.

Theorem 1.4 [35] . The following statements are equivalent for an arc *(u, v)* of a fuzzy graph $G : (\sigma, \mu)$.

(1) *(u,v)* is a fuzzy bridge

(2) *(u,v)* is not a weakest arc of any cycle in *G*.

Theorem 1.5 [34]. An arc (a node) is a fuzzy bridge (a fuzzy cutnode) iff there exists a partition V of nodes into subsets U, W, and X such that all nodes $u \in U$ and $w \in W$, the arc (the node) is on every strongest u - w path.

Theorem 1.6 [31]. Let $G : (\sigma, \mu)$ be a fuzzy graph with $V = \{ v_1, v_2, \dots\dots\dots v_n\}$ and let C be the cycle $v_1, v_2, \dots\dots\dots v_n, v_1$. If $\mu^* \supseteq C$ and for every arc $(v_j, v_k) \in \mu^* - C$, $\mu(v_j, v_k) <$ Max $\{ \mu(v_i, v_{i+1}) : i = 1,2,3,\dots\dots\dots n \}$, where $v_{n+1} = v_1$, then either μ is a constant on C or G has a fuzzy bridge.

Theorem 1.7 [35] . Let G be a connected fuzzy graph. Then G is a fuzzy tree if and only if in any cycle of G, there is an arc (u,v) such that $\mu(u,v) < \mu'^{\infty}(u,v)$, where the prime denotes the deletion of the arc (u,v).

Theorem 1.8 [35] . Let G be a connected fuzzy graph. If there is atmost one strongest path between any two nodes of G, then G is a fuzzy tree.

Theorem 1.9 [35] . If G is a fuzzy tree then arcs of F are the fuzzy bridges of G.

Now, regarding the blocks in fuzzy graphs [35], if between every two nodes u,v of G there exist two strongest paths that are disjoint, then G is a block and the converse is not true.

Bhattacharya [3] has extended the definitions of eccentricity and center based on the metric in fuzzy graphs defined in [35], and the inequality $r(G) \leq d(G) \leq 2r(G)$ also has been proved.

Automorphism of fuzzy graphs has been studied by Bhattacharya and Bhutani and they have shown how to associate a fuzzy graph with a group as the group of automorphism of fuzzy graphs [3,5].

One can also attempt to compute $\mu^{\infty}(u, v)$ using the concept of fuzzy matrix A of a fuzzy graph $G : (\sigma, \mu)$ where the rows and columns are indexed by the set V of nodes and the (u, v) entry of A is $\mu(u, v)$ $\forall u \neq v$ and $\mu(u, u) = \sigma(u)$. The matrix product $A.A = A^2$ is defined where the usual addition and multiplication of real numbers are replaced by maximum and minimum respectively. Higher powers A^k are defined recursively. It can be shown that $\mu^i(u, v)$ is the (u,v) entry of A^i $\forall u, v$ and there exists some k such that $A^k = A^{k+1}$ where

$k =$ Max {length of $P(u,v)$: P is a shortest strongest u-v path} [4], [44]. An algorithm to find the connectedness matrix of a fuzzy graph is in [41].

Yeh and Bang's [44] approach for the study of fuzzy graphs were motivated by its applicability to pattern classification and clustering analysis. They worked more with the fuzzy matrix of a fuzzy graph, introduced concepts like vertex connectivity $\Omega(G)$, edge connectivity $\lambda(G)$ and established the fuzzy analogue of Whiteney's theorem. They also proved that for any three real numbers a, b, c such that $0 < a \leq b \leq c$, there exists a fuzzy graph G with $\Omega(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$. Techniques of fuzzy clustering analysis can also be found in [44].

The concepts of connectedness and acyclicity levels were introduced for fuzzy graphs [10] and several fuzzy tree definitions which are consistent with cut - level representations were given in [10]. Introducing the notion of fuzzy chordal graphs, Craine. W. L.[8] has obtained the fuzzy analogue of the Gilmore and Hoffman characterization of interval graphs and also that of Fulkerson and Gross.

J.N.Mordeson and P.S. Nair [30] have introduced the notions of union, join, cartesian product and composition of fuzzy graphs and have studied some basic properties.

Applications of fuzzy graphs to database theory[19], to problems concernig the group structure [40] and also to chemical structure research [43] are found in literature.

To expand the application base, the notion of fuzzy graphs have been generalized to fuzzy hypergraphs also[12], [13], [14], [15].

Zimmermann [47] has discussed some properties of fuzzy graphs. The book [33] by Mordeson and Nair entitled " Fuzzy graphs and Fuzzy hypergraphs "is an excellent source for research in fuzzy graphs and fuzzy hypergraphs.

Fuzzy graphs have also been discussed in [6], [7], [9], [11], [16], [22], [26], [27],[32], [39] and [42].

1.4 Summary of the Thesis

This thesis consists of five chapters including this introductory one. In this thesis an attempt to study more on the basic concepts in fuzzy graphs given by Rosenfeld [35] such as fuzzy bridges, fuzzy cutnodes, fuzzy trees, blocks and metric concepts in fuzzy graphs has been made. Also, we modify the definition of the complement of a fuzzy graph and some of its properties are studied.

In the second chapter we have studied in detail the notions of fuzzy bridges, fuzzy cutnodes and fuzzy trees and various interconnections. We call for convenience, an arc, and a node of $G : (\sigma, \mu)$, a bridge and a cutnode of $G : (\sigma, \mu)$ if they are the bridge and cutnode of G^* respectively. Note that a bridge and a cutnode of G^* is a fuzzy bridge and a fuzzy cut node of $G : (\sigma, \mu)$, respectively.

One can see that identification of fuzzy bridges and fuzzy cutnodes is not easy. We observe that if an arc (u, v) is a fuzzy bridge then it is the unique strongest u - v path and the converse holds only in fuzzy trees. Also if G^* is a cycle then all arcs of G except the weakest are fuzzy bridges.

Some significant differences from crisp theory are

(1) existence of a fuzzy bridge need not imply existence of a fuzzy cutnode.

(2) a complete fuzzy graph can have atmost one fuzzy bridge.

(3) there are fuzzy graphs with diametrical nodes, as fuzzy cutnodes (Chapter 4).

(4) a node can be a fuzzy cutnode of both G and its complement (Chapter 5).

Next we present a sufficient condition for a node to be a fuzzy cutnode as a common node of atleast two fuzzy bridges. This also becomes necessary in the cases when (1) G is a cycle and (2) G is a fuzzy tree.

Now, using the concept of maximum spanning tree we characterise fuzzy bridges and fuzzy cutnodes in connected fuzzy graphs as follows.

Theorem 1. An arc (u, v) is a fuzzy bridge of G if and only if (u, v) is in every maximum spanning tree T of G.

Corollary. If $G : (\sigma, \mu)$ is a connected fuzzy graph with $|V| = n$ then G has atmost $n-1$ fuzzy bridges.

Theorem 2. A node is a fuzzy cutnode of G if and only if it is an internal node of every maximum spanning tree T of G.

Corollary. Every fuzzy graph has atleast two nodes which are not fuzzy cutnodes.

In the second part of this chapter we concentrate on fuzzy trees.

Theorem 3. Let $G : (\sigma, \mu)$ be a fuzzy tree and $G^* \neq K_l$. Then G is not complete.

Theorem 4. If G is a fuzzy tree, then internal nodes of F are the fuzzy cutnodes of G.

Corollary. A fuzzy cutnode of a fuzzy tree is the common node of atleast two fuzzy bridges.

Next, using the concept of fuzzy bridges we characterize fuzzy trees as follows.

Theorem 5. $G : (\sigma, \mu)$ is a fuzzy tree if and only if the following are equivalent.

(1) An arc (u, v) is a fuzzy bridge

(2) $\mu^{\infty}(u, v) = \mu(u, v)$.

In this proof we establish that a maximum spanning tree T of G is the required fuzzy spanning subgraph F for G to be a fuzzy tree and that arcs of T are the fuzzy bridges of G, which leads to another characterization of fuzzy trees.

Theorem 6. A fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree.

Corollary . If $G : (\sigma, \mu)$ is a fuzzy tree with $|V| = n$, then G has $n - 1$ fuzzy bridges.

Mordeson [31] has defined a cycle C as a fuzzy cycle if it has more than one weakest arc and proved that a cycle is a fuzzy cycle iff it is not a fuzzy tree. We present a sufficient condition for a fuzzy graph G to be a fuzzy tree .

Theorem 7. Let $G : (\sigma, \mu)$ be a connected fuzzy graph with no fuzzy cycles. Then G is a fuzzy tree.

Third chapter deals with blocks in fuzzy graphs. We observe that block may have more than one fuzzy bridge and that no two fuzzy bridges in a block can have a common node. Also it follows that a complete fuzzy graph is a block.

Now, recall that when a fuzzy bridge is removed from a fuzzy graph, the strength of connectedness between some pair of nodes of G is reduced. We have some interesting observations regarding the reduction of strength of connectedness when G is a fuzzy tree or a block.

Theorem 8. If G is a fuzzy tree, then removal of any fuzzy bridge reduces the strength of connectedness between its end nodes and also between some other pair of nodes .

In the fourth chapter, we discuss some metric aspects of fuzzy graphs. We introduce the notion of a self centered fuzzy graph. We denote by $\langle C(G)\rangle$, the center of a connected fuzzy graph $G : (\sigma, \mu)$, the fuzzy sub graph induced by the central nodes of G. A connected fuzzy graph is self centered if $\langle C(G)\rangle$ is isomorphic to G.

Theorem 11. A connected fuzzy graph $G : (\sigma, \mu)$ is self centered if $\mu^{\infty}(u, v) = \mu(u, v)$ for all u, v in V and $r(G) = \dfrac{1}{\mu(u,v)}$ where $\mu(u, v)$ is least.

Corollary. A complete fuzzy graph is self centered and $r(G) = \dfrac{1}{\sigma(u)}$ where $\sigma(u)$ is least.

As a consequence, there exists self centered fuzzy graph of radius c for each real number $c > 0$. Also, for any two real numbers a, b such that $0 < a \leq b \leq 2a$, there exists a fuzzy graph G such that $r(G) = a$ and $d(G) = b$.

An obvious necessary condition for a fuzzy graph to be self centered is that each node is eccentric and examples are given to show that this is not sufficient.

Note that in the crisp case, cycles C_n are self centered with $r(C_n) = n/2$, if n is even and $r(C_n) = (n-1)/2$ if n is odd. We investigate this property in a fuzzy graph

$G : (\sigma, \mu)$ where G^* is a cycle and a sufficient condition for G to be self centered depending on various values of n is also obtained.

Analogous to Hedetniemi's construction in the crisp case we prove that every fuzzy graph H can be embedded as the central subgraph of a fuzzy graph G. Also if H is connected with diameter d, we construct G with $r(G) = d$, and $d(G) = 2d$.

A similar problem for fuzzy trees is also discussed. If H is a fuzzy tree with diameter d, then there exists a fuzzy tree G such that $\langle C(G) \rangle \approx H$. Note that even if H is not a fuzzy tree, this gives another construction of G such that $\langle C(G) \rangle \approx H$. It is noted that center of a fuzzy tree need not be a fuzzy tree.

In the last chapter we mention some drawbacks in the definition of complement of a fuzzy graph given in [30] and suggest a new definition. We study the properties of G and its complement $\overline{G}$ and prove that the automorphism group of G and $\overline{G}$ are identical. Distinct from crisp theory, we observe that a node can be a fuzzy cutnode of both G and $\overline{G}$.

If G $\approx \overline{G}$, then we call G, a selfcomplementary fuzzy graph and independent necessary and sufficient conditions for a fuzzy graph G to be self complementary are obtained.

Theorem 12. Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then

$$\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

Theorem 13. Let $G : (\sigma, \mu)$ be a fuzzy graph. If $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \forall\ u,v \in V$, then G is self complementary.

In the second part of this chapter, we study some operations on fuzzy graphs and prove that complement of the union two fuzzy graphs is the join of their complements and complement of the join of two fuzzy graphs is the union of their complements. Finally we discuss the complement of the composition of two fuzzy graphs.

The study of fuzzy graphs made in this thesis is far from being complete. We conclude the thesis with some suggestions for further study. We sincerely hope that the wide ranging applications of graph theory and the interdisciplinary nature of fuzzy set theory, if properly blended together could pave a way for a substantial growth of fuzzy graph theory. ∎

Chapter 2

FUZZY BRIDGES, FUZZY CUTNODES AND FUZZY TREES

The first part of this chapter deals with fuzzy bridges and fuzzy cutnodes . A sufficient condition for a node to be a fuzzy cutnode is obtained which becomes also necessary in the case of fuzzy trees. A characterization of fuzzy cutnode is obtained for fuzzy graphs G such that G^* is a cycle. Some significant differences from the crisp theory are pointed out. Note that , bridges and cutnodes of the crisp graph G^* are fuzzy bridges and fuzzy cutnodes of the fuzzy graph G respectively. Next we present a necessary condition for an arc (u, v) to be a fuzzy bridge and prove that this condition is also sufficient in the case of fuzzy trees. Also fuzzy bridges and fuzzy cutnodes are characterized using maximum spanning trees. Consequently it is shown that every fuzzy graph has atleast two nodes which are not fuzzy cutnodes and that a fuzzy graph with $|V| = n$ has atmost $n - 1$ fuzzy bridges.

