STUDIES ON THE INTERACTION OF TWO LEVEL ATOMS WITH QUANTIZED ELECTROMAGNETIC FIELD

Thesis submitted to

Cochin University of Science and Technology

in partial fulfilment of the requirements
for the award of the degree of

DOCTOR OF PHILOSOPHY

by

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May 2014

Studies on the interaction of two level atoms with quantized electromagnetic field.

PhD thesis in the field of Quantum Optics

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to my family & friends

CERTIFICATE

Certified that the work presented in this thesis is a bonafide research work done by Mr. Priyesh K. V. under my guidance in the Department of Physics, Cochin University of Science and Technology, Kochi, India - 682022, and has not been included in any other thesis submitted previously for the award of any degree.

Kochi-22	Prof. Ramesh Babu T.
May, 2014	(Supervising Guide)

CERTIFICATE

This is to certify that all the relevant corrections and modifications suggested by the audience during the Pre-Synopsis Seminar and recommendations by the Doctoral Committee of the candidate have been incorporated in the thesis entitled "Studies on the interaction of two level atoms with quantized electromagnetic field" by Mr. Priyesh K. V.

Kochi-22	Prof. Ramesh Babu T.
May, 2014	(Supervising Guide)

DECLARATION

I hereby declare that the work presented in this thesis is based on the original research work done by me under the guidance of Prof. Ramesh Babu T., Department of Physics, Cochin University of Science and Technology, Kochi, India - 682022, and has not been included in any other thesis submitted previously for the award of any degree.

Kochi-22
May, 2014

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Preface

QUANTUM optics is an emerging field in physics which mainly deals with the interaction of atoms with quantised electromagnetic fields. The two level system(TLS) approximation, where the atom is assumed to have only two levels, viz. ground and excited states, is a very significant model which gives great insight into the complex nature of the atom field interaction and this model could make many interesting predictions. Jaynes-Cummings Model (JCM) is a key model among them, which describes the interaction between a two level atom and a single mode radiation field. Purely quantum mechanical phenomena such as spontaneous emission, Rabi oscillations in the case of photon Fock state-atom interaction, collapses and revivals in population inversion in the case of coherent light atom interactions, have been predicted by JCM. Recent advances in the experimental realizations agree very well with the explanations given by JCM.

Recently there is noticeable progress in realizing frequency chirped laser pulses and it open up a way for the fine tuned interaction between photons and atoms. These developments attracted many researchers to the study on the interaction of atoms with frequency varying photon field. It is reported that there are significant changes in the evolution of atom field system especially in the evolution of atom-field probability amplitudes due to the time dependency of the field frequency. In our work we investigate the various possibilities of manipulating and modifying the atom filed state probabilities during their interaction, by applying a time variation in the field frequency.

The qubit analogue of a two level atom and photon system makes its role countable in Quantum Information Processing(QIP). In the current setting, data transmission from one point to other is one of the main challenges in the realization of quantum computers. Many systems have been proposed in this area for effective transmission and manipulation of quantum information. A coupled cavity array is a subject of discussion in the present time as it can serve the purpose of quantum data transmission and can be modelled as a controllable many body system. We also studied the various possibilities of quantum state transfer between atoms in a coupled cavity system.

Chapter 1 begins with a brief history of light, atom and their interactions. The quantization of single mode electromagnetic field inside a cavity is explained using Maxwell's equations. The photon Fock states(number states) are introduced as the eigenstates of the quantized field Hamiltonian. Properties of annihilation and creation operators are examined in detail in this chapter. This chapter contains the properties of coherent states and also discusses the characteristics of squeezed state. The two level atom approximation and its importance are also discussed in this chapter.

Chapter 2 discusses the interaction between atoms and electomagnetic fields. The interaction between light and matter is described using classical, semi-classical and the fully quantum mechanical formulations. The interaction between a two level atom and quantized single mode field is studied using Jayness Cummings model(JCM). The evolution of atom field system during their interaction for various initial photon distributions are examined. At the end of this chapter some extensions to the JCM and a brief review of the recent literatures on two level atom field interaction are included.

Chapter 3 suggest a method to manipulate the population inversion due to interaction and control the randomness in it, by applying a time dependence on the frequency of the interacting squeezed field. A detailed study of the dependence of population inversion on the applied sinusoidal frequency modulation parameters are done and presented in this chapter. As a continuation to this, the dynamics of interaction between a two level atom and electromagnetic field with a phase shifted sinusoidally varying frequency is also included in this chapter. The change in behaviour of the population inversion due to the presence of a phase factor in the applied frequency variation is explained in this chapter.

Chapter 4 describes the interaction between two level atom and electromagnetic field in nonlinear Kerr medium. The frequency of the field is set to be fluctuating and phase shifted. This chapter also comprises of the evolution of the von Newman entropy of the system, which is a direct measure of the entanglement between the atom and field. The cases, with and without frequency fluctuations in the interacting field frequency is described. It is noted that the entanglement between the atom and field can be controlled by varying the period of the field frequency fluctuations.

Chapter 5 deals with atomic and field state evolution in a coupled cavity system. A coupled cavity system is a series of cavities coupled together via photon hopping and each cavity contains a two level atom. This chapter focus on the exchange of atomic (or field) probability amplitudes between the cavities in a two cavity system. The analytical expressions for the time variation of atomic and field probability amplitudes, population inversion, are obtained and the results are presented in this chapter. The evolution of atom(field) probability amplitudes in a coupled cavity system with Kerr non-linearity is also studied in this section and the effect of susceptibility on the atomic state transfer between cavities is explained.

Chapter 6 summarises the main results and major findings in the theses.

Publications related to the work presented in the thesis:

In refereed journals

1. "Control of collapse revival phenomenon using a time varying squeezed field", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *Journal of Nonlinear Opt. Phy. & Mat.*, Vol. 20, No. 2 (2011).

2. "Effect of Phase Shifted Frequency Modulation on Two Level Atom-Field Interaction", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *Commun. Theor. Phys.* 57 (2012).

3. "Evolution of atom-field probability in a coupled cavity system", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *Journal of Nonlinear Opt. Phy. & Mat.* Vol. 22, No. 3 (2013).

4. "Dynamics of atom-field probability amplitudes in a coupled cavity system with Kerr non-linearity", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *AIP Conf. Proc.* 1576, 87 (2014).

In conferences/seminars

1. "Studies on coupled cavity array", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *National conference in advances in Physics-2012*, Indian Institute of Technology, Roorkee, India.

2. "Dynamics of atom-field probability amplitudes in a coupled cavity system with Kerr nonlinearity", **K. V. Priyesh** and Ramesh Babu Thayyullathil, *Biennial meeting of Indian Society of Atomic, Molecular Physics 2013*, IISER Kolkata, India.

Acknowledgements

It is a great pleasure to thank all those people who influenced my life during the past few years and contributed in different ways to this thesis.

Foremost, I would like to express my deepest sense of gratitude to my supervisor Prof. Ramesh Babu T., who offered his continuous advice and encouragement throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about his students, give enough freedom to work. His patience, motivation, enthusiasm and logical means of thinking helped me in all the time of my research.

I extend my sincere thanks to the present Head, Department of Physics, Prof. B. Pradeep and former Heads, for providing me the necessary facilities for my research. I would like to express my sincere gratitude to Prof. M. Sabir, my Doctoral Committee member for the helps and inspiration that I have received. I also thank Prof. V. C. Kuriakose for his advice and concern towards me. I convey my gratitude to Titus sir, and Raveendranath sir, for their friendly advice and candid discussions. I am grateful to all my teachers of the Department of Physics, for their support and encouragement right from my post graduation days. I am also thankful to all the office and library staff of the Department of Physics for all the help and cooperation.

A sound backup organization is important for pulling through and remaining sane in the research study. I was lucky to be in the middle of good, caring friends in a fun filled environment. Sanal is always there to care me as a brother through out my life in the university and Nijo is a constant source of support and encouragement for me. Thanks to

Arun for a wonderful friendship. I have greatly enjoyed the company of Tharanath. I would wish to bring up the names of Rajeshmon and Subin here. Thanks to Sasankan for a grand friendship. I am too thankful to Bhavya for a great friendship and I thankfully recall the wonderful company of Sudheep. Thanks to Rajesh C. S. for his jovial jokes and pleasing comments. I would like to remark here the names of Sajan, Shijeesh, Sagar, Abhilash, Titu, Manoj, Sreejith, Anand, Aravind, Santhosh, Deepu, Vikas and Vinod. I thank all friends in the department for the nice time I experienced with them.

I am much indebted to my fellows in the theory group for all the wonderful time I dealt with them. I am very much thankful to Jishnu for his timely helps and Shaju sir for his support. I also thank Nima chechi, Saneesh, Lini, Prasobh, Prasia, Anoop, Praseetha for their nice company. I am thankful to Rijeesh, Navaneeth, Naseef and Sijith the "neutrino guys".

Thanks to the comrades Sreejesh and Anees for a tender friendly relationship. Call back here the names of Binu P Paul, Vinod Sir and Balu here.

I thankfully acknowledge the financial assistance given by University Grants Commission (UGC), India under BSR-RFSMS Scheme.

I state my profound gratitude from my deep heart to my parents, sister and my in laws for all the love, patience and support they afforded me throughout these years.

Lastly, and most importantly, I would like to thank my wife Anju for her love without which I could not make this thesis. Her support, encouragement, quiet patience and unwavering love were undeniably the bedrock upon which the past five years of my life have been made. A *particular thanks to my little daughter "Totto" for the divine guidance in her innocent smile, which makes me move forward always.*

Priyesh K. V.

