MAXIMAL COMPACTNESS, MINIMAL HAUSDORFFNESS AND REVERSIBILITY OF FRAMES AND A STUDY ON AUTOMORPHISM GROUP OF FRAMES

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Certificate

This is to certify that the thesis entitled ' **Maximal compactness, minimal Hausdorffness and reversibility of frames and a study on automorphism group of frames** ' submitted to the Cochin University of Science and Technology by Mr. Jayaprasad.P.N. for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by him under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

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Declaration

I, Jayaprasad.P.N., hereby declare that this thesis entitled '**Maximal compactness, minimal Hausdorffness and reversibility of frames and a study on automorphism group of frames**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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To

My Wife

and in loving memory of

my Father and Mother

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"It is better to perform one's own duties imperfectly than to master the duties of another. By fulfilling the obligations he is born with, a person never comes to grief."

(Bhagavad Gita)

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MAXIMAL COMPACTNESS, MINIMAL HAUSDORFFNESS AND REVERSIBILITY OF FRAMES AND A STUDY ON AUTOMORPHISM GROUP OF FRAMES

Contents

Notations used

- ΩX: openset lattice of the topological space X;
- $L - A$: set theoretic difference;
- 1_L : the top of the frame L;
- 0_L : the bottom of the frame L;
- $A[b]$: singly generated frame extension;
- a^* : pseudo-complement of a in L;
- a^c : complement of a in L;
- A^c : set theoretic complement;
- $\uparrow a$: principal filter generated by a;
- $\downarrow a$: principal ideal generated by a;
- $\uparrow A$: union of all principal filters in A;
- $\downarrow A$: union of all principal ideals in A;
- $D(L)$: set of all dense elements of L;
- $\mathcal{D}(F)$: frame corresponding to downset functor $\mathcal{D}$;
- $\bigoplus_{i \in I} A_i$: frame coproduct of the frames A_i;

- $Fr\,\langle G \mid R \rangle$: frame presented by generators G and relations R;
- $A(T)$: group of frame isoomorphisms on the frame T;
- $A(X)$: group of homeomorphisms on the topological space X;
- $|X|$: cardinality of the set X;
- $\mathcal{P}(X)$: power set of the set X;
- f^* : localic map corresponding to the frame homomorphism f;

Abbreviation used

CCE	:	Closed and Compact Equivalent;
MRC	:	Maximal Relative to Compactness;
$C - C$	:	Closed and Compact;
Frm	:	Category of frames and frame homomorphisms;
Loc	:	Category of locales and localic maps;
Sp	:	Category of all topological spaces and continuous maps;

Chapter 1

Introduction and Preliminaries

1.1 Introduction

Hausdorff [15] was the first mathematician who studied continuity properties in topological spaces using the notion of openset as primitive. Till then, topological space was known to possess a lattice of open subsets, though the lattice-theoretic principles were not used there. The application of lattice-theoretic ideas to study about topological spaces began with the work of the American Mathematician Marshall Stone on the topological representation of Boolean algebras [45] [46] and distributive lattices [47]. Two revolutionary ideas originated in his paper. The first one proved the importance of ideals in lattice theory through the result that Boolean algebras is a certain type of ring called Boolean ring. The next one was the link between topology and lattice theory through the following famous theorem.

Stone's Representation Theorem

Every Boolean algebra is isomorphic to the Boolean algebra of open-closed sets of a totally disconnected compact Hausdorff space.

Thus Stone's representation theorem revealed that topological spaces can be constructed from purely algebraic thing such as a Boolean algebra. His work gave the motivation for employing lattice theory to solve geometrical problems.

The first person who employed this idea was Henry Wallman [49] , an American mathematician, where he used lattice-theoretic ideas to construct "Wallman compactification" of a T_1 topological space. A few years later an American logician McKinsey and a Polish mathematician Tarski [28] [29] made a study of the "algebra of topology". The first text book which presented topology from the lattice-theoretic viewpoint was written by the German Mathematician Nöbeling [30]. Charles Ehresmann [13] and his student Jean Bénabou [4] made remarkable changes in the study of topological spaces from lattice-theoretic viewpoint. He remarked that a lattice with right distributive property(finite meets distribute over arbitrary joins) should be studied as a "generalized topological space" irrespective of it being the openset lattice of some topological space [14]. These "generalized topological space" are called "local lattices" by them. The works of Dona and Seymour Papert [31] [32] at about the same time seconded the same aspect of study of topological spaces.

The term *frame* was the contribution of C.H.Dowker. Frame theory is lattice theory applied to topology. This approach takes the lattice of opensets as the basic notion. In other words, it is a pointfree topology where one investigates typical properties of lattices of opensets that can be expressed without reference to points. C.H. Dowker and Dona Papert Strauss extended many results in topology to these “generalized spaces” [8] to [12]. One may think of frames as “generalized spaces”. According to Isbell, the word “generalized” is imprecise because arbitrary spaces are not determined by their lattices of open sets. In 1972, through the paper [18], J.R.Isbell pointed out the need for a seperate terminology for the dual category of frames. The objects of this dual category are named as “locales” by him and they are actually the “generalized spaces”. The letters $X, Y, Z, \ldots$ are usually used to represent a locale. The frame corresponding to the locale X is denoted by ΩX.

The notion of sublocales(quotient frames) have been studied by Dowker and Papert and Isbell. The term sublocale was introduced by Isbell. Sublocales of a given locale X correspond to quotient frames of ΩX. Compactness and connectedness are notions traditionally defined in terms of properties of the lattice of open sets of a space. Hence the task of defining them in frames is easy. But those definitions depending on points of a space cannot be carried out to frames as they are free of points. Hence the classical T_1 axiom for spaces cannot be adopted since it mentions points. But there are various alternative definitions in use. The “unorderedness axiom” is an example.

For localic version of the Hausdorff axiom there are many candidates. Of these, the most accepted is the one due to Isbell. He defined a locale L as Hausdorff if L can be regrded as a closed sublocale of the localic product $L \oplus L$. The only drawback of this axiom is that it is not equivalent to the classical Hausdorff axiom for spaces because a space X may be closed in the localic product $X \oplus X$ without being closed in the topological product $X \times X$. For this reason Isbell called locales satisfying this axiom strongly Hausdorff locales. C.H.Dowker and D.Strauss[10], H. Simmons[44], P.T.Johnstone and S.H.Sun[24] and J.Paseka[35] are other mathematicians who defined alternatives for Hausdorff axiom for locales. They have the advantage that they coincide with the classical Hausdorff axiom on space, but are not satisfactory in other respects.

The regularity axiom for spaces, though involve points, is frame theoretic. This is becuse, in topological spaces the regularity axiom says that each open set is a union of open sets whose closures it contains. The same is defined for locales and it implies all other seperation axioms considered. The stronger seperation axioms complete regularity and normality are straightforward for locales.

Frame theory has the advantage that many results in topology requiring *Axiom of Choice* or some of its variants can be proved without its use. Examples are Tychonoff theorem[21], the construction of Stone-Cech compactification[1] or the construction of Samuel compactification[3]. Sometimes the frame situation differs from the classical one. For example, coproducts of paracompact locales are *paracompact*[18] while products of paracompact spaces are not necessarily paracompact. Another example is that coproducts of regular frames preserve the *Lindelöf property*[12] while product of regular spaces do not.

We wish to give a brief description of five important problems settled in topology by eminent mathematicians for which the frame counterpart we discuss in this thesis.

The concept of *simple extension* of a topology was studied by Norman

Levine [26]. If (X, τ) is a topological space, then

$$\tau(A) = \{O \cup (O' \cap A) : O, O' \in \tau\}$$

where $A \notin \tau$ is called a simple extension of τ. He studied the conditions under which the simple extension of a topological space with a specified topological property also holds that property. In his paper, it is proved that the simple extension $(X, \tau(A))$ of a compact topological space (X, τ) is compact if A^c is compact in (X, τ). Also if (X, τ) is regular(completely regular or normal), then $(X, \tau(A))$ is regular(completely regular or normal) provided $A^c \in \tau$.

A topological space (X, τ) is said to be *maximal compact* if it is compact and there is no strictly stronger topology on X which is compact. In 1948, A.Ramanathan[42] proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed sets. Also, E. Hewitt[16] proved that a compact Hausdorff space is maximal compact as well as minimal Hausdorff. Topological spaces in which closed subspaces coincide with compact subspaces was studied by N.Levine [27]. He called such spaces C-C spaces. In this paper, it is proved that if the product topology is C-C, then each component is C-C. The converse of this need not be true. Here he proved that $X \times X$ is C-C if and only if X is C-C and Hausdorff. It is also proved that a C-C space is necessarily compact and T_1.

The concept of minimal topologies was introduced by A.S.Parhomeko [33]. A topological space (X, τ) is said to be *minimal Hausdorff* if it is Hausdorff and there is no strictly weaker topology on X that is Hausdorff. As remarked in the above paragraph, compact Hausdorff spaces are always minimal Hausdorff. The following characterization for minimal Hausdorff topological spaces was given in [6] in terms of convergence of filters.

A necessary and sufficient condition that a Hausdorff space (X, τ) be minimal Hausdorff is that τ satisfies the following property:

1. Every open filter-base has an adherent point;
2. If an open filter-base has a unique adherent point, then it converges to this point.

It is also shown that a Hausdorff space which satisfies condition (2) also satisfies condition (1) and such a space is minimal Hausdorff. Also a compact Hausdorff space is minimal Hausdorff[5] and the converse need not be true.

The concept of *reversibility* in spaces was studied by M.Rajagopalan and A.Wilanski [38]. A topological space (X, τ) is called *reversible* if it has no strictly stronger topology τ^* such that (X, τ) and (X, τ^*) are homeomorphic. Equivalently, it has no strictly weaker topology τ^* such that (X, τ) and (X, τ^*) are homeomorphic. Then it is proved that a space is reversible if and only if each continuous bijection of the space onto itself is a homeomorphism. It is also proved that the finite product of reversible spaces is reversible if and only if each component is reversible. The concept of reversibility has also been extended to fuzzy topological space[19] by T.P.Johnson and to partially ordered sets[25] by Michal Kukiela.

De Groot[7] proved that any group is isomorphic to the group of homeomorphisms of topological space. A related problem is to determine the subgroups of the group of permutations of a fixed set X which can be group of homeomorphisms of (X, τ) for some topology τ on X. This problem was solved by P.T.Ramachandran[39] in topology and by T.P.Johnson[20] in fuzzy topology. It is proved in topology that the subgroup of the group of permutations on X containing two elements can represent the group of homeomorphisms of a topological space for some topology. But for a finite set X with $|X| \geqslant 3$ has no topology for which the group of homeomorphisms is the alternating group of permutations on X. Also it is proved that for a set X with $|X| \geqslant 3$ there is no nontrivial proper normal subgroup of the group of permutations on X which is the group of homeomorphisms for some topology.

One can do research in pointfree topology in two ways. The first is the contravariant way where research is done in the category **Frm** but the ultimate objective is to obtain results in **Loc**. The other way is the covariant way to carry out research in the category **Loc** itself directly. According to Johnstone [23], "frame theory is lattice theory applied to topology whereas locale theory is topology itself". The most part of this thesis is written according to the first view. In this thesis, we make an attempt to study about

1. the frame counterparts of *maximal compactness, minimal Hausdorffness and reversibility*,
2. the automorphism groups of a finite frame and its relation with the subgroups of the permutation group on the generator set of the frame.

Chapter 1 contains a quick review of the preliminary materials required to read and understand this thesis.

The concept of *singly generated extension* of a frame was introduced by

B.Banaschewski [2]. In *chapter 2*, we study some problems concerned with the singly generated extension of a frame. As the first step, we conducted an analoguous study on singly generated extension of a frame, following N.Levine. We obtained the conditions underwhich the singly generated extension of a frame possessing a specified frame isomorphic property also holds that property. The frame isomorphic properties studied are *compactness, regularity, complete regularity and normality.*

In *chapter 3*, we introduce the concept of *CCE frames* - frames in which closed sublocales are exactly the compact sublocales- analoguous to that in topology. As an application of the theorem that gives the condition for preserving compactness under singly generated extension of a frame, we have characterized CCE frames as maximal compact frames. We have also discussed some properties and characterizations of such frames in this chapter.

In *chapter 4*, we introduce the concept of *minimal Hausdorff frames* and obtained a partial characterization for them in terms of convergence of filters in frames. Some other properties of minimal Hausdorff frames are also discussed.

In *chapter 5*, we proceed to introduce *reversibility in frames.* The association between reversible spatial frames and the corresponding topological spaces is also studied here. A characterization for reversible frames is proved. Also, it is proved that a frame which is maximal or minimal with respect to some frame isomorphic property is reversible and conversely. Reversibility in frames can be used as a tool for solving some problems related to reversible topological spaces. As an application, we solved a problem put forward by M.Rajagopalan and A.Wilansky in [38] using "reversibility" in frames.

The *final chapter* presents a study on the *automorphism group finite frames*. Here, we describe how a subgroup of the permutation group of a given generator set of a finite frame can be extended to the automorphism group of that frame. Consequently, we found some permutation subgroups of the permutation group of all generators of a frame always extend to the automorphism group of that frame for some relation set and some never for any relation set.

1.2 Categorical concepts

Certain concepts in one branch of Mathematics have remarkable resemblence to those from other branches in Mathematics. For example, the concept of a homeomorphism in Topology has resemblance with the concept of an isomorphism in Groups or a bijection in Set Theory. The theory of Categories seeks to isolate what is common to these various branches of Mathematics. The theory is useful because it puts construction in one branch into a broader perspective and inspires similar constructions in other branches.

Definition 1.2.1. A *category* consists of the following data:

- Objects: $A, B, C, \ldots,$
- Arrows: $f, g, h, \ldots,$
- For each arrow f there are given objects $\mathrm{dom}(f)$ called the *domain*

and codom(f) called the *codomain*. We write $f : A \to B$ to indicate that $A = dom(f)$ and $B = codom(f)$,

- Given arrows $f : A \to B$ and $g : B \to C$ with $codom(f) = dom(g)$ there is given an arrow $g \circ f : A \to C$ called the *composite* of f and g,

- For each object A there is given an arrow $1_A : A \to A$ called the *identity arrow* of A.

These data are required to satisfy the following laws:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B, g : B \to C, h : C \to D$.

- Unit: $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

A category is anything that satisfies this definition.

Definition 1.2.2. The opposite or dual category $\mathbf{C}^{\mathbf{op}}$ of a category $\mathbf{C}$ has the same objects as $\mathbf{C}$ and an arrow $f : C \to D$ is an arrow $f : D \to C$. That is $\mathbf{C}^{\mathbf{op}}$ is just $\mathbf{C}$ with all of the arrows formally turned around.

Definition 1.2.3. In any category $\mathbf{C}$, an arrow $f : A \to B$ is called a monomorphism if given any $g, h : C \to A$, we have $f \circ g = f \circ h$ implies $g = h$.

Definition 1.2.4. In any category $\mathbf{C}$, an arrow $f : A \to B$ is called an epimorphism if given any $i, j : B \to D$, we have $i \circ f = j \circ f$ implies $i = j$.

Functors are means of passing from one category to another and they resemble functions in many respects. Often in Mathematics, it happens that to each object of a category , we can associate an object of another category which reflects the properties of the original object. The advantage of such an association is that information about one category can lead to information about another category. The basic features of a category are compositions and identity arrows. Thus it is natural to require that they must be preserved under transaction from one category to another.

Definition 1.2.5. A *covariant functor* between two categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping $F : \mathbf{C} \to \mathbf{D}$ of objects to objects and arrows to arrows in such a way that

1. $F(f : A \to B) = F(f) : F(A) \to F(B)$.
2. $F(g \circ f) = F(g) \circ F(f)$.
3. $F(1_A) = 1_{F(A)}$.

Definition 1.2.6. A *contravariant functor* between two categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping $F : \mathbf{C} \to \mathbf{D}$ of objects to objects and arrows to arrows in such a way that

1. $F(f : A \to B) = F(f) : F(B) \to F(A)$.
2. $F(g \circ f) = F(f) \circ F(g)$.
3. $F(1_A) = 1_{F(A)}$.

Definition 1.2.7. Let $F, G : \mathbf{A} \to \mathbf{B}$ be any two covariant functors. A natural transformation τ from F to G denoted by $\tau : F \to G$ is a map that assigns to each object A of **A** an arrow $\tau_A : FA \to GA$ in such a way that for each arrow $f : A \to A'$, we have $\tau_{A'} \circ Ff = Gf \circ \tau_A$.

Definition 1.2.8. Let F,G: $\mathbf{C} \to \mathbf{D}$ be any two functors. Let $\eta : id_B \to GF$ and $\epsilon : FG \to id_A$ be two natural transformations satisfying

1. $G \stackrel{\eta G}{\to} GFG \stackrel{G\epsilon}{\to} G = G \stackrel{id_G}{\to} G$.

2. $F \stackrel{F\eta}{\to} FGF \stackrel{\epsilon F}{\to} G = F \stackrel{id_F}{\to} F$.

In this situation we say that F is a left adjoint for G and G is a right adjoint for F. The natural transformation $\eta : id_B \to GF$ is called the *unit* and the natural transformation $\epsilon : FG \to id_A$ is called the *co-unit*.

Definition 1.2.9. A map $f : L \to M$ between partially ordered sets L and M is said to be monotone if for all $a, b \in L, a \leqslant b \Rightarrow f(a) \leqslant f(b)$. Let $f : L \to M$ and $g : M \to L$ be monotone maps between partially ordered sets L and M. We say that the pair(f, g) is a *Galois connection* if for all $a \in L, b \in M, f(a) \leqslant b \Leftrightarrow a \leqslant g(b)$. The above condition is also equivalently given as $fg(b) \leqslant b$ and $gf(a) \geqslant a$ for all $a \in L, b \in M$. If such a situation exists f will be the left adjoint of g and g will be the right adjoint of f.

Remark 1.2.1. The following points are useful.

- g is uniquely determined by f and the other way is also true,
- f preserves existing joins and g preseves existing meets,

- each join preserving map $f : L \to M$ is a left adjoint and each meet preserving map $g : M \to L$ is a right adjoint, with L and M being complete lattices.

1.3 Order theoretic concepts

Definition 1.3.1. A set L with a binary relation "$\leqslant$" satisfying the following conditions is called a partially ordered set.

1. $a \leqslant a$,
2. $a \leqslant b,\ b \leqslant a$ implies $a = b$,
3. $a \leqslant b,\ b \leqslant c$ implies $a \leqslant c$.

where $a,\ b\, and\, c \in L$

Definition 1.3.2. An element $x \in A \subseteq L$ is called *minimal* if $a \in A$, $a \leqslant x$ implies $a = x$. If L has a unique minimal element, then it is called the *least element(bottom)* of L denoted by 0_L.