In the second part we discuss fuzzy trees. The concept of maximum spanning tree plays a key role in the characterization of fuzzy trees. A fuzzy graph is a

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Some results of this chapter are included in the paper " A characterization of fuzzy trees", Information Sciences, 113, 293 – 300(1999) and also in the book "Fuzzy graphs and Fuzzy Hypergraphs", J.N.Mordeson and P.S.Nair, Physica Verlag(2000).

a fuzzy tree if and only if it has a unique maximum spanning tree. A sufficient condition for a fuzzy graph to be a fuzzy tree is also obtained using the concept of fuzzy cycle.

2.1 Fuzzy Bridges and Fuzzy Cutnodes

The notion of strength of connectedness plays a significant role in the structure of fuzzy graphs. When a fuzzy bridge (fuzzy cutnode) [Definitions 1.22 & 1.23] is removed from a fuzzy graph, the strength of connectedness between some pair of nodes is reduced rather than a disconnection as in the crisp case. Note that weakest arcs of cycles cannot be fuzzy bridges [Theorem 1.4] and it follows that if G is a fuzzy graph such that G^* is a cycle, then all arcs except the weakest are fuzzy bridges. Moreover we have,

Theorem 2.1. Let $G : (\sigma, \mu)$ be a fuzzy graph and let (u, v) be a fuzzy bridge of G. Then $\mu^{\infty}(u, v) = \mu(u, v)$.

Proof : Suppose that (u, v) is a fuzzy bridge and that $\mu^{\infty}(u, v)$ exceeds $\mu(u, v)$. Then there exists a strongest $u - v$ path with strength greater that $\mu(u, v)$ and all arcs of this strongest path have strength greater than $\mu(u, v)$. Now, this path together with the arc (u, v) forms a cycle in which (u, v) is the weakest arc, contradicting that (u, v) is a fuzzy bridge[Theorem 1.4].`

Remark 2.1. It follows from Theorems 1.4 and 2.1 that an arc (u, v) is a fuzzy bridge if and only if it is the unique strongest u - v path. However, the converse of Theorem 2.1

is not true. In the following fuzzy graph (Fig 2.1), (u_2, u_4) and (u_3, u_4) are the only fuzzy bridges and $\mu^{\infty}(u_1, u_2) = \mu(u_1, u_2) = 0.4 = \mu^{\infty}(u_1, u_3) = \mu(u_1, u_3)$, but (u_1, u_2) and (u_1, u_3) are not fuzzy bridges. The condition for the converse to be true is discussed in Theorem 2.11.

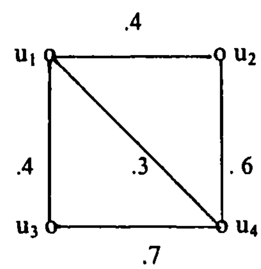


Fig. 2.1

We first observe that the identification of fuzzy cutnodes [Definition 1.23] is not easy. In the next theorem, we characterize fuzzy cutnodes in G such that G^* is a cycle and then present a sufficient condition for a node to be a fuzzy cutnode in the general case.

Theorem 2.2. Let $G : (\sigma, \mu)$ be a fuzzy graph such that G^* is a cycle. Then, a node is a fuzzy cutnode of G if and only if it is a common node of two fuzzy bridges.

Proof : Let w be a fuzzy cutnode of G. Then there exists u and v, distinct from w, such that w is on every strongest $u - v$ path which is unique since G^* is a cycle and it follows that all its arcs are fuzzy bridges. Thus w is a common node of two fuzzy bridges.

Conversely, let w be a common node of two fuzzy bridges (u, w) and (w, v). Then both (u, w) and (w, v) are not the weakest arcs of G[Theorem 1.4]. Also, the path from u to v not containing the arcs (u, w) and (w, v) has strength less than $\mu(u, w) \wedge \mu(w, v)$. Thus the strongest $u - v$ path is the path u, w, v and $\mu^{\infty}(u, v) = \mu(u, w) \wedge \mu(w, v)$. Hence w is a fuzzy cutnode.

In general we have,

Theorem 2.3. Let $G : (\sigma, \mu)$ be a fuzzy graph and let w be a common node of atleast two fuzzy bridges, then w is a fuzzy cutnode.

Proof : Let (u_1, w) and (w, u_2) be two fuzzy bridges. Then there exists some u, v such that (u_1, w) is on every strongest $u - v$ path. If w is distinct from u and v it follows that w is a fuzzy cutnode. Next, suppose one of v, u is w so that (u_1, w) is on every strongest $u - w$ path or (w, u_2) is on every strongest $w - v$ path. If possible let w be not a fuzzy cutnode. Then between every two nodes, distinct from w, there exists atleast one strongest path not containing w. In particular there exists atleast one strongest path P, joining u_1 and u_2 not containing w. This path together with (u_1, w) and (w, u_2) forms a cycle. Now we have the following two cases.

Case 1. u_1, w, u_2 is not a strongest path.

Then, clearly either (u_1, w) or (w, u_2) or both become the weakest arcs of the cycle which contradicts that (u_1, w) and (w, u_2) are fuzzy bridges.

Case 2. u_1, w, u_2 is also a strongest path joining u_1 to u_2.

Then, $\mu^{\infty}(u_1, u_2) = \mu(u_1, w) \wedge \mu(w, u_2)$, the strength of P. Thus , arcs of P are atleast as strong as $\mu(u_1, w)$ and $\mu(w, u_2)$ which implies that (u_1, w), (w, u_2) or both are the weakest arcs of the cycle , which again is a contradiction.

Remark 2.2. The condition in the above theorem is not necessary . In Fig 2.2 , w is the fuzzy cutnode ; (u, w) and (v, x) are the only fuzzy bridges, in Fig. 2.3, w is the fuzzy cutnode and (u, w) is the only fuzzy bridge and in Fig 2.4, w is the fuzzy cutnode and no arc is a fuzzy bridge. But the converse of Theorem 2.3 holds in fuzzy trees[Corollary to Theorem 2.10].

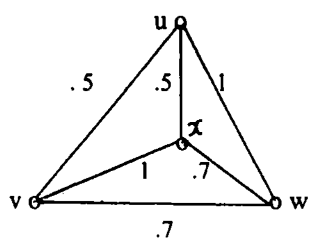


Fig.2.2

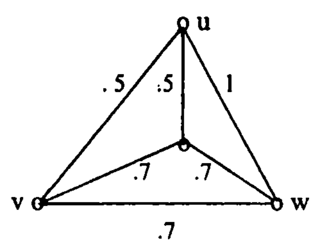


Fig.2.3

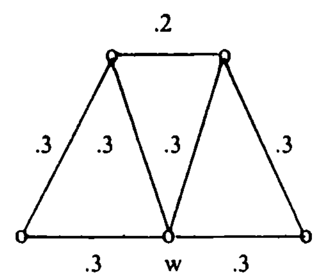


Fig.2.4

Remark 2.3. As distinct from crisp graph theory ,there are fuzzy graphs with fuzzy bridges and having no fuzzy cutnodes . In Fig. 2.5, (u, w) and (v, x) are the fuzzy bridges and no node is a fuzzy cutnode, where $0 < a < b \leq 1$.

Fig 2.5

Lemma 2.1[5] . If $G : (\sigma, \mu)$ is a complete fuzzy graph then $\mu^{\infty}(u, v) = \mu(u, v)$.

Lemma 2.2[5]. A complete fuzzy graph has no fuzzy cutnodes.

Remark 2.4. From lemma 2.1 we have in a complete fuzzy graph that each arc (u, v) is a strongest $u - v$ path . But the converse does not hold as we see in the Fig. 2.6. Also it follows from Lemma 2.2 that if in $G : (\sigma, \mu)$, $\mu^{\infty}(u, v) = \mu(u, v)$ for all u, v , then G has no fuzzy cutnodes.

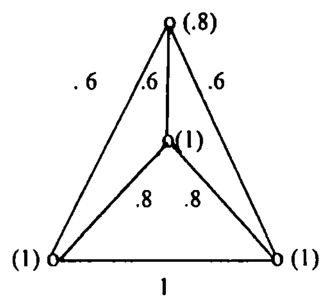


Fig 2.6

Note that a fuzzy graph with a fuzzy bridge need not have fuzzy cutnodes (Fig.2.5) and a complete fuzzy graph has no fuzzy cutnodes[Lemma 2.2]. But we have,

Theorem 2.4. A complete fuzzy graph has atmost one fuzzy bridge.

Proof : Let $G : (\sigma, \mu)$ be a complete fuzzy graph with $|V| = 3$. Then G can have atmost one fuzzy bridge by Theorem 2.3 and Lemma 2.2. Now , let $|V| \geq 4$ and let u_1, u_2, u_3 and u_4 be any four nodes of G. With out loss of generality , let u_1 be such that $\sigma(u_1)$ is least among $\sigma(u_i)$'s, $i = 1,2,3,4$. Then (u_1, u_2), (u_1, u_3) and (u_1, u_4) are not fuzzy bridges, they being the weakest arcs of some cycle in the fuzzy subgraph induced by u_1, u_2, u_3, u_4. Now the arcs (u_2, u_3), (u_2, u_4) and (u_3, u_4) are adjacent to each other and it follows that atmost one of them can be a fuzzy bridge .

Moreover we have,

Theorem 2.5. Let $G : (\sigma, \mu)$ be a complete fuzzy graph with $|V| = n$. Then G has a fuzzy bridge if and only if there exists an increasing sequence $\{ t_1, t_2, \ldots\ldots\ldots\ldots t_{n-1}, t_n \}$ such that $t_{n-2} < t_{n-1} \leq t_n$ where $t_i = \sigma(u_i)\ \forall\ i = 1,2,\ldots\ldots\ldots\ldots,n$. Also, the arc (u_{n-1}, u_n) is the fuzzy bridge of G.

Proof : Assume that $G : (\sigma, \mu)$ is a complete fuzzy graph and that G has a fuzzy bridge (u, v). Now, $\mu(u,v) = \sigma(u) \wedge \sigma(v)$. With out loss of generality, let

$\sigma(u) \leq \sigma(v)$, so that $\mu(u, v) = \sigma(u)$. Also, note that (u, v) is not a weakest arc of any cycle in G. Now required to prove that $\sigma(u) > \sigma(w)\ \forall\ w \neq v$. On the contrary assume that there is atleast one node $w \neq v$ such that $\sigma(u) \leq \sigma(w)$. Now consider the cycle C : u, v, w, u. Then $\mu(u, v) = \mu(u, w) = \sigma(u)$ and

$$\mu(v,w) = \begin{cases} \sigma(v), \text{ if } \sigma(u) = \sigma(v) \text{ or if } \sigma(u) < \sigma(v) \le \sigma(w) \\ \sigma(w) \text{ if } \sigma(u) < \sigma(w) < \sigma(v). \end{cases}$$

In either case the arc (u, v) becomes a weakest arc of the cycle which contradicts our assumption that (u, v) is a fuzzy bridge.

Conversely, let $t_1 \le t_2 \le .... \le t_{n-2} < t_{n-1} \le t_n$ and $t_i = \sigma(u_i)\ \forall\ i$.

Claim: Arc (u_{n-1}, u_n) is the fuzzy bridge of G.

Now, $\mu(u_{n-1}, u_n) = \sigma(u_{n-1}) \wedge \sigma(u_n) = \sigma(u_{n-1})$ and clearly by hypothesis, all other arcs of G will have strength strictly less than $\sigma(u_{n-1})$. Thus the arc (u_{n-1}, u_n) is not a weakest arc of any cycle in G and hence is the fuzzy bridge.

Example : G_1 and G_2 (Fig. 2.7) are complete fuzzy graphs where G_1 has no fuzzy bridges. The increasing sequence $\{ t_i \}$ in G_2 is { .2, .5, 1, 1} and (u_3, u_4) is the fuzzy bridge.