1 Introduction

1.1 Light

Light always has a distinctiveness in the continuing endeavour of mankind to understand nature. Evidently there are sequential progresses in inferring the nature and behaviour of light more scientifically. In 300 BC Euclid postulated, in his book named *optica*, that light travels in straight line. Newton, who made so many fundamental contributions to optics, in his *Hypotheses of light*(1675) modelled that light is composed of small particles called *corpuscles*. According to his theory corpuscles emitted in all directions from a source of light and this concept could explain the phenomenon called reflection. During the same time Christian Huygens worked out a mathematical wave theory of light in 1678, and published it in his *Treatise on light* in 1690. After that in 1800 Thomas Young, through his famous double slit experiment, showed that light waves interfere like sound waves as predicted by the wave theory. Also Fresnel interpreted diffraction using the ideas of wave theory. In 1863 Maxwell introduced the concept of unification of electricity and magnetism through the famous Maxwell's equations, based on which light can be understood as the waves of electric and magnetic fields propagating through space. In the explanation of the black body radiation spectrum Max plank pointed out that radiation is emitted and absorbed not continuously but in small discrete units called quanta. Einstein strengthen the concept of discrete quanta in his explanation to photoelectric effect and his theory won Nobel Prize in 1921.

1.2 Atom

Similar to the case of light, there are number of models which attempt to explain the atomic structure. The first of that kind is the 'Plum Pudding Model' put forward by J. J. Thomson in 1904. In this model atom is composed of electrons surrounded by a soup of positive charge to balance the negative charge of electrons. This model was disproved by the gold foil experiment of Hans Geiger and Ernest Marsden. After this in 1911, Ernest Rutherford introduced Rutherford atom model, where the atom is made up of a central charge surrounded by a cloud of orbiting electrons. Later in 1913, Neils Bohr came up with a model of atom now known as Bohr atom model. According to this theory atom is a small, positively charged nucleus surrounded by electrons that travel in circular orbits around the nucleus, the attraction provided by electrostatic forces. An electron can orbit the nucleus only on allowed orbits determined by a discrete value of energy. Electrons can only gain and lose energy by jumping from one allowed orbit to another, absorbing or emitting electromagnetic radiation with a frequency ν, which corresponds to the energy difference of the levels, according to the Planck relation,

$$E_2 - E_1 = h\nu, \tag{1.1}$$

where h is the Planck constant. In an atom electrons from a lower energy level is excited to a higher energy level by absorbing a quanta of radiation equivalent to the energy difference between the two energy levels. Electron from a higher energy level jumps to a lower energy level by spontaneously emitting a quanta of radiation.

1.3 Quantum optics

The formulation of quantum theory of light came in 1920's after the birth of quantum mechanics. The word 'photon' was coined by Gilbert Lewis in 1926[1] for the quanta of light and an year later Dirac published a seminal paper on the quantum theory of radiation[2]. Following these developments there were many attempts to study the optical spectra of atoms and also to understand the quantum effects directly associated with the light itself. The modern subject of quantum optics was born in an effective manner in 1956 with the work of Hanbury Brown and Twiss to measure the fluctuations in the light intensity on short time-scales[3]. This opened the door to more sophisticated experiments on photon statistics, which eventually leads to the observation of optical phenomena with no classical counter part.
The invention of laser, a device which emits light through the process of optical amplification caused by the stimulated emission of electromagnetic radiation, in 1960 led to new interest in this subject. It was hoped that the properties of the laser light would be considerably different from those of conventional sources, but these attempts again turned out to be negative. Glauber in 1963 described a new state which appears particularly well suited for the discussion of experiments performed with light beams whether coherent or incoherent[4]. These states have different statistical properties to those of classical light and is known as coherent states. The experimental confirmation of these non-classical properties was given by Kimble, Dagenais, and Mandel in 1977 when they demonstrated photon antibunching for the first time[5]. They also pointed out that, unlike photoelectric bunching, which can be given a semiclassical interpretation, antibunching is understandable only in terms of a quantized electromagnetic field.

Later Slusher et al. successfully generated squeezed states of the electromagnetic field by nondegenerate four-wave mixing due to sodium atoms in an optical cavity[6].
In the recent years, the subject has expanded to include the associated disciplines of quantum information processing and controlled light-matter interactions. The work of Aspect and co-workers starting from 1981 onwards may be considered as landmark in this area. They used the entangled photons from an atomic cascade to demonstrate violations of Bell's inequality[7, 8], there by decidedly showing how quantum optics can be applied to other branches of physics. Since then, there have been growing number of examples of the use of quantum optics in the ever widening applications.

1.4 Quantization of electromagnetic field

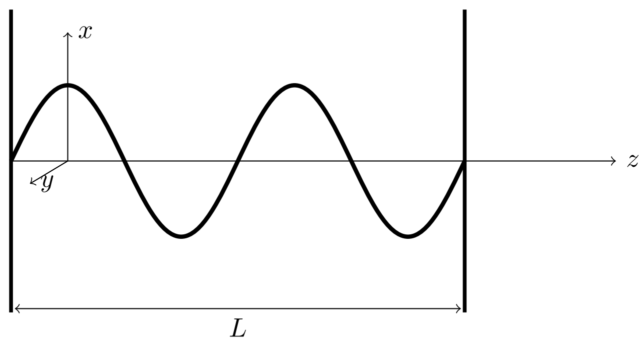


Figure 1.1: Electromagnetic field inside a one dimensional cavity

Here we consider electromagnetic radiation field in a one dimensional cavity along the z axis. The walls of the cavity are perfectly conducting, where the field vanishes, are at $z = 0$ and $z = L$. The electric field inside the cavity will form a standing wave as shown in the Fig.1.1. Assuming that there are no sources of radiation inside the cavity, the corresponding Maxwell's equations in SI units are,

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \frac{-\partial \boldsymbol{B}}{\partial t} \\ \nabla \times \boldsymbol{B} &= \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \\ \nabla . \boldsymbol{B} &= 0 \\ \nabla . \boldsymbol{E} &= 0 \end{aligned} \tag{1.2}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields respectively, ϵ_0 and μ_0 are the free space permittivity and permeability respectively and $\mu_0 \epsilon_0 = 1/c^2$ where c is the speed of light in vacuum. Let the electric field be linearly polarized in the x direction *i.e.,* $\boldsymbol{E}(\boldsymbol{r}, t) = \hat{\boldsymbol{e}}_x E_x(z, t)$, where $\hat{\boldsymbol{e}}_x$ is the unit vector along the x-axis. A single mode field obeying Maxwell's equations given in Eq. (1.2) and satisfying the boundary conditions is given by

$$E_x(z, t) = \left(\frac{2\omega^2}{V \epsilon_0} \right)^{\frac{1}{2}} q(t) \sin(kz) \tag{1.3}$$

Here ω is the frequency of the mode and $k = \omega/c$ is the wave number associated to it. Now, in order to satisfy the boundary conditions, the possible values of ω must be; $\omega_m = c\,(m\pi/L)$, where $m = 1, 2, ...$ We assume that ω in Eq. (1.3) is one of these frequencies and ignore the rest for convenience. The effective volume of the cavity $V = LA$ (A is the transverse area of the optical resonator) and $q(t)$ is a time dependent factor having the dimension of length. From Eq. (1.3) and using Maxwell's equation the magnetic field inside the cavity is $\boldsymbol{B}(\boldsymbol{r}, t) = \hat{\boldsymbol{e}}_y B_y(z, t)$, where

$$B_y(z, t) = \left(\frac{\mu_0 \epsilon_0}{k} \right) \left(\frac{2\omega^2}{V \epsilon_0} \right)^{\frac{1}{2}} \dot{q}(t) \cos(kz) \tag{1.4}$$

Here $q(t)$ and $\dot{q}(t)$ respectively will play the role of a generalized coordinate and momentum, for a "particle" of unit mass, i.e., $p(t) = \dot{q}(t)$. The classical field energy, or Hamiltonian H, of the single mode field is

$$\begin{aligned} H &= \frac{1}{2}\int dV \left[\epsilon_0 \boldsymbol{E}^2(\boldsymbol{r},t) + \frac{1}{\mu_0}\boldsymbol{B}^2(\boldsymbol{r},t)\right] \\ &= \frac{1}{2}\int dV \left[\epsilon_0 E_x^2(z,t) + \frac{1}{\mu_0}B_y^2(z,t)\right] \end{aligned} \tag{1.5}$$

and for a single mode field given in Eq. (1.3) we have

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) \tag{1.6}$$

Now one can compare the single mode electromagnetic field Hamiltonian to a harmonic oscillator Hamiltonian of unit mass, where the electric and magnetic fields, apart from some scalar factors, play the role of generalized coordinate and momentum. Taking the operator correspondence of $q(t)$ and $p(t)$, we may write the commutation relation in the form

$$[\hat{q}, \hat{p}] = i\hbar\hat{I} \tag{1.7}$$

In the following discussion we will omit the identity operator, $\hat{I}$ whenever no confusion is possible. Now the Eqs. (1.3) and (1.4) becomes

$$\hat{E}_x(z,t) = \left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} \hat{q}(t)\sin(kz) \tag{1.8}$$

$$\hat{B}_y(z,t) = \left(\frac{\mu_0\epsilon_0}{k}\right)\left(\frac{2\omega^2}{V\epsilon_0}\right)^{\frac{1}{2}} \hat{p}(t)\cos(kz) \tag{1.9}$$

and the corresponding Hamiltonian is

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right) \tag{1.10}$$

The operators $\hat{q}$ and $\hat{p}$ are hermitian and corresponds to observable quantities. However it is convenient to introduce the nonhermitian (nonobservable) annihilation($\hat{a}$) and creation($\hat{a}^\dagger$) operators such that