Definition 1.3.3. An element $x \in A \subseteq L$ is called *maximal* if $a \in A$, $x \leqslant a$ implies $x = a$. If L has a unique maximal element, then it is called the *greatest element(top)* of L denoted by 1_L.

Definition 1.3.4. An element $x \in L$ is called an *upperbound* of $A \subseteq L$, if for all $a \in A$, we have $a \leqslant x$. The least element of the set of all upperbounds of A in L, if it exists, is called the *least upper bound(supremum)* of A. It is denoted by $\bigvee A$.

Definition 1.3.5. An element $x \in L$ is called a *lowerbound* of $A \subseteq L$, if for all $a \in A$, we have $x \leqslant a$. The greatest element of the set of all lowerbounds of A in L, if it exists, is called the *greatest lower bound(infimum)* of A. It is denoted by $\bigwedge A$.

Definition 1.3.6. A partially ordered set L in which for every pair of elements a and b there exists the supremum $a \vee b$ and the infimum $a \wedge b$ is called a *lattice*. A partially ordered set L for which every set $A \subseteq L$ has the supremum $\bigvee A$ and the infimum $\bigwedge A$ exist in L is called a *complete lattice*.

Definition 1.3.7. A lattice L is distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ which is equivalent to $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Definition 1.3.8. A map $f : L \to M$ where L, M are partially ordered sets is called *monotone(order preserving)* if $a \leqslant_L b \Rightarrow f(a) \leqslant_M f(b)$ for all a, $b \in L$. If f is bijective and its inverse f^{-1} is also monotone, then it is called an *order isomorphism*.

Definition 1.3.9. Let L be any partially ordered set and $A \subseteq L$ be any subset. Then A is called the *downset* of L generated by A if $\downarrow A = A$ where $\downarrow A = \{x \in L : \textit{there exists}\ a \in A, x \leqslant a\}$

Definition 1.3.10. Let L be any partially ordered set and $A \subseteq L$ be any subset. Then A is called the *upset* of L generated by A if $\uparrow A = A$ where $\uparrow A = \{x \in L : \textit{there exists}\ a \in A, x \geqslant a\}$

Definition 1.3.11. $\uparrow a = \uparrow \{a\}$ and $\downarrow a = \downarrow \{a\}$ are called the principal filter and the principal ideal generated by a respectively.

Definition 1.3.12. Let L be a lattice. Then an element $a \in L$ is an *atom*, if $0_L \leqslant x \leqslant a$ implies $x = 0_L$ *or* $x = a$.

Definition 1.3.13. Let L be a lattice. Then an element $a \in A$ is a *dual atom*, if $a \leqslant x \leqslant 1_L$ implies $x = a$ *or* $x = 1_L$.

Definition 1.3.14. Let L be a lattice with least element 0_L and let $a \in L$. The *pseudocomplement* of a, denoted by a^*, is the one satisfying $x \wedge a = 0_L$ *if and only if* $x \leqslant a^*$.

The following remark provides some useful rules on pseudocomplements.

Remark 1.3.1. Let L be any lattice with top 1_L and bottom 0_L, then

1. ${0_L}^* = 1_L$, ${1_L}^* = 0_L$,
2. $a \leqslant b$ *implies* $b^* \leqslant a^*$,
3. $a \leqslant a^{**}$,
4. $a^* = a^{***}$.

Definition 1.3.15. Let L be a distributive lattice with greatest element 1_L and least element 0_L. The complement a^c of an element $a \in L$ is the one satisfying $a \wedge a^c = 0_L$ *and* $a \vee a^c = 1_L$.

Definition 1.3.16. A *Boolean Algebra* is a distributive lattice with 0_L and 1_L in which every element has a complement.

1.4 Frames and Locales

Definition 1.4.1. A *frame* is a complete lattice L in which the infinite distributive law $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in S\}$ holds for all $a \in L, S \subseteq L$.

Definition 1.4.2. A map $f : L \to M$ between frames L, M satisfying for every $a_i, a, b \in L$

$$f(\bigvee_i a_i) = \bigvee_i f(a_i) \tag{1.1}$$
$$f(a \wedge b) = f(a) \wedge f(b) \tag{1.2}$$

is called a *frame homomorphism*. A bijective frame homomorphism is called a *frame isomorphism*.

Definition 1.4.3. An element $a \in L$ is said to be *dense* if $a^* = 0_L$.

Remark 1.4.1. The category whose objects are frames and morphisms are frame homomorphisms is denoted by **Frm**. The dual category $\mathbf{Frm}^{op}$ is referred to as the category of locales denoted by **Loc**. The objects of this category are known as locales and as objects they are same as that of frames. These two categories differ only in morphisms. The morphisms in **Loc**, called localic maps(continuous maps) are frame homomorphisms when considered in the opposite direction.

Remark 1.4.2. The category of topological spaces and continuous maps is denoted by **Sp**.

Definition 1.4.4. The functor $\Omega : \mathbf{Sp} \to \mathbf{Frm}$ maps objects and arrows as below

(1) a topological space (X, ΩX) is mapped to its frame of open sets ΩX,
(2) for an arrow $f : X \to Y$, the corresponding arrow in **Frm** is given by $\Omega(f) : \Omega Y \to \Omega X$ where $\Omega(f)(U) = f^{-1}(U)$ where $U \in \Omega Y$.

Theorem 1.4.1. *The functor Ω : **Sp** $\to$ **Frm** is a contravariant functor.*

A point of a frame L is a frame homomorphism $h : L \to \mathbf{2}$, where $\mathbf{2}$ is the two element boolean algebra. We denote by ΣL the set of all points of L. For $a \in L$, set $\Sigma_a = \{h : L \to \mathbf{2} : h(a) = 1\}$ and $\tau = \{\Sigma_a : a \in L\}$.

Theorem 1.4.2. *$(\Sigma L, \tau)$ is a topological space.*

Theorem 1.4.3. *Let L be any frame. Let ΣL be the set of all points of L. For each object L in **Frm**, Σ maps that object to the topological space $(\Sigma L, \tau)$. For a frame homomorphism $h : L \to M$, define the mapping $\Sigma h : \Sigma M \to \Sigma L$ by $(\Sigma h)(\alpha) = \alpha \circ h$. Then Σ : **Frm** $\to$ **Sp** is a contravariant functor.*

Theorem 1.4.4. *Σ : **Frm** $\to$ **Sp** is right adjoint to Ω : **Sp** $\to$ **Frm***

Definition 1.4.5. A frame L is said to be *spatial* if it is isomorphic to ΩX for some set X.

1.5 Subframes and Sublocales

Definition 1.5.1. A subset of a frame which is closed under the same finite meets and arbitrary joins in that frame is called a *subframe*. Thus it is clear that every subframe includes the top and bottom of the frame of which it is a subframe.

Definition 1.5.2. A *nucleus* on a locale A is a map $j : A \to A$ satisfying the following conditions

1. $j(a \wedge b) = j(a) \wedge j(b)$,
2. $a \leqslant j(a)$,
3. $j(j(a)) \leqslant j(a)$.

for all $a, b \in A$.

Remark 1.5.1. Define $A_j = \{a \in A : j(a) = a\}$. Then A_j is a frame and $j : A \to A_j$ is a frame homomorphism.

Definition 1.5.3. A *sublocale* of the frame L is an onto frame homomorphism $h : L \to M$ where M is a frame.

A sublocale can also be defined in another way as follows.

Definition 1.5.4. A *sublocale* of a locale A is a subset of the form A_j, for some nucleus j. The infima in the sublocale coincide with those of A and the suprema given by $\bigvee' x_i = j(\bigvee x_i)$.

Definition 1.5.5. The sublocale given by the nucleus $j_1 : A \to \uparrow a$ defined by $x \to a \vee x$ for any $a \in A$ is called a *closed sublocale* and is denoted by $\mathbf{c}(a)$.

Definition 1.5.6. The sublocale given by the nucleus $j_2 : A \to \downarrow a$

defined by $x \to a \wedge x$ for any $a \in A$ is called an *open sublocale* and is denoted by $\mathbf{o}(a)$.

An open sublocale $\mathbf{o}(b)$ can also be defined by $\mathbf{o}(b) = \{x \in A : b \to x\} = \{x \in A : b \to x = x\}$.

Definition 1.5.7. A sublocale A_j of A is said to be *dense* if it contains 0_A.

The following remark is given as an exercise in page 51[22].

Remark 1.5.2. For any sublocale A_j of A, there is a unique A_k called the *closure* of A_j, such that A_j is a dense sublocale of A_k and A_k is a closed sublocale of A.

By the above remark, the closure of a sublocale is defined as follows.

Definition 1.5.8. The *closure* of a sublocale K of L is the unique sublocale $\overline{K}$ of L satisfying,

1. $\overline{K}$ is a closed sublocale of L,
2. K is dense in $\overline{K}$

Definition 1.5.9. A lattice A is a *Heyting Algebra* if and only if for every $a, b \in A$ there is an element $a \to b$ satisfying $c \leqslant a \to b$ if and only if $c \wedge a \leqslant b$.

Definition 1.5.10. A *cover* in a frame L is a subset S of L with $\bigvee S = 1_L$.

1.6 Coproducts in frames

The construction of coproducts in frames was first presented in [12].

Definition 1.6.1. Let $R \subseteq A \times A$ be an arbitrary binary relation on a frame A. An element $s \in A$ is *R-saturated* if

$$aRb \Rightarrow (a \wedge c \leqslant s \Leftrightarrow b \wedge c \leqslant s)\ forall\ a, b, c \in A$$

Remark 1.6.1. Let A/R denotes the set of all saturated elements of A. Define $\nu : A \to A/R$ by $\nu(a) = \nu_R(a) = \bigwedge \{s \in A : a \leqslant s\}$ where s is saturated. Then ν is a surjective frame homomorphism.

For a semilattice A, define $\mathcal{D}(A) = \{U \subseteq A : \phi \neq U = \downarrow U\}$. Then $(\mathcal{D}(A), \subseteq)$ is a frame. Define $\lambda_A : A \to \mathcal{D}A$ by $\lambda_A(a) = \downarrow a$ which is a semilattice homomorphism between them. Let $A_i, i \in I$ be frames. Set $\prod'_{i\in I} A_i = \{(a_i)_{i\in I} \in \prod_{i\in I} A_i : a_i = 1 \text{ for all but finitely many i } \} \bigcup \{(0)_{i\in I}\}$ Define $\gamma_j : A_j \to \prod'_{i\in I} A_i$ by setting

$$(\gamma_j(a))_i = \begin{cases} a & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Consider the frame $\mathcal{D}(A)$ where $A = \prod'_{i\in I} A_i$.
$R = \left\{(\lambda_A\gamma_j(\bigvee_{m\in M} a_m), \bigvee_{m\in M} \lambda_A\gamma_j(a_m)) : j \in I, a_m \in A_j\right\}$ where M is any set is a relation.

Definition 1.6.2. The frame $\bigoplus_{i\in I} A_i = \mathcal{D}(\prod'_{i\in I} A_i)/R$, containing all R-saturated elements of the frame $\mathcal{D}(\prod'_{i\in I} A_i)$ is called the *frame coproduct*.

Definition 1.6.3. The mapping $\nu : \mathcal{D}(\prod'_{i\in I} A_i) \to \bigoplus_{i\in I} A_i$ be as defined above. The maps $p_j = \nu \circ \lambda \circ \gamma_j : A_j \to \bigoplus_{i\in I} A_i$ are frame homomorphisms, called *coproduct injections*. Also $\bigwedge_{i\in I} p_i(a_i) = \oplus(a_i)_{i\in I}$. Thus the set of all elements of the form $\oplus(a_i)_{i\in I}$ is a join basis of $\oplus(A_i)_{i\in I}$. The right adjoint of p_j denoted by ${p_j}^*$ is called the *projection* of the locale $\bigoplus_{i\in I} A_i$ to the locale A_j.

Remark 1.6.2. The set

$$\mathbf{O} = \{(a_i)_{i\in I} \in \prod\nolimits'_{i\in I} A_i : \textit{ there exists } i,\ a_i = 0\}$$

is saturated. It is the least element of the frame coproduct $\bigoplus_{i\in I} A_i$.

Theorem 1.6.1. *The set* $\oplus_{i\in I} a_i = \downarrow (a_i)_{i\in I} \cup \mathbf{O}$ *is saturated for any* $(a_i)_{i\in I} \in \prod'_{i\in I} A_i$.

Corollary 1.6.2. *If* $\oplus_{i\in I} a_i \leqslant \oplus_{i\in I} b_i$ *and* $a_i \neq 0$ *for all* i, *then* $a_i \leqslant b_i$ *for all* i.

Remark 1.6.3. Also note that $a \oplus b = \downarrow (a, b) \cup \mathbf{O}$ where $\mathbf{O} = \{(x, y) : x = 0 \textit{ or } y = 0\} = 0_{L\oplus L}$, is the bottom of $L \oplus L$.

Theorem 1.6.3. *Let* L_i, $i = 1, 2$ *be frames and* $a_i \in L_i$. *Then* $\downarrow a_1 \oplus \downarrow a_2 = \downarrow (a_1 \oplus a_2)$.

The following theorem is proved in [36].

Theorem 1.6.4. *For each* $U \in \oplus_{i\in I} L_i$, *the set* $U = \bigvee \{\oplus_{i\in I} a_i : \oplus_{i\in I} a_i \leqslant U\}$. *That is* $\oplus_{i\in I} a_i$ *forms the join basis for* $\bigoplus_{i\in I} L_i$

We state the following result proved in [22] for proving some results in this chapter.

Theorem 1.6.5. *Let $X_i, i \in I$ be family of spaces. Then $\oplus_i(\Omega X_i)$ is isomorphic to $\Omega(\oplus X_i)$ if and only if it is a spatial frame.*

1.7 Special Frames

At first we define the concept of ideals and filters in a frame.

Definition 1.7.1. [22] An *ideal* in a frame L is a nonempty subset I with the property that $0_L \in I$, $a \leqslant b \in I$ implies $a \in I$, and $a \vee b \in I$ whenever a and b are in I.

Definition 1.7.2. [22] A *filter* in a frame L is a nonempty subset F with the property that $0_L \notin F$, $a \geqslant b \in F$ implies $a \in F$, and $a \wedge b \in F$ whenever a and b are in F. An *ultrafilter* is a maximal filter.

We state the following results proved in [22] for proving some results in this chapter.

Theorem 1.7.1. *let I be an ideal of a lattice A, and F a filter disjoint from I. Then there exists an ideal M of A which is maximal amongst those containing I and disjoint from F.*

Theorem 1.7.2. *Let F be a filter in a distributive lattice A, and I an ideal which is maximal amongst those disjoint from F. Then I is prime.*

Now we give the definitions for some special frames which we encounter in the coming chapters.

Definition 1.7.3. [22] A frame L is said to be *compact* if each subset A of L with $\bigvee A = 1_L$ has a finite subset $B \subseteq A$ with $\bigvee B = 1_L$.

Definition 1.7.4. [34] A frame L is called an *almost compact* frame if whenever $\bigvee \{x_i : i \in I\} = 1_L$ then there exists a finite subset $K \subseteq I$ of the index set I such that $(\bigvee \{x_i : i \in K\})^{**} = 1_L$ where "$*$" denotes the pseudo-complementation operator in L.

Theorem 1.7.3. *[34]*

1. *A compact frame is almost compact.*
2. *A frame L is not almost compact if and only if an ideal Q in L exists such that $Q \subseteq S_L = \{l \in L : l^* = 0_L\}$ and $\bigvee Q = 1_L$.*

Theorem 1.7.4. *[22] The product of compact locales is compact.*

Definition 1.7.5. [36] Let $a, b \in L$. A frame L is said to be *subfit(conjunctive)* if $a \nleqslant b \Rightarrow$ there exists c such that $a \vee c = 1_L \neq b \vee c$.

Definition 1.7.6. [36] Let $a, b \in L$. The relation $a \prec b$ holds if $a^* \vee b = 1_L$ where $*$ denotes the pseudocomplementation operator in L.

Theorem 1.7.5. *[36] Let L be any frame. Then the following rules hold in L.*

1. *$0_L \prec a \prec 1_L$ for any $a \in L$.*
2. *$a \prec b$ implies $a \leqslant b$.*

3. $x \leqslant a \prec b \leqslant y$ *implies* $x \prec y$.

4. *If* $a \prec b$, *then* $b^* \prec a^*$.

5. *If* $a \prec b$, *then* $a^{**} \prec b$.

6. *If* $a_i \prec b_i$ *for* $i = 1,\ 2,$ *then* $a_1 \vee a_2 \prec b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

Definition 1.7.7. [36] The relation $a \prec\prec b$ holds if there are $x_r \in L$ for r dyadic rational in the interval (0,1) such that $x_0 = a, x_1 = b$ and $x_r \prec x_s$ for $r < s$.

Theorem 1.7.6. *[36] Let L be any frame. Then the following rules hold in L.*

1. $0_L \prec\prec a \prec\prec 1_L$ *for any* $a \in L$.

2. $a \prec\prec b$ *implies* $a \leqslant b$.

3. $x \leqslant a \prec\prec b \leqslant y$ *implies* $x \prec\prec y$.

4. *If* $a \prec\prec b$, *then* $b^* \prec\prec a^*$.

5. *If* $a \prec\prec b$, *then* $a^{**} \prec\prec b$.

6. *If* $a_i \prec\prec b_i$ *for* $i = 1, 2,$ *then* $a_1 \vee a_2 \prec\prec b_1 \vee b_2$ *and* $a_1 \wedge a_2 \prec\prec b_1 \wedge b_2$.

Definition 1.7.8. [36] The frame L is said to be a *regular* frame if $a = \bigvee \{x \in L / x \prec a\}$ for all $a \in L$.

Theorem 1.7.7. *[22] A compact regular locale is spatial.*

Theorem 1.7.8. *[22] The product of regular locales is regular.*

Definition 1.7.9. [36] The frame L is said to be *completely regular* if $a = \bigvee \{x \in L / x \prec\prec a\}$ for all $a \in L$.

Definition 1.7.10. [36] A frame L is *normal* if whenever $a \vee b = 1_L$ for $a, b \in L$, there exist $u, v \in L$ with $u \wedge v = 0_L$ such that $u \vee b = 1_L$ and $a \vee v = 1_L$.

We take the definition for Hausdorff frame as given by Isbell [18] throughout this thesis.

Definition 1.7.11. A frame A is called a *Hausdorff frame* if for any $U \in A \oplus A$, the codiagonal $\nabla : A \oplus A \to A$ defined by

$$\nabla(U) = \bigvee \{a \wedge b : (a, b) \in U\}$$

is a closed sublocale.

Remark 1.7.1. For any frame L,

1. Set $d_L = \bigvee \{x \oplus y : x \wedge y = 0_L\} \in L \oplus L$.
2. Also note that $\nabla (a \oplus b) = a \wedge b$ where $a, b \in L$.