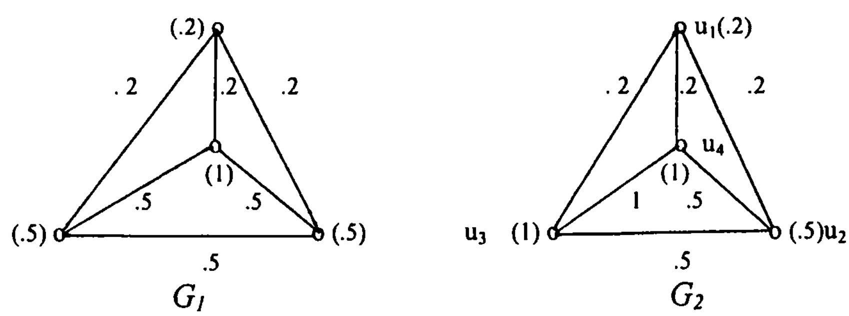


Fig.2.7

Now using the concept of maximum spanning tree of a fuzzy graph [Definition 1.21] we present a characterization of fuzzy bridge and fuzzy cutnode. Also , in a (crisp) graph G^*, note that each spanning tree is a maximum spanning tree . The following are characterizations of fuzzy bridge and fuzzy cutnode ,which are obvious in crisp case.

Theorem 2.6. An arc (u, v) is a fuzzy bridge of $G : (\sigma, \mu)$ if and only if (u, v) is in every maximum spanning tree of G.

Proof : Let (u, v) be a fuzzy bridge of G. Then arc (u, v) is the unique strongest $u - v$ path and hence is in every maximum spanning tree of G.

Conversely, let (u, v) be in every maximum spanning tree T of G and assume that (u, v) is not a fuzzy bridge. Then (u, v) is a weakest arc of some cycle in G and $\mu^{\infty}(u, v) > \mu(u, v)$, which implies that (u, v) is in no maximum spanning tree of G.

Remark 2.5. From Theorem 2.6, it follows that arcs not in T are not fuzzy bridges of G and we have,

Corollary. If $G : (\sigma, \mu)$ is a connected fuzzy graph with $|V| = n$ then G has atmost $n - 1$ fuzzy bridges.

Theorem 2.7. A node w is a fuzzy cutnode of $G : (\sigma, \mu)$ if and only if w is an internal node of every maximum spanning tree of G.

Proof : Let w be a fuzzy cutnode of G. Then there exist u, v distinct from w such that w is on every strongest $u - v$ path. Now each maximum spanning tree of G contains unique strongest $u - v$ path and hence w is an internal node of each maximum spanning tree of G.

Conversely, let w be an internal node of every maximum spanning tree. Let T be a maximum spanning tree and let (u, w) and (w, v) be arcs in T. Note that the path u, w, v is a strongest $u - v$ path in T. If possible assume that w is not a fuzzy cutnode. Then between every pair of nodes u, v there exist atleast one strongest
$u - v$ path not containing w. Consider one such $u - v$ path P which clearly contain arcs not in T. Now, with out loss of generality, let $\mu^{\infty}(u, v) = \mu(u, w)$ in T. Then arcs in P have strength $\geq \mu(u, w)$. Removal of (u, w) and adding P in T will result in another maximum spanning tree of G for which w is an end node, which contradicts our assumption.

Remark 2.6. It follows from Theorem 2.7 that the end nodes of a maximum spanning tree T of G are not fuzzy cutnodes of G. This results in the following corollary.

Corollary . Every fuzzy graph has atleast two nodes which are not fuzzy cutnodes of G.

However in Chapter 4 ,we see that there are fuzzy graphs with diametrical nodes, nodes which have maximum eccentricity, as fuzzy cutnodes, distinct from crisp graph theory.

2.2 Fuzzy Trees

Rosenfeld [35]has proved that if there exists a unique strongest path joining any two nodes in G then G is a fuzzy tree[Definition 1.24] and the converse does not hold. In Fig. 2.8, G is a fuzzy tree and $P_1 : x ,u, v, w, y$ & $P_2 : x, v, w, y$ are two strongest $x - y$ paths with $\mu^{\infty}(x, y) = .5$, of which P_2 is in F. Also, note that if G^* is a tree, then F is G itself and maximum spanning tree T of G is also G. In general we observe that a maximum spanning tree T of a fuzzy tree G is the required fuzzy spanning subgraph and that T is unique for a fuzzy tree[Theorem 2.12].

G Fig.2.8 *F*

Lemma 2.3[35] If (τ, ν) is a partial fuzzy subgraph of (σ, μ) . Then for all u, v,

$\nu^{\infty}(u, v) \leq \mu^{\infty}(u, v)$.

Theorem 2.8. If $G : (\sigma, \mu)$ is a fuzzy tree and $G^* : (\sigma^*, \mu^*)$ is not a tree, then there exists atleast one arc (u, v) in μ^* for which $\mu(u, v) < \mu^\infty(u, v)$.

Proof : If G is a fuzzy tree, then by definition there exists a fuzzy spanning subgraph $F : (\sigma, \nu)$, which is a tree and $\mu(u, v) < \nu^\infty(u, v)$ for all arcs (u, v) not in F. Also, $\nu^\infty(u, v) \leq \mu^\infty(u, v)$ by lemma2.3. Thus $\mu(u, v) < \mu^\infty(u, v)$ for all (u, v) not in F, and by hypothesis there exists atleast one arc (u, v) not in F, which completes the proof.

Theorem 2.9. Let $G : (\sigma, \mu)$ be a fuzzy tree and $G^* \neq K_1$. Then G is not complete.

Proof : If possible let G be a complete fuzzy graph. Then $\mu^\infty(u, v) = \mu(u, v)$ for all u, v [lemma 2.1]. Now G being a fuzzy tree, $\mu(u, v) < \nu^\infty(u, v)$, for all (u, v) not in F. Thus $\mu^\infty(u, v) < \nu^\infty(u, v)$, contradicting lemma 2.3.

Remark 2.7 Rosenfeld has proved that if G is a fuzzy tree, then arcs of F are the fuzzy bridges of G [Theorem 1. 9] and thus F is unique [35]. In the next theorem we characterize fuzzy cutnodes of a fuzzy tree.

Theorem 2.10. If G is a fuzzy tree, then internal nodes of F are the fuzzy cutnodes of G.

Proof : Let w be any node in G which is not an end node of F. Then by Theorem 1.9, it is the common node of atleast two arcs in F which are fuzzy bridges of G and by

Theorem 2.3, w is a fuzzy cutnode. Also, if w is an end node of F, then w is not a fuzzy cutnode; for if so , there exists u, v distinct from w such that w is on every strongest $u - v$ path and one such path certainly lies in F. But w being an end node of F , this is not possible.

With reference to the Remark 2.2,we have,

Corollary. A fuzzy cutnode of a fuzzy tree is the common node of atleast two fuzzy bridges.

We have seen that the condition that an arc (u, v) is a strongest $u - v$ path is only necessary for it to be a fuzzy bridge[Theorem 2.1]. In the case of fuzzy trees this condition becomes sufficient also and we have the following characterization of fuzzy trees.

Theorem 2.11. $G : (\sigma, \mu)$ is a fuzzy tree if and only if the following are equivalent.

(1) Arc (u, v) is a fuzzy bridge

(2) $\mu^{\infty}(u, v) = \mu(u, v)$.

Proof : Let $G : (\sigma, \mu)$ be a fuzzy tree and let (u, v) be a fuzzy bridge. Then $\mu^{\infty}(u, v) = \mu(u, v)$[Theorem2.1]. Now let (u, v) be an arc of G such that $\mu^{\infty}(u, v) = \mu(u, v)$. If G* is a tree, then clearly (u, v) is a fuzzy bridge ; otherwise, it follows from Theorem 2.8 that (u, v) is in F and it is a fuzzy bridge.[Theorem 1.9].

Conversely, assume that (1) ⇔ (2). Construct a maximum spanning tree $T:(\sigma, \nu)$ for G[4]. If (u, v) is in T, by an algorithm [4] , $\mu^{\infty}(u, v) = \mu(u, v)$ and hence (u, v) is a fuzzy bridge. Now these are the only fuzzy bridges of G ; for, if possible let (u', v') be a fuzzy bridge of G which is not in T. Consider a cycle C consisting of (u', v') and the unique u' - v' path in T . Now, the arcs of this u'- v' path being fuzzy bridges they are not weakest arcs of C and hence (u', v') must be the weakest arc of C and hence cannot be a fuzzy bridge[Theorem1.4].

Moreover, for all arcs (u', v') not in T, we have $\mu(u', v') < \nu^{\infty}(u', v')$; for, if possible let $\mu(u', v') \geq \nu^{\infty}(u', v')$. But $\mu(u', v') < \mu^{\infty}(u', v')$ (strict inequality holds, since (u', v') is not a fuzzy bridge). So, $\nu^{\infty}(u', v') < \mu^{\infty}(u', v')$ which gives a contradiction, since $\nu^{\infty}(u', v')$ is the strength of the unique u'- v' path in T and by an algorithm in [4] , $\mu^{\infty}(u', v') = \nu^{\infty}(u', v')$. Thus T is the required spanning subgraph F, which is a tree and hence G is a fuzzy tree.

Remark 2.8. It follows from the proof of Theorem 2.11 that arcs of the maximum spanning tree T are the fuzzy bridges of the fuzzy tree G and thus we have,

Theorem 2.12. A fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree.

Using Theorems 2.6 and 2.7, the theorems 1.9 and 2.10 follow from theorem 2.12.

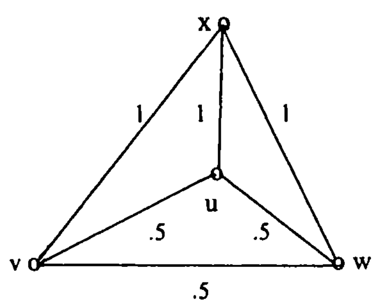
Note that if G is a fuzzy graph on n nodes then the maximum number of fuzzy bridges in G is $n - 1$ [Corollary to Theorem 2.6] and it follows from Remark 2.8 that a fuzzy tree on n nodes have $n - 1$ fuzzy bridges.

Mordeson [31] has defined a cycle C as a fuzzy cycle if it has more than one weakest arc and proved that a cycle is a fuzzy cycle iff it is not a fuzzy tree. In the following theorem , we consider the case of a general fuzzy graph.

Theorem 2.13. Let $G : (\sigma, \mu)$ be a connected fuzzy graph with no fuzzy cycles. Then G is a fuzzy tree.

Proof : If G^* has no fuzzy cycles then G^* is a tree and G is a fuzzy tree. So assume that G has cycles and by hypothesis no cycle is a fuzzy cycle. ie. every cycle in G will have exactly one weakest arc in it. Remove the weakest arc (say) e in a cycle C of G. If there are still cycles in the resulting fuzzy graph, repeat the process, which will eventually results in a fuzzy subgraph, which is a tree, and which is the required spanning subgraph F. Hence the theorem.

Remark 2.9. Converse of the above theorem is not true. In Fig.2.9, G is a fuzzy tree and u, v, w, u is a fuzzy cycle.



G

Fig. 2.9

Chapter 3

BLOCKS IN FUZZY GRAPHS

The concepts of blocks and trees are antonyms of each other with reference to bridges and cutnodes in the crisp case. But, it was observed by Rosenfeld that a block in a fuzzy graph may have a fuzzy bridge. We observe that a block in a fuzzy graph may have more than one fuzzy bridge and a comparative study of the reduction of the strength of connectedness when a fuzzy bridge is removed from a fuzzy tree and from a block is made. Rosenfeld has also pointed out that if between every two nodes there exist two strongest paths that are internally disjoint, then G is a block and that the converse does not hold. We observe that the converse is true for a block G with no fuzzy bridges and then blocks are characterized.

3.1 Blocks and Fuzzy Bridges

As pointed out in Remark 2.3, there are fuzzy graphs with fuzzy bridges and having no fuzzy cutnodes . So it is natural that there are blocks in fuzzy graphs [Definition 1.26] with fuzzy bridges. In Fig. 3.1, arcs (u, v) and (x, y) are fuzzy bridges(Also see Figs. 1.5 and 2.5).

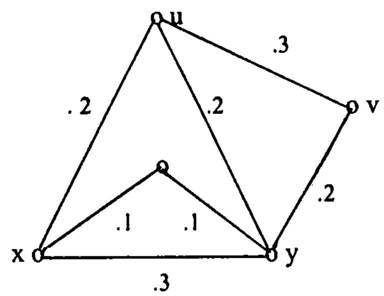


Fig.3.1

A block with fuzzy bridges.

Remark 3.1. No two fuzzy bridges in a block can have a common node[Theorem 2.3]. Also, a complete fuzzy graph is a block.

Recall the definition of a fuzzy bridge [Definition 1.22]. We have some interesting observations regarding the reduction of strength of connectedness depending on the structure of G. The cases when G is a fuzzy tree and G is a block are discussed now.

Theorem 3.1. If G is a fuzzy tree then removal of any fuzzy bridge reduces the strength of connectedness between its end nodes and also between some other pair of nodes .

Proof : Let $G : (\sigma, \mu)$ be a fuzzy tree and let (u, v) be a fuzzy bridge of G. Then (u, v) is an arc of the maximum spanning tree T of G [Theorems 2.6 & 2.12] and T has a unique strongest path joining every pair of nodes. So, removal of (u, v) reduces the strength of connectedness between some other pair of nodes u_i, v_i where u_i is adjacent to u and v_i is adjacent to v, if (u, v) is an internal arc of T, and $u_i = u$ or $v_i = v$ otherwise.