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q} + i\hat{p}\right) \tag{1.11}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega\hat{q} - i\hat{p}\right) \tag{1.12}$$

and the electric and magnetic field respectively becomes,

$$\hat{E}_x(z,t) = E_0\left(\hat{a} + \hat{a}^\dagger\right)\sin(kz) \tag{1.13}$$

$$\hat{B}_y(z,t) = B_0\frac{1}{i}\left(\hat{a} - \hat{a}^\dagger\right)\cos(kz) \tag{1.14}$$

where $E_0 = \sqrt{(\hbar\omega/\epsilon_0)}$ and $B_0 = (\mu_0/k)\sqrt{(\epsilon_0\hbar\omega^3/V)}$ represent respectively the electric and magnetic field per photon. Operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the commutation relation

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \tag{1.15}$$

and the Hamiltonian operator takes the form

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{1.16}$$

To study the time evolution of the operator $\hat{a}$, we have the Heisenberg's equation

$$\begin{aligned}\frac{d\hat{a}}{dt} &= \frac{i}{\hbar}\left[\hat{H}, \hat{a}\right]. \\ &= -i\omega\hat{a},\end{aligned} \tag{1.17}$$

which has the solution

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t} \tag{1.18}$$

and by taking the Hermitian conjugate of Eq. (1.18) we have,

$$\hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \tag{1.19}$$

Thus Eqs.(1.18) and (1.19) represent the time evolution of annihilation and creation operators.

1.4.1 Number state

The operator product $\hat{a}^\dagger\hat{a}$ has a special significance and is called the number operator, which we denote as $\hat{n}$. Let $|n\rangle$ denotes the simultaneous eigenstates of $\hat{n}$ and the Hamiltonian in Eq. (1.10) corresponding to a single mode field with the energy eigenvalue E_n such that

$$\hat{H}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{1.20}$$

multiplying the Eq. (1.20) from left by $\hat{a}^\dagger$ gives,

$$\hbar\omega\left(\hat{a}^\dagger\hat{a}^\dagger\hat{a} + \frac{1}{2}\hat{a}^\dagger\right) = E_n\hat{a}^\dagger|n\rangle \tag{1.21}$$

Using the commutation relation given in Eq. (1.15) we can rewrite Eq. (1.21) as

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\left(\hat{a}^\dagger|n\rangle\right) = (E_n + \hbar\omega)\left(\hat{a}^\dagger|n\rangle\right) \tag{1.22}$$

which corresponds to the eigenvalue equation for the state $\left(\hat{a}^\dagger|n\rangle\right)$ with the energy eigenvalue $E_n + \hbar\omega$. This means that when creation operator $\hat{a}^\dagger$ acts on state $|n\rangle$ it creates a quantum of energy, $\hbar\omega$. Similarly if we multiply Eq. (1.20) with $\hat{a}$ and using the commutation relation in Eq. (1.15) we obtain,

$$\hat{H}\left(\hat{a}|n\rangle\right) = (E_n - \hbar\omega)\left(\hat{a}|n\rangle\right) \tag{1.23}$$

which implies that the operator $\hat{a}$ destroys or annihilates one quantum of energy or one photon, the eigenstate $\hat{a}|n\rangle$ possessing the energy eigenvalue $E_n - \hbar\omega$. Repeating the procedure on Eq. (1.23) will result in the lowering of the energy eigenvalue by integer multiples of $\hbar\omega$. But the energy of the harmonic oscillator must always be positive so there must be a minimum energy eigenvalue $E_{(n=0)} > 0$ with the corresponding eigenstate $|0\rangle$ such that

$$\hat{a}|0\rangle = 0 \tag{1.24}$$

Thus the eigenvalue equation for the ground state $|0\rangle$ is

$$\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle \tag{1.25}$$

so that the lowest energy eigenvalue is the so called zero point energy $\hbar\omega/2$. Since $E_{n+1} = E_n + \hbar\omega$, the possible energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, ... \tag{1.26}$$

For the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ we have

$$\hat{n}|n\rangle = n|n\rangle \tag{1.27}$$

These number states must be normalized such that $\langle n|n\rangle = 1$. For the state $\hat{a}|n\rangle$, we have

$$\hat{a}|n\rangle = c_n|n-1\rangle \tag{1.28}$$

where c_n is a constant to be determined. By taking the inner product of $\hat{a}|n\rangle$ with itself we get

$$\begin{aligned} \left(\langle n|\hat{a}^\dagger\right)(\hat{a}|n\rangle) &= \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n \\ &= \langle n-1|c_n^* c_n|n-1\rangle = |c_n^2| \end{aligned} \tag{1.29}$$

Thus $|c_n^2| = n$,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{1.30}$$

Similarly we can show that

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{1.31}$$

The number state $|n\rangle$ can be generated by the repeated action of the creation operator,$\hat{a}^\dagger$ on the ground state $|0\rangle$.

$$|n\rangle = \frac{\left(\hat{a}^\dagger\right)^n |0\rangle}{\sqrt{n!}} \tag{1.32}$$

Since $\hat{H}$ and $\hat{n}$ are Hermitian operators, the number states, $|n\rangle$ forms an orthonormal complete set of states, i.e.,

$$\langle n|n'\rangle = \delta_{nn'} \tag{1.33}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}. \tag{1.34}$$

The only nonvanishing matrix elements of the annihilation and creation operators are

$$\begin{aligned}\langle n-1|\hat{a}|n\rangle &= \sqrt{n}\langle n-1|n-1\rangle = \sqrt{n} \\ \langle n+1|\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\langle n+1|n+1\rangle = \sqrt{n+1}\end{aligned} \tag{1.35}$$

1.4.2 Quantum fluctuations of a single mode field

In the previous section we have seen that the number state, $|n\rangle$ is a state of definite energy equal to $\hbar\omega(n+1/2)$ but the expectation value of the electric field operator for this number state is zero, since

$$\langle n|\hat{E}_x(z,t)|n\rangle = E_0 \sin(kz)\left[\langle n|\hat{a}|n\rangle + \langle n|\hat{a}^\dagger|n\rangle\right] = 0, \tag{1.36}$$

Thus the mean field is zero. We have the equation Eq. (1.5) for the total energy, from which the mean of square of the field, which contributes to the energy density, is not zero, i.e.,

$$\begin{aligned}\langle n|\hat{E}_x^2(z,t)|n\rangle &= E_0^2 \sin^2(kz)\langle n|\hat{a}^{\dagger^2} + \hat{a}^2 + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger|n\rangle \\ &= \sin(kz)E_0^2 \sin^2(kz)\langle n|\hat{a}^{\dagger^2} + \hat{a}^2 + 2\hat{a}^\dagger\hat{a} + 1|n\rangle \\ &= 2E_0^2 \sin^2(kz)\left(n + \frac{1}{2}\right).\end{aligned} \tag{1.37}$$

For any operator $\hat{A}$, the fluctuation is characterized by its variance given by

$$\left\langle \left(\Delta\hat{A}\right)^2 \right\rangle = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 \tag{1.38}$$

or by the standard deviation $\Delta A = \sqrt{\langle(\Delta A)^2\rangle}$, which is sometimes referred to as the uncertainty of the observable represented by the operator $\hat{A}$. Applying it for the electric field operator, $\hat{E}$ for the number state, $|n\rangle$:

$$\langle(\Delta\hat{E}_x(z,t))^2\rangle = \left\langle \hat{E}_x^2(z,t) \right\rangle - \langle\hat{E}_x(z,t)\rangle^2 \tag{1.39}$$

we get

$$\Delta E_x = \sqrt{2E_0}\sin(kz)\left(n + \frac{1}{2}\right)^{\frac{1}{2}} \tag{1.40}$$

Note that even when $n = 0$, the field has non zero fluctuations, the so-called "vacuum fluctuations".
Since

$$\begin{aligned}\left[\hat{n}, \hat{E}_x\right] &= E_0 \sin(kz)\left[\hat{a}^\dagger\hat{a}\left(\hat{a}+\hat{a}^\dagger\right) - \left(\hat{a}+\hat{a}^\dagger\right)\hat{a}^\dagger\hat{a}\right] \\ &= E_0\sin(kz)(\hat{a}^\dagger - \hat{a}),\end{aligned} \tag{1.41}$$

the number operator does not commute with the electric field operator. We also know from uncertainty principle for any operators $\hat{A}$, $\hat{B}$ and $\hat{C}$, satisfying $\left[\hat{A}, \hat{B}\right] = \hat{C}$ that $\Delta A\ \Delta B \geq \frac{1}{2}|\langle C\rangle|$. Applying this inequality for the operators $\hat{n}$ and $\hat{E}_x$ we get

$$\Delta n\ \Delta E_x \geq \frac{1}{2}E_0|\sin(kz)||\left\langle \hat{a}^\dagger - \hat{a} \right\rangle|. \tag{1.42}$$

which means that if the field is accurately known, number of photons will be uncertain.