The following result is proved in [36].

Theorem 1.7.9. *A compact Hausdorff locale is regular.*

For proof of the following theorem, see [36].

Theorem 1.7.10. *A frame L is Hausdorff if and only if for any $a,\ b \in L,\ a \oplus b \leqslant ((a \wedge b) \oplus (a \wedge b)) \vee d_L$.*

The following theorem tells which of the above concepts are extensions of the classical ones in topology and for proofs refer [36].

Theorem 1.7.11. *[22] Let $(X, \Omega X)$ be any topological space. Then $(X, \Omega X)$ is*

1. *compact if and only if ΩX is compact,*
2. *regular if and only if ΩX is regular,*
3. *completely regular if and only if ΩX is completely regular,*
4. *normal if and only if ΩX is normal.*

In contrary to the above, Hausdorffness is not an extension of the classical Hausdorff axiom in topology. The following theorem gives only a sufficient condition. We end up this section with the following theorem from [22].

Theorem 1.7.12. *If ΩX is Hausdorff, then the T_0 topological space $(X, \Omega X)$ is Hausdorff. The converse need not be true.*

Chapter 2

Singly Generated Extension of Frames

2.1 Introduction

The construction of enlarging the topology of a given space by adding a new open set is a familiar type of topological construction. This can be algebraically viewed as extending a given frame by adjoining a new element. The conditions under which the simple extension of a topological space having a specified topological property also holds that property was studied by N.Levine[26]. The same notion called singly generated extension in frames was introduced by B. Banaschewski[2]. In this chapter, we discuss

Some results of this chapter are included in the following paper.
P.N., Jayaprasad : On Singly Generated Extension of a Frame, Accepted for publication in Bulletin of Allahabad Mathematical Society, 2013.

the conditions under which a singly generated extension of the frame A possesses a property that is already owned by the frame A.
We start with the definition of *singly generated extension of a frame* due to Banaschewski appeared in his paper[2].

Definition 2.1.1. A frame M is called a *singly generated extension* of a frame A if A is a subframe of M, and M is generated by A and some $b \in M$. We write $M = A[b]$.

The next theorem and the first remark is due to B. Banaschewski[2].

Theorem 2.1.1. *Let L be any frame. Let A be a subframe of L. Let $b \in L - A$. Then $A[b] = \left\{a \vee (a^{'} \wedge b) : a, a^{'} \in A\right\}$ where $\vee$ and $\wedge$ are respectively the join and meet operations in L is a subframe of L.*

Remark 2.1.1. We can take $a \leqslant a^{'}$ in the above description of $A[b]$ because $a \vee (a^{'} \wedge b) = a \vee (a \wedge b) \vee (a^{'} \wedge b) = a \vee ((a \vee a^{'})) \wedge b)$.

Remark 2.1.2. If $x \in A[b]$, then $x \vee b = 1$ because there exists $a_1, a_2 \in A$ such that $x = a_1 \vee (a_2 \wedge b)$, so that $x \vee b = a_1 \vee (a_2 \wedge b) \vee b = a_1 \vee b = a_1 \vee (1 \wedge b) \in A[b]$.

The following corollary is proved in [36]

Corollary 2.1.2. *Every frame is isomorphic to a subframe of a complete Boolean algebra.*

The results proved in this chapter require the existence of complement of the element b added to the frame A which in general need not happen. Let ϕ represents the frame isomorphism which makes $A[b]$ isomorphic to a subframe of a Boolean frame B according to *corollary* 2.1.2, then $\phi(A[b]) = \{\phi(a) \vee (\phi(a') \wedge \phi(b)) : a, a' \in A\}$ will be a subframe of the Boolean frame B which is the singly generated extension of the frame $\phi(A)$ in B on adding the element $\phi(b)$ in $B - \phi(A)$. The complement of $\phi(b)$ exists in B as B is a Boolean frame. Even if the complement of the added element "b" does not exist in the given frame "A", we can consider the frame isomorphic copy of "A" and the image of the element "b" under the frame isomorphism ϕ in the embedded Boolean frame B. Now the situation is what we discuss here and can determine whether the singly generated extension $\phi(A[b])$ in the Boolean frame preserves the specified frame isomorphic property when $\phi(b)$ is added and if it is so, then definitely the singly generated extension of A on adding b also preserves the frame isomorphic property.

Theorem 2.1.3. *Let A be any subframe of L. Let $b \in L - A$ and let $A[b]$ be the singly generated extension of A in L. Then $b^c \in A$ if and only if $b^c \in A[b]$.*

Proof. If $b^c \in A$, then $b^c = b^c \vee (0 \wedge b) \in A[b]$, by definition of $A[b]$.
Conversely assume that $b^c \in A[b]$.
Then, $b^c = a \vee (a' \wedge b)$ where $a \leqslant a'$ and $a, a' \in A$.
Now, $1 = b \vee b^c = [a \vee (a' \wedge b)] \vee b = a \vee [(a' \wedge b) \vee b] = a \vee b$.
Also $0 = b \wedge b^c = b \wedge [a \vee (a' \wedge b)] = (b \wedge a) \vee (b \wedge a')$. Thus $a \wedge b = 0$.
Hence $b^c = a \in A$. $\square$

In the following sections we proceed to investigate whenever A has a frame isomorphic property **p**, under what conditions $A[b]$ also has the property **p**.

2.2 Singly Generated Extension and Compactness

In this section, we find the conditions under which a compact subframe A of the frame L still remains compact when extended singly to a frame by adding a single element $b \in L - A$. Every subframe of a compact frame is compact but a sublocale of a compact frame need not be compact.A closed sublocale of a compact frame is compact. A topological space is compact if and only if the frame of opens is compact, by *Theorem* 1.7.11. Thus compactness in frames is equivalent to compactness in topology when the frame is a spatial frame.
We introduce the following definition for further discussion.

Definition 2.2.1. Let L be any frame and A be a subframe. An element $b \in L$ is said to be *compact relative to the subframe* A if for every $S \subseteq A$ with $\bigvee S \geqslant b$, there exists $F \subseteq S$ with F finite and $\bigvee F \geqslant b$.

Theorem 2.2.1. *Let A be a subframe of the frame L and $b \in L - A$ be complemented in L. Consider the following statements about $A[b]$.*
(1) $A[b]$ is compact.
(2) b^c is compact relative to $A[b]$.
Then, the following statements hold.
(a) Statement (1) implies statement (2).
(b) If A is compact, then (1) and (2) are equivalent.

Proof. (a) Suppose that (1) holds and let $S \subseteq A[b]$ with $\bigvee_{A[b]} S \geqslant b^c$. Then $(\bigvee_{A[b]} S) \vee b = 1$, which implies $\bigvee_{A[b]} \{s \vee b : s \in S\} = 1$. Since $s \vee b \in A[b]$ by remark 2, for every $s \in S$, the compactness of $A[b]$ implies that there are finitely many elements $s_1, s_2, \dots s_m$ in S such that $b \vee s_1 \vee s_2 \vee \dots \vee s_m = 1$. This implies that $s_1 \vee s_2 \vee \dots \vee s_m \geqslant b^c$, whence b^c is compact relative to $A[b]$.

(b) Assume that A is compact and b^c is compact relative to $A[b]$. Let $S = \{s_i : i \in I\}$ be a cover of $A[b]$. For each $i \in I$, let $s_i = a_i \vee (b_i \wedge b)$ where $a_i \leqslant b_i$ with $a_i, b_i \in A$. Since $s_i = a_i \vee (b_i \wedge b) \leqslant b_i \vee (b_i \wedge b) = b_i$ for each i, we have $\bigvee b_i = 1$. By compactness of A, there is a finite $J \subseteq I$ such that $\bigvee_{A[b]} \{b_i : i \in J\} = 1$. Since b^c is compact relative to $A[b]$, there is a finite $K \subseteq I$ such that $b^c \leqslant \bigvee_{A[b]} \{s_i : i \in K\}$. Set $H = J \cup K$, and note that H is a finite subset of I. Since $\bigvee_{A[b]} b_i = 1$, we have that $b = \bigvee_{A[b]} \{b_i \wedge b : i \in H\}$ and hence

$$\begin{aligned} 1 = b \vee b^c &\leqslant \bigvee\nolimits_{A[b]} \{b_i \wedge b : i \in H\} \vee \bigvee\nolimits_{A[b]} \{s_i : i \in H\} \\ &= \bigvee\nolimits_{A[b]} \{s_i \vee (b_i \wedge b) : i \in H\} \\ &= \bigvee\nolimits_{A[b]} \{s_i : i \in H\} \end{aligned}$$

This shows that $A[b]$ is compact. □

We can derive *Theorem 6* of [26] as a simple corollary of the above theorem.

Corollary 2.2.2. *Let (X, τ) be a compact topological space and let $A \notin \tau$. Then $(X, \tau(A))$, the simple extension in the sense of [26], is compact if and only if A^c is compact in (X, τ).*

Proof. A topological space is compact if and only if the frame of its open sets is compact, by *Theorem* 1.7.11. Now the proof follows from

Theorem 2.1.3 and *Theorem* 2.2.1. □

2.3 Singly Generated Extension and Regularity

In this section, we discuss the separation axioms reguarity and complete regularity in connection with singly generated extension of a frame. It is known that a regular frame is always Hausdorff. Also a completely regular frame is always regular. The condition under which the singly generated extension $A[b]$ is Hausdorff provided A is Hausdorff is that there exists $c \in A$ such that $c \vee b, c \wedge b \in A$. This is proved by B. Banaschewski in [2]. We examine the conditions under which a regular(completely regular) frame is again reguar(completely regular) when extended by adding a single element. The following lemma finds application in the proof of the main result in this section.

Lemma 2.3.1. Let A be any subframe of the frame L. Let $b \in L - A$ be complemented in L and let $b^c \in A$. Denote by $\preceq$ and $\preceq\preceq$ the rather below and the completely below relations in $A[b]$.Then the following statements hold.
(a) $x \prec a$ and $y \prec a'$ in A imply $x \vee (y \wedge b) \preceq a \vee (a^{'} \wedge b)$.
(b) $x \prec\prec a$ and $y \prec\prec a'$ in A imply $x \vee (y \wedge b) \preceq\preceq a \vee (a^{'} \wedge b)$.

Proof. Let $p = a \vee (a^{'} \wedge b)$.
(a) Since $x \prec a, y \prec a^{'}$ we have

$$x^{*} \vee a = 1, y^{*} \vee a^{'} = 1 \tag{2.1}$$

where x^* and y^* are the pseudocomplements of x and y in A respectively. Let $\times$ denotes the pseudocomplementation with respect to $A[b]$. Since $x^* \leqslant x^\times$, $y^* \leqslant y^\times$, from equation 2.1 we have

$$x^\times \vee a = 1, y^\times \vee a^{'} = 1 \tag{2.2}$$

Then $x^\times \vee p = 1$ using equation 2.2. Also $y^\times \vee p = y^\times \vee a \vee (a^{'} \wedge b) \geqslant (y^\times \vee a^{'}) \wedge (y^\times \vee b) = y^\times \vee b$ by equation 2.2.
$A[b]$ satisfies the De Morgan's law $(\alpha \vee \beta)^\times = \alpha^\times \wedge \beta^\times$ on pseudocomplements and $(\alpha \wedge \beta)^\times \geqslant \alpha^\times \vee \beta^\times$ because $A[b]$ is a distributive pseudocomplemented lattice. Now $[x \vee (y \wedge b)]^\times \vee p \geqslant (x^\times \vee p) \wedge (y^\times \vee b^c \vee p) = y^\times \vee b^c \vee p \geqslant y^\times \vee b^c \vee b = 1$. So $[x \vee (y \wedge b)]^\times \vee p = 1$. Thus $x \vee (y \wedge b) \preceq p$.

(b) Take $x_0 = x, x_1 = a$ and $y_0 = y, y_1 = a^{'}$. Set $p_0 = x_0 \vee (y_0 \wedge b) = x \vee (y \wedge b)$ and $p_1 = x_1 \vee (y_1 \wedge b) = p$. Let l, m are two dyadic rational numbers in (0,1) with $l < m$. Then by definition of $x \prec\prec a$, we have $x_l, x_m \in A$ with $x_l \prec x_m$. Similarly $y \prec\prec a^{'}$, gives $y_l, y_m \in A$ with $y_l \prec y_m$. Under the assumption $b^c \in A$, repeating steps in (a), $x_l \prec x_m$ and $y_l \prec y_m$ implies $x_l \vee (y_l \wedge b) = p_l \preceq p_m = x_m \vee (y_m \wedge b)$. Thus $x \vee (y \wedge b) \preceq\preceq p$. □

Theorem 2.3.1. *Let A be a regular subframe of the frame L. Let $b \in L - A$ with the complement b^c of b exists in L. Then $A[b]$ is regular if $b^c \in A$.*

Proof. Let $p \in A[b]$. Then $p = c \vee (d \wedge b)$ where $c, d \in A$ and of course $c \leqslant d$. Since A is regular, we can write $c = \bigvee_A \{x \in A : x \prec c\}$ and

$d = \bigvee_A \{x \in A : x \prec d\}$. Now

$$\begin{aligned} p &= \left(\bigvee_A \{x \in A : x \prec c\}\right) \vee \left(\bigvee_A \{y \in A : y \prec d\} \wedge b\right) \\ &= \left(\bigvee_{A[b]} \{x \in A : x \prec c\}\right) \vee \left(\bigvee_{A[b]} \{y \wedge b : y \prec d, y \in A\}\right) \\ &= \bigvee_{A[b])} \{x \vee (y \wedge b) : x \prec c, y \prec d, x, y \in A\}. \\ &= \bigvee_{A[b]} \{x \vee (y \wedge b) : x \vee (y \wedge b) \preceq p\} \end{aligned}$$

using *Lemma* 2.3.1. Hence $A[b]$ is regular. □

We can derive *Theorem 2* of [26] as a simple corollary of the above theorem.

Corollary 2.3.2. *Let (X, τ) be a regular topological space and $A \notin \tau$. Then $(X, \tau(A))$, the simple extension in the sense of [26], is regular if $A^c \in \tau$.*

Proof. A topological space is regular if and only if the frame of its open sets is regular. Now the proof follows from *Theorem* 2.3.1. □

Theorem 2.3.3. *Let A be a completely regular subframe of the frame L. Let $b \in L - A$ with the complement b^c of b exists in L. Then $A[b]$ is completely regular if $b^c \in A$.*

Proof. Let $p \in A[b]$. Then $p = c \vee (d \wedge b)$ where $c, d \in A$ where $c \leqslant d$. Since A is completely regular, we can write $c = \bigvee_A \{x \in A : x \prec\!\prec c\}$ and

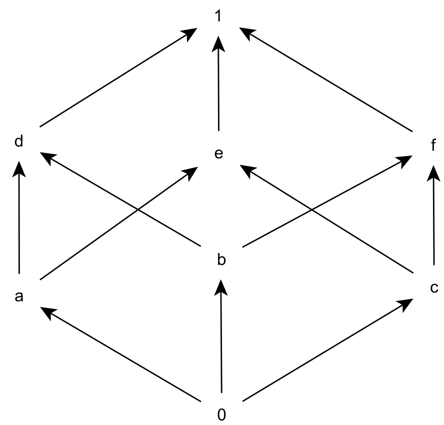

Figure 2.1:

$d = \bigvee_A \{x \in A : x \prec\prec d\}$. Now

$$\begin{aligned} p &= \left(\bigvee_A \{x \in A : x \prec\prec c\}\right) \vee \left(\bigvee_A \{y \in A : y \prec\prec d\} \wedge b\right) \\ &= \left(\bigvee_{A[b]} \{x \in A : x \prec\prec c\}\right) \vee \left(\bigvee_{A[b]} \{y \wedge b : y \prec\prec d, y \in A\}\right) \\ &= \bigvee_{A[b]} \{x \vee (y \wedge b) : x \prec\prec c, y \prec\prec d, x, y \in A\}. \\ &= \bigvee_{A[b]} \{x \vee (y \wedge b) : x \vee (y \wedge b) \preceq\preceq p\} \end{aligned}$$

using *Lemma* 2.3.1. Hence $A[b]$ is completely regular. □

We can derive *Theorem 4* of [26] as a simple corollary of the above theorem.

Corollary 2.3.4. *Let (X, τ) be a completely regular topological space and $A \notin \tau$. Then $(X, \tau(A))$, the simple extension in the sense of*

[26], is completely regular if $A^c \in \tau$.

Proof. A topological space is completely regular if and only if the frame of its open sets is completely regular. Now the proof follows from *Theorem* 2.3.3. $\square$

The following example shows that if the condition in the above theorems is dropped, then the singly generated extension need not be regular.

Example 2.3.1. Consider a subframe $A = \{0, a, f, 1\}$ of the frame L in Fig.1. A is completely regular. Now consider the element $b \in L - A$. Then the singly generated extension of the frame A on adding b is $A[b] = \{0, a, b, d, f, 1\}$ where $b^c = e$ is not in A. It is easy to see that $A[b]$ is not regular, because the only nonzero element with the property $x \prec d$ in the subframe A is f and hence d cannot be written as $d = \bigvee_A \{x : x \prec d\}$.

2.4 Singly Generated Extension and Normality

Theorem 2.4.1. *Let A be a normal subframe of the frame L. Let $b \in L - A$ with the complement b^c of b exists in L. Then $A[b]$ is normal if and only if the open quotient $\downarrow b^c = \{b^c \wedge a : a \in A\}$ is normal.*

Proof. Suppose $A[b]$ is normal
We show that $\downarrow b^c$ is normal. Let $x, y \in \downarrow b^c$ where $x \vee y = b^c = 1_{\downarrow b^c}$. Let $x = b^c \wedge x_0, y = b^c \wedge y_0$ where $x_0, y_0 \in A$. Now $b^c = x \vee y =$

$(b^c \wedge x_0) \vee (b^c \wedge y_0) = b^c \wedge (x_0 \vee y_0) \leqslant x_0 \vee y_0$. Then $(x_0 \vee b) \vee (y_0 \vee b) \geqslant b \vee b^c = 1$. Thus $(x_0 \vee b) \vee (y_0 \vee b) = 1$. Since $A[b]$ is normal, there exist $u, v \in A[b]$ with $u \wedge v = 0$ and $x_0 \vee b \vee v = 1, u \vee y_0 \vee b = 1$. Let $u = u_1 \vee (u_2 \wedge b)$ where $u_1 \leqslant u_2, u_1, u_2 \in A$ and let $v = v_1 \vee (v_2 \wedge b)$ where $v_1 \leqslant v_2, v_1, v_2 \in A$. Then $u \wedge v = 0$ implies $u_1 \wedge v_1 = 0$. Also
$1 = x_0 \vee b \vee v = x_0 \vee b \vee [v_1 \vee (v_2 \wedge b)] = x_0 \vee v_1 \vee [b \vee (v_2 \wedge b)] = x_0 \vee v_1 \vee b$
Similarly $y_0 \vee u_1 \vee b = 1$. Thus

$$x_0 \vee v_1 \vee b = 1, y_0 \vee u_1 \vee b = 1, u_1 \wedge v_1 = 0 \tag{2.3}$$

Let $v_1 \wedge b^c = p, u_1 \wedge b^c = q$. Then $p, q \in \downarrow b^c$ because $u_1, v_1 \in A$. Now $p \wedge q = (v_1 \wedge b^c) \wedge (u_1 \wedge b^c) = b^c \wedge u_1 \wedge v_1 = 0$ from equation 2.3. Also $x \vee p = x \vee (v_1 \wedge b^c) = (b^c \wedge x_0) \vee (b^c \wedge v_1) = (x_0 \vee v_1) \wedge b^c = (x_0 \vee v_1 \vee b) \wedge b^c = 1 \wedge b^c = b^c = 1_{\downarrow b^c}$ from equation 2.3. Similarly $y \vee q = 1_{\downarrow b^c}$. Thus $\downarrow b^c$ is normal.