Theorem 3.2. If $G : (\sigma, \mu)$ is a block with atleast one fuzzy bridge then removal of any fuzzy bridge reduces the strength of connectedness only between its end nodes.

Proof : Let $G : (\sigma, \mu)$ be a block and (u, v) be a fuzzy bridge of G . Assume on the contrary that removal of (u, v) reduces the strength of connectedness between some other pair of nodes u_1 and v_1.

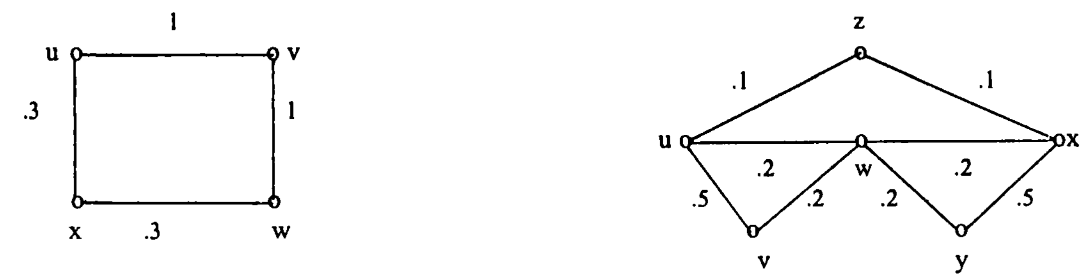
Case I : Both u_1 and v_1 are distinct from u and v.

Without loss of generality let $u_1 \neq u$ and, by assumption , every strongest u_1 - v_1 path contains the arc (u, v) . Then, clearly removal of either u or v reduces the strength of connectedness between u_1 and v_1, which shows that u and v are fuzzy cutnodes of G contradicting that G is a block.

Case II : One of u, v is u_1 or v_1 .

Let $u_1 = u$ and $v_1 \neq v$. Then as before removal of v reduces the strength of connectedness between u_1 and v_1 showing that v is a fuzzy cutnode of G and similarly if $v_1 = v$ and $u_1 \neq u$ then u becomes a fuzzy cutnode , both contradict the hypothesis that G is a block. Thus the only possibility is that $u_1 = u$ and $v_1 = v$ and hence the theorem.

Remark 3.2. The conditions in Theorems 3.1 and 3.2 are not sufficient . G_1 (Fig. 3.2) has two fuzzy bridges *(u, v)* and *(v, w)*, removal of each of which reduces the strength of connectedness between *u* and *w* also but G_1 is not a fuzzy tree. Now *(u, v)* and *(x, y)* are the fuzzy bridges in G_2 removal of each of which does not reduce the strength of connectedness between any pair of nodes other than that between their end nodes, but G_2 is not a block, as *w* is a fuzzy cutnode of G_2.



G_1 G_2

Fig. 3.2

3.2 Characterization of Blocks

The concept of strongest paths play a major role in the study of blocks. It is known that, if between every two nodes *u, v* of *G*, there exist two strongest paths that are internally disjoint , then *G* is a block but the converse does not hold[35]. Recall that, if an arc *(u, v)* is a fuzzy bridge then it is the unique strongest $u - v$ path[Remark 2.1].

Theorem 3.3 . The following statements are equivalent for a fuzzy graph $G : (\sigma, \mu)$.

1. G is a block.
2. Any two nodes u and v such that (u, v) is not a fuzzy bridge are joined by two internally disjoint strongest paths.
3. For every three distinct nodes of G, there is a strongest path joining any two of them not containing the third.

Proof :

$1 \Rightarrow 2$.

Let $G : (\sigma, \mu)$ be a block. Let u and v be any two nodes such that $\mu (u, v) \geq 0$ and (u, v) is not a fuzzy bridge. If there exists a unique strongest $u - v$ path of length ≥ 2, then the nodes on this path other than u and v are fuzzy cutnodes of G. Hence there exist more than one strongest $u - v$ path. If these strongest $u - v$ paths are internally disjoint then we are done. Note that all strongest $u - v$ paths do not have a common node , if so, that node becomes a fuzzy cutnode. So consider the following cases.

Case I

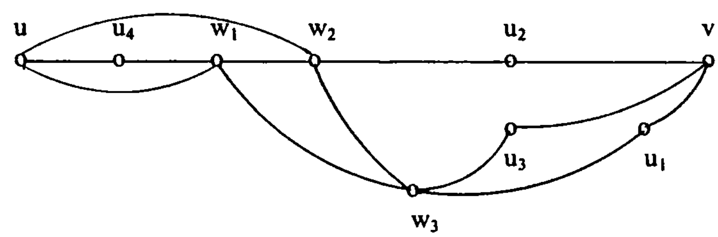


Fig 3.3

Let $P_1 : u - w_2 - w_3 - u_1 - v$, $P_2 : u - u_4 - w_1 - w_2 - u_2 - v$ and $P_3 : u - w_1 - w_3 - u_3 - v$ be strongest u - v paths. Let w_2 be the last common node of P_1 and P_2 (Fig.3.3). Then $u - w_2$ subpath in P_1 together with $w_2 - u_2 - v$ subpath in P_2 is a path (say) P disjoint from P_3.

Claim : P is a strongest $u - v$ path.

Let e_1 , e_2 , and e_3 be weakest arcs in P_1 , P_2 and P_3 respectively and let $\mu(e_1) = \mu(e_2) = \mu(e_3) = \mu^{\infty}(u, v)$. Then e_1 should be in u - w_2 subpath of P_1 or e_2 should be in $w_2 - u_2 - v$ subpath of P_2 ; for if not , then strength of $P > \mu^{\infty}(u, v)$, contradiction . Hence P is a strongest $u - v$ path.

Case II

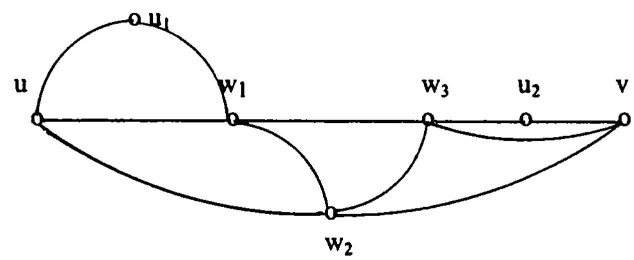


Fig 3.4

Let $P_1 : u - u_1 - w_1 - w_2 - v$, $P_2 : u - w_1 - w_3 - u_2 - v$ and $P_3 : u - w_2 - w_3 - v$ be strongest $u - v$ paths. Let w_2 be the first common node of P_1 and P_3. Then $u - w_2$ subpath in P_3 together with $w_2 - v$ subpath in P_1 is a path disjoint from P_2. As in Case I it can be proved that P is a strongest $u - v$ path.

Case III

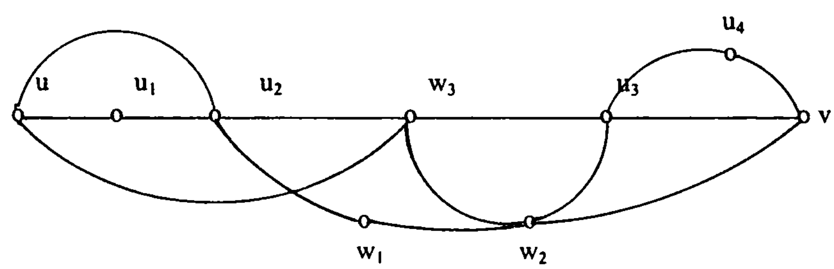


Fig. 3.5

Let $P_1 : u - u_2 - w_1 - w_2 - u_3 - u_4 - v$, $P_2 : u - u_1 - u_2 - w_3 - u_3 - v$ and $P_3 : u - w_1 - w_3 - w_2 - v$ be strongest $u - v$ paths. Let w_1 and w_2 be the first and last common nodes of P_1 and P_3 respectively. Then $u - w_1$ subpath in P_3 and $w_1 - w_2$ subpath in P_1 together with $w_2 - v$ subpath in P_3 will give a strongest $u - v$ path disjoint from P_2.

2 ⇒ 3

Let $u \neq v \neq w$ be any three nodes of *G*. Choose any two (say) *u* and *v*. If arc *(u, v)* is a fuzzy bridge it is the strongest $u - v$ path and 3 holds. So assume *(u, v)* is not a fuzzy bridge. Now by 2 , there exist two internally disjoint strongest $u - v$ paths and hence *w* cannot be in both.

3 ⇒ 1

If possible let *w* be a fuzzy cutnode of *G*. Then by definition there exist *u, v* different from *w* such that *w* is on every strongest $u - v$ path . But this contradicts 3.

Remark 3.3. We observe that the other equivalent conditions [1], [17] of a block cannot as such be extended to fuzzy graphs. ∎

Chapter 4

METRIC IN FUZZY GRAPHS

In this chapter some metric aspects of fuzzy graphs are discussed, focussing more on center problems. The notion of a self centered fuzzy graph is introduced and a sufficient condition for a for a fuzzy graph to be self centered is obtained, from which it also follows that complete fuzzy graphs are self centered. As a consequence, for each real number $c > 0$, there exists a self centered fuzzy graph of diameter c. Also, for any two real numbers a, b such that $a \leq b \leq 2a$, there exists a fuzzy graph G such that $r(G) = a$ and $d(G) = b$. Using the concept of eccentric nodes, a necessary condition for a fuzzy graph to be self centered is obtained and sufficient conditions for a fuzzy graph G such that G^* is a cycle to be self centered are also given.

Analogous to the Hedetniemi's construction in the crisp case, it is proved that every fuzzy graph H can be embedded as the central subgraph of a fuzzy graph G. In particular if H is connected with diameter d, the resulting fuzzy graph G has radius d and diameter $2d$. A similar problem for fuzzy trees is also discussed.

Some results of this chapter are included in the paper " Some metric aspects of fuzzy graphs", Proceedings of the Conference on Graph Connections, Allied Publishers, (1999), 111-114.

4.1 Self centered fuzzy graphs

Recall the definition of a metric in fuzzy graphs[Definition 1.27], which infact models the idea that "stronger" is the relation , "lesser" is the "distance". We call a fuzzy graph G self centered if $\langle C(G)\rangle$ is isomorphic to G [Definition 1.28]. The following theorem gives a sufficient condition for a connected fuzzy graph to be self centered.

Theorem 4.1 . A connected fuzzy graph $G : (\sigma, \mu)$ is self centered if

$\mu^{\infty}(u, v) = \mu(u, v)$ for all u, v in V and $r(G) = \dfrac{1}{\mu(u,v)}$ where $\mu(u, v)$ is least.

Proof : By assumption $G^*: (\sigma^*, \mu^*)$ is a complete graph. Also, the arc (u, v) is a strongest u - v path. It follows that weight of the weakest arc in any other strongest u - v path is $\mu(u, v)$ and hence μ - length of a strongest u - v path is atleast $\dfrac{1}{\mu(u,v)}$.

Now let $P : u = u_0, u_1, \ldots\ldots\ldots\ldots u_n = v$ be any u - v path which is not strongest. Then strength of P is strictly less than $\mu(u, v)$. So $\dfrac{1}{\mu(u,v)} < \dfrac{1}{Strength\ of\ P}$. Thus μ - length of P is strictly greater than $\dfrac{1}{\mu(u,v)}$ and hence $\delta(u, v) = \dfrac{1}{\mu(u,v)}$.

$$\text{Now } e(u) = max_v\ \delta(u, v) = \max_v \frac{1}{\mu(u,v)} = \frac{1}{Min_v\, \mu(u,v)}$$

Claim : $e(v_i) = e(v_j)$ *for all* $v_i \neq v_j$

If not, let $e(v_i) < e(v_j)$ (1)

and let u_i and u_j be such that $e(v_i) = \dfrac{1}{\mu(v_i, u_i)}$ and $e(v_j) = \dfrac{1}{\mu(v_j, u_j)}$. (Note that u_i may or may not be equal to u_j). Consider the path $P : v_j, v_i, u_j$. Then,

$\mu(v_j, v_i) \geq \mu(v_i, u_i)$ *and* $\mu(v_i, u_j) \geq \mu(v_i, u_i)$.

So, $\mu(v_j, v_i) \wedge \mu(v_i, u_j) \geq \mu(v_i, u_i) > \mu(v_j, u_j)$ by (1)

i.e.strength of $P > \mu(v_j, u_j)$.

i.e.strength of a v_j - u_j path exceeds $\mu(v_j, u_j)$ which contradicts our assumption that every arc is a strongest path. Interchanging i and j, a similar argument holds and thus $e(v_i) = e(v_j)$ for all $v_i \neq v_j$. Hence G is self centered.