1.4.3 Quadrature operators of single mode field

The time dependence of the creation and annihilation operators is given in Eqs. (1.18) and (1.19). Considering this, the expression for the electric field operator given by Eq. (1.13) becomes

$$\begin{aligned}\hat{E}_x &= E_0\left[\hat{a}(0)e^{-i\omega t} + \hat{a}^\dagger(0)e^{i\omega t}\right]\sin(kz) \qquad (1.43)\\ &= E_0\left\{\left[\hat{a}(0)+\hat{a}^\dagger(0)\right]\cos(\omega t) + i\left[\hat{a}(0)-\hat{a}^\dagger(0)\right]\sin(\omega t)\right\}\sin(kz)\end{aligned}$$

We now define the so called quadrature operators

$$\hat{X}_1 = \frac{1}{2}\left[\hat{a}(0)+\hat{a}^\dagger(0)\right] \qquad (1.44)$$

$$\hat{X}_2 = \frac{1}{2i}\left[\hat{a}(0)-\hat{a}^\dagger(0)\right] \qquad (1.45)$$

Now in terms of these field quadrature operators the electric field can be rewritten as:

$$\hat{E}_x(t) = 2E_0\sin(kz)\left[\hat{X}_1\cos(\omega t)+\hat{X}_2\sin(\omega t)\right]. \qquad (1.46)$$

From the above expression one can infer that $\hat{X}_1$ and $\hat{X}_2$ are associated to the field amplitudes oscillating out of phase with each other by 90°; they are in quadrature. Using Eqs. (1.11) and (1.12) in Eqs. (1.44) and (1.45) it is clear that $\hat{X}_1$ and $\hat{X}_2$ play the roles of position and momentum but they are scaled to be dimensionless. The Eqs. (1.44) and (1.45) yield,

$$\hat{a}(0) = \hat{X}_1 + i\hat{X}_2 \qquad (1.47)$$

$$\hat{a}^\dagger(0) = \hat{X}_1 - i\hat{X}_2. \qquad (1.48)$$

The quadrature operators, $\hat{X}_1$ and $\hat{X}_2$ satisfy the commutation relation

$$\left[\hat{X}_1,\hat{X}_2\right] = \frac{i}{2} \qquad (1.49)$$

and now by using the uncertainty principle it follows that:

$$\left\langle \left(\Delta\hat{X}_1\right)^2\right\rangle \left\langle \left(\Delta\hat{X}_2\right)^2\right\rangle \geq \frac{1}{16}. \tag{1.50}$$

For the number state $|n\rangle$ we have $\left\langle n|\hat{X}_1|n\right\rangle = 0 = \left\langle n|\hat{X}_2|n\right\rangle$ but

$$\begin{aligned} \left\langle n|\hat{X}_1^2|n\right\rangle &= \frac{1}{4}\left\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n\right\rangle \\ &= \frac{1}{4}(2n+1) \end{aligned}$$

where $\hat{a}(0) \equiv \hat{a}$ and $\hat{a}^\dagger(0) \equiv \hat{a}^\dagger$. Similarly for $\hat{X}_2$ we get

$$\left\langle n|\hat{X}_2^2|n\right\rangle = \frac{1}{4}(2n+1) \tag{1.51}$$

It is clear from the Eqs. (1.51) and (1.51) that, for number state the uncertainties in both the quadratures are the same. For vacuum state $|0\rangle$, from Eqs. (1.51) and (1.51) we get

$$\left\langle \left(\Delta\hat{X}_1\right)^2\right\rangle_{vac} = \frac{1}{4} = \left\langle \left(\Delta\hat{X}_2\right)^2\right\rangle_{vac}. \tag{1.52}$$

That is the quadrature uncertainty is minimum for the vacuum state.

1.5 Photon Distributions

In this section we discuss about various photon distributions and their characteristics.

1.5.1 Coherent state

Coherent states are the eigenstates of the annihilation operator and also are minimum uncertainty states.

Consider a photon state $|\psi\rangle$ which is a superposition of number state. We know that the electric field and magnetic field operators can be written as the linear combinations of creation and annihilation operators, which is given in Eqs. (1.13 and (1.14) as

$$\hat{E}_x(z,t) = E_0\left(\hat{a}+\hat{a}^\dagger\right)\sin(kz) \tag{1.53}$$

$$\hat{B}_y(z,t) = B_0\frac{1}{i}\left(\hat{a}-\hat{a}^\dagger\right)\cos(kz) \tag{1.54}$$

In order to have a non zero expectation value for these field operators we are required to have states which are linear combinations of number states. For example, one such possible state is

$$|\psi\rangle = C_n|n\rangle + C_{n\pm1}|n\pm1\rangle \tag{1.55}$$

where $|C_n|^2 + |C_{n\pm1}|^2 = 1$. The replacement of $\hat{a}$ and $\hat{a}^\dagger$ by continues variables produces a classical field. A unique way to make this replacement is to seek the eigenstates of the annihilation operator. These states are denoted as $|\alpha\rangle$ and satisfy the relation

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1.56}$$

where α in general is a complex number, otherwise arbitrary. Taking the complex conjugate of the above equation, we have

$$\langle\alpha|\hat{a} = \alpha^*\langle\alpha|. \tag{1.57}$$

The states $|\alpha\rangle$ are the "right" eigenstates of $\hat{a}$ with eigenvalues α and $\langle\alpha|$ are the "left" eigenstates of $\hat{a}^\dagger$ with eigenvalues α^*.

Since the number states $|n\rangle$ form a complete set, expanding the coherent state in terms of the number states we can write

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle \tag{1.58}$$

Operating $\hat{a}$ on both sides of the above equation and using Eq. (1.56), we obtain

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} C_n\sqrt{n}|n-1\rangle = \alpha\sum_{n=0}^{\infty} C_n|n\rangle \tag{1.59}$$

and equating the coefficients of $|n\rangle$ on both sides we get,

$$C_n\sqrt{n} = \alpha C_{n-1} \tag{1.60}$$

or

$$\begin{aligned} C_n &= \frac{\alpha}{\sqrt{n}}C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}C_{n-2} = ... && (1.61) \\ &= \frac{\alpha^n}{\sqrt{n!}}C_0. && (1.62) \end{aligned}$$

Thus

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{1.63}$$

The value of C_0 can be found by using the normalization of $|\alpha\rangle$

$$\begin{aligned} \langle\alpha|\alpha\rangle &= 1 = |C_0|^2 \sum_n \sum_{n'} \frac{\alpha^{*^n}\alpha^{n'}}{\sqrt{n!n'!}}\langle n|n'\rangle && (1.64) \\ &= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2} && (1.65) \end{aligned}$$

i.e.,

$$C_0 = \exp\left(-\frac{1}{2}\ |\alpha|^2\right) \tag{1.66}$$

Now the normalized coherent states are:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.67}$$

The expectation value of the electric field operator,

$$\hat{E}_x(\boldsymbol{r},t) = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}} \left[\hat{a}e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)} - \hat{a}^\dagger e^{-i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}\right], \tag{1.68}$$

for the state $|\alpha\rangle$ is

$$\langle\alpha|\hat{E}_x(\boldsymbol{r},t)|\alpha\rangle = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}} \left[\alpha e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)} - \alpha^* e^{-i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}\right]. \tag{1.69}$$

Since α is a complex quantity it can written in the form $\alpha = |\alpha|e^{i\theta}$

$$\langle\alpha|\hat{E}_x(\boldsymbol{r},t)|\alpha\rangle = 2|\alpha|\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}}\sin(\omega t - \boldsymbol{k}.\boldsymbol{r} - \theta). \tag{1.70}$$

which looks like the solution of a classical field equations. Similarly we can find the expectation value of the operator $\hat{E}_x^2(\boldsymbol{r},t)$; given by

$$\begin{aligned}
\langle\alpha|\hat{E}_x^2(\boldsymbol{r},t)|\alpha\rangle &= -\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)\times \\
&\langle\alpha|\left[\hat{a}^2 e^{2i(\boldsymbol{k}.\boldsymbol{r}-\omega t)} + \hat{a}^{\dagger 2} e^{-2i(\boldsymbol{k}.\boldsymbol{r}-\omega t)} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right]|\alpha\rangle \\
= &\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)\left\{2|\alpha|^2\left[1+\cos 2(\omega t - \boldsymbol{k}.\boldsymbol{r} - \theta)\right]+1\right\} \\
= &\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)\left[1+4|\alpha|^2\sin\left(\omega t - \boldsymbol{k}.\boldsymbol{r} - \theta\right)\right]
\end{aligned} \tag{1.71}$$

To find the fluctuations in the field we use the variance given by

$$\langle(\Delta\hat{E}_x(z,t))^2\rangle = \left\langle\hat{E}_x^2(z,t)\right\rangle - \langle\hat{E}_x(z,t)\rangle^2 \tag{1.72}$$

Using Eqs. (1.70) and (1.71) in Eq. (1.72) we get

$$\left\langle(\Delta\hat{E}_x(z,t))^2\right\rangle = \frac{\hbar\omega}{2\epsilon_0 V}\ , \tag{1.73}$$

which are identical to those for a vacuum state. The coherent state is nearly a classical like state because it not only yields the correct form for the field expectation values, given in Eq. (1.71) but also have only the noise of the vacuum as in Eq. (1.73).