Conversely assume that $\downarrow b^c$ is normal. Let $x, y \in A[b]$ with $x \vee y = 1$. Let $x = c \vee (d \wedge b)$ where $c \leqslant d$; $c, d \in A$ and $y = e \vee (f \wedge b)$ where $e \leqslant f$; $e, f \in A$. Now
$1 = x \vee y = [c \vee (d \wedge b)] \vee [e \vee (f \wedge b)] = (c \vee e \vee d \vee f) \wedge (c \vee e \vee b) = (d \vee f) \wedge (c \vee e \vee b)$ which yield

$$d \vee f = 1, c \vee e \vee b = 1 \tag{2.4}$$

Since A is normal, there exist $u, v \in A$ such that

$$u \wedge v = 0, d \vee v = 1, u \vee f = 1 \tag{2.5}$$

Consider $c \wedge b^c, (e \vee b) \wedge b^c$ in $\downarrow b^c$. Now $(c \wedge b^c) \vee [(e \vee b) \wedge b^c] = (c \vee e \vee b) \wedge b^c = b^c = 1_{\downarrow b^c}$ using equation 2.4. Since $\downarrow b^c$ is normal there exist $p, q \in \downarrow b^c, p = p_0 \wedge b^c, q = q_0 \wedge b^c$ and $p_0, q_0 \in A$ with $p \wedge q = 0, b^c = (c \wedge b^c) \vee q = (c \wedge b^c) \vee (q_0 \wedge b^c) = (c \vee q_0) \wedge b^c, b^c = [(e \vee b) \wedge b^c] \vee p = (e \wedge b^c) \vee p = (e \wedge b^c) \vee (p_0 \wedge b^c) = (e \vee p_0) \wedge b^c$. Thus we have $c \vee q_0 \geqslant b^c, e \vee p_0 \geqslant b^c$. Hence $c \vee q_0 \vee b \geqslant b^c \vee b = 1, e \vee p_0 \vee b \geqslant b^c \vee b = 1$. Thus we get

$$c \vee q_0 \vee b = 1, e \vee p_0 \vee b = 1 \tag{2.6}$$

Set $\alpha = q \vee (v \wedge b), \beta = p \vee (u \wedge b)$. It is clear that $\alpha, \beta \in A[b]$.
Now $\alpha \wedge \beta = [q \vee (v \wedge b)] \wedge [p \vee (u \wedge b)] = (q \wedge p) \vee (p \wedge v \wedge b) \vee (q \wedge u \wedge b) \vee (u \wedge v \wedge b) = (p_0 \wedge b^c \wedge v \wedge b) \vee (q_0 \wedge b^c \wedge u \wedge b) = 0$, since $p \wedge q = u \wedge v = 0$. Also

$$\begin{aligned}
x \vee \alpha &= [c \vee (d \wedge b)] \vee [q \vee (v \wedge b)] \\
&= (c \vee q) \vee [(d \vee v) \wedge b] \\
&= (c \vee q \vee d \vee v) \wedge (c \vee q \vee b) \\
&= (d \vee q \vee v) \wedge (c \vee q \vee b) \\
&= (d \vee (q_0 \wedge b^c) \vee v) \wedge (c \vee (q_0 \wedge b^c) \vee b) \\
&= (d \vee v \vee q_0) \wedge (d \vee v \vee b^c) \wedge (c \vee q_0 \vee b) \wedge (c \vee b \vee b^c) \\
&= 1
\end{aligned}$$

using equations 2.5 and 2.6.
Similarly we can show that $y \vee \beta = 1$. Hence $A[b]$ is normal. □

We can derive *Theorem 5* of [26] as a simple corollary of the above theorem.

Corollary 2.4.2. *Let (X, τ) be a normal topological space and $A \notin \tau$, $A^c \in \tau$. Then $(X, \tau(A))$, the simple extension in the sense of [26], is normal if and only if $(A^c, \tau \cap A^c)$ is normal.*

Proof. A topological space is normal if and only if the frame of its open sets is normal. Now the proof follows from *Theorem* 2.4.1. $\square$

Chapter 3

Maximal Compact Frames

3.1 Introduction

In topological spaces, a closed subspace of a compact space is compact and a compact subspace of a Hausdorff space is closed. Thus in a compact Hausdorff space, closed subspaces coincide with compact subspaces. In 1948 A. Ramanathan [42] proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed sets. A topological space in which the closed subspaces are precisely the compact subspaces are called *C-C Spaces.* N.Levine made a study on such spaces [27] and proved that product of maximal compact spaces are not necessarily maximal compact . In this chapter, we extend some of his results into

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Some results of this chapter are included in the following paper.
P.N., Jayaprasad and Johnson, T.P. : On Frames with Closed Sublocales Equivalent to Compact Sublocales, Submitted.

the frame theoretic set up. The following are some of the results proved in [27].

Theorem 3.1.1. *If (X, τ) is a compact Hausdorff space, then τ is M.R.C.*

Theorem 3.1.2. *Suppose that (X, τ) is a topological space. Then (X, τ) is C-C if and only if τ is M.R.C.*

Theorem 3.1.3. *Let (X, τ) be a topological space. If (X, τ) is C-C, then it is compact and T_1.*

Theorem 3.1.4. *Let (X, τ) be a topological space and let $(X \times X, \tau^*)$ be the cartesian product of (X, τ) with itself. Then $(X \times X, \tau^*)$ is C-C if and only if (X, τ) is C-C and Hausdorff.*

3.2 Closed and Compact Equivalent Frames

It is known that[22] every closed sublocale of a compact locale is compact and every compact sublocale of a regular locale is closed. Hence in compact regular locales closed sublocales coincide with compact sublocales. We try to answer when does a closed sublocale equivalent to a compact sublocale. This leads to our definition for what is named as CCE frames. We also formulate some characterizations of CCE frames(locales).

Definition 3.2.1. A frame A is called a *Closed and Compact Equivalent Frame(CCE Frame)* if the closed sublocales of A coincide with the compact sublocales of A.

Any compact regular frame is a CCE frame as closed sublocales of a compact locale are compact and compact sublocales of a regular locale are closed. In topology, compact Hausdorff spaces are maximal compact[27]. We, therefore , examine whether such frames possess this maximality. We introduce the following definition.

Definition 3.2.2. A frame A is said to be *maximal relative to compactness(M.R.C.)* if,

1. A is compact,
2. if A is a proper subframe of the frame L, then L is not compact.

Lemma 3.2.1. Let the frame $A \subseteq L$ be M.R.C. Then $\uparrow_L a$ is compact for $a \in A$ if and only if $\uparrow_L a$ contains no element of $L - A$.

Proof. Suppose that $\uparrow_L a$ is compact
Let $b \in L - A$ and $b \in \uparrow_L a$. Consider the singly generated extension $A[b]$ of the frame A by adding the element b. We prove that $A[b]$ is compact. Let $S \subseteq A[b]$ with $\bigvee S = 1$. Let $S = \left\{a_i \vee (a_i{}' \wedge b) : a_i, a_i{}' \in A, a_i \leqslant a_i{}', i \in I\right\}$. Now $1 = \bigvee S = (\bigvee a_i) \vee [(\bigvee a_i{}') \wedge b] = (\bigvee a_i{}') \wedge [(\bigvee a_i) \vee b] = (\bigvee a_i{}') \wedge [\bigvee (a_i \vee b)]$. Thus $\bigvee_{i \in I} a_i{}' = 1$ and $\bigvee_{i \in I}(a_i \vee b) = 1$. Since A is compact, there exists a finite subset $J_1 \subseteq I$ with $\bigvee_{j_1 \in J_1} a_{j_1}{}' = 1$. Also $a_i \vee b \geqslant b \geqslant a$ and hence $a_i \vee b \in \uparrow_L a$. Since $\uparrow_L a$ is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2}(a_{j_2} \vee b) = 1$. Set $J = J_1 \bigcup J_2$ and $F = \left\{a_j \vee (a_j{}' \wedge b) : j \in J\right\}$. Clearly $F \subseteq S$ and F is finite. Then $\bigvee F = (\bigvee a_j) \vee [(\bigvee a_j{}') \wedge b] = (\bigvee a_j{}') \wedge (\bigvee (a_j \vee b)) = 1 \wedge 1 = 1$, because $J_1 \subseteq J, J_2 \subseteq J$ and $\bigvee_{j_1 \in J_1} a_{j_1}{}' = 1, \bigvee_{j_2 \in J_2}(a_{j_2} \vee b) = 1$. Hence $A[b]$ is compact. Since $A \subset A[b]$ and A is M.R.C., this leads to a contradiction

as there is no strictly larger frame that is compact. Thus our assumption that $b \in L - A$ and $b \in \uparrow_L a$ is wrong. Thus $\uparrow_L a$ contains no element of $L - A$.
Conversely suppose that $\uparrow_L a$ contains no element of $L - A$.
Then $\uparrow_L a = \uparrow_A a$. But $\uparrow_A a$ is compact as it is a closed sublocale of A which is compact being M.R.C. Thus $\uparrow_L a$ is compact for $a \in A$. $\square$

We state the following definition due to J.Paseka and B.$\breve{S}$marda[34] for proving the next result.

Definition 3.2.1. Define $F_C = \{a \in L : \uparrow a \text{ is compact in L }\}$. Then the locale generated by the set $\{(l, 0_L) : l \in L\} \bigcup \{(a, 1) : a \in F_C\}$ is defined as L_{F_C}. L_{F_C} is a compact locale called the *one point compactification* [34] of L.

The next thoeorem is proved in [36]

Theorem 3.2.2. *An image of a compact sublocale $S \subseteq L$ under a localic map $f : L \to M$ is compact.*

Now we state and prove the main theorem in connection with CCE Frames.

Theorem 3.2.3. *Let A be any frame. Then it is a CCE frame if and only if it is M.R.C.*

Proof. Assume that A is a CCE Frame. Suppose A is not M.R.C. Then there exists a frame B such that $A \subset B$ and B is compact. Let us assume that these two frames are subframes of a boolean frame L(Otherwise we can consider the isomorphic copies of them in a Boolean frame, according to *Corollary* 2.1.2 that any frame is isomorphic to a subframe of a complete Boolean algebra). Let $b \in B - A$ with b^c exists in L. Consider the singly generated extension $A[b]$ of the frame A by adding the element b. Clearly $A[b] \subseteq B$. Since B is compact and every subframe of a compact frame is compact, we have $A[b]$ is compact. Then by *Lemma* 2.2.1, b^c is compact relative to $A[b]$ and hence relative to A. Hence $\downarrow b^c$ is compact regarding it as a locale itself.

Case 1: Suppose $b^c \in A$

Define $\mathbf{o}(b^c) = \{x \in A : b^c \to x\} = \{x \in A : b^c \to x = x\}$. We know that $\mathbf{o}(b^c)$ is a sublocale and we claim that it is compact in A.

For, it is the image of $\downarrow b^c$ regarded as a locale under the localic map obtained as the adjoint of the frame homomorphism $j : A \to \downarrow b^c$ defined by $x \to b^c \wedge x$ and since $\downarrow b^c$ is compact as a locale, $\mathbf{o}(b^c)$ is compact in A, by *Theorem* 3.2.2. Now we prove that $\mathbf{o}(b^c)$ is not closed in A. Suppose that $\mathbf{o}(b^c)$ is closed in A. Then there exists $y \in A$ such that $\mathbf{o}(b^c) = \uparrow_A y$. Then $y \in \uparrow_A y = \mathbf{o}(b^c)$. Since $0 \in \mathbf{o}(b^c)$, we have $0 \in \uparrow_A y$ and hence $y = 0$. Then $\mathbf{o}(b^c) = A$. Thus $b^c = 1$ and hence $b = 0$. But $b \in B - A$ and hence $b \neq 0$. Thus we get a contradiction and hence $\mathbf{o}(b^c)$ is not closed in A. Thus the sublocale K is compact but not closed in A. This is a contradiction to the assumption that A is a CCE Frame. Thus A is M.R.C. in this case.

Case 2: Suppose $b^c \notin A$

Let $p = \bigwedge\{x \in A : x \geqslant b^c\}$. Then $p \neq 1$. For, if $p = 1$, then the only element $x \in A$ with $x \geqslant b^c$ is 1. Then the filter $F = \uparrow_A b^c$ is disjoint from the ideal $I = \{x \in A : x \leqslant b^c\}$ in L as $b^c \notin A$. Now, by *Theorem* 1.7.1, there exists a maximal ideal $M \subseteq A$ containing I and disjoint from F. Then, by *Theorem* 1.7.2, M is a prime ideal. Now $b^c \wedge b = 0 \in M$. Since $b^c \notin A$ and M is a prime ideal, $b \in A$, which is not true. Hence $p \neq 1$.

Consider $\downarrow_A p$. We prove that $\downarrow_A p$ is compact but not closed in A. For, it needs to prove that p is compact relative to A. Let $\bigvee S = p$ where $S \subseteq A$. Then $\bigvee S \geqslant b^c$ since $p \geqslant b^c$. Since b^c is compact relative to A, there exists a finite subset $F \subseteq S$ with $\bigvee F \geqslant b^c$. Hence $\bigvee F \in \{x \in A : x \geqslant b^c\}$ and we get $\bigvee F \geqslant p$. Also $F \subseteq S$ and hence $\bigvee F \leqslant p$. Combining we get $\bigvee F = p$ where $F \subseteq S$ is finite. Hence p is compact relative to A.
Now $\mathbf{o}(p)$ can be proved to be a compact sublocale but not closed, by repeating the proof in case 1 with b^c replaced by p.
Thus we have a compact but not closed sublocale of A and this contradict the fact that A is a CCE Frame. Hence in this case A must be M.R.C.

Assume that A is M.R.C. in a frame L. Since every closed sublocale of a compact frame is compact, it needs to prove that every compact sublocale of A is closed in A. Let $K \subseteq A \subseteq L$ where K is a compact sublocale of A. Assume the contrary that K is not closed in A.
Case 1: K is closed in L.
Then there exists $\alpha \in L - A$ such that $K = \uparrow_L \alpha$. Consider the singly generated extension $A[\alpha]$ of the frame A by adding the element α. We prove that $A[\alpha]$ is compact. Let $S \subseteq A[\alpha]$ with $\bigvee S = 1$.
Let $S = \left\{a_i \vee (a_i{}' \wedge \alpha) : a_i, a_i{}' \in A, a_i \leqslant a_i{}', i \in I\right\}$

$$\begin{aligned}
\bigvee S &= (\bigvee a_i) \vee [(\bigvee a_i{}') \wedge \alpha] \\
\bigvee S &= (\bigvee a_i{}') \wedge [(\bigvee a_i) \vee \alpha] \\
1 &= (\bigvee a_i{}') \wedge [\bigvee (a_i \vee \alpha)]
\end{aligned}$$

Thus $\bigvee_{i \in I} a_i{}' = 1$ and $\bigvee_{i \in I}(a_i \vee \alpha) = 1$. Since A is compact, there exists

a finite subset $J_1 \subseteq I$ with $\bigvee_{j_1 \in J_1} {a_{j_1}}' = 1$.
Also $a_i \vee \alpha \geqslant \alpha$ and hence $a_i \vee \alpha \in \uparrow_L \alpha = K$. Since $\uparrow_L \alpha = K$ is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2 \in J_2} (a_{j_2} \vee \alpha) = 1$. Set $J = J_1 \bigcup J_2$ and $F = \{a_j \vee ({a_j}' \wedge \alpha) : j \in J\}$. Clearly $F \subseteq S$ and F is finte. Then

$$\begin{aligned}\bigvee F &= (\bigvee a_j) \vee [(\bigvee {a_j}') \wedge \alpha] \\ &= (\bigvee {a_j}') \wedge (\bigvee (a_j \vee \alpha)) \\ &= 1 \wedge 1 \\ &= 1\end{aligned}$$

because $J_1 \subseteq J, J_2 \subseteq J$ and $\bigvee_{j_1 \in J_1} {a_{j_1}}' = 1, \bigvee_{j_2 \in J_2} (a_{j_2} \vee \alpha) = 1$. Hence $A[\alpha]$ is compact. Since $A \subset A[\alpha]$ and A is M.R.C., this leads to a contradiction as there is no strictly larger frame that is compact. Thus our assumption that K is not closed in A is wrong. Thus K must be closed in A in this case.
Case 2: Assume that K is not closed in L.
Let $\overline{K}$ be the closure of K by *Definition* 1.5.8, which is a unique closed sublocale of L such that K is dense in it. Since $\overline{K}$ is closed in L, there exists $\beta \in L - A$ such that $\overline{K} = \uparrow_L \beta$. Consider the singly generated extension $A[\beta]$ of the frame A by adding the element β. We prove that $A[\beta]$ is compact. Let $S \subseteq A[\beta]$ with $\bigvee S = 1$.

Let $S = \{a_i \vee (a_i{}' \wedge \beta) : a_i, a_i{}' \in A, a_i \leqslant a_i{}', i \in I\}$

$$\begin{aligned} \bigvee S &= (\bigvee a_i) \vee [(\bigvee a_i{}') \wedge \beta] \\ \bigvee S &= (\bigvee a_i{}') \wedge [(\bigvee a_i) \vee \beta] \\ 1 &= (\bigvee a_i{}') \wedge [\bigvee (a_i \vee \beta)] \end{aligned}$$

Thus $\bigvee_{i\in I} a_i{}' = 1$ and $\bigvee_{i\in I}(a_i \vee \beta) = 1$. Since A is compact, there exists a finite subset $J_1 \subseteq I$ with $\bigvee_{j_1\in J_1} a_{j_1}{}' = 1$. Since A is M.R.C., L is not compact. So we consider the one point compactification L_{F_C} of L in *Definition* 3.2.1.