Remark 4.1 The condition in the above theorem is not necessary for a fuzzy graph to be self centered as seen from the following example.

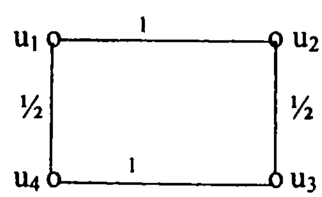


Fig. 4.1

In Fig 4.1, $e(u_i) = 3$, i = 1,2,3,4. But, $\mu^{\infty}(u_1, u_3) = \mu^{\infty}(u_2, u_4) = ½$ and

$\mu(u_1, u_3) = \mu(u_2, u_4) = 0.$

Note that in a complete fuzzy graph $\mu^{\infty}(u, v) = \mu(u, v)$ for all u, v and it follows that a complete fuzzy graph is self centered and $r(G) = \dfrac{1}{\sigma(u)}$ where $\sigma(u)$ is the least. We prove this result independently.

Theorem 4.2. A complete fuzzy graph is self centered and $r(G) = \dfrac{1}{\sigma(u)}$ where $\sigma(u)$ is the least.

Proof : Let $G :(\sigma, \mu)$ be a complete fuzzy graph. Choose some $u \in \sigma^*$ such that $\sigma(u)$ is least. Let $v \neq u$ and , $\mu(u , v) = \sigma(u) \Lambda \sigma(v) = \sigma(u)$. Also $\delta(u, v) = \dfrac{1}{\sigma(u)}$ for all v, since any other $u - v$ path will have one of its arc with strength $\sigma(u)$ and thus its μ-length will exceed $\dfrac{1}{\sigma(u)}$.

Now $\mu(v, w) = \sigma(v) \Lambda \sigma(w) \geq \sigma(u)$ and hence $\dfrac{1}{\mu(v,w)} \leq \dfrac{1}{\sigma(u)}$ for all v, w.

Now, if $P : v, v_1, w$ is a $v - w$ path of length 2 ,then

$$\ell(P) = \frac{1}{\mu(v,v_1)} + \frac{1}{\mu(v_1,w)} \leq \frac{2}{\sigma(u)}.$$

In general, any $v - w$ path of length n has μ - length $\leq n / \sigma(u)$ and thus

$$\delta(v,w) \le \frac{1}{\sigma(u)} \forall v,w.$$

Now, $e(u) = max\ \delta(u, v) = 1 / \sigma(u)$ and $e(v) = max\ \delta(v, w) = 1 / \sigma(u)$ for all $v \neq u$.

Thus G is self centered with $r(G) = \dfrac{1}{\sigma(u)}$.

Corollary. For each real number $c > 0$, there exists a self centered fuzzy graph with radius c.

Theorem 4.3. For any two real numbers a, b such that $0 < a \le b \le 2a$, there exists a fuzzy graph G such that $r(G) = a$, $d(G) = b$.

Proof : The proof is by construction. Let $V = \{u, v, w\}$ with $\mu(u, v) = 1/a$, $\mu(v, w) = 1/a$ and $\mu(u, w) = 1/b$. Then $e(v) = a$ and $e(u) = e(w) = b$.

A necessary condition for a fuzzy graph to be self centered is obtained in the following theorem using the concept of eccentric nodes. An eccentric node [17] of a node v, is a node v^* such that $e(v) = \delta(v, v^*)$.

Theorem 4.4. If $G: (\sigma, \mu)$ is a self centered fuzzy graph, then each node of G is eccentric.

Remark 4.2. This condition is not sufficient. In the following example each node is eccentric but G is not self centered.

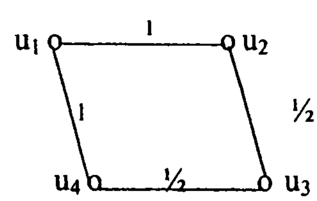


Fig. 4.2

$e(u_1) = 3 = e(u_3)$, $e(u_2) = 2 = e(u_4)$ and

$u_1{}^* = u_3,\quad u_2{}^* = u_4, u_3,\quad u_3{}^* = u_1,\quad u_4{}^* = u_2, u_3.$

Note that in the crisp case, cycles are self centered with $r(C_n) = n / 2$ if n is even and $r(C_n) = (n - 1) / 2$ if n is odd.

Theorem 4.5. Let G be a fuzzy graph such that $G^* \approx C_n$ with arcs $e_i = (u_i, u_{i+1})$, $i = 1,2,\ldots\ldots\ldots,n-1,n$. Let $0 < t \le s \le 1$. Then G is self centered if

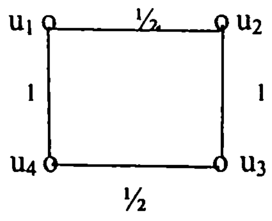
1. $\mu(e_i) = t,\ i = 1,3,5,\ldots\ldots,n-1$ and $\mu(e_i) = s,\ i = 2,4,6,\ldots\ldots,n$ when n is even
2. $\mu(e_i) = t,\ i = 1,3,5,\ldots\ldots,n-2, n$ and $\mu(e_i) = s,\ i = 2,4,6,\ldots\ldots,n-1$ when n is odd and $n = 4k-1$
3. $\mu(e_i) = t,\ i = 1,3,5,\ldots\ldots,n-2,$ and $\mu(e_i) = s,\ i = 2,4,6,\ldots\ldots,n-1,n$ when n is odd and $n = 4k+1$

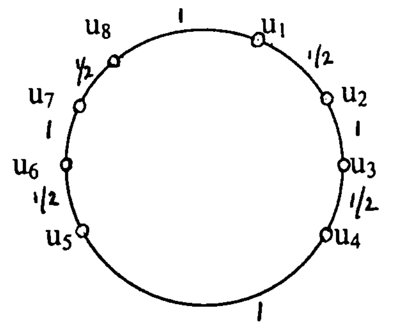
Also,

$$r(G) = \begin{cases} \frac{n}{4}\left[\frac{1}{t}+\frac{1}{s}\right], & n = 4k,\ k = 1,2,3,\ldots\ldots \\ \frac{1}{s}+\frac{n-2}{4}\left[\frac{1}{t}+\frac{1}{s}\right], & n = 4k+2,\ k = 2,4,6,\ldots\ldots \\ \frac{1}{t}+\frac{2}{s}+\frac{n-6}{4}\left[\frac{1}{t}+\frac{1}{s}\right], & n = 4k+2,\ k = 1,3,5,\ldots\ldots \\ \frac{n+1}{4}\left[\frac{1}{t}\right]+\frac{n-3}{4}\left[\frac{1}{s}\right], & n = 4k-1,\ k = 1,2,3,\ldots\ldots \\ \frac{n-1}{4}\left[\frac{1}{t}+\frac{1}{s}\right], & n = 4k+1,\ k = 1,2,3,\ldots\ldots \end{cases}$$

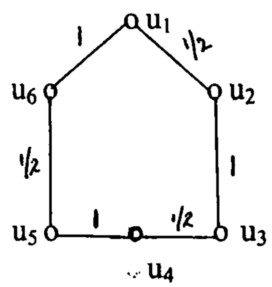
Illustration : Take $t = ½$ and $s = 1$.

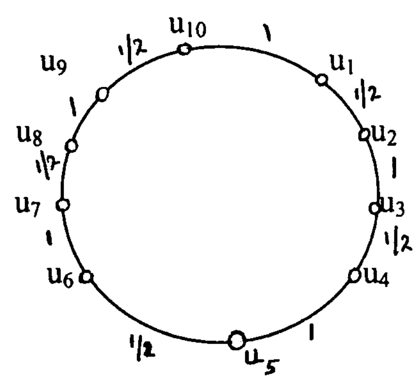
Case 1. n is even and $n = 4k$. $r(C_4) = 3$ and $r(C_8) = 6$.



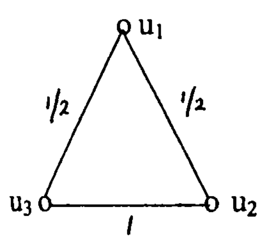


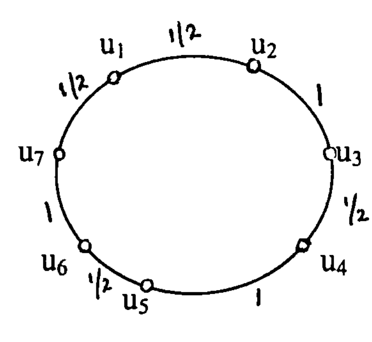
Case 2. n is even and $n = 4k + 2$ where $k = 1, 2$. $r(C_6) = 4$, $r(C_{10}) = 7$.



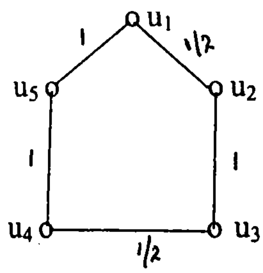


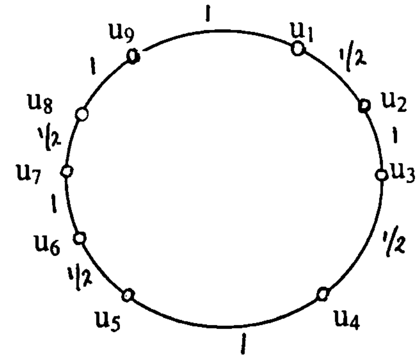
Case 3. n is odd and $n = 4k - 1$, where $k = 1,2$. $r(C_3) = 2$, $r(C_7) = 5$.





Case 4. n is odd and $n = 4k + 1$, where $k = 1,2$. $r(C_5) = 3$, $r(C_9) = 6$.





4.2 Two Constructions

In this section, we shall consider the construction of a fuzzy graph G such that $\langle C(G) \rangle \approx H$.

Theorem 4.6 Let $H = (\sigma', \mu')$ be a fuzzy graph . Then there exists a fuzzy graph $G : (\sigma, \mu)$ such that $\langle C(G) \rangle$ is isomorphic to H.

Proof : Let $c = \wedge\, \sigma'(u)$. Construct $G: (\sigma, \mu)$ as follows.

Take new nodes u_1 , u_2, v_1, v_2 and put $\sigma^* = \sigma'^* \cup \{ u_1 , u_2, v_1, v_2\}$, where

$\sigma = \sigma'$ for all u in H, $\mu = \mu'$ for all (u, v) in H and $\sigma(u_i) = \sigma(v_i) = t$ $(t \le c)$, $i = 1,2$; $\mu(u_1, u_2) = \mu(v_2 , v_1) = t$ and $\mu(u_2, w) = \mu(w , v_2) = t$ for all w in H. Then clearly $G : (\sigma, \mu)$ is a fuzzy graph and $e(w) = 2/t$ for all w in H, $e(u_2) = 3/t = e(v_2)$ and

$e(u_1) = 4/t = e(v_1)$. Thus $\langle C(G)\rangle \approx H$ with $r(G) = 2/t$ and $d(G) = 4/t$.

Theorem 4.7: Let $H = (\sigma', \mu')$ be a connected fuzzy graph with diameter d. Then there exist a connected fuzzy graph $G : (\sigma, \mu)$ such that $\langle C(G)\rangle$ is isomorphic to H. Also $r(G) = d$ and $d(G) = 2d$.

Proof : Construct $G : (\sigma, \mu)$ from H as follows: Take two nodes u and v with $\sigma(u) = \sigma(v) = 1/d$ and join all nodes of H to both u and v with $\mu(u, w) = \mu(v, w) = 1/d$ for all w in H.

Put $\sigma = \sigma'$ for all nodes in H and $\mu = \mu'$ for all arcs in H.

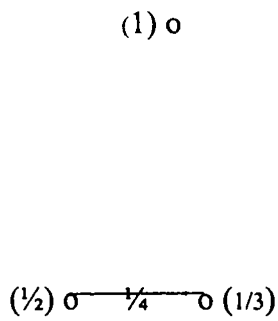
Claim: $G : (\sigma, \mu)$ is a fuzzy graph.

First note that $\sigma(u) \le \sigma(w)$ for all w in H; for, if possible let $\sigma(u) > \sigma(w)$ for atleast one node w in H. Then $1/d > \sigma(w)$.