Now let us check the uncertainties in quadratures for the coherent field. For that consider the field quadrature operators defined in the previous section, Eqs. (1.44) and (1.45):

$$\begin{aligned}
\hat{X}_1 &= \frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right) \\
\hat{X}_2 &= \frac{1}{2}\left(\hat{a}-\hat{a}^\dagger\right)
\end{aligned} \tag{1.74}$$

In order to find the uncertainties of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ in the state $|\alpha\rangle$ we have,

$$\begin{aligned}\langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}\langle\alpha|\hat{a}+\hat{a}^\dagger|\alpha\rangle \\ &= \frac{1}{2}\left(\alpha+\alpha^*\right)\end{aligned} \tag{1.75}$$

and

$$\begin{aligned}\langle\alpha|\hat{X}_1^2|\alpha\rangle &= \frac{1}{4}\langle\alpha|\hat{a}^2+\hat{a}^{\dagger^2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}|\alpha\rangle \\ &= \frac{1}{4}\left(\alpha^2+\alpha^{*^2}+2\alpha\alpha^*+1\right)\end{aligned} \tag{1.76}$$

similarly for $\hat{X}_2$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2i}\left(\alpha-\alpha^*\right) \tag{1.77}$$

$$\langle\alpha|\hat{X}_1^2|\alpha\rangle = \frac{-1}{4}\left(\alpha^2+\alpha^{*^2}-2\alpha\alpha^*-1\right) \tag{1.78}$$

and the uncertainty in each quadrature is given by:

$$\left(\langle\Delta\hat{X}_1\rangle\right)^2 = \langle\hat{X}_1^2\rangle - \langle\hat{X}_1\rangle^2 \tag{1.79}$$

and using Eqs. (1.75) - (1.79) we obtain

$$\left(\Delta\hat{X}_1\right)^2 = \frac{1}{4} = \left(\Delta\hat{X}_2\right)^2 \tag{1.80}$$

which means that coherent states are having the same uncertainties in both quadratures.

$$\text{i.e}\quad \left(\Delta\hat{X}_1\right)^2 = \left(\Delta\hat{X}_2\right)^2 = \frac{1}{4} \tag{1.81}$$

Thus for coherent state uncertainties in quadrature operators($\hat{X}_1$ and $\hat{X}_2$) are the same and also the uncertainty product is minimum. It is also to be noted that the uncertainty of quadrature operators for coherent state is same as that of vacuum. The expectation value of the photon number operator $\hat{n}$ in the state $|\alpha\rangle$ is

$$\overline{n} = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2. \tag{1.82}$$

Thus $|\alpha|^2$ is just the average photon number of the field. To calculate the fluctuations of the photon number we need to calculate

$$\begin{aligned}\langle\alpha|\hat{n}^2|\alpha\rangle &= \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle \\ &= |\alpha|^4 + |\alpha|^2 = \overline{n}^2 + \overline{n}\end{aligned} \tag{1.83}$$

and thus

$$\Delta n = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = \sqrt{\overline{n}} \tag{1.84}$$

which is the characteristic of a Poisson distribution. For a measurement of the number of photons, the probability of detecting n photons in the coherent state is

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{1.85}$$

$$= e^{-\overline{n}}\,\frac{\overline{n}^n}{n!}, \tag{1.86}$$

which is a Poisson distribution with a mean of $\overline{n}$. The probability versus photon number is plotted in Fig. 1.2 for an average photon number, $\bar{n} = 25$. The fractional uncertainty in the photon number is

$$\frac{\Delta n}{n} = \frac{1}{\sqrt{\overline{n}}} \tag{1.87}$$

which decreases with increasing $\overline{n}$.

1.5.2 Squeezed photon distribution

In the previous section we have discussed the coherent light with minimum and equal quadrature uncertainties in both quadratures

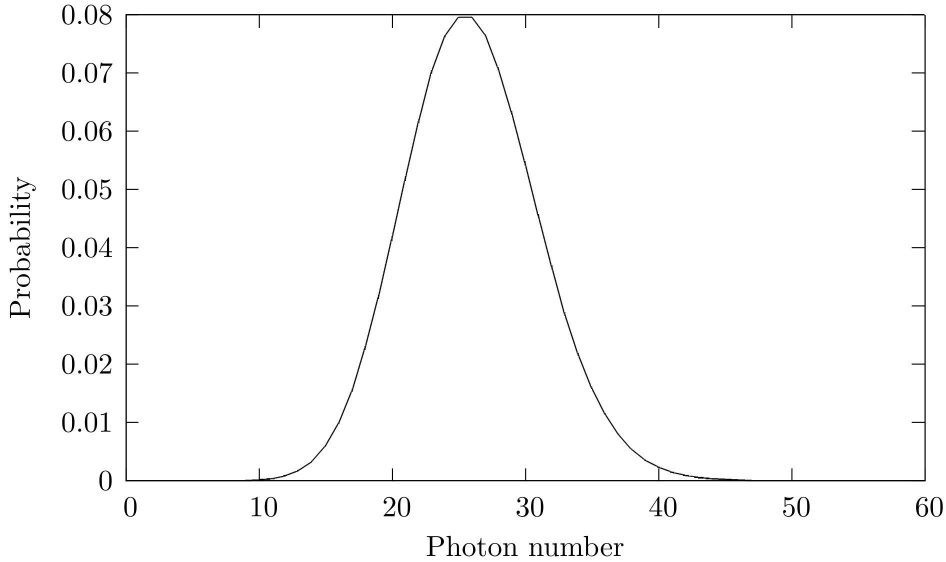


Figure 1.2: Coherent distribution with average photon number=25

as given in Eq. (1.81). For squeezed light the uncertainty in both the quadratures are not the same, but their product has a minimum value. We know that for any two operators $\hat{A}$ and $\hat{B}$ satisfying the commutation relation $[\hat{A}, \hat{B}] = i\hat{C}$ has the property

$$\left\langle (\Delta\hat{A})^2 \right\rangle \left\langle (\Delta\hat{B})^2 \right\rangle \geq \frac{1}{4} \left\langle (\Delta\hat{C})^2 \right\rangle . \tag{1.88}$$

A state of the system is said to be squeezed if either

$$\left\langle (\Delta\hat{A})^2 \right\rangle < \frac{1}{2} \left| \langle \hat{C} \rangle \right| \qquad \text{or} \qquad \left\langle (\Delta\hat{B})^2 \right\rangle < \frac{1}{2} \left| \langle \hat{C} \rangle \right| \tag{1.89}$$

In the case of quadrature squeezing, we take $\hat{A} = \hat{X}_1$ and $\hat{B} = \hat{X}_2$, with $\hat{X}_1$ and $\hat{X}_2$ being the quadrature operators and thus $\hat{C} = \hat{I}/2$. That is the quadrature squeezing exists whenever

$$\left\langle (\Delta\hat{X}_1)^2 \right\rangle < \frac{1}{4} \qquad \text{or} \qquad \left\langle (\Delta\hat{X}_2)^2 \right\rangle < \frac{1}{4} \tag{1.90}$$

Mathematically, a squeezed state is generated through the action of a "squeeze" operator defined as,

$$\hat{S}(\xi) = \exp\left[\frac{1}{2}\left(\xi^* a^2 - \xi a^{\dagger^2}\right)\right], \tag{1.91}$$

on vacuum, where $\xi = re^{i\theta}$; r is known as the squeeze parameter and $0 \leq r < \infty$ and $0 \leq \theta \leq 2\pi$. The operator $\hat{S}(\xi)$ creates or annihilates photons in pairs. It is a kind of a two photon generalization of the displacement operator used to define the coherent states of a single mode field. A more general squeezed state may be obtained by applying the displacement operator, $\hat{D}(\alpha)$ to $\hat{S}(\xi)$ given by

$$|\alpha\ \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle, \tag{1.92}$$

where the displacement operator is defined as

$$D(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right). \tag{1.93}$$

Obviously for $\xi = 0$ we obtain the coherent state. A squeezed vacuum state, where $\alpha = 0$ can be represented as,

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{m=0}^{\infty}(-1)^m\frac{\sqrt{2m!}}{2^m m!}e^{im\theta}(\tanh r)^m|2m\rangle \tag{1.94}$$

and the probability of detecting $2m(m = 0, 1, 2..)$ photons in the squeezed vacuum state is

$$P_{2m} = |\langle 2m|\xi\rangle|^2 = \frac{(2m)!}{2^{2m}(m!)^2}\frac{(\tanh r)^{2m}}{\cosh r}. \tag{1.95}$$

Similarly the probability of finding any odd number$(2m + 1)$ of photons in the squeezed vacuum state is zero. i.e.,

$$P_{2m+1} = |\langle 2m + 1|\xi\rangle|^2 = 0. \tag{1.96}$$

Thus the photon probability distribution for a squeezed vacuum state is oscillatory, vanishing it for the case of all odd number of photons. When the coherent part is non zero, i.e., $\alpha \neq 0$, we obtain the general representation for the photon squeezed state given by,

$$\begin{aligned} |\alpha, \xi\rangle &= \frac{1}{\sqrt{\cosh r}} \exp\left[\frac{-1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*^2} e^{i\theta} \tanh r\right] \\ &\times \sum_{n=0}^{\infty} \frac{\left[\frac{1}{2} e^{i\theta} \tanh r\right]^{n/2}}{\sqrt{n!}} H_n\left[\gamma\left(e^{i\theta}\sinh(2r)\right)^{-1/2}\right]|n\rangle \end{aligned} \tag{1.97}$$

And the corresponding probability of finding n photons in the state is given by,

$$\begin{aligned} P_n &= |\langle n|\alpha, \xi\rangle|^2 \\ &= \frac{\left(\frac{1}{2}\tanh r\right)^n}{n! \cosh r} \exp\left[-|\alpha|^2 - \frac{1}{2}\left(\alpha^{*^2} e^{i\theta} + \alpha^2 e^{-i\theta}\right)\tanh r\right] \\ &\times \left| H_n\left[\gamma\left(e^{i\theta}\sinh(2r)\right)^{-1/2}\right]\right|^2. \end{aligned} \tag{1.98}$$

The distribution is plotted in the Fig. 1.3 for an average photon number 25.