Claim: $A \subseteq F_C$

Let $a \in A$. Since $\uparrow_A a$ is a closed sublocale of A, it is compact in A. Since A is M.R.C., by *Lemma* 3.2.1, all the compact upsets $\uparrow_A a$ in A are same as compact upsets $\uparrow_L a$ in L. Thus for each $a \in A$, $\uparrow_L a$ is compact in L. Hence $A \subseteq F_C$.

Now $a_i \vee \beta \in L$ and $a_i \in F_C$. Hence by definition of L_{F_C}, we have $(a_i \vee \beta, 0) \vee (a_i, 1) = (a_i \vee \beta, 1) \in L_{F_C}$. Now

$$\begin{aligned} \bigvee_{i\in I}(a_i \vee \beta, 1) &= (\bigvee_{i\in I}(a_i \vee \beta), 1) \\ &= (1, 1) \end{aligned}$$

Since L_{F_C} is compact, there exists a finite subset $J_2 \subseteq I$ with $\bigvee_{j_2\in J_2}(a_{j_2} \vee \beta, 1) = (1,1)$ and hence $\bigvee_{j_2\in J_2}(a_{j_2} \vee \beta) = 1$. Set $J = J_1 \bigcup J_2$ and $F = \{a_j \vee (a_j{}' \wedge \beta) : j \in J\}$. Clearly $F \subseteq S$ and F is finite. As seen before, $\bigvee F = 1$. Hence $A[\beta]$ is compact. Since $A \subset A[\beta]$ and A is

M.R.C., this leads to a contradiction as there is no strictly larger frame that is compact. Thus our assumption that K is not closed in A is wrong. Thus K must be closed in A in this case.
Thus every compact sublocale of A is closed. Hence A is a CCE Frame. $\square$

Corollary 3.2.4. *Every compact regular frame is maximal relative to compactness.*

Proof. Every closed sublocale of a compact frame is compact. Also every compact sublocale of a regular frame is closed. Thus every compact regular frame is CCE and hence M.R.C. $\square$

Corollary 3.2.5. *Let A be any compact frame. Then no subframe of A is regular.*

Proof. Every subframe of a compact frame is compact. If such a frame becomes regular, then by *Corollary* 3.2.4, it is M.R.C. which is a contradiction as A is compact. $\square$

Corollary 3.2.6. *The topological space $(X, \Omega X)$ is a C-C space if and only if ΩX is a CCE Frame.*

Proof. Assume that $(X, \Omega X)$ is a C-C space. Then it is M.R.C by *Theorem* 3.1.1. Then ΩX is M.R.C.

Conversely, if ΩX is a CCE Frame, then it is M.R.C. by *Theorem* 3.2.3. Hence $(X, \Omega X)$ is M.R.C and thus a C-C space by *Theorem* 3.1.1. $\square$

Corollary 3.2.7. *Let A be a spatial CCE Frame. Then it is compact and subfit.*

Proof. Since A is CCE Frame, it is compact. Since A is a CCE Frame, by *Corollary* 3.2.6, the topological space which corresponds to A will be a C-C space and hence compact and T_1 by *Theorem* 3.1.3. Since frame of opens of a T_1 topological space is subfit, we have A is subfit. □

Example 3.2.1. Let (X, τ) be a cofinite topological space. Then it is compact and T_1 but not a C-C space.Then the frame τ is subfit as the frame of opens of a T_1 topological space is subfit. τ is also compact. Now by *Corollary* 3.2.6, τ is not a CCE frame

Corollary 3.2.8. *If A is a compact Hausdorff frame, then it is a CCE frame.*

Proof. A compact Hausdorff frame is regular by *Theorem* 1.7.9. Since a compact regular frame is spatial by *Theorem* 1.7.7, A is spatial. Also the topological space corresponding to such a frame is Hausdorff. Thus the topological space corresponding to the frame A is compact Hausdorff. Then by *Theorem* 3.1.1, it is M.R.C and hence a C-C space by *Theorem* 3.1.2. Now by *Corollary* 3.2.6, A is a CCE Frame. □

The following is an example of a CCE frame which is compact but not Hausdorff.

Example 3.2.2. Let $(R, \Omega R)$ be the space of rationals with the relative topology and let $(R, \Omega R^*)$ be the one point compactification of $(R, \Omega R)$. Then it is proved in [27] that $(R, \Omega R^*)$ is not Hausdorff but it is a C-C space. Since $(R, \Omega R^*)$ is not Hausdorff, the frame ΩR^* is not a Hausdorff frame, as the topological space representing a Hausdorff spatial frame is Hausdorff. Again by *Corollary* 3.2.6, the frame ΩR^* is a CCE frame as $(R, \Omega R^*)$ is a C-C space.

Theorem 3.2.9. *Let A be a non spatial CCE Frame. Then it cannot be subfit.*

Proof. Since A is a CCE Frame, it is compact. If A is subfit, then by *Theorem 2.11* of[18], a compact subfit frame is spatial, which is a contradiction. $\square$

3.3 CCE Frame and Coproduct

Most of the topological properties are preserved under the act of taking product. In this section, we are seeking whether this is true in the case of CCE Frames.

Theorem 3.3.1. *Let $\{A_i : i \in I\}$ be a non empty family of non empty compact frames and let A be the frame coproduct. If C_j is a compact sublocale of A_j, then ${p_j}^{*-1}(C_j)$ is compact in A.*

Proof. Take $A_j = C_j$ in $\bigoplus_{i \in I} A_i$, then we have ${p_j}^*(\bigoplus_{i \in I} A_i) = C_j$. Thus ${p_j}^{*-1}(C_j) = \bigoplus_{i \in I} A_i$ where $A_j = C_j$. Since C_j is a compact sublocale and all the other $A_i's$ are compact, by Tychonoff theorem for locales, $\bigoplus_{i \in I} A_i$ is compact. Hence ${p_j}_*^{-1}(C_j)$ is compact in A. $\square$

We state the Kuratowski-Mrowka Theorem for locales[36] for proving the next result.

Theorem 3.3.2. *A locale L is compact if and only if the product projection $p^* : L \oplus M \to M$(the coproduct injection $p : M \to L \oplus M$ in frame language) is closed for every locale M*

Theorem 3.3.3. *Let $\{A_i : i \in I\}$ be a non empty family of non empty compact frames and let A be the frame coproduct. If A is a CCE Frame, then each A_i is a CCE Frame.*

Proof. If C_j is closed in A_j, then it is compact as a closed sublocale of a compact frame is compact. Now suppose that C_j is a compact sublocale of A_j. Then by *Theorem* 3.3.1, $C = {p_j}^{*-1}(C_j)$ is compact in A and hence it is closed in A as A is a CCE Frame. Then ${p_j}^*(C) = C_j$ and since A is compact by Tychonoff theorem for locales C_j is closed as the projections of A to A_i being closed maps by *Theorem* 3.3.2. Hence each A_i is a CCE Frame. □

The converse of the above result need not be true. We prove this through the next theorem which tells that the coproduct of a compact frame A with itself is a CCE frame if and only if A is a CCE frame that is strongly Hausdorff.Hence, if the condition strong Hausdorffness is dropped, then the coproduct may not be a CCE frame.

Theorem 3.3.4. *Let A be any compact frame and let $A \oplus A$ be the coproduct of A with itself. Then $A \oplus A$ is a CCE Frame if and only if A is CCE and Hausdorff.*

Proof. Suppose that $A \oplus A$ is a CCE Frame. Then A is a CCE Frame by *Theorem* 3.3.3. Let $\Delta(U) = \bigvee \{a \wedge b : (a, b) \in U\}$ where $U \in A \oplus A$. Then $\Delta : A \oplus A \to A$ called the codiagonal is a surjective frame homomorphism and hence its right adjoint Δ^* is a sublocale map by which A is a sublocale of $A \oplus A$. Since A is a CCE Frame, it is compact. Hence A is a compact sublocale of $A \oplus A$. Since $A \oplus A$ is a CCE Frame, A is closed in $A \oplus A$. Thus A is Hausdorff as the diagonal Δ embeds A as a closed sublocale of $A \oplus A$, by definition of a Hausdorff frame.

Assume that A is a CCE Frame and Hausdorff. A compact Hausdorff frame is regular by *Theorem* 1.7.9. Since a compact regular frame is spatial by *Theorem* 1.7.7, A is spatial. Then the topological space corresponding to A is a C-C space by *Corollary* 3.2.6. Also the space corresponding to a regular frame is regular and hence Hausdorff. Thus A is a C-C space that is Hausdorff and hence by *Theorem* 3.1.4, the product topological space is C-C. Then by *Corollary* 3.2.6, the product topology is a CCE Frame. Since A is compact and regular, by *Theorem* 1.7.4 and *Theorem* 1.7.8, $A \oplus A$ is compact and regular. Then, by *Theorem* 1.7.7, $A \oplus A$ is spatial. Now, by *Theorem* 1.6.5, $A \oplus A$ is isomorphic to the product topology. Hence $A \oplus A$ is a CCE Frame. □

We know that every subframe of a compact frame is compact. But every sublocale of a compact locale need not be compact. It happens when the sublocale becomes a closed sublocale. We know that a sublocale is different from a subframe. A sublocale is quotient frame and hence it cannot be regarded as subframes of the frame. Hence a CCE Frame can have a sublocale which in its own respect may become a CCE Frame. In the next theorem, we prove that the above situation occurs when the sublocale is closed.

Theorem 3.3.5. *Let A be a CCE Frame. A sublocale K of A is CCE if and only if K is closed in A.*

Proof. Suppose that the sublocale K of A is CCE. Then it is a compact sublocale of A and hence closed, as A is a CCE Frame.

Conversely, suppose that K is a closed sublocale of A. Then K is a compact sublocale of A as it is CCE. Thus any closed sublocale of K is compact. Now assume that K_1 is a compact sublocale of K. Then it is a compact sublocale of A. Since A is a CCE Frame K_1 is closed in A. Therefore $K_1 = \uparrow_A a$ where $a \in A$. Since K_1 is a sublocale of K, we have $\uparrow_A a = \uparrow_K a$. Hence K_1 is closed in K. Thus K is a CCE locale. $\square$

Chapter 4

Minimal Hausdorff Frames

4.1 Introduction

The concept of minimal topologies was introduced by A.S.Parhomeko [33]. He proved that compact Hausdorff spaces are minimal Hausdorff. Later E.Hewitt [16] proved that compact Hausdorff spaces are maximal compact as well as minimal Hausdorff. A.Ramanathan [40] [41] proved the existence of noncompact minimal Hausdorff spaces and characterized all minimal Hausdorff spaces.

The Hausdorffness axiom for frames is not yet successfully defined as to become an extension or equivalent of the classical Hausdorff axiom for topological spaces. Many forms of it were defined by Dowker and Strauss[8], A. Pultr[37], J.Rosický and B. Šmarda[43] and by Isbell[18]. We know that locales are the categorical extension of topological spaces. In this chapter, we introduce the notion of *minimal Hausdorff frames* and try to formulate some characterizations of such frames.

4.2 Filters in Frames

At first, we give some preliminary definitions and theorems used to prove the results in this chapter.

Definition 4.2.1. [36] A filter P in a frame L is called a *completely prime filter* if for every $S \subseteq L$ with $\bigvee S = a \in P$, it follows that $S \bigcap P \neq \phi$.

The concept of convergence of filters in frames was introduced by S.S.Hong [17]. The following definitions and proof of theorems are given in [17].

Definition 4.2.2. [17] A filter F in a frame L is said to be *convergent* if for any cover S of L, F meets S.

Definition 4.2.3. [17] A filter F in a frame L is said to be *clustered* if for any cover S of L,

$$sec\, F = \{x \in L : \ for\ all\ a \in F, a \wedge x \neq 0_L\}$$

meets S.

Definition 4.2.4. [17] A subset B of L is a base for L if for any $x \in L$ there exists a subset C of B with $\bigvee C = x$.

Remark 4.2.1. [17]

1. Every completely prime filter in a frame L is convergent and a convergent filter F in L is clustered.

2. A filter containing a convergent filter is convergent and a filter contained in a clustered filter is clustered.

3. A maximal filter in a frame L is convergent if and only if it is clustered, because $secF = F$ for a maximal filter F.

4. A filter F in a frame L is convergent(clustered) if and only if for any subset $C \subseteq B$ with $\bigvee C = 1_L$, B a base for L, $F(secF)$ meets C.

The following results on filters are helpful in proving some of our results in this chapter.

Theorem 4.2.1. *[17] For a frame L, the following are equivalent:*

1. *L is almost compact.*

2. *Every filter in L is clustered.*

3. *Every maximal filter in L is convergent.*

Theorem 4.2.2. *[17] For a regular frame L, the following are equivalent:*

1. *L is compact.*

2. *Every filter in L is clustered.*

3. Every maximal filter in L is convergent.

An element $x \in L$ is *dense*, if $a \wedge x \neq 0$, for all $0 \neq x \in L$. That is x is dense if and only if $x^* = 0$. Denote by $D(L)$, the set of all dense elements of L. Then by the following theorem, $D(L) = \{l \in L : l^* = 0\}$ is a filter in L.

Theorem 4.2.3. *$D(L)$ is a filter in L.*

Proof. Let $u \in L$ and $x \leqslant u$ where $x \in D(L)$. Then $u^* \leqslant x^* = 0$ yields $u^* = 0$ and $u \in D(L)$ follows. Now let $u, v \in D(L)$. Then $(u \wedge v)^* = \bigvee \{y \in L : y \wedge u \wedge v = 0\}$ Now $y \wedge u \wedge v = 0 \Rightarrow y \wedge u \leqslant v^* \Rightarrow y \wedge u \leqslant 0 \Rightarrow y \leqslant u^* \Rightarrow y = 0$ Therefore $(u \wedge v)^* = 0$ and hence $u \wedge v \in D(L)$. Also $1^* = 0$ and thus $1 \in D(L)$. Hence the theorem. □

Recall from the paper[17] by S.S.Hong that a filter F in L clusters if and only if $\bigvee \{x^* : x \in F\} < 1$. Since $\bigvee \{x^* : x \in D(L)\} < 1$, $D(L)$ clusters. Also from the same paper recall that a frame L is almost compact if and only if every filter in L clusters.

Theorem 4.2.4. *A frame in which every clustered filter converges is almost compact.*

Proof. Let L be such a frame and suppose, for contradiction, that L is not almost compact. Then by *Theorem* 1.7.3, $L - D(L)$ is a cover of L. Since $D(L)$ misses this cover, $D(L)$ does not converge. But this is a contradiction because $D(L)$ clusters. □

4.3 Minimal Hausdorff Frames

The concept of minimal Hausdorff spaces was studied by M.P.Berri. The following theorem[[6], pp.110,111] gives a characterization for such topological spaces in terms of convergence of filters.

Theorem 4.3.1. *A necessary and sufficient condition that a Hausdorff space (X, τ) be minimal Hausdorff is that τ satisfies the following property:*

1. *Every open filter-base has an adherent point;*
2. *If an open filter-base has a unique adherent point, then it converges to this point.*

In this section we introduce minimal Hausdorff frames and find some partial characterizations for them analoguously.
The following convention is adopted for defining what is called a *proper subframe*. We know that $\{0, 1\}$ is a subframe of any frame and it is Hausdorff. A *proper subframe* here means a frame which is a strict subframe of the frame under consideration other than the trivial frame $\{0, 1\}$.

Definition 4.3.1. A frame L is said to be *minimal Hausdorff* if L is Hausdorff and no *proper subframe* of L is Hausdorff.

The four element frame $B_4 = \{0, a, b, 1\}$ where $a \,||\, b$, $a \vee b = 1$, $a \wedge b = 0$ is a Hausdorff frame as it is regular and is minimal Hausdorff by definition. Then any Boolean frame other than B_4 is not minimal Hausdorff since any such frame contains B_4 as a subframe. Thus B_4 is the only finite frame that is minimal Hausdorff.

When we come to infinite spatial frames, there are Hausdorff frames containing no Boolean frames as a proper subframe. For example, the set of all real numbers with usual topology, denoted by $(R, \Omega R)$, is regular and hence the frame ΩR is regular. Since every regular frame is Hausdorff, ΩR is Hausdorff. But R is a connected space and there are no *open and closed sets* other than R and ϕ. Hence ΩR contains no Boolean frame as a proper subframe. Also $(R, \Omega R)$ is coarser than discrete topological space. Later in this chapter, we can prove that ΩR is not minimal Hausdorff. This reveals the presence infinite spatial Hausdorff frames which contains Hausdorff frames other than Boolean ones. The task of verifying minimal Hausdorffness is very complex in infinite case as verifying the presence of Hausdorff frames in such frames is practically very difficult. Hence we need some characterizations for minimal Hausdorffness in frames.

Remark 4.3.1. Let F be any bounded meet semilattice on L. Then

$$\mathcal{D}(F) = \{A \subseteq F : \phi \neq A = \downarrow_L A\}$$

where $\downarrow_L A = \{x \in L : x \leqslant u, u \in A\}$ is a frame under set inclusion. Let $A,\ B \in \mathcal{D}(F)$. Then $\downarrow A = A$ and $\downarrow B = B$. Therefore $\downarrow A \cap B = A \cap B$. Now $\downarrow (A \cap B) \oplus (A \cap B) = (A \cap B) \oplus (A \cap B)$.

Remark 4.3.2. It is known that $d_{\mathcal{D}(F)} = \bigvee \{U \oplus V : U \cap V = \{0\}\}$. $\downarrow d_{\mathcal{D}(F)} = \downarrow \bigvee \{U \oplus V : U \cap V = \{0\}\} = \bigvee \{\downarrow (U \oplus V) : U \cap V = \{0\}\} = \bigvee \{\downarrow U \oplus \downarrow V : U \cap V = \{0\}\} = \bigvee \{U \oplus V : U \cap V = \{0\}\} = d_{\mathcal{D}(F)}$.

Remark 4.3.3. $\downarrow [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)} = [(A \cap B) \oplus (A \cap B)] \vee d_{\mathcal{D}(F)}$, by remarks 4.3.1 and 4.3.2.

Remark 4.3.4. Let $A, B \in D(F)$, Then $x \in A \cap B \Rightarrow x \in \{a \wedge b : a \in A,\ b \in B\}$. Conversely if $a \in A$, $b \in B$ and since $a \wedge b \leqslant a$, b, we have $a \wedge b \in \downarrow_L A = A$, $a \wedge b \in \downarrow_L B = B$. Thus $a \wedge b \in A \cap B$ and hence $A \cap B = \{a \wedge b : a \in A,\ b \in B\}$.