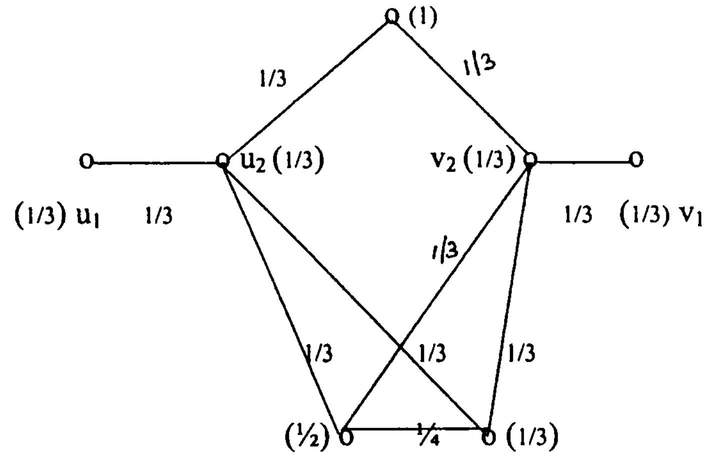
ie: $d < \dfrac{1}{\sigma(w)} \le \dfrac{1}{\mu(w,w')}$, where the last inequality holds for every w' in H, since H is a fuzzy graph. i.e. $\dfrac{1}{\mu(w,w')} > d$ for all w' in H, which contradicts that $d(H) = d$. Therefore $\sigma(u) \le \sigma(w)$ for all w in H and $\mu(u, w) = \sigma(u) \wedge \sigma(w) = \sigma(u) = 1/d$. Similarly, $\mu(v, w) = \sigma(v) \wedge \sigma(w) = 1/d$ for all w in H. Thus $G : (\sigma, \mu)$ is a fuzzy graph.

Also, $e(w) = d$ for all w in H and $e(u) = e(v) = \dfrac{1}{\mu(u,w)} + \dfrac{1}{\mu(w,v)} = 2d$. Thus $r(G)=d$, $d(G) = 2d$ and $\langle C(G)\rangle$ is isomorphic to H.

Example 1.



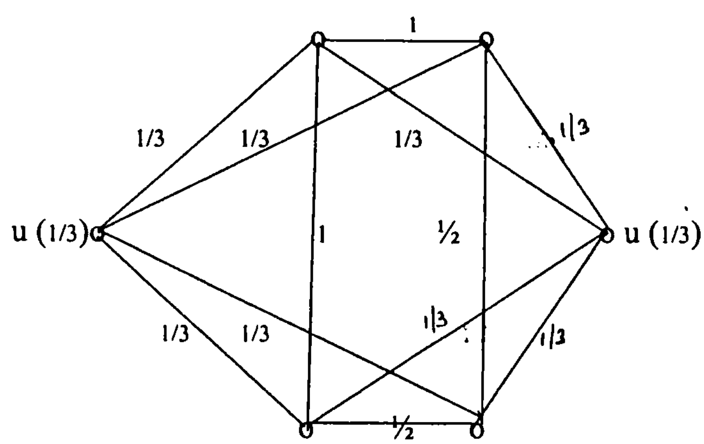
$H = (\sigma^{/}, \mu^{/})$



G : (σ, μ) where ⟨C(G)⟩ = H.

Example 2.

Let H = (σ^{1}, μ^{1}) be the fuzzy graph in Fig 4.2 where d (H) = 3. Then G: (σ, μ) such that ⟨C(G)⟩ = H and r(G) = d is constructed as follows.



4.3. Center of a Fuzzy Tree

It is well known that center of a tree is either K_1 or K_2. But, for a fuzzy tree it need not be so. (Fig. 4.3). In fact, there are self centered fuzzy trees. (Fig. 4.4).

$G:\ (\sigma, \mu)$ Fig.4.3 $\langle C(G) \rangle$

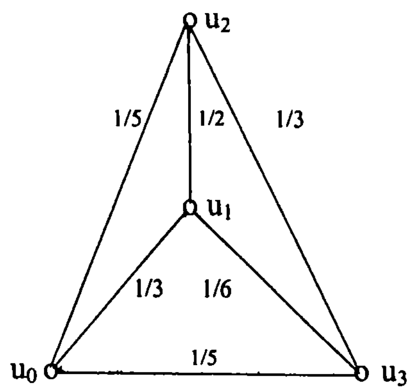


Fig 4.4

$G:\ (\sigma, \mu)$ is a self centered fuzzy tree with $e(u_i) = 5,\ i = 0,1,2,3.$

Theorem 4.8. Let $H : (\sigma', \mu')$ be a fuzzy tree with diameter d. Then there exists a fuzzy tree $G : (\sigma, \mu)$ such that $\langle C(G) \rangle$ is isomorphic to H.

Proof : Put $t = max \{\sigma'(w): \; w \in (\sigma')^*\}$. Construct $G : (\sigma, \mu)$ from $H: (\sigma', \mu')$ as follows:

Take two nodes u and v with $\sigma(u) = \sigma(v) = 1/d$ and join all nodes in H to both u and v. Let w and w' be any two nodes in H.

Put $\mu(u, w) = 1/d; \; \mu(w, v) = \dfrac{1}{d+\frac{1}{t}}$

$\mu(u, w') = \dfrac{1}{d+\frac{1}{t}}$; $\mu(w', v) = 1/d$ and put $\dfrac{1}{d+\frac{1}{t}}$ as the strength of all the other new arcs.

Also, put $\sigma = \sigma'$ for all nodes in H and $\mu = \mu'$ for all arcs in H.

Claim 1. $G : (\sigma, \mu)$ is a fuzzy graph.

As in the proof of Theorem 4. 7, $\sigma(u) \leq \sigma(w)$ and $\sigma(v) \leq \sigma(w)$ for all nodes w in H.

So $\mu(u, w) = \sigma(u) \wedge \sigma(w)$ and $\mu(w', v) = \sigma(w') \wedge \sigma(v)$. Also, since $\dfrac{1}{d+\frac{1}{t}} < \dfrac{1}{d}$,

$\mu(w, v) < \sigma(w) \wedge \sigma(v)$; $\mu(u, w') < \sigma(u) \wedge \sigma(w')$ and the inequality holds for all the other new arcs. Hence $G : (\sigma, \mu)$ is a fuzzy graph.

Claim 2. $(C(G))$ is isomorphic to H.

Note that $\mu(w_i, w_j) \leq t$ for every arc (w_i, w_j) in H. ie. $\dfrac{1}{t} \leq \dfrac{1}{\mu(w_i, w_j)}$ (2)

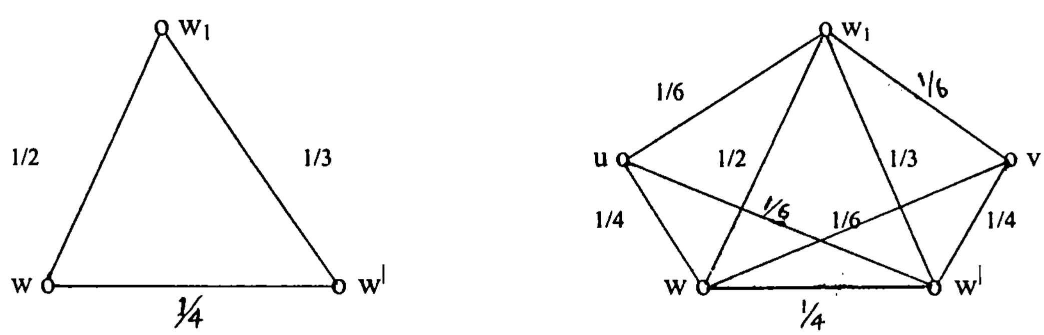
Now, $u^* = v$, $v^* = u$ and $e(v) = e(u) = \dfrac{1}{(\frac{1}{d})} + \dfrac{1}{\frac{1}{d+\frac{1}{t}}} = 2d + \dfrac{1}{t}$

Also, $w^* = v$, $(w^{/})^* = u$,

$e(w) = e(w^{/}) = \dfrac{1}{\frac{1}{d+{}^{1}/_{t}}}$ =d + $^{1}/_{t}$ and all other nodes in H have eccentricity equal to $d + {}^{1}/_{t}$ by (2), with u and v as their eccentric nodes. Thus each node in H is a central node of G with $r(G) = d + 1/t$, $d(G) = 2d + 1/t$ and $\langle C(G) \rangle$ is isomorphic to H.

Finally we claim that $G : (\sigma, \mu)$ is a fuzzy tree; for, H being a fuzzy tree, it has a spanning subgraph F_H, which is a tree, satisfying the requirements. Now, F_H together with the arcs (u, w) and $(w^{/}, v)$ is the required spanning subgraph of G.

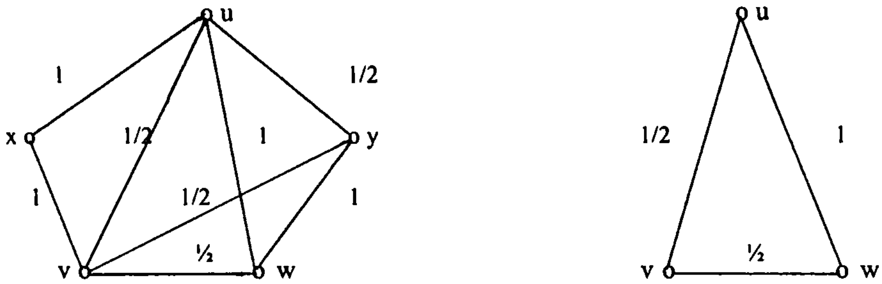
Illustration:



$H : (\sigma^{/}, \mu^{/})$ Fig. 4.5 $G : (\sigma, \mu)$

$d(H) = 4$, $t = ½$, $e(u) = e(v) = 10$, $e(w) = e(w^{|}) = e(w_{|}) = 6$.

Remark 4.3 : The center of a fuzzy tree need not be a fuzzy tree.



$G : (\sigma, \mu)$ Fig. 4.6 $(C(G))$

There are fuzzy graphs with diametrical nodes (nodes with maximum eccentricity) as fuzzy cutnodes. See u_3 and u_5 in the following example.

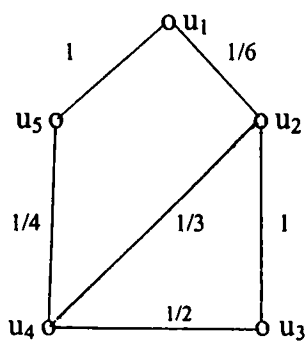


Fig. 4.7

$e(u_1) = e(u_2) = e(u_3) = e(u_5) = 7$ and $e(u_4) = 5$. ∎

Chapter 5

SOME OPERATIONS ON FUZZY GRAPHS

Mordeson [30] has defined the complement of a fuzzy graph $G : (\sigma, \mu)$ as a fuzzy graph $G^c: (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$ and $\mu^c(u, v) = 0$ if $\mu(u, v) > 0$ and $\mu^c(u, v) = \sigma(u) \wedge \sigma(v)$ otherwise. It follows from this definition that G^c is a fuzzy graph even if G is not and that $(G^c)^c = G$ if and only if G is a strong fuzzy graph [Definition 1.16]. Also, automorphism group of G and G^c are not identical. These observations motivate us to modify the notion of complement of a fuzzy graph. Some properties of self complementary fuzzy graphs are also studied. We also show that automorphism group of G and its complement $\overline{G}$ are identical.

In the second part of this chapter we consider some operations on fuzzy graphs and prove that complement of the union of two fuzzy graphs is the join of their complements and the complement of the join of two fuzzy graphs is the union of their complements. Finally we prove that complement of the composition of two strong fuzzy graphs is the composition of their complements. We conclude this chapter with a discussion on some open problems.

5.1 Complement of a Fuzzy Graph

We first illustrate the drawbacks in the definition of complement of a fuzzy graph mentioned above. In Fig. 5.1, $(G^c)^c \neq G$ and note that they are identical provided G is a strong fuzzy graph[Definition1.16].

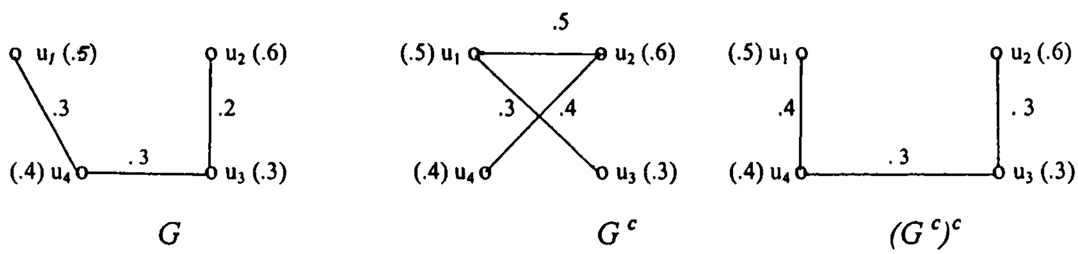


Fig 5.1

Now, consider the fuzzy graph G and G^c in Fig. 5.2. The automorphism group of G consists of two maps h_1 and h_2 where h_1 is the identity map and h_2 is given by the permutation $(v_1)\ (v_2\ v_4)\ (v_3)$. But the automorphism group of G^c consists of four maps h_1, h_2, h_3 and h_4 where h_1 and h_2 are automorphisms of G and h_3 and h_4 are given by

$h_3 = (v_1\ v_3)\ (v_2)\ (v_4)$ and $h_4 = (v_1\ v_3)\ (v_2\ v_4)$.