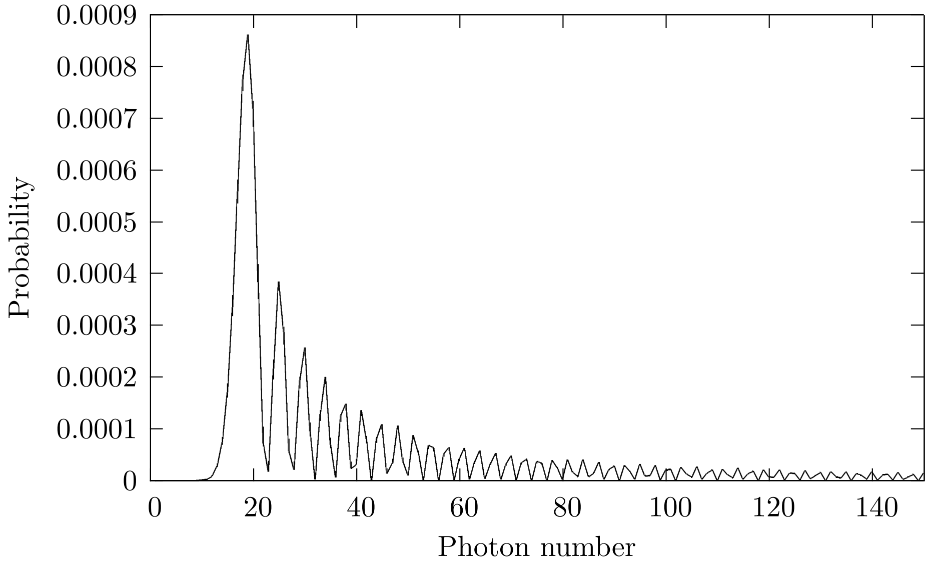


Figure 1.3: Squeezed photon distribution with average photon number 25, squeezing parameter $r = 4$ and $\psi - \theta/2 = 0$

One of the most important type of squeezed states are the photon number states that we introduced in Section 1.4.1. These are states of perfectly defined photon number n, which implies $\Delta n = 0$ and a completely undefined phase. This contrasts with coherent states which have larger photon number fluctuations ($\Delta n = \sqrt{n}$) but have a better defined phase.

2
Atom field interactions

2.1 Introduction

In general the interaction between electromagnetic field and atoms is described using three different approaches. They are the classical, semi classical and quantum mechanical formulations. In classical treatment atom is considered as Hertzian dipoles and light as waves where as in semiclassical method atom is quantized and light is still waves and finally in quantum mechanical formulation both atoms and light are considered to be quantized. The concept of photons, the discrete energy units of electromagnetic radiation, is adopted to explain the interaction between atom and field in the fully quantum mechanical treatment. The following sections discuss the interaction between atom and electromagnetic field using classical, semiclasical and fully quantum mechanical approaches.

2.2 Classical theory of atom field interaction

In classical theory, the electric field of the electromagnetic radiation can be represented as a sinusoidal wave in the form

$$\vec{E}(t) = E_0 \vec{e} \cos(\omega t) \tag{2.1}$$

and the atoms is considered as it consists of a massive nucleus in the center and charged electrons around it. The interaction between atom and radiation field can be explained by taking the analogue of a mass(electron) connected to a spring(attached to the nucleus). This spring is contracted and extended as it interacts with field. When the spring extends the energy from the electromagnetic field get stored in it(absorption of radiation) and is then released when the spring contracts(radiation emission). Thus the interaction can be represented in the form of a driven harmonic oscillator as

$$\frac{d^2x}{dt^2} = -\omega_0^2 x + \frac{F}{m} \tag{2.2}$$

where F is the force on the electron due to the field given by

$$F = qE_0 \cos(\omega t) \tag{2.3}$$

The solution of Eq. (2.2) is of the form

$$x(t) = A\cos(\omega_0 t - \phi_0) + \frac{qE_0}{m\left(\omega^2 - \omega_0^2\right)}\cos(\omega t) \tag{2.4}$$

where ϕ_0 is the initial phase of the oscillator and A is a constant. From Eq. (2.4) it is clear that due to the driving force the response of the atom is to oscillate with a frequency, ω of the driving field and the amplitude of oscillation is maximum when the field is at resonance i.e., when the driving frequency, ω is the same as the natural oscillator frequency, ω_0. In order to treat the resonance in a systematic way we have to include damping effect in Eq. (2.4).

2.3 Atom-field interaction Hamiltonian

The Hamiltonian of an electron bound to the nucleus is the sum of kinetic energy and the usual electrostatic Coulomb potential, $V(r)$, which binds the electron to the nucleus. It can be written as

$$\hat{H}_0 = \frac{1}{2m}\hat{\mathbf{P}}^2 + V(r). \tag{2.5}$$

In the configuration space representation the canonical momentum, $\vec{P}$, is an operator; $\hat{P} = -i\hbar\bigtriangledown$ and $\hat{\mathbf{r}}|\mathbf{r}\rangle = \mathbf{r}|\mathbf{r}\rangle$. The corresponding electron wave functions are given by $\psi(\vec{\mathbf{r}}, t) = \langle\vec{\mathbf{r}}|\psi\rangle$. Let $|k\rangle$ be the eigenstates of the the Hamiltonian $\hat{H}_0$ with corresponding energy eigenvalues E_k. We assume that these energy eigenstates satisfy the time independent Schrödinger equation, i.e.,

$$\hat{H}_0|k\rangle = E_k|k\rangle \tag{2.6}$$

and we assume that E_k, $|k\rangle$ are known. Now consider the following electromagnetic field due to the radiation

$$\mathbf{E}(\mathbf{r}, t) = -\bigtriangledown\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \tag{2.7}$$

$$\mathbf{B}(\mathbf{r}, t) = \bigtriangledown \times \mathbf{A}(\mathbf{r}, t), \tag{2.8}$$

where $\mathbf{A}(\mathbf{r}, t)$ and $\Phi(\mathbf{r}, t)$ respectively are the vector and scalar potential. These electric and magnetic fields in Eqs.(2.7) and (2.8) are invariant under the local gauge transformations such that,

$$\Phi'(\mathbf{r}, t) = \Phi(\mathbf{r}, t) - \frac{\partial \chi(\mathbf{r}, t)}{\partial t}, \tag{2.9}$$

$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r}, t) + \bigtriangledown\chi(\mathbf{r}, t). \tag{2.10}$$

Let us now consider the case when an atom with the Hamiltonian given in Eq. (2.5) interact with the electromagnetic field represented by Eqs. (2.7) and (2.8). The corresponding modified Hamiltonian is

$$\hat{H}(\mathbf{r}, t) = \frac{1}{2m}\left[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)\right]^2 - e\Phi(\mathbf{r}, t) + V(r), \tag{2.11}$$

where $-e$ is the charge of the electron and $e > 0$. Now the time dependent Schrödinger equation in the coordinate space becomes

$$\left\{\frac{1}{2m}\left[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r}, t)\right]^2 - e\Phi(\mathbf{r}, t) + V(r)\right\}\Psi(\mathbf{r}, t) = i\hbar\frac{\partial \Psi(\mathbf{r}, t)}{\partial t}. \tag{2.12}$$

The above Eq. (2.12) represents the interaction of an atomic electron with the given external electromagnetic field. In order to solve Eq. (2.12), we introduce an unitary operator, $\hat{R}$ such that $\Psi'(\mathbf{r},t) = \hat{R}\Psi(\mathbf{r},t)$. Now by choosing

$$\hat{H}' = \hat{R}\hat{H}\hat{R}^{\dagger} + i\hbar\frac{\partial\hat{R}}{\partial t}\hat{R}^{\dagger}, \tag{2.13}$$

we can write

$$\hat{H}'(\mathbf{r},t)\Psi'(\mathbf{r},t) = i\hbar\frac{\partial\Psi'(\mathbf{r},t)}{\partial t}. \tag{2.14}$$

Let the unitary operator in Eq. (2.13) be $\hat{R} = \exp\left(-ie\chi(\mathbf{r},t)/\hbar\right)$, then $\hat{H}'$ takes the form

$$\begin{aligned}
\hat{H}' &= \hat{R}\hat{H}\hat{R}^{\dagger} + i\hbar\frac{\partial\hat{R}}{\partial t}\hat{R}^{\dagger} \\
&= e^{[-ie\chi(\mathbf{r},t)/\hbar]}\left\{\frac{1}{2m}\left[\hat{\mathbf{P}} + e\mathbf{A}(\mathbf{r},t)\right]^2 - e\Phi(\mathbf{r},t) + V(r)\right\} \\
&\quad \times e^{[ie\chi(\mathbf{r},t)/\hbar]} \\
&\quad + i\hbar\frac{\partial\left[\exp\left(-ie\chi(\mathbf{r},t)/\hbar\right)\right]}{\partial t}\exp\left(ie\chi(\mathbf{r},t)/\hbar\right)
\end{aligned} \tag{2.15}$$

Taking $\hat{\boldsymbol{P}} = -i\hbar\triangledown$ we get,

$$\begin{aligned}
\hat{H}' &= \frac{1}{2m}\left\{\hat{\boldsymbol{P}} + e\left[\mathbf{A}(\mathbf{r},t) + \triangledown\chi(\mathbf{r},t)\right]\right\}^2 \\
&\quad - e\left[\Phi(\mathbf{r},t) - \frac{\partial\chi(\mathbf{r},t)}{\partial t}\right] + V(r).
\end{aligned} \tag{2.16}$$

Using the gauge invariance given in Eqs. (2.9) and (2.10) we obtain

$$\hat{H}' = \frac{1}{2m}\left[\hat{\mathbf{P}} + e\mathbf{A}'\right]^2 - e\Phi' + V(r). \tag{2.17}$$