Theorem 4.3.2. *Let L be a Hausdorff frame and let F be a bounded meet semilattice in L. Then the frame $(\mathcal{D}(F), \subseteq)$ is Hausdorff.*

Proof. Since $\{a \oplus b : a \in A,\ b \in A\}$ forms a join basis for $A \oplus B$, $\{(a \wedge b) \oplus (a \wedge b) : a \in A,\ b \in B\}$ is a join basis for $[(A \cap B) \oplus (A \cap B)]$, by *Remark* 4.3.4.
If $U \cap V = \{0\}$ where $U, V \in \mathcal{D}(F)$, as proved before, we have

$$\{0\} = U \cap V = \{u \wedge v : u \in U,\ v \in V\} \Leftrightarrow u \wedge v = 0,\ \forall\, u \in U,\ v \in V$$

Thus $U \oplus V$ where $U \cap V = \{0\}$ has join basis $\{u \oplus v : u \wedge v = 0,\ u \in U,\ v \in V\}$.
Since L is Hausdorff, for any $a, b \in L$,

$$a \oplus b \leqslant [(a \wedge b) \oplus (a \wedge b)] \vee d_L \tag{4.1}$$

where $d_L = \bigvee_{L \oplus L} \{x \oplus y : x \wedge y = 0\}$.
Now $[(a \wedge b) \oplus (a \wedge b)] \vee_{L \oplus L} d_L$

$$\begin{aligned} &= [(a \wedge b) \oplus (a \wedge b)] \vee_{L \oplus L} \left[\bigvee\nolimits_{L \oplus L} \{x \oplus y : x \wedge y = 0\}\right] \\ &= \bigvee\nolimits_{L \oplus L} \{[(a \wedge b) \oplus (a \wedge b)] \vee (x \oplus y) : x \wedge y = 0\} \end{aligned}$$

Then $[(A\cap B)\oplus(A\cap B)]\vee_{\mathcal{D}(F)\oplus\mathcal{D}(F)}d_{\mathcal{D}(F)}$

$$\begin{aligned}
&= [(A\cap B)\oplus(A\cap B)]\vee_{\mathcal{D}(F)\oplus\mathcal{D}(F)}\left[\bigvee\nolimits_{\mathcal{D}(F)\oplus\mathcal{D}(F)}\{U\oplus V : U\cap V=\{0\}\}\right]\\
&= \bigvee\nolimits_{\mathcal{D}(F)\oplus\mathcal{D}(F)}\{[(A\cap B)\oplus(A\cap B)]\vee_{\mathcal{D}(F)\oplus\mathcal{D}(F)}(U\oplus V) : U\cap V=\{0\}\}\\
&= \bigvee\nolimits_{L\oplus L}\left\{\bigvee\nolimits_{L\oplus L}\{[(a\wedge b)\oplus(a\wedge b)]\vee(x\oplus y) : x\wedge y=0\}\right\}\\
&= \bigvee\nolimits_{L\oplus L}\{[(a\wedge b)\oplus(a\wedge b)]\vee d_L : a\in A,\ b\in B\}
\end{aligned}$$

Hence $[(a\wedge b)\oplus(a\wedge b)]\vee d_L$ forms a join basis for $[(A\cap B)\oplus(A\cap B)]\vee d_{\mathcal{D}(F)}$. Since $[(A\cap B)\oplus(A\cap B)]\vee d_{\mathcal{D}(F)}$ is a downset, we have

$$[(a\wedge b)\oplus(a\wedge b)]\vee d_L\in[(A\cap B)\oplus(A\cap B)]\vee d_{\mathcal{D}(F)}$$

and hence $a\oplus b\in[(A\cap B)\oplus(A\cap B)]\vee d_{\mathcal{D}(F)}$ by equation 4.1. Thus $A\oplus B\subseteq(A\cap B)\oplus(A\cap B)\vee d_{\mathcal{D}(F)}$. Hence $\mathcal{D}(F)$ is Hausdorff, by theorem 1.7.10. □

We state the following theorem from [17] which we use to prove our next result.

Theorem 4.3.3. *A filter F is convergent if and only if for any $C\subseteq B$ with $\bigvee C=1$ where B a base for L, F meets C.*

The following result is proved in [36].

Theorem 4.3.4. *Let F be a semilattice and L be a frame. Let $f:F\to L$ be a semilattice homomorphism(L viewed, for a moment, as the semilattice$(L,\wedge,1)$). Then there exists precisely one frame homomorphism $h:\mathcal{D}F\to L$ such that $h\circ\lambda_F=f$ where $\lambda_F:F\to\mathcal{D}F$ defined by*

$\lambda_F(x) = \downarrow x$.

In the rest of this chapter, a filter means any filter other than $\{1\}$ unless stated otherwise.

Lemma 4.3.1. Let L be any Hausdorff frame which contains a clustered filter that is not convergent. Then there exists a proper subframe of L which is Hausdorff.

Proof. Let F' be the clustered filter in L that is not convergent. Then $F = F' \cup \{0\}$ is a bounded meet semilattice in L. By *Theorem* 4.3.4, for every meet semilattice homomorphism $f : F \to L$, there is exactly one frame homomorphism $h : \mathcal{D}(F) \to L$ such that $h \circ \lambda_F(a) = f(a)$, namely the mapping given by $h(A) = \bigvee \{f(a) : a \in A\}$. Take $f = i : F \to L$,the inclusion map, then $h(A) = \bigvee A$.
Claim: $h : \mathcal{D}(F) \to L$ is not onto.
Suppose $h : \mathcal{D}(F) \to L$ is onto. Then for any $x \in L$ there exists $A \in \mathcal{D}(F)$ such that $h(A) = x$. That means for every $x \in L$ there exists $A \subseteq F$ such that $\bigvee A = x$. Thus F is a base for the frame L. Then for any subset C of F with $\bigvee C = 1$, C meets F', as $F = F' \cup \{0\}$. Then by *Theorem* 4.3.3, the filter F' is convergent, a contradiction. Also if $h(\mathcal{D}(F)) = \{0, 1\}$, then $h \circ \lambda_F(F) \subseteq h(\mathcal{D}(F)) = \{0, 1\}$. But $h \circ \lambda_F = f = i$. Therefore $i(F) \subseteq \{0, 1\}$. That is $F \subseteq \{0, 1\}$. Then $F = \{1\}$, a contradiction as F is a non trivial filter. Thus $h(\mathcal{D}(F)) \neq \{0, 1\}$. Thus $h(\mathcal{D}(F))$ is a proper subframe of L. Since h is a frame homomorphism and $D(F)$ is hausdorff by *Theorem* 4.3.2, $h(\mathcal{D}(F))$ is proper subframe of L that is Hausdorff. □

The following theorem gives a partial characterization of minimal Hausdorff frames in terms of convergence of filters.

Theorem 4.3.5. *If frame L is minimal Hausdorff, then every clustered filter in L converges.*

Proof. Suppose that L is minimal Hausdorff. If there exists a clustered filter that does not convergent, then by *Lemma* 4.3.1, there exists a proper subframe of L that is Hausdorff, contradicting the minimality of L. □

Corollary 4.3.6. *Every filter in a minimal Hausdorff frame is clustered.*

Proof. The proof follows from *Theorem* 4.3.5 and *Theorem* 4.2.4. □

Corollary 4.3.7. *A minimal Hausdorff frame is almost compact.*

Proof. By *Theorem* 4.2.1, if every filter in a frame is clustered, then it is almost compact. □

The converse of the above corollary need not be true, as a finite Boolean frame other than B_4 is compact and hence almost compact but not minimal Hausdorff.

Corollary 4.3.8. *Let L be a minimal Hausdorff frame then the clustered filters in L are exactly the convergent filters in L.*

Proof. If L is minimal Hausdorff, then every clustered filter is convergent by *Theorem* 4.3.5. Also every convergent filter is clustered. □

We know that Hausdorffness in frames is not equivalent to Hausdorffness defined in topological spaces, but an imitation of the classical Hausdorffness axiom in topological spaces. Also, if ΩX is a Hausdorff frame, then $(X, \Omega X)$ need not be a Hausdorff space by the following example.

Example 4.3.1. Let $X = \{a, b, c, d\}$ and $\Omega X = \{X, \phi, \{a, b\}, \{c, d\}\}$. Then ΩX is a Hausdorff frame, but $(X, \Omega X)$ is not a Hausdorff space.

The space $(X, \Omega X)$ is Hausdorff when X is a T_0 space. In a similar way, minimal Hausdorffness also differs from that in the topological spaces. Like the case of Hausdorff axiom, minimal Hausdorffness can be regarded as an imitation of that defined in topological spaces, by the following theorem.

Theorem 4.3.9. *Let $(X, \Omega X)$ be a T_0 topological space. If ΩX is a minimal Hausdorff frame, then $(X, \Omega X)$ is a minimal Hausdorff space.*

Proof. Let $\mathcal{B}$ be an open filter-base in the topological space X having a unique cluster point p. The filter $\mathcal{F}$ in the frame ΩX generated by $\mathcal{B}$ will be the minimal filter containing $\mathcal{B}$. Let $\mathcal{A}$ be a cover of the frame ΩX. Then it is an open cover of X and let G be the open neighbourhood of p in $\mathcal{A}$. Since p is a cluster point of $\mathcal{B}$, the neighbourhood G intersects every element of $\mathcal{B}$ and consequently every member of $\mathcal{F}$. Therefore $G \in sec\,\mathcal{F}$. Thus $\mathcal{A}$ intersects $sec\,\mathcal{F}$ and hence $\mathcal{F}$ is clustered in ΩX. Since ΩX is minimal Hausdorff, by *Theorem* 4.3.5, $\mathcal{F}$ is convergent in ΩX. Then every cover of the frame ΩX intersects $\mathcal{F}$. Let $\mathcal{U}$ be the filter with base $\mathcal{B}$ in the topological space X. Since it contains $\mathcal{F}$, every open cover of X intersects $\mathcal{U}$. Hence $\mathcal{U}$ is covergent in X. Suppose that $\mathcal{U}$ converges to a point $x \neq p$. The open filter-base $\mathcal{C}$ which contains all sets that are the finite intersection of elements of $\{B \cup N_p : B \in \mathcal{B}, N_p \in \mathcal{N}_p\}$ where $\mathcal{N}_p$ is the open neighbourhood system at p gives a filter that converges to p and containing the filter $\mathcal{U}$ which converges to $x \neq p$. Thus we have a filter containing two distinct limit points which is not possible in X as it is a Hausdorff space. Hence $\mathcal{U}$ converges to p and consequently the open filter-base $\mathcal{B}$ must converge to p. Now by *Theorem* 4.3.1, a Hausdorff space in which every open filter-base having a unique cluster point converges to that point is minimal Hausdorff. Hence $(X, \Omega X)$ is minimal Hausdorff. □

Corollary 4.3.10. *The set of all real numbers with usual topology is denoted by* $(R, \Omega R)$. *Then the frame* ΩR *is not minimal Hausdorff.*

Proof. $(R, \Omega R)$ is T_0 and is not minimal Hausdorff. Hence ΩR cannot be a minimal Hausdorff frame. □

Remark 4.3.5. The converse of the above theorem need not be true. Let $X = \{a, b, c\}$. Consider the topological space $(X, \mathcal{P}(X))$ where $\mathcal{P}(X)$ is the power set of X. It is a minimal Hausdorff space as it is compact and Hausdorff. But the frame $\mathcal{P}(X)$ is a Boolean frame containing B_4 and hence not a minimal Hausdorff frame.

Remark 4.3.6. A compact Hausdorff topological space is minimal Hausdorff. But a compact Hausdorff frame need not be a minimal Hausdorff frame, by the example in the above remark.

Remark 4.3.7. In the category **Sp** of all topological spaces and continuous mappings, a compact Hausdorff space is minimal Hausdorff. But in the category **Frm** of all frames and frame homomorphisms, this need not happen. The frame in the example provided in *Remark* 4.3.5 is a compact frame, but not minimal Hausdorff.

Chapter 5

Reversible Frames

5.1 Introduction

The concept of *reversibility* in spaces was studied by M.Rajagopalan and A.Wilanski [38]. A topological space (X, τ) is called *reversible* if it has no strictly stronger topology τ^* such that (X, τ) and (X, τ^*) are homeomorphic. Equivalently, it has no strictly weaker topology τ^* such that (X, τ) and (X, τ^*) are homeomorphic. A characterization for reversible topological space given in [38] is that a space is reversible if and only if each continuous bijection of the space onto itself is a homeomorphism. It is also proved that the finite product of reversible space is reversible if and only if each component is reversible. The concept of reversibility has

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Some results of this chapter are included in the following paper.
P.N., Jayaprasad and Johnson, T.P. : Reversible Frames, Journal of Advanced Studies in Topology,Vol.3, No.2(2012),7-13.

also been extended to fuzzy topological space[19] by T.P.Johnson and to partially ordered sets[25] by Michal Kukiela.

The concept of reversibility is recently introduced in partially ordered sets by Michal Kukiela in [25]. In this chapter, we introduce the concept of *reversible frames*. The relation between reversible spatial frames and reversible topological spaces is also studied here. Frame theoretic methods are useful in solving problems related to topology. As an example, in the last section, we prove that the product of a two point discrete topological space and a reversible topological space is reversible which is posed by M.Rajagopalan and A.Wilansky[38].

In this chapter and in the next, we adopt the representation of a frame using the method of presentation of algebras. The method is explained in chapter 4 of [48]. We describe the way in which a finite frame A can be presented algebraically in four steps.

- Let G be a set of subbasic elements called generators.
- Derive all possible joins and meets of these generators.
- R contains certain axiomatic relations to hold between expressions of step2. They can be of the form $e_1 \leqslant e_2$ or $e_1 = e_2$, as both are interconvertible.
- Using these relations and frame laws, when any two expressions of step2 are equal, defines an equivalence relation on step2 expressions.

We write the frame presented in this way as $A = Fr\langle G \mid R\rangle$ which stands for the algebra which is a frame A presented by the set of generators G and the set of relations R as explained in [48]. The generator set can be any set. But here we insist that it must be a subset of the given frame. Then we present the given frame using that generator set and a relation set. Here the frame laws(axioms defining the frame) and relations deductible from them are not included in the relation set R. They are assumed by the notation Fr itself. We can form different frames on the same generator set G by imposing different relation sets.

5.2 Reversible Frames

We begin by introducing the notion of *reversible frames*.

Definition 5.2.1. A frame A is *reversible*, if every order preserving self bijection on A is a frame isomorphism.

We introduce a new definition called *strict extension of the relation set R* as defined below.

Definition 5.2.2. A relation set R^* is said to be the *strict extension of R*, if R^* contains all relations in R and at least one relation other than those in R which is not derivable from the relations in R. We denote $R \subset R^*$ when R^* is a strict extension of R.

The following result is proved in [25].

Theorem 5.2.1. *A poset $(P, \leqslant)$ is reversible if and only if there is no strict extension $\leqslant^*$ of $\leqslant$ such that $(P, \leqslant)$ is isomorphic to $(P, \leqslant^*)$*

(equivalently, if $\leqslant$ is not a strict extension of any order $\leqslant_*$ such that $(P, \leqslant)$ is isomorphic to $(P, \leqslant_*)$).

The following theorem characterises reversible frames. In this theorem, $G \subset G^*$ means the usual set theoretic inclusion.

Theorem 5.2.2. *A frame $A = Fr\langle G \mid R\rangle$ is reversible if and only if there exists no strict extension R^* of the relation set R and $G \subset G^*$ such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G^* \mid R^*\rangle$ are isomorphic.*
Equivalently there exists no relation set R^* with $R^* \subset R$ and $G^* \subset G$ such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G^* \mid R^*\rangle$ are isomorphic.

Proof. Suppose $A = Fr\langle G \mid R\rangle$ is reversible. Assume that R^* is a strict extension of R ($R \subset R^*$) and $G \subset G^*$ such that $\phi : Fr\langle G \mid R\rangle \to Fr\langle G^* \mid R^*\rangle$ is a frame isomorphism. Then one of them can be embedded properly in the other as a subframe and they are isomorphic through ϕ. Thus by *Theorem* 5.2.1, A is not reversible as a poset. Hence there exists an order preserving bijection that is not a poset isomorphism and consequently it is not a frame isomorphism. This is a contradiction, as A is reversible.
Conversely, assume that there is no strict extension R^* of the relation set R ($R \subset R^*$) and $G \subset G^*$ such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G^* \mid R^*\rangle$ are isomorphic. Let $f : A \to A$ be any order preserving bijection. Since every element of A is the arbitrary join of finite meets of generators in G, we have $x = \vee_{i\in I} \wedge_{j\in J} x_{ij}$ where $x_{ij} \in G$, $x \in A$ and I, J are index sets with J finite. Since G and R are fixed $f\mid_G$ is an order preserving bijection from $G \to G^*$, where G^* is another generator set of the frame A, as order preserving bijection maps generators to generators. Define a new frame

as $A^* = Fr\langle G^* \mid R^* \rangle$ where

$$R^* = R \cup \{f(x) = \vee_{i \in I} \wedge_{j \in J} f(x_{ij}), x_{ij} \in G\}$$

Clearly R^* is a strict extension of R. Define $P(x) = \vee_{i \in I} \wedge_{j \in J} f(x_{ij})$ from A to A^*.
Since A^* is a complete lattice, P is well defined. Let $x \neq y$ where $x = \vee_{i \in I} \wedge_{j \in J} x_{ij}$ and $y = \vee_{l \in L} \wedge_{k \in K} y_{lk}$ with $x_{ij}, y_{lk} \in G$. If $P(x) = P(y)$, then $\vee_{i \in I} \wedge_{j \in J} f(x_{ij}) = \vee_{l \in L} \wedge_{k \in K} f(y_{lk})$ where L, K are index sets with K finite and hence $f(x) = f(y)$ by the definition of R^*. Thus we get $x = y$ as f is one to one, which is a contradiction. Hence P is one to one.
Let $y = \vee_{i \in I} \wedge_{j \in J} y_{ij} \in A^*$. Define $x = \vee_{i \in I} \wedge_{j \in J} f^{-1}(y_{ij})$ which is in A and $P(x) = y$. Thus P is a bijection on A. Also P preserves finite meet and arbitrary join by the nature of the definition and by frame laws. Thus $A = Fr\langle G \mid R \rangle$ and $A^* = Fr\langle G^* \mid R^* \rangle$ are isomorphic via P with R^* being a strict extension of R. Hence $G = G^*$, $R = R^*$, by assumption. Thus $A = A^*$.
Let $x = \vee_{i \in I} \wedge_{j \in J} x_{ij}$ with $x_{ij} \in G$. Then $P(x) = \vee_{i \in I} \wedge_{j \in J} f(x_{ij}) = f(x)$ on A^* by its definition and hence $P \equiv f$ on A.
Thus $f : A \to A$ is a frame isomorphism. □

Now we establish a connection between reversible spatial frames and reversible topological spaces using the above theorem.