Fig 5.2

Definition 5.1. The complement of a fuzzy graph $G : (\sigma, \mu)$ is the fuzzy graph $\overline{G} : (\overline{\sigma}, \overline{\mu})$ where $\overline{\sigma} \equiv \sigma$ and $\overline{\mu}\ (u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v)$ for all u, v in V.

Example.

G $\overline{G}$

Fig 5.3

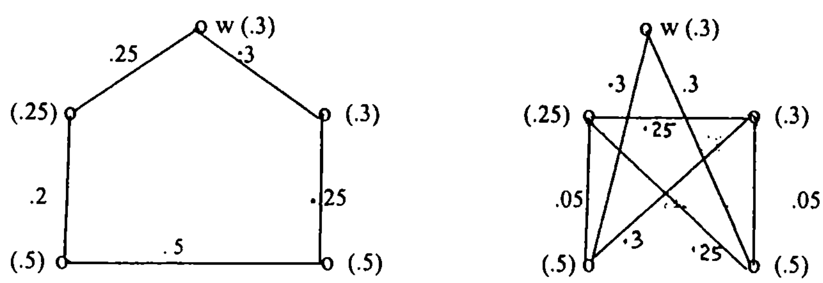
We have

$$\overline{\overline{\sigma}} = \overline{\sigma} = \sigma \text{ and } \quad \overline{\overline{\mu}}(u,v) = \overline{\sigma}(u) \wedge \overline{\sigma}(v) - \overline{\mu}(u,v)$$
$$= \sigma(u) \wedge \sigma(v) - ((\sigma(u) \wedge \sigma(v)) - \mu(u,v))$$
$$= \mu(u,v)\ \forall u,v.$$

Hence $\overline{\overline{G}} = G$.

Remark 5.1. A node can be a fuzzy cut node of both G and $\overline{G}$.

In Fig.5.4, w is a fuzzy cutnode of G and $\overline{G}$.



G $\overline{G}$

Fig. 5.4

Definition 5.2. A fuzzy graph G is self complementary if $G \approx \overline{G}$..

In the following theorems we present a necessary and then a sufficient condition for a fuzzy graph to be self complementary.

Theorem 5.1. Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then

$$\sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

Proof : Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then there exists an isomorphism $h : V \to V$ such that

$$\overline{\sigma}\, h(u) = \sigma(u) \quad \forall\, u \in V \text{ and}$$
$$\overline{\mu}(h(u), h(v)) = \mu(u,v) \forall\, u, v \in V.$$

Now by definition of $\overline{G}$, we have,

$$\overline{\mu}(h(u), h(v)) \quad = \overline{\sigma}(h(u)) \wedge \overline{\sigma}(h(v)) - \mu(h(u), h(v))$$

$$\text{ie.,} \quad \mu(u,v) \quad = \quad \sigma(u) \wedge \sigma(v) - \mu(h(u), h(v))$$

$$\text{ie.,} \sum_{u \neq v} \mu(u,v) + \sum_{u \neq v} \mu(h(u), h(v)) = \sum_{u \neq v} \sigma(u) \wedge \sigma(v))$$

$$\text{ie.,} \qquad 2 \sum_{u \neq v} \mu(u,v) = \sum_{u \neq v} \sigma(u) \wedge \sigma(v))$$

$$\text{ie.,} \qquad \sum_{u \neq v} \mu(u,v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \wedge \sigma(v)).$$

Hence the theorem.

Remark 5.2. If $G : (V,E)$ is a self complementary (crisp) graph, then from Theorem 5.1, it follows that $2m = \frac{n(n-1)}{2}$ where $m = |E|$ and $n = |V|$ which is equivalent to the result that every self complementary (crisp) graph has $4k$ or $4k+1$ nodes for some k.

Remark 5.3. The condition given in Theorem 5.1 is not sufficient. In the following example (Fig. 5.5), G is not isomorphic to $\overline{G}$ but,

$$\sum_{u \neq v} \mu(u,v) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1 \text{ and}$$
$$\frac{1}{2}\sum_{u \neq v} \sigma(u) \wedge \sigma(v) = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} + 1\right] = 1.$$

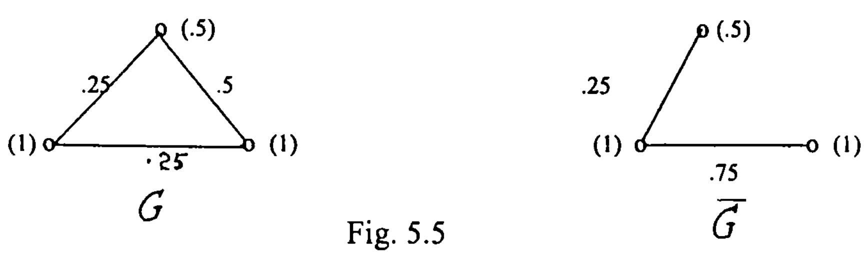


Fig. 5.5

Theorem 5.2. Let $G:(\sigma, \mu)$ be a fuzzy graph. If $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \forall\, u,v \in V$, then G is self complementary.

Proof: Let $G:(\sigma, \mu)$ be a fuzzy graph such that $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \forall u,v \in V$. Then $G \approx \bar{G}$ under the identity map on V.

Remark 5.4. The condition in Theorem 5.2 is not necessary. In the following example (Fig. 5.6) $G \approx \overline{G}$ where the isomorphism $h : V \rightarrow V$ is given by $h(u) = v$, $h(v) = x$, $h(w) = u$, $h(x) = w$, $h(y) = y$.

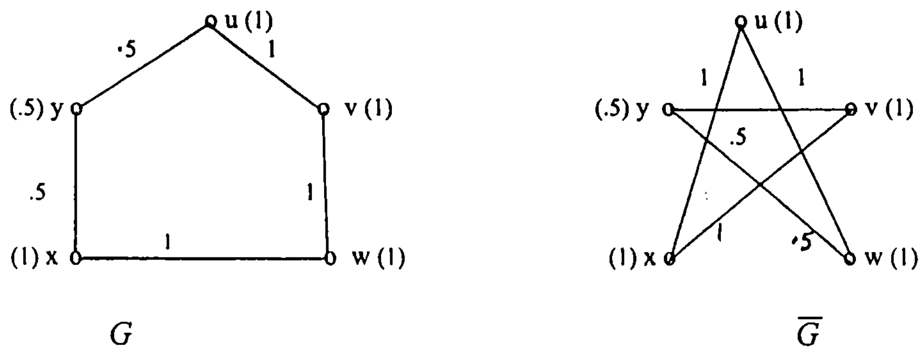


Fig. 5.6

Note that if h is an automorphism of G then h^{-1} is also an automorphism of G, for ;

$\sigma(h^{-1}(u)) = \sigma(u') = \sigma(h(u')) = \sigma(u)$ *and*

$\mu(h^{-1}(u), h^{-1}(v)) = \mu(u', v') = \mu(h(u), h(v)) = \mu(u, v)$.

Similarly if h and g are automorphism of G then their composition $h \circ g$ is also an automorphism of G [3] and we have,

Theorem 5.3[3]. The set of all automorphisms of a fuzzy graph G is a group under set-theoretic composition of maps as the binary operation.

Notation: We denote by ΓG, the group of automorphisms of the fuzzy graph G.

Theorem 5.4. Let $G : (\sigma, \mu)$ be a fuzzy graph. Then the automorphism groups of G and $\overline{G}$ are identical.

Proof : Note that if $h \in \Gamma G$, then $h \in \Gamma\overline{G}$, for ;

$h : V \to V$ is bijective ,

$$\overline{\sigma}(h(u)) = \sigma(h(u)) = \sigma(u) = \overline{\sigma}(u) \text{ and}$$

$$\overline{\mu}(h(u),h(v)) = (\sigma\, h(u) \wedge \sigma\, h(v)) - \mu(h(u),h(v))$$

$$= (\sigma(u) \wedge \sigma(v)) - \mu(u,v)$$

$$= \overline{\mu}(u,v) \quad \forall\ u,v \in V.$$

Hence the theorem.

In the following example (Fig. 5.7), $\Gamma G = \Gamma\overline{G} = \{ I, h \}$, where I is the identity map and h is given by the permutation $(v_1, v_3)(v_2, v_4)$.

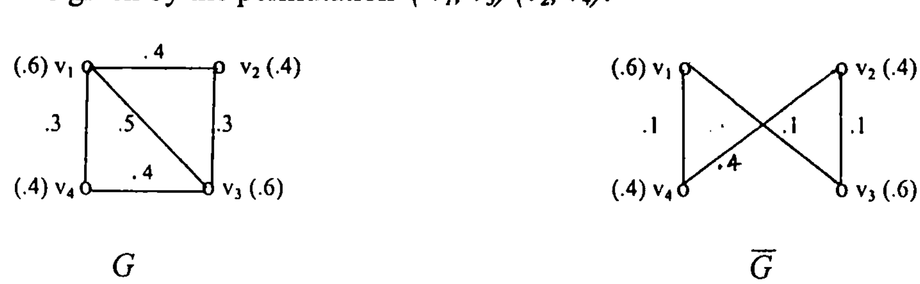


Fig. 5.7

5.2 Operations on Fuzzy Graphs

The operations on (crisp) graphs such as union join, cartesian product and composition are extended to fuzzy graphs [Definitions 1.29 – 1.32] and some of their properties are studied in [30]. `. In the following discussions an arc between two nodes u and v is denoted by uv rather than (u, v), because in the cartesian product of two graphs, a node of the graph is infact, an ordered pair.

Theorem 5.5 Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two fuzzy graphs. Then

1) $\overline{G_1 + G_2} \approx \overline{G}_1 \cup \overline{G}_2$ and

2) $\overline{G_1 \cup G_2} \approx \overline{G}_1 + \overline{G}_2$

Proof : We shall prove that the identity map is the required isomorphism.

1) *Let* $I : V_1 \cup V_2 \to V_1 \cup V_2$ *be the identity map.*

To prove $\overline{\sigma_1 + \sigma_2}\,(u) = \overline{\sigma}_1 \cup \overline{\sigma}_2\,(u)$

and $\overline{\mu_1 + \mu_2}\,(uv) = \overline{\mu}_1 \cup \overline{\mu}_2\,(uv)$

$$\overline{\sigma_1 + \sigma_2}\,(u) = (\sigma_1 + \sigma_2)\,(u) \quad \textit{by definition of complement}$$

$$= \begin{cases} \sigma_1(u) \textit{ if } u \in V_1 \\ \sigma_2(u) \textit{ if } u \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\sigma}_1(u) \textit{ if } u \in V_1 \\ \overline{\sigma}_2(u) \textit{ if } u \in V_2 \end{cases}$$

$$= \overline{\sigma}_1 \cup \overline{\sigma}_2(u).$$

$$\overline{\mu_1+\mu_2}\,(uv)=(\sigma_1+\sigma_2)\,u \;\wedge\; (\sigma_1+\sigma_2)v-(\mu_1+\mu_2)uv$$

$$=\begin{cases}(\sigma_1\cup\sigma_2)\;u\;\;\wedge(\sigma_1\cup\sigma_2)\;v-(\mu_1\cup\mu_2)uv\;\textit{if}\;\; uv\in E_1\cup E_2\\(\sigma_1\cup\sigma_2)\;u\;\;\wedge(\sigma_1\cup\sigma_2)v-\sigma_1(u)\wedge\sigma_2(v),\;\; uv\in E'(\textit{Definition}\,1.)\end{cases}$$

$$=\begin{cases}\sigma_1(u)\;\;\wedge\sigma_1(v)-\mu_1(uv),\;\textit{if}\;uv\in E_1\\\sigma_2(u)\;\;\wedge\sigma_2(v)-\mu_2(uv),\;\textit{if}\;uv\in E_2\\\sigma_1(u)\;\;\wedge\sigma_2(v)-\sigma_1(u)\wedge\sigma_2(v)\,\textit{if}\;uv\in E'\;\;\textit{where}\,u\in V_1,v\in V_2\end{cases}$$

$$=\begin{cases}\overline{\mu}_1(uv),uv\in E_1\\\overline{\mu}_2(uv),uv\in E_2\\0\qquad,uv\in E'\end{cases}$$

$$=\overline{\mu}_1\cup\overline{\mu}_2(uv).$$

2) $\overline{G_1\cup G_2}\approx\overline{G}_1+\overline{G}_2$.

Let $I:V_1\cup V_2\to V_1\cup V_2$ *be the identity map*.