If we choose radiation gauge, for which $\Phi = 0$ and $\mathbf{A}$ satisfies the transversality condition $\triangledown.\mathbf{A} = 0$, the Hamiltonian $\hat{H}$ given in Eq. (2.11) and $\hat{H}'$ given in Eq. (2.17) can be re written as

$$H = \frac{\hat{\mathbf{P}}^2}{2m} + \frac{e}{m}\mathbf{A}.\hat{\mathbf{P}} + \frac{e^2}{2m}\mathbf{A}^2 + V(r), \tag{2.18}$$

$$\hat{H}'(\mathbf{r},t) = \frac{1}{2m}\left[\hat{\mathbf{P}} + e\left(\mathbf{A} + \nabla\chi\right)\right]^2 + e\frac{\partial\chi}{\partial t} + V(r). \tag{2.19}$$

When there are no sources of radiation near the atom the vector potential, $\mathbf{A}$ satisfies the wave equation, i.e.,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \tag{2.20}$$

The solution of this wave equation, Eq. (2.20) will in the form given by

$$\mathbf{A} = \mathbf{A}_0 e^{i(k.\mathbf{r}-\omega t)} + \text{c.c.}, \tag{2.21}$$

where $\vec{k}$ is the wave vector of the radiation, $|\vec{k}| = 2\pi/\lambda$ and c.c denote the complex conjugation. Here in the cases we considered, $|\vec{r}|$ is of typical atomic dimensions(few Angstroms) and λ the wave lengths of electromagnetic radiation comes in the range of few hundred nanometers(400-700 nm). In this case $\mathbf{k}.\mathbf{r} << 1$, so that over the extent of an atom, the vector potential is spatially uniform i.e., $\mathbf{A}(\mathbf{r},t) \approx \mathbf{A}(t)$. This approximation is the well known "dipole approximation".

By choosing the gauge function, $\chi(\mathbf{r},t) = -\mathbf{A}(t).\mathbf{r}$ we get

$$\begin{aligned} \nabla\chi(\mathbf{r},t) &= -\mathbf{A}(t), \\ \frac{\partial\chi}{\partial t}(\mathbf{r},t) &= -\mathbf{r}.\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{r}.\mathbf{E}(t). \end{aligned} \tag{2.22}$$

Applying the results, Eq. (2.22) in the expression for $\hat{H}'$, Eq. (2.19) becomes

$$\hat{H}' = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\mathbf{r}.\mathbf{E}(t). \tag{2.23}$$

According to the equation (2.23) within the dipole approximation there is only one interaction term. The quantity $-e\mathbf{r}$ is the electric dipole moment, $\mathbf{d} = -e\mathbf{r}$. In general for unspecified representation the dipole moment is an operator, $\hat{\mathbf{d}}$. Using this convention we can write the Hamiltonian in the dipole approximation as

$$\hat{H}' = \hat{H}_0 - \hat{\mathbf{d}}.\mathbf{E}(t) \tag{2.24}$$

So we have considered the interaction Hamiltonian for the atom field interaction, which is valid for both classical and quantum fields. Now we will examine the differences in the way an atom behaves when interacting with classical or quantum fields.

2.4 Interaction of a quantized atom with classical field: perturbative analysis

Consider the case where an atom is driven by classical sinusoidal electric field, $\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega t)$, where ω is the frequency of the radiation field. In the dipole approximation the state of the atom obey the time dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \left(H_0 + H^{(I)} \right) |\psi(t)\rangle, \tag{2.25}$$

where $H^{(I)} = -\hat{\mathbf{d}}.\mathbf{E}(t)$. Now the state vector $|\psi(t)\rangle$ of the atomic system can be expanded in terms of the unperturbed atomic states $|k\rangle$ as

$$|\psi(t)\rangle = \sum_k C_k(t) e^{-iE_k t/\hbar} |k\rangle, \tag{2.26}$$

since $\langle \psi(0)|\psi(0)\rangle = 1$ the amplitudes $C_k(t)$, which depends on time, satisfy the normalization condition

$$\sum_k |C_k(t)|^2 = 1. \tag{2.27}$$

Substituting Eq. (2.26) in the time dependent Schrödinger equation (2.25) and by taking the scalar product with $\langle l|e^{iE_l t/\hbar}$ we get

$$\dot{C}_l(t) = -\frac{i}{\hbar}\sum_k C_k(t)\langle l|\hat{H}^{(I)}|k\rangle e^{i\omega_{lk}t} \tag{2.28}$$

where $\omega_{lk} = (E_l - E_k)/\hbar$ is the transition frequency between lth and kth levels of the atom. We assume that the only state which is initially populated is $|i\rangle$ such that $C_j(0) = \delta_{ij}$. As time goes forward, population will be lost from the initially populated state $|i\rangle$ and will be increased in some initially unpopulated state $|f\rangle$. The probability for the atom to make a transition from state $|i\rangle$ to state $|f\rangle$ in time t is equal to the probability of the atom being in state $|f\rangle$ at time t. i.e.,

$$P_{i\to f}(t) = |\langle f|\psi(t)\rangle|^2 = |C_f(t)|^2. \tag{2.29}$$

The time dependent perturbation theory can be used to find the transition amplitudes if the driving force is weak. Which means $|\mathbf{E}|$ is small, or the transition amplitude from state $|i\rangle$ to $|f\rangle$, i.e., $|\langle f|\hat{\mathbf{d}}.\hat{\mathbf{E}}_0|i\rangle$ is small. Now by following the standard procedures of the time dependent perturbation method we can get the first order equation for the transition amplitudes as

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(I)}(t')e^{i\omega_{fi}t'}C_i^{(0)}(t'). \tag{2.30}$$

We know that the states with opposite parity will have non zero matrix elements for the dipole moment operator $\hat{\mathbf{d}}$. i.e.,

$$H_{ii}^{(I)} = -\langle i|\vec{d}.\vec{E}(t)|i\rangle = 0 \tag{2.31}$$

Applying Eq. (2.31) in Eq. (2.30) we get

$$C_i^{(t)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{ii}^{(I)}(t')C_i^{(0)}(t') = 0. \tag{2.32}$$

Thus the amplitude of the initial state will have a vanishing first order correction. Therefore, up to first order we can take $C_i(t) = C_i^{(0)}(t) = 1$ and for $f \neq i$ we have

$$C_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt' H_{fi}^{(I)}(t') e^{i\omega_{fi}t'} \tag{2.33}$$

Using $H^{(I)} = -\hat{\mathbf{d}}.\mathbf{E}_0 \cos\omega t$ we get

$$\begin{aligned} C_f^{(1)}(t) &= \frac{1}{2\hbar}(\hat{\mathbf{d}}.\mathbf{E}_0)_{fi} \\ &\times \left\{ \frac{\left(e^{i(\omega+\omega_{fi})t} - 1\right)}{\left(\omega + \omega_{fi}\right)} - \frac{\left(e^{-i(\omega-\omega_{fi})t} - 1\right)}{\left(\omega - \omega_{fi}\right)} \right\} \end{aligned} \tag{2.34}$$

where $\left(\hat{\mathbf{d}}.\mathbf{E}_0\right)_{fi} = \langle f|\hat{\mathbf{d}}.\mathbf{E}_0|i\rangle$. In the near resonance cases, that is the frequency of the radiation(ω) is approximately equal to the atomic transition frequency(ω_{fi}), the second term clearly dominates the first. In such a resonant interaction we may drop the "antiresonant" first term making the "rotating wave approximation" (RWA).i.e.,

$$P_{i\to f}^{(1)}(t) = \left| C_f^{(1)}(t) \right|^2 = \frac{|\left(\hat{\mathbf{d}}.\mathbf{E}_0\right)_{fi}|^2}{\hbar^2}\frac{\sin^2(\Delta t/2)}{\Delta^2} \tag{2.35}$$

where $\Delta = \omega - \omega_{fi}$ is the "detuning" between the radiation field and atomic transition. For nonzero Δ it oscillates with time with the maximum transition probability

$$\left(P_{i\to f}^{(1)}\right)_{\max} = \frac{|\left(\hat{\mathbf{d}}.\mathbf{E}_0\right)_{fi}|^2}{\hbar^2}\frac{1}{\Delta^2}. \tag{2.36}$$

2.5 Interaction of an atom with quantized field

A single mode electromagnetic field in free space in the dipole approximation can be represented as

$$\hat{\boldsymbol{E}}(t) = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}} \boldsymbol{e}\left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right). \tag{2.37}$$

In the Schrodinger picture we may avoid the time dependence and the electric field becomes

$$\hat{\boldsymbol{E}} = i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{\frac{1}{2}} \boldsymbol{e}\left(\hat{a} - \hat{a}^\dagger\right) \tag{2.38}$$

The total Hamiltonian of the atom-field system can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}^{(I)}, \tag{2.39}$$

where $\hat{H}_0 = \hat{H}_{\text{atom}} + H_{\text{field}}$ and $H^{(I)}$ is the interaction Hamiltonian given by

$$\hat{H}^{(I)} = -\hat{\boldsymbol{d}}.\hat{\boldsymbol{E}} = -i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)(\boldsymbol{d}.\boldsymbol{e})\left(\hat{a} - \hat{a}^\dagger\right) = -\boldsymbol{d}.\boldsymbol{\varepsilon}_0\left(\hat{a} - \hat{a}^\dagger\right), \tag{2.40}$$

where $\boldsymbol{\varepsilon}_0 = i\sqrt{\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)}\, \boldsymbol{e}$.