Theorem 5.2.3. *A topological space $(X, \Omega X)$ is reversible if and only if the frame of open sets ΩX is reversible.*

Proof. Suppose $(X, \Omega X)$ is reversible. If the frame ΩX is not reversible, then by *Theorem* 5.2.2, there exists a frame ΩX^* such that $\Omega X \subset \Omega X^*$(or reversely) and ΩX is isomorphic to ΩX^* through a frame isomorphism f. Thus $f \in Mor(\Omega X^*, \Omega X)$ in **Frm**. Then the functor Σ from **Frm** $\rightarrow$ **Sp** given by *Theorem* 1.4.3, sends f to the homeomorphism $\overline{f} : (X, \Omega X) \rightarrow (X, \Omega X^*)$ in **Sp**, contradicting the reversibility of $(X, \Omega X)$. Conversely assume that the frame of open sets ΩX is reversible. If $(X, \Omega X)$ is not reversible, then there exists a strictly larger topology ΩX^* on X such that $(X, \Omega X)$ and $(X, \Omega X^*)$ are homeomorphic through a homeomorphism f. Thus $f \in Mor((X, \Omega X^*), (X, \Omega X))$ in **Sp**. Then the functor $\Omega :$ **Sp** $\rightarrow$ **Frm** according to *Theorem* 1.4.1, assigns $\Omega f : \Omega X^* \rightarrow \Omega X$ to f, is a continuous map between the locales ΩX^* and ΩX. Then the corresponding frame homomorphism $\Omega f^* : \Omega X \rightarrow \Omega X^*$ is a frame isomorphism as functors maps equivalence in categories to the same in the other category, contradicting the reversibility of ΩX. □

Corollary 5.2.4. *A spatial frame A is reversible if and only if the topological space* $(ptA, \Omega ptA)$ *is reversible.*

Theorem 5.2.3 provides sufficiently many examples of reversible frames from topology. We know that the real line with usual topology is reversible as a topological space. Hence ususal topology is a reversible frame. We present the frame in the form of presentation of frames.

Example 5.2.1. The frame of reals denoted by $\Omega R = Fr\langle G \mid S\rangle$ where $G = \{(p, q)/p, q \in Q\}$ and S contains the relations $(p, q) \wedge (r, t) = (p \vee r, q \wedge t), (p, q) \vee (r, t) = (p, t)$ when $p \leqslant r < q \leqslant t, (p, q) = \bigvee\{(r, t)/p < r < t < q\}, 1 = \bigvee\{(p, q)/p, q \in Q\}$ is reversible.

Now we provide an example for a non-reversible frame.

Example 5.2.2. Consider example 4 due to M Rajagopalan in [38]. In this example, the notation (p,q) for intervals will refer to intervals of irrational numbers, even if p,q are rational. The interval $I = (-1, 1)$ as a subspace of the Euclidean space is not reversible. This is proved there by producing a strictly larger topology on (-1,1) by adjoining the set [u,1) where u is an arbitrary irrational in (-1,1) such that the two topological spaces are homeomorphic.

The frame of opens of this topological space can be presented as a frame by $Fr\langle G \mid R\rangle$ where $G = \{(p, q) : -1 < p, q < 1 \in Q\}$ and R contains the relations $(p, q) \wedge (r, s) = (p \vee r, q \wedge s), (p, q) \vee (r, s) = (p, s)$ when $p \leqslant r < q \leqslant s, (p, q) = \bigvee\{(r, s) : p < r < s < q\}, 1 = \bigvee\{(p, q) : -1 < p, q < 1 \in Q\}$. Then the strengthened topology can be presented by the frame with generator set $G^* = G \cup \{[u, 1) : u \in (-1, 1) \: is \: irrational\}$ and relation set R^* contains R and a new relation $\cup \{[u, 1) = \bigwedge\{(x, 1) : x < u < 1\}\}$ which is not derivable from the relations in R. Thus there exists $R \subset R^*$ such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G^* \mid R^*\rangle$ are isomorphic through the frame isomorphism induced by the homeomorphism that makes the above topological spaces homeomorphic. Thus the frame of open sets of the topological space presented in the above example is not reversible.

Some results that are not generally true in topological spaces are the same in frames also in view of *Theorem* 5.2.2.

Remark 5.2.1. A subframe of a reversible frame need not be reversible. This is because the set of reals R with discrete topology is reversible but the space R with topology, containing the opensets in Euclidean topology together with singleton sets $\{x\}$ where $x < 0$ and $x \in R$, is not reversible[see eg.3,[38]]. Then by *Theorem* 5.2.2 the frame of opensets of the former space is reversible and the frame of opensets of the latter space is a nonreversible subframe of that frame.

Remark 5.2.2. The product of two reversible frames need not be reversible. This is because the product of a countable discrete space with a reversible space is not reversible by *Corollary* in section 5 of [38]. Hence the corresponding frame of opensets of the product space is not reversible by *Theorem* 5.2.2 where as the frame of opensets of the component spaces are reversible.

It is already known that, in topological spaces, reversibility is a topological property. In the next theorem we state that, in frames, reversibility is a frame isomorphic property.

Theorem 5.2.5. *Let A and B be two isomorphic frames. If A is reversible, then B is also reversible.*

Remark 5.2.3. A finite frame is a finite poset and is reversible as a poset by [25]. Thus a finite frame is reversible as a frame.

A Boolean algebra is said to be atomic if each of its elements is a join of atoms. A Boolean frame is spatial if and only if it is atomic. Also it is known that spatial Boolean frames correspond to the discrete spaces. Thus non-atomic complete Boolean algebras are examples of non-spatial locales. In the next result, we prove that all Boolean frames are reversible. We state the following theorem from [25] for proving the next result.

Theorem 5.2.6. *Let A, B be Boolean lattices (complemented, distributive lattices) and $f : A \to B$ an order preserving bijection. Then f is an isomorphism.*

Theorem 5.2.7. *A Boolean frame L is reversible.*

Proof. Let f be any order preserving bijection from the Boolean frame L to itself. Let A be an arbitrary subset of L. Let $b = (\bigvee A)^c$. Therefore

$$b \vee (\bigvee A) = 1; \qquad b \wedge (\bigvee A) = 0 \tag{5.1}$$

Since a Boolean lattice is reversible as a poset by *Theorem* 5.2.6, f is a lattice isomorphism preserving complementation. Now from *equation* 5.1,

$$f(b) \vee f(\bigvee A) = 1; \qquad f(b) \wedge f(\bigvee A) = 0 \tag{5.2}$$

Since $f(a^c) = (f(a))^c$, we have

$$\bigvee f(A) \wedge f(b) = 0 \tag{5.3}$$

Consider

$$\begin{aligned} \bigvee f(A) \vee f(b) &= \bigvee\{f(a) \vee f(b) : a \in A\} \\ &= \bigvee\{f(a \vee b) : a \in A\} \\ &= r \qquad\qquad (say) \end{aligned}$$

Now $f(a \vee b) \leqslant r$ for all $a \in A$, thus $a \vee b \leqslant f^{-1}(r)$ for all $a \in A$. Hence $\bigvee A \leqslant f^{-1}(r)$ and $(\bigvee A) \vee b \leqslant f^{-1}(r)$. Therefore $f((\bigvee A) \vee b) \leqslant r$ and hence $f(1) = 1 \leqslant r$. Hence

$$\bigvee f(A) \vee f(b) = 1 \tag{5.4}$$

Thus from *equations* 5.2, 5.3 and 5.4, $f(\bigvee A)$ and $\bigvee f(A)$ are complements of $f(b)$. Therefore $f(\bigvee A) = \bigvee f(A)$. Hence f is a frame isomorphism. $\square$

5.3 Reversibility and Maximal Frames

In topology, spaces that are maximal or minimal with respect to some specified topological property are reversible. For example, a compact Hausdorff space is maximal compact and minimal Hausdorff. Such a space is an example of a reversible topological space. Thus, maximal(minimal) topological spaces are related to reversible topological spaces. In the last two chapters, we have seen frames that are maximal or minimal with respect to some specified frame isomorphic property. We expect frames that are maximal or minimal with respect to some specified frame isomorphic property to be reversible and definitely the answer is positive. In the following theorem, we have used *maximality of the relation set* R with respect to a frame isomorphic property p in the sense that no strict extension R^* of the relation set R can be found such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G \mid R^*\rangle$ both have that frame isomorphic property p. Similarly minimality of the relation set R with respect to a frame isomorphic property p in the sense that R is not the strict extension of any relation set R^* such that the frames $Fr\langle G \mid R\rangle$ and $Fr\langle G \mid R^*\rangle$ both have that frame isomorphic property p.

Theorem 5.3.1. *Let $A = Fr\langle G \mid R\rangle$ be a frame. Then the following are equivalent.*

(i) $A = Fr\langle G \mid R\rangle$ *is reversible*

(ii) R *is maximum for some frame isomorphic property*

(iii) R *is minimum for some frame isomorphic property*

Proof. (i) $\Rightarrow$ (ii)
Let $A = Fr\langle G \mid R\rangle$ is reversible.
We define property p as follows.
A frame $Fr\langle G \mid K\rangle$ is said to have property p if there exists an order preserving bijection from $A = Fr\langle G \mid R\rangle$ to $Fr\langle G \mid K\rangle$.
Clearly A has property p. Assume that R^* is a relation set such that $A^* = Fr\langle G \mid R^*\rangle$ has the property p and $R \subset R^*$. Then there exists an order preserving bijection $f : Fr\langle G \mid R\rangle \to Fr\langle G \mid R^*\rangle$. Since every element of A is the arbitrary join of finite meets of generators in G, we have $x = \vee_{i\in I} \wedge_{j\in J} x_{ij}$ where $x_{ij} \in G$ and I, J are index sets with J finite.
Define a new frame as $B = Fr\langle G \mid R_B\rangle$ where

$$R_B = R^* \bigcup \{f(x) = \vee_{i\in I} \wedge_{j\in J} f(x_{ij}), x_{ij} \in G\}$$

Clearly R_B is a strict extension of R^*. Define $P(x) = \vee_{i\in I} \wedge_{j\in J} f(x_{ij})$ from A to B.
Since B is a complete lattice P is well defined. Now using the similar steps in *Theorem 1.2.1*, P is a frame isomorphism, $R^* = R_B, A^* = B$ and $P \equiv f$. Thus $f : Fr\langle G \mid R\rangle \to Fr\langle G \mid R^*\rangle$ is a frame isomorphism. This

contradicts the reversibility of R. Hence R must be maximal.
(ii) $\Rightarrow$ (i)
Suppose that R is maximal for some frame isomorphic property p. Let f be an order preserving bijection from $A = Fr\langle G \mid R\rangle$ to itself. Define

$$R^* = R \bigcup \{f(\vee e_i) = \vee\{f(e_i) : e_i \in G\}, f(e_i \wedge e_j) = f(e_i) \wedge f(e_j); e_i, e_j \in G\}$$

Clearly $Fr\langle G \mid R\rangle$ and $Fr\langle G \mid R^*\rangle$ are isomorphic through f.
Hence $Fr\langle G \mid R^*\rangle$ has property p. Thus R^* is also maximal with respect to p and hence $R \subseteq R^*$. Also, by assumption R is maximal and hence $R^* \subseteq R$. Thus $R = R^*$ and $f : Fr\langle G \mid R\rangle \to Fr\langle G \mid R = R^*\rangle$ is a frame isomorphism. Therefore $A = Fr\langle G \mid R\rangle$ is reversible.

In order to prove that condition (i) and condition (iii) are equivalent a similar proof will work with property p redefined - A frame $Fr\langle G \mid K\rangle$ is said to have property p if there exists an order preserving bijection from $A = Fr\langle G \mid K\rangle$ to $Fr\langle G \mid R\rangle$. $\square$

5.4 Application

In this section, we prove a question posed by M.Rajagopalan and A.Wilanski in connection with the reversibility of the product of the two point discrete space and a reversible topological space in [38]. We use the frame theoretic approach to find a positive answer to that question.
The product of two reversible topological spaces need not be reversible. This is also true for reversible posets. But there are occassions where the product of two reversible posets can become reversible. The next theorem discuss such a situation.

Theorem 5.4.1. *Let (L,$\leqslant_1$) and (K,$\leqslant_2$) be two reversible posets with least elements 0_L and 0_K respectively. Then $(L \times K, \leqslant)$ where $\leqslant$ is the product partial order is reversible as a poset.*

Proof. Suppose that $L \times K$ is not reversible as a poset. Then by *Theorem* 5.2.1 there exists a strict extension $\leqslant^*$ of $\leqslant$ such that $(L \times K, \leqslant)$ is isomorphic to $(L \times K, \leqslant^*)$. Let $\phi : (L \times K, \leqslant) \to (L \times K, \leqslant^*)$ be the isomorphism. Let ${\leqslant_1}^*$ and ${\leqslant_2}^*$ be the component partial orders of $\leqslant^*$ such that (a,b) $\leqslant^*$ (c,d) in $(L \times K, \leqslant^*)$ if and only if $a \leqslant_1{}^* c$ in (L,${\leqslant_1}^*$) and b ${\leqslant_2}^*$ d in (K,${\leqslant_2}^*$). Also, let $\leqslant_1$ and $\leqslant_2$ be the component partial orders of $\leqslant$ such that (a,b) $\leqslant$ (c,d) in $(L \times K, \leqslant)$ if and only if $a \leqslant_1 c$ in (L,$\leqslant_1$) and b $\leqslant_2$ d in (K,$\leqslant_2$). Clearly (L,${\leqslant_1}^*$) and (K,${\leqslant_2}^*$) are extensions of (L,$\leqslant_1$) and (K,$\leqslant_2$) respectively with at least one of them a strict extension because otherwise the extension $\leqslant^*$ of $\leqslant$ would not be a strict extension. Now we prove that each of these extensions are isomorphic with the corresponding old one.

Define $f : (L, \leqslant_1) \to (L, {\leqslant_1}^*)$ as $f(x) = y$ where y is chosen from $\phi(x, 0_K) = (y, 0_K)$, as ϕ preserves minimal elements.

Claim 1: f is one to one and onto.

Let $f(x) = x_1$ and $f(y) = y_1$ where $\phi(x, 0_K) = (x_1, 0_K)$ and $\phi(y, 0_K) = (y_1, 0_K)$. Now $f(x) = f(y)$ implies $x_1 = y_1$ which gives $(x_1, 0_K) = (y_1, 0_K)$. That is $(x, 0_K) = (y, 0_K)$ and hence $x = y$. Also let $y \in L$. Then $(y, 0_K) \in (L \times K, \leqslant^*)$ and since ϕ is an isomorphism $\phi(x, 0_K) = (y, 0_K)$ for some unique $x \in L$ which by definition of f gives $f(x) = y$, proves ontoness.

Claim 2: f is order preserving.

Let $x \leqslant_1 y$. Then $(x, 0_K) \leqslant (y, 0_K)$ in $(L \times K, \leqslant)$. Taking the images of x and y under f as in claim 1 and applying ϕ to $(x, 0_K) \leqslant (y, 0_K)$, we get

$(x_1, 0_K) \leqslant^* (y_1, 0_K)$ which gives $x_1 {\leqslant_1}^* y_1$ in $(L, {\leqslant_1}^*)$. Hence $f(x) {\leqslant_1}^* f(y)$.
Claim 3: f^{-1} is order preserving.
Let $y_1, y_2 \in L$ with $y_1 {\leqslant_1}^* y_2$. Then $(y_1, 0_K) \leqslant^* (y_2, 0_K)$. Then there exist $x_1, x_2 \in L$ such that $\phi(x_1, 0_K) = (y_1, 0_K)$ and $\phi(x_2, 0_K) = (y_2, 0_K)$ where $f(x_1) = y_1$ and $f(x_2) = y_2$. Since ϕ^{-1} is order preserving, $(x_1, 0_K) \leqslant (x_2, 0_K)$ in $(L \times K, \leqslant)$ and hence $x_1 \leqslant_1 x_2$ in (L,$\leqslant_1$). Thus $f^{-1}(y_1) \leqslant_1 f^{-1}(y_2)$.
Thus f is a poset isomorphism.
In a similar way we can construct an isomorphism $g : (K, \leqslant_2) \to (K, {\leqslant_2}^*)$ as $g(x) = y$ where y is chosen from $\phi(0_L, x) = (0_L, y)$. Thus either or both of $(L, \leqslant_1)$ and $(K, \leqslant_2)$ are not reversible, by *Theorem* 5.2.1, a contradiction. □

Using the above theorem, we now prove that the product of two reversible frames under product partial order is again a reversible frame.

Theorem 5.4.2. *Let $L = \{0_L, a, b, 1_L\}$ be the frame with 0_L as the least element, 1_L as the greatest element and a,b are non-comparable elements. Let K be any reversible frame. Then (L×K,$\leqslant$) where $\leqslant$ is the product partial order is reversible.*

Proof. By *Theorem* 5.4.1, $(L \times K, \leqslant)$ is reversible as a poset. So it needs to prove that every order preserving self bijection on $(L \times K, \leqslant)$ preserves arbitrary join. Let $f : (L \times K, \leqslant) \to (L \times K, \leqslant)$ be an order preserving bijection. Since $(L \times K, \leqslant)$ is reversible as a poset, f is a lattice isomorphism. Write

$$L \times K = \bigcup_{l \in L} I_l$$

where $I_l = \{(l, x) \colon x \in K\}$. Each I_l is a subframe of $(L \times K, \leqslant)$ because each I_l is isomorphic to K through the frame isomorphism $g \colon I_l \to K$ defined by $g(l, x) = x, l \in L$ and $x \in K$. Each I_l is reversible since K is reversible. Since f is order preserving and a,b are non-comparable f maps I_l to I_l according to either of the following cases.

1. $f(I_l) = I_l, \forall l \in L$

2. $f(I_l) = I_l, l = 0_L, 1_L,\ f(I_a) = I_b,\ f(I_b) = I_a$

In case (1) define mappings $f_l(x)$ from K to K by $f_l(x) = y$ where y is chosen via the order preserving bijection $f(l, x) = (l, y), l \in L$ and $x, y \in K$
In case (2) define mappings by $f_l(x)$ by $f_l(x) = y$ where y is chosen from $f(l, x) = (l, y), l = 0_L, 1_L$.
Similarly $f_a(x)$ and $f_b(x)$ from K to K are defined by $f_a(x) = y$ where y is chosen from $f(a, x) = (b, y)$ and $f_b(x) = y$ where y is chosen from $f(b, x) = (a, y)$. Each f_l is one to one and onto. Also each f_l and is an order preserving bijection on K as f is an order preserving bijection on $(L \times K, \leqslant)$. Since K is reversible, each f_l is a frame isomorphism. $f\mid_{I_l}$ is a frame isomorphism in case (1) and case (2).
Claim:$f : (L \times K, \leqslant) \to (L \times K, \leqslant)$ is a frame isomorphism.
Let A be any subset of (L×K,⩽). Write

$$A = \bigcup_{l \in L} A_l$$

where $A_l \subseteq I_l$ and $l \in L$. Then,

$$\begin{aligned}
f(\bigvee A) &= f[(\bigvee A_{0L}) \vee (\bigvee A_a) \vee (\bigvee A_b) \vee (\bigvee A_{1L})] \\
&= f(\bigvee A_{0L}) \vee f(\bigvee A_a) \vee f(\bigvee A_b) \vee f(\bigvee A_{1L}) \\
&= [\bigvee f(A_{0L})] \vee [\bigvee f(A_a)] \vee [\bigvee f(A_b)] \vee [\bigvee f(A_{1L})] \\
&= \bigvee f(A)
\end{aligned}$$

since f is a lattice isomorphism and since $f\mid_{I_l}$ is a frame isomorphism. □

In [38], M.Rajagopalan and A.Wilanski suggested a number of problems in reversible topological spaces that are still unanswered. The first problem they suggested is given as follows.