To prove $\overline{\sigma_1\cup\sigma_2}(u)=(\overline{\sigma}_1+\overline{\sigma}_2)(uv)$
and $\overline{\mu_1\cup\mu_2}\,(uv)=(\overline{\mu}_1+\overline{\mu}_2)(uv)$.

$$\overline{\sigma_1\cup\sigma_2}(u)=(\sigma_1\cup\sigma_2)(u)$$
$$=\begin{cases}\sigma_1(u),\textit{if}\;u\in V_1\\\sigma_2(u),\textit{if}\;u\in V_2\end{cases}$$

$$=\begin{cases}\overline{\sigma}_1(u),\textit{if}\;u\in V_1\\\overline{\sigma}_2(u),\textit{if}\;u\in V_2\end{cases}$$

$$=(\overline{\sigma}_1\cup\overline{\sigma}_2)(u)$$
$$=(\overline{\sigma}_1+\overline{\sigma}_2)(u).$$

$$\overline{\mu_1 \cup \mu_2}(uv) = (\sigma_1 \cup \sigma_2)u \wedge (\sigma_1 \cup \sigma_2)v - (\mu_1 \cup \mu_2)(uv)$$

$$= \begin{cases} \sigma_1(u) \wedge \sigma_1(v) - \mu_1(uv), \textit{if } uv \in E_1 \\ \sigma_2(u) \wedge \sigma_2(v) - \mu_2(uv), \textit{if } uv \in E_2 \\ \sigma_1(u) \wedge \sigma_2(v) - 0, \textit{when } u \in V_1, v \in V_2. \end{cases}$$

$$= \begin{cases} \overline{\mu}_1(uv), \textit{if } uv \in E_1 \\ \overline{\mu}_2(uv), \textit{if } uv \in E_2 \\ \sigma_1(u) \wedge \sigma_2(v), \textit{if } \ u \in V_1, v \in V_2 \end{cases}$$

$$= \begin{cases} (\overline{\mu}_1 \cup \overline{\mu}_2)uv \textit{ if } uv \in E_1 \textit{ or } E_2 \\ \sigma_1(u) \wedge \sigma_2(v) \textit{ if } uv \in E^1 \end{cases}$$

$$= (\overline{\mu}_1 + \overline{\mu}_2)(uv).$$

Remark 5.5. Note that if G is a strong fuzzy graph, then $\overline{G}$ is also strong, for, let $uv \in \mu^*$, then

$$\overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv) = \sigma(u) \wedge \sigma(v) - \sigma(u) \wedge \sigma(v) = 0$$

and if $uv \notin \mu_1^*$ then

$$\overline{\mu}(uv) = \sigma(u) \wedge \sigma(v) - \mu(uv) = \sigma(u) \wedge \sigma(v) - 0 = \sigma(u) \wedge \sigma(v).$$

Theorem 5.6. Let $G_1: (\sigma_1, \mu_1)$ and $G_2: (\sigma_2, \mu_2)$ be two strong fuzzy graphs. Then $G_1 \circ G_2$ is a strong fuzzy graph and $\overline{G_1 \circ G_2} \approx \overline{G}_1 \circ \overline{G}_2$.

Proof : Let $G_1 \circ G_2 = G : (\sigma, \mu)$ where $\sigma = \sigma_1 \circ \sigma_2$, $\mu = \mu_1 \circ \mu_2$

and $G^* = (V, E)$ where $V = V_1 \times V_2$, $E = \{ (u, u_2)(u, v_2) : u \in V_1, u_2v_2 \in E_2 \} \cup \{ (u_1, w)(v_1, w) : w \in V_2, u_1v_1 \in E_1 \} \cup \{ (u_1, u_2)(v_1, v_2) : u_1v_1 \in E_1, u_2 \neq v_2 \}$.

Now,

1) $\mu(u, u_2)(u, v_2) = \sigma_1(u) \wedge \mu_2(u_2, v_2)$ [Definition 1.32]

$= \sigma_1(u) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$, G_2 being strong

$= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(v_2))$

$= (\sigma_1 \circ \sigma_2)(u, u_2) \wedge (\sigma_1 \circ \sigma_2)(u, v_2)$

$= \sigma(u, u_2) \wedge \sigma(u, v_2)$

2) $\mu(u_1, w)(v_1, w) = \sigma_2(w) \wedge \mu_1(u_1, v_1)$ [Definition 1.32]

$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(w)$, G_1 being strong

$= (\sigma_1(u_1) \wedge \sigma_2(w)) \wedge (\sigma_1(v_1) \wedge \sigma_2(w))$

$= (\sigma_1 \circ \sigma_2)(u_1, w) \wedge (\sigma_1 \circ \sigma_2)(v_1, w)$

$= \sigma(u_1, w) \wedge \sigma(v_1, w)$

and

3) $\mu(u_1, u_2)(v_1, v_2) = \mu_1(u_1, v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$[Definition 1.32]

$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$, G_1 being strong

$= (\sigma_1(u_1) \wedge \sigma_2(u_2)) \wedge (\sigma_1(v_1) \wedge \sigma_2(v_2))$

$= (\sigma_1 \circ \sigma_2)(u_1, u_2) \wedge (\sigma_1 \circ \sigma_2)(v_1, v_2)$

$= \sigma(u_1, u_2) \wedge \sigma(v_1, v_2)$.

Thus from 1, 2 and 3 it follows that G is a strong fuzzy graph.

Next, to prove $\overline{G_1 \circ G_2} = \overline{G}_1 \circ \overline{G}_2$.

Let $\overline{G}:(\sigma,\overline{\mu})=\overline{G_1\circ G_2}$, $\overline{\mu}=\overline{\mu_1\circ\mu_2}$ $,\overline{G}^*=(V,\overline{E})$;

$\overline{G}_1:(\sigma_1,\overline{\mu}_1)$, $\overline{G}_1^*=(V_1,\overline{E}_1)$;

$\overline{G}_2:(\sigma_2,\overline{\mu}_2)$, $\overline{G}_2^*=(V_2,\overline{E}_2)$ *and*

$\overline{G}_1\circ\overline{G}_2:(\sigma_1\circ\sigma_2,\overline{\mu}_1\circ\overline{\mu}_2)$.

Now, the various types of arcs (say) *e*, joining the nodes of *V* are the following and it suffices to prove that $\overline{\mu_1\circ\mu_2}=\overline{\mu}_1\circ\overline{\mu}_2$ in each case.

Case 1. $e=(u,u_2)(u,v_2)$, $u_2v_2\in E_2$

Then $e\in E$ and G being strong $\overline{\mu}(e)=0$. *Also*, $\overline{\mu}_1\circ\overline{\mu}_2(e)=0$, since $u_2v_2\notin\overline{E}_2$.

Case II. $e=(u,u_2)(u,v_2)$, $u_2\neq v_2$ and $u_2v_2\notin E_2$

Here $e\notin E$, so $\mu(e)=0$.

Now,

$\overline{\mu}(e)=\sigma(u,u_2)\wedge\sigma(u,v_2)$

$=(\sigma_1(u)\wedge\sigma_2(u_2))\wedge(\sigma_1(u)\wedge\sigma_2(v_2))$ and since

$u_2v_2\in\overline{E}_2$ *we have*,

$$\begin{aligned}\overline{\mu}_1\circ\overline{\mu}_2(e)&=\sigma_1(u)\wedge\overline{\mu}_2(u_2v_2)\\&=\sigma_1(u)\wedge\sigma_2(u_2)\wedge\sigma_2(v_2),\ \overline{G}_2\ \textit{being strong}\\&=\overline{\mu}(e).\end{aligned}$$

Case III. $e=(u_1,w)(v_1,w)$, $u_1v_1\in E_1$

Then $e\in E$, *so* $\overline{\mu}(e)=0$ as in Case 1.

Also, since $u_1v_1\notin\overline{E}_1$, *we have* $\overline{\mu}_1\circ\overline{\mu}_2(e)=0$.

Case IV. $e=(u_1,w)(v_1,w)$, $u_1v_1\notin E_1$

Here $e \notin E$, hence $\mu(e) = 0$ and

$\bar{\mu}(e) = \sigma(u_1, w) \wedge \sigma(v_1, w)$

$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(w)$ and since $u_1v_1 \in \bar{E}_1$

we have,

$$\bar{\mu}_1 \circ \bar{\mu}_2(e) = \sigma_2(w) \wedge \bar{\mu}_1(u_1v_1)$$

$$= \sigma_2(w) \wedge \sigma_1(u_1) \wedge \sigma_1(v_1), \bar{G}_1 \text{ being strong}$$

$$= \bar{\mu}(e).$$

Case V. $e = (u_1, u_2)(v_1, v_2), u_1v_1 \in E_1$ and $u_2 \neq v_2$

Here $e \in E$, so $\bar{\mu}(e) = 0$ as in Case 1.

Also, since $u_1v_1 \notin \bar{E}_1$, we have $\bar{\mu}_1 \circ \bar{\mu}_2(e) = 0$.

Case VI. $e = (u_1, u_2)(v_1, v_2), u_1v_1 \notin E_1$ and $u_2 \neq v_2$

Then $e \notin E$, hence $\mu(e) = 0$

Thus $\bar{\mu}(e) = \sigma(u_1, u_2) \wedge \sigma(v_1, v_2)$

$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$

and since $u_1v_1 \in \bar{E}_1$, we have ,

$$\bar{\mu}_1 \circ \bar{\mu}_2 = \bar{\mu}_1(u_1v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$$

$$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), \bar{G}_1 \text{ being strong}$$
$$= \bar{\mu}(e).$$

Case VII. $e = (u_1, u_2)(v_1, v_2), u_1v_1 \notin E_1, u_2v_2 \notin E_2$

Here $e \notin E$, hence $\mu(e) = 0$

Thus $\bar{\mu}(e) = \sigma(u_1, u_2) \wedge \sigma(v_1, v_2)$

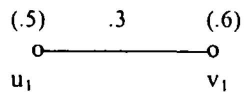
$= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2)$

Now, $u_1v_1 \in \overline{E}_1$ and if $u_2 = v_2 = w$, then we have Case IV.

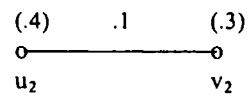
Next , if $u_1v_1 \in \overline{E}_1$ and if $u_2 \neq v_2$, then we have Case VI.

Thus from Cases I to VI, it follows that $\overline{G_1 \circ G_2} \approx \overline{G}_1 \circ \overline{G}_2$.

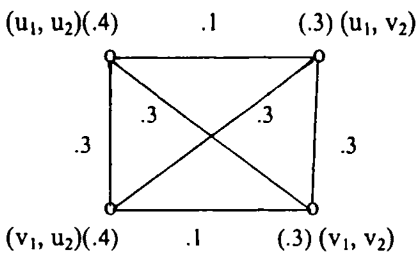
Remark 5.6. In general $\overline{G_1 \circ G_2} \neq \overline{G}_1 \circ \overline{G}_2$. Consider the following example in which G_1 and G_2 are not strong.



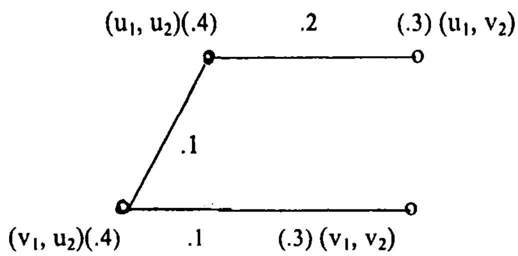
G_1



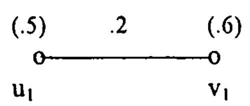
G_2



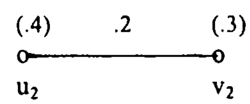
$G_1 \circ G_2$



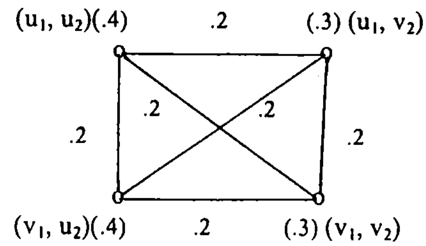
$\overline{G_1 \circ G_2}$



$\overline{G}_1$



$\overline{G}_2$



$\overline{G}_1 \circ \overline{G}_2$

Thus $\overline{G_1 \circ G_2} \neq \overline{G}_1 \circ \overline{G}_2$.

5.3 Conclusion and Suggestions for Further Study

In this thesis an attempt to develop the properties of basic concepts in fuzzy graphs such as fuzzy bridges , fuzzy cutnodes, fuzzy trees and blocks in fuzzy graphs have been made.

The notion of complement of a fuzzy graph is modified and some of its properties are studied. Since the notion of complement has just been initiated, several properties of G and $\overline{G}$ available for crisp graphs can be studied for fuzzy graphs also.

We have mainly focussed on fuzzy trees defined by Rosenfeld. In [10], several other types of fuzzy trees are defined depending on the acyclicity level of a fuzzy graph. We have observed that there are selfcentered fuzzy trees. However, which fuzzy trees are selfcentered is yet to be analyzed. The center problems can also be carried over to other types of fuzzy trees mentioned above.

Identification of blocks of a fuzzy graph is still an open problem ; solving which may lead to the study of fuzzy block graphs, fuzzy cut point graphs etc. The study of the parameter $C(G)$ – the connectedness level of a fuzzy graph [10] can be done on G and $\overline{G}$. ■

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