The atom and field states are quantized and the states of the combined system will be the direct products of states of both the systems. That is, the state $|i\rangle$ of the system may write as $|i\rangle = |a\rangle|n\rangle$, where $|a\rangle$ is the initial state of the atom with energy E_a and field contains n number of photons. The interaction of the quantized field with the atom causes emission or absorption of a photon and results in following final states

$$|f_1\rangle = |b\rangle|n-1\rangle \quad \text{or} \tag{2.41}$$

$$|f_2\rangle = |b\rangle|n+1\rangle, \tag{2.42}$$

where $|b\rangle$ is another atomic state with energy E_b. These initial and final states of the system holds the energy

$$|i\rangle = |a\rangle|n\rangle \quad \Rightarrow \quad E_i = E_a + n\hbar\omega, \tag{2.43}$$

$$|f_1\rangle = |b\rangle|n-1\rangle \quad \Rightarrow \quad E_{f1} = E_b + (n-1)\hbar\omega, \tag{2.44}$$

$$|f_2\rangle = |b\rangle|n+1\rangle \quad \Rightarrow \quad E_{f2} = E_b + (n+1)\hbar\omega. \tag{2.45}$$

The matrix elements of the interaction Hamiltonian corresponding to the initial and final states are

$$\begin{aligned} \langle f_1|\hat{H}^{(I)}|i\rangle &= \langle b, n-1|\hat{H}^{(I)}|a, n\rangle \\ &= -(\boldsymbol{d}.\boldsymbol{\varepsilon}_0)_{ba}\sqrt{n}, \qquad \text{(absorption)} \end{aligned} \tag{2.46}$$

$$\begin{aligned} \langle f_2|\hat{H}^{(I)}|i\rangle &= \langle b, n+1|\hat{H}^{(I)}|a, n\rangle \\ &= (\boldsymbol{d}.\boldsymbol{\varepsilon}_0)_{ba}\sqrt{n+1}, \qquad \text{(emission)} \end{aligned} \tag{2.47}$$

where

$$(\boldsymbol{d}.\boldsymbol{\varepsilon}_0)_{ba} = \langle a|\hat{\boldsymbol{d}}|b\rangle.\boldsymbol{\varepsilon}_0 \equiv \boldsymbol{d}_{ab}.\boldsymbol{\varepsilon}_0 \tag{2.48}$$

The factor $\boldsymbol{d}_{ab} = \langle a|\hat{\boldsymbol{d}}|b\rangle$ gives the dipole matrix element between the states $|a\rangle$ and $|b\rangle$ of the atom. Now look at the expression for the matrix elements of absorption given in Eq. (2.46). Here when $n = 0$ there will not be any absorption just as one might expect. This is in agreement with the case of a classical driving field - no field, no transitions. But in case of emission according to Eq. (2.47), transitions may occur even when no photons are present. This phenomenon is called spontaneous emission which has no classical counter part. When $n > 0$, the emission of an additional photon is called stimulated emission, which is the underlying principle of the operation of a laser.

2.6 Two level atom approximation

The quantum treatment of the interaction between light and atoms is usually developed in terms of the two-level atom approximation. This approximation is applicable when electromagnetic field interacts with atom and the frequency of the field is comparable with one of the optical transition of the atom. *i.e*,

$$E_2 - E_1 \approx \hbar\omega \tag{2.49}$$

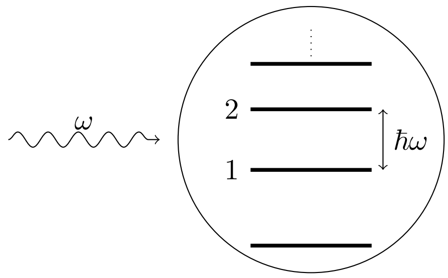


Figure 2.1: The two-level atom approximation. When the light angular frequency ω coincides with one of the optical transitions of the atom, we have a resonant interaction between that transition and the light field. We can therefore neglect the other levels of the atom, which only weakly couples with the light because they are off-resonance.

In an atom there will be many quantum levels with different energy and between these levels many optical transitions are possible. But in the two-level atom approximation we only consider the specific transition that satisfies Eq. (2.49) and ignore all the other levels and we label the lower and upper levels as 1 and 2 respectively as shown in Fig. 2.1. The physical basis for the two-level approximation is the fact that we are dealing with a resonance process. According to the classical picture of light atom interactions, the light beam stimulate dipole oscillations in the atom, which then re-radiate at the same frequency. If the light frequency coincides with the natural frequency of the atom, the magnitude of the dipole oscillations will be large and the interaction between the atom and the light will be strong. In a similar way, if the light frequency is far off from the natural frequency of the atom (i.e. off-resonance), then the magnitude of the induced oscillations will be small and in these cases the effect of light-atom interaction will be negligible. That is, for resonant cases the effect of interaction between the light and the atom is very much stronger in comparison with the case of off-resonant interaction and it is a good approximation to ignore the latter. There are cases in which the presence of the off-resonant levels may become important indirectly. For example, when the atom is in level 2, it could make transitions to other lower levels in addition to level 1. One can include damping terms in the equations in order to incorporate such levels. A useful inference can also be made between the properties of two-level atoms and those of spin 1/2 particles in presence of a magnetic field.

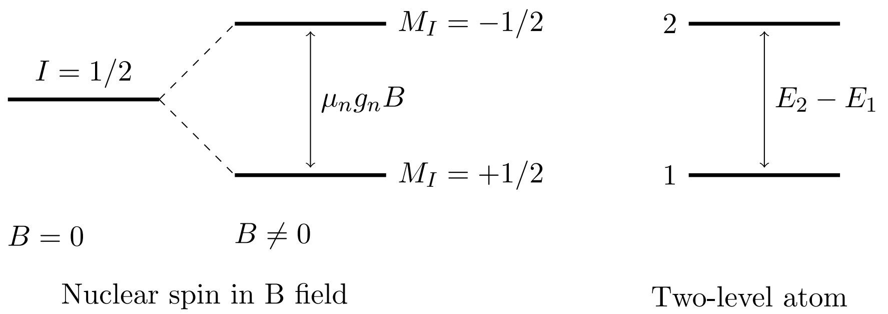


Figure 2.2: Splitting of a spin 1/2 system in a magnetic field of strength B along the z-axis (a) is formally equivalent to a two-level atom (b). Here g_N is the nuclear g-factor and μ_N is the nuclear Bohr magneton

Here Fig. 2.2 shows the splitting of a spin 1/2 system into a doublet in the presence of a magnetic field of strength B due to the Zeeman effect. The Zeeman-split levels are formally equivalent to the two-level atom as in Fig. 2.2. The reason for making the analogy is that the theory of the resonant interaction between microwave radiation and the Zeeman-split nuclear spin states had been formulated in the 1940's to explain a whole range of nuclear magnetic resonance (NMR) phenomena. With the invention of the laser in 1960, the same types of phenomena were soon observed in two-level atomic systems at optical frequencies. A lot of physics can be investigated by two level atom approximation and many recent experiments agree fairly well with a description given by the so-called Jaynes-Cumming model(JCM) describing a two-level atom interacting with a single mode radiation field[9–11].

2.6.1 The Rabi Model

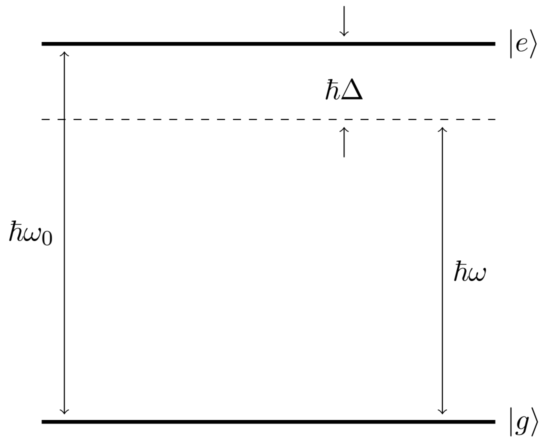


Figure 2.3: Energy level diagram for a two-level atom driving with a near resonant classical field of frequency ω. The resonant frequency between the two atomic levels is ω_0 and the detuning $\Delta = \omega_0 - \omega$

So far we have discussed the interaction between an atom and electromagnetic field using the perturbation theory. In the perturbation method, it is obvious to assume that the change in initial atomic population is very small. But in reality the probability of population transfer is very high when a strong electromagnetic field of frequency very near to the resonance frequency of the atomic levels, interacts with the atom. In such cases the perturbation theory is inadequate to study the interaction. Also in these types of interactions only two dominant states need to be considered. So we can very well adopt the concept of two level atom approximation. In the following section

we study this category of interaction and in semiclassical approach which is known as Rabi model.
Let the two states in the atom are $|g\rangle$ and $|e\rangle$ with respective energies E_g and E_e. The energy difference of these two levels may characterized by the transition frequency $\omega_0 = (E_e - E_g)/\hbar$. This frequency is near to the frequency ω of the interacting electromagnetic field. The interaction Hamiltonian from Eq. (2.24) we have;

$$\hat{H}^{(I)}(t) = -\boldsymbol{d}.\,(\boldsymbol{E}_0 \cos\omega t) = \hat{V}_0 \cos\omega t \tag{2.50}$$

where $\hat{V}_0 = -\boldsymbol{d}.\boldsymbol{E}_0$.
The state vector of the atom at any time t can be written in terms of the two atomic states $|g\rangle$ and $|e\rangle$ as,

$$|\psi(t)\rangle = C_g(t)e^{-iE_gt/\hbar}|g\rangle + C_e(t)e^{-iE_gt/\hbar}|e\rangle \tag{2.51}$$

Now substituting the total Hamiltonian, $\hat{H} = \hat{H}_0 + \hat{V}_0 \cos\omega t$, and state vector in time dependent Schrödinger equation we arrive at a set of coupled equations for the amplitudes C