"Is the product of a two-point discrete space and a reversible space reversible ?"

The answer to the question is positive and we provide a proof for the above stated problem through the following theorem.

Theorem 5.4.3. *Let X be the two point discrete topological space and Y be any reversible topological space. Then the product topological space $X \times Y$ is reversible.*

Proof. The frame of open sets of X is the frame L in *Theorem* 5.4.2. Let K be the frame of open sets of the topological space Y. Since Y is reversible as a topological space, K is reversible as a frame, by *Theorem* 5.2.3. Therefore the frame of open sets L × K of the product topological space $X \times Y$ is reversible by *Theorem* 5.4.2. The product topology on

$X \times Y$ coincides with L × K, by *Theorem* 1.6.5. Thus X × Y is reversible by *Theorem* 5.2.3. □

Chapter 6

Automorphism Group of Finite Frames

6.1 Introduction

De Groot[7] proved that any group is isomorphic to the autohomeomorphism group of some connected bicompact Hausdorff space. We can prove that any group is isomorphic to the automorphism group of a frame, using the foresaid result. We prove that any frame isomorphism on a frame with generator set and relation set both fixed can be obtained as a unique extension of some permutation on the generator set to the frame as a frame isomorphism. Thus the group of automorphisms on a frame is induced

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Some results of this chapter are included in the following paper.
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by a subgroup of the group of all permutations of the generator set in the above sense. A related problem is to determine the subgroups of the group of all permutations of the generator set of a frame that can induce the automorphism group of the frame for some relation set. This problem is the point free version of the problems solved by P.T.Ramachandran[39] in topology and T.P.Johnson[20] in fuzzy topological space.

6.2 Automorphism Group and Generators of a Frame

As stated in the previous chapter, we represent a given frame A in the form of presentation of frames $A = Fr\langle G \mid R\rangle$. The generator set can be any set. But here we insist that G is a subset of A whose elements generate the elements of A as join of finite meets. For example, the frame of reals ΩR is generated by $G = \{(p,q)/p,q \in Q\}$ under the relation set R containing the relations

1. $(p,q) \wedge (r,t) = (p \vee r, q \wedge t)$ where $p \leqslant r < q \leqslant t$

2. $(p,q) \vee (r,t) = (p,t)$ where $p \leqslant r < q \leqslant t$

3. $(p,q) = \bigvee\{(r,t)/p < r < t < q\}$

4. $1 = \bigvee\{(p,q)/p,q \in Q\}$

Here G is a subset of the frame of reals.

In the following theorem, we prove that every frame isomorphism on a frame $A = Fr\langle G \mid R\rangle$ with generator set G and relation set R both fixed can be obtained as the extension of a unique permutation on G to A as a frame isomorphism.

Theorem 6.2.1. *Let $A = Fr\langle G \mid R\rangle$ be a frame with generator set G and relation set R both fixed. Then every frame isomorphism on A is the extension of a unique permutation on G to A.*

Proof. Let f be a frame isomorphism on A. Define $p : G \to G$ by $p(x) = f(x), x \in G$.
Since G and R are fixed and a frame isomorphism maps generators to generators, p is a bijection on G. Now we extend p on G to P on A by defining P as follows.
Let $x \in A$. Since every element of A is the arbitrary join of finite meets of generators in G, we have $x = \bigvee_{i\in I} \bigwedge_{j\in J} x_{ij}$ where $x_{ij} \in G$ and I, J are index sets with J finite. Define $P(x) = \bigvee_{i\in I} \bigwedge_{j\in J} p(x_{ij})$.
Since A is a complete lattice P is well defined. Let $x \neq y$ where $x = \bigvee_{i\in I} \bigwedge_{j\in J} x_{ij}$ and $y = \bigvee_{l\in I} \bigwedge_{k\in J} y_{lk}$ with $x_{ij}, y_{lk} \in G$. If $P(x) = P(y)$, then $\bigvee_{i\in I} \bigwedge_{j\in J} p(x_{ij}) = \bigvee_{l\in I} \bigwedge_{k\in J} p(y_{lk})$.
That is, $\bigvee_{i\in I} \bigwedge_{j\in J} f(x_{ij}) = \bigvee_{l\in I} \bigwedge_{k\in J} f(y_{lk})$ and hence $f(x) = f(y)$ since f is a frame isomorphism. Thus we get $x = y$ as f is one to one, which is a contradiction. Hence P is one to one.

Let $y = \bigvee_{i\in I} \bigwedge_{j\in J} y_{ij} \in A$. Define $x = \bigvee_{i\in I} \bigwedge_{j\in J} p^{-1}(y_{ij})$ which is in A and $P(x) = y$. Thus P is a bijection on A.

Let $x = \bigvee_{i\in I} \bigwedge_{j\in J} x_{ij}$ with $x_{ij} \in G$. Then $P(x) = \bigvee_{i\in I} \bigwedge_{j\in J} f(x_{ij}) = f(x)$ and hence $P \equiv f$ on A.

To prove uniqueness, assume that f and g are two distinct frame isomorphisms on A that agree on G. Since f and g are distinct, there exists at least one $x \in A$ such that $f(x) \neq g(x)$. Let $x = \bigvee_{i\in I} \bigwedge_{j\in J} x_{ij}$ where $x_{ij} \in G$. Now $f(x) = f(\bigvee_{i\in I} \bigwedge_{j\in J} x_{ij}) = \bigvee_{i\in I} \bigwedge_{j\in J} f(x_{ij}) = \bigvee_{i\in I} \bigwedge_{j\in J} g(x_{ij}) = g(x)$, as f and g agree on G. Thus we get a contradiction and consequently the extension is unique. □

In view of the above theorem and the fact that no two permutations yield the same frame isomorphism on A, the automorphism group of a frame is induced by some subgroup of the group of all permutations on its generator set G. Here we determine some of those subgroups of the group of all permutations on G that always induce or never induce the automorphism group of the frame A.
We state the following theorem due to De Groot[7]

Theorem 6.2.2. *Every group is isomorphic to the autohomeomorphism group of some connected bicompact Hausdorff space.*

Theorem 6.2.3. *The group of homeomorphisms A(X) of a Hausdorff topological space (X,T) is isomorphic to the group of frame isomorphisms A(T) of the frame T under set inclusion.*

Proof. Define $\psi : A(X) \to A(T)$ by $\psi(f) = f^*$ where f^* is the frame isomorphism on T induced by the homeomorphism f on X. That is,

$$f^*(U) = f(U)$$

First, we prove that ψ is one to one. Let $f \neq g$ be two homeomorphisms on X. Then there exists at least one $x \in X$ such that $f(x) \neq g(x)$. Let $u = f(x)$ and $v = g(x)$. Since X is Hausdorff there exist disjoint open sets G_u containing u and G_v containing v of X. Define $G = f^{-1}(G_u) \cap g^{-1}(G_v)$. Clearly G is an open set containing x. Also $f(G) \subseteq G_u$ and $g(G) \subseteq G_v$. Since G_u and G_v are disjoint $f(G) \neq g(G)$. Thus $f^* \neq g^*$ and hence ψ is one to one.

Now we prove that ψ is onto. For, consider $f^* : T \to T$ as a morphism in **Frm**. Then by *Theorem* 1.4.3 we get a morphism $f : X \to X$ in **Sp** which will induce the mapping f^* on T. Thus $\psi(f) = f^*$ and hence onto.
$g \circ f(U) = g[f^*(U)] = g^*[f^*(U)] = g^* \circ f^*(U)$.
Thus $(g \circ f)^* = g^* \circ f^*$. Therefore $\psi(g \circ f) = \psi(g) \circ \psi(f)$ and hence ψ is a group isomorphism. □

Corollary 6.2.4. *Let X be any set with $|X| = n$. Then the frame $(\mathcal{P}(X), \subseteq)$ has frame isomorphism group isomorphic to S_n.*

Proof. The topological space $(X, \mathcal{P}(X))$ is Hausdorff. Since its topology is discrete, the group of homeomorphisms is S_n. Hence the frame $(\mathcal{P}(X), \subseteq)$ has frame isomorphism group isomorphic to S_n, by *Theorem* 6.2.3. □

Theorem 6.2.5. *Every group is isomorphic to the frame isomorphism group of some frame.*

Proof. Let G be any group. Now by *Theorem* 6.2.2, there exists a connected bicompact Hausdorff topological space (X,T) whose group of autohomeomorphisms A(X) is isomorphic to G. Since X is Hausdorff, the frame isomorphism group of the frame T is isomorphic to A(X) by *Theorem* 6.2.3. Thus the group of frame isomorphisms of the frame T is isomorphic to G. $\square$

6.3 Automorphism Group of Finite Frames

We know that every frame can be presented by means of set of generators G and set of relations R. Keeping the generator set fixed, we can impose different relation sets on G. Each time we get a frame and we examine the frame isomorphism group of the resulting frame. As a result, we can see that, some permutation groups on the generator set can be the frame isomorphism group of a frame on any generator set for some fixed relation set. But there are some permutation subgroups on the generator set that can never be the frame isomorphism group of a frame on generated by the same generator set for any relation set. Note that this does not contradict the representation *Theorem* 6.2.5 as it tells that the particular permutation subgroup of the generatorset G itself cannot be the frame isomorphism group of the frame generated by elements of G for any relationset but definitely there is some other frame isomorphic to the frame isomorphism group of that frame.
Through the following theorems in this section, we provide examples of permutation subgroups of the generator set of a frame that can always represent the frame isomorphism groups of such frames.

Theorem 6.3.1. *For every finite generator set G, a relation set R exists so that the frame $A = Fr\langle G \mid R\rangle$ has automorphism group induced by $\{e\}$ where e is the identity permutation on G.*

Proof. Let $G = \{a_1, a_2, \ldots, a_n\}$. Take $R = \{a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n\}$. Clearly $A = Fr\langle G \mid R\rangle$ is the chain of length *n+1* and hence its automorphism group contains only the identity automorphism on A which is induced by the identity permutation on G. □

Theorem 6.3.2. *For every finite generator set G, a relation set R exists so that the frame $A = Fr\langle G \mid R\rangle$ has automorphism group induced by the two element subgroup of the group of all permutations on G.*

Proof. Let $G = \{a_1, a_2, \ldots, a_n\}$.
Take $R = \{a_1 \leqslant a_2 \leqslant \ldots \leqslant a_{i-1} \leqslant a_i, a_{i+1} \leqslant a_{i+2} \leqslant a_{i+3} \leqslant \ldots \leqslant a_n\}$. Let $f : G \to G$ be defined by

$$f(a_j) = \begin{cases} a_j & \text{if } j \neq i,\ i+1 \\ a_{i+1} & \text{if } j = i \\ a_i & \text{if } j = i+1 \end{cases}$$

Clearly $f^2 = e$ where e is the identity permutation on G. It is evident that the group of frame isomorphisms on A contains f and e only which is induced by the two element subgroup of the group of all permutations on G □

Theorem 6.3.3. *For every finite generator set G, a relation set R exists so that the frame $A = Fr\langle G \mid R\rangle$ has automorphism group induced by the group of all permutations on G.*

Proof. Let $G = \{a_1, a_2, \ldots, a_n\}$. Take $R = \phi$. Then A is a free distributive lattice on n generators and hence represents the discrete topology on G. Then by *Corollary* 6.2.4, the frame isomorphism group of A is isomorphic to S_n. Thus by *Theorem* 6.2.1, every permutation on G induce an automorphism on A. □

Corollary 6.3.4. *For every finite generator set G having n elements, a relation set R exists so that the frame $A = Fr\langle G \mid R\rangle$ has automorphism group induced by the group of all permutations on $m < n$ elements of G.*

Proof. Take $G = \{a_1, a_2, \ldots, a_n\}$ and
$R = \{a_1, a_2, \ldots, a_m \leqslant a_{m+1} \leqslant \ldots \leqslant a_n\}$. □

Through the following theorems in this section, we provide examples of such permutation subgroups of the generator set of a frame that never represent the frame isomorphism groups of such frames for any relation set.

Theorem 6.3.5. *For every finite generator set G, there is no relation set R that presents a frame having automorphism group induced by a cyclic subgroup of order $n \geqslant 3$ of the group of all permutations on G.*

Proof. Let $G = \{a_1, a_2, \ldots, a_n\}$. Suppose that there exists a relation set R such that some cyclic subgroup of order $n \geqslant 3$ of the group of all permutations on G induces the automorphism group of $A = Fr\langle G \mid R\rangle$. Let it be generated by the permutation $p = (a_1, a_2, \ldots, a_m)$ where $m \geqslant 3$. If $R = \phi$, then automorphism group of A would be $S_n(G), n \geqslant 3$ which is not cyclic. So R is non empty. We argue that the relation set must contain a relation involving some of the generators appearing in p. If not, there are no relations in R involving generators in p. Then using *Corollary* 6.3.4, we can show that the automorphism group of A is induced by $S_m(G), n \geqslant 3$ which is not cyclic.

Now suppose that $a_i \leqslant a_j$ is a relation in R where both a_i, a_j are distinct generators in p. This is because if the relations are equations they can be converted into inequations and can be further splitted into basic form using frame laws which end up with relations of the assumed type. Suppose that $i < j$ and let $j - i = r$. Since the group generated by p induces the automorphism group of A, p^r induces an automorphism on A. Since $a_i \leqslant a_j$, we have $p^r(a_i) \leqslant p^r(a_j)$. That is $a_j \leqslant a_{j+r}$. Repeatedly applying p^r on both sides of the new inequation each time, we get $a_i \leqslant a_j, a_j \leqslant a_{j+r}, a_{j+r} \leqslant a_{j+2r}, \ldots$. Since the generator set is finite and the cyclic group is of order m, the sequence will end up with a_i after m such operations. Thus $a_i \leqslant a_j$ and $a_j \leqslant a_i$. Hence $a_i = a_j$, which is a contradiction. Thus all possible cases lead to a contradiction. Hence existence of such a relation set R is not possible. □

Theorem 6.3.6. *For every finite generator set G, there is no relation set R that presents a frame having automorphism group induced by a non trivial proper normal subgroup of the group of all permutations on G.*

Proof. Case I: $n \leqslant 4$

The non trivial proper normal subgroups of the group of all bijections on G are the alternating group on G and the Klein 4 group when $n = 4$ and alternating group only when $n = 3$. The Klein 4 group on $G = \{a, b, c, d\}$ is $\{(a, b)\,(c, d)\,, (a, c)\,(b, d)\,, (a, d)\,(b, c)\,, e\}$.
If the automorphism group of $A = Fr\langle G \mid R\rangle$ is induced by this group for some R, then $R = \phi$ because none of the relations $a \leqslant b, a \leqslant c, a \leqslant d, b \leqslant c, b \leqslant d, c \leqslant d$ can be in R. For, if $a \leqslant b$ is in R, then the frame isomorphism induced by the permutation containing the transposition (a, b) applied on $a \leqslant b$ gives $b \leqslant a$ and hence $a = b$, which is a contradiction. The other

relations do also give such contradiction.
Then by *Theorem* 6.3.3, the automorphism group of A would be induced by the group of all permutations on G which is a contradiction.
The alternating group on G does not induce the automorphism group of A in this case can be proved using the same technique in the proof of the following case.

Case II: $n \geqslant 5$.

Let $G = \{a_1, a_2, \ldots, a_n\}$.
Suppose there exists a relation set R such that the group of automorphisms of $A = Fr\langle G \mid R\rangle$ is induced by the alternating group on G. If $R = \phi$, then automorphism group of A would be induced by S_n which is not the case. So $R \neq \phi$. Now suppose that R contains a relation $a_i \leqslant a_j$ where both a_i, a_j are distinct generators in G. Since $|G| \geqslant 5$, we can find a generator $a_k \neq a_i, a_j$ and $p = (a_i, a_j, a_k)$ induces an automorphism on A. Now $p(a_i) \leqslant p(a_j)$.That is $a_j \leqslant a_k$. Similarly $a_k \leqslant a_i$.Thus $a_i, a_j = a_k$ which is a contradiction. $\square$

Concluding remarks and suggestions for further study

It is known that, for a given topological property R and a set X, the set of all topologies on X denoted by R(X) which have property R is a partially ordered set by set inclusion. A topological space (X, T) is *maximal R(R-maximal)* if T is a maximal element in R(X). Also a topological space (X, T) is *minimal R(R-minimal)* if T is a minimal element in R(X). The concept of minimal topologies was first introduced in 1939 by A.S.Parhomenko when he showed that compact Hausdorff spaces are minimal Hausdorff. Four years later E. Hewitt proved that compact Hausdorff spaces are maximal compact as well as minimal Hausdorff. In 1948, A. Ramanathan proved that a topological space is maximal compact if and only if its compact subsets are precisely the closed sets. Following this, many maximal and minimal topological spaces holding properties such as regular, completely regular, normal, countability axioms, separation axioms etc. were analysed fom this point of view.

The concept of maximality and minimality in frames holding a specified frame isomorphic property has not studied so far. Here we have made an attempt to start such a study. It is known that a topological space is maximal or minimal with respect to a given topological property if and only if it is reversible. We introduced the analoguous concept of reversible frames here and established a connection with maximal or minimal frames possessing a given frame isomorphic property.

We started our study with the introduction of maximal compact frames and then proved a characterization for such frames. Using this characterization, we have shown that a compact regular frame is maximal compact. Also the concept of minimal Hausdorff frames is introduced with a partial characterization. A minimal Hausdorff frame is almost compact. Also a minimal Hausdorff frame has non trivial convergent filters coinciding with clustered filters. Such results are useful in finding maximal and minimal frames. Also frame theoretic methods are useful in solving problems from topology. For example, as an application of reversible frames, we have proved that the product of a two point discrete space with a reversible space is again reversible.

There are still remaining many frame isomorphic properties such as regularity, complete regularity, normality etc. to be analyzed in the context of maximal or minimal frames.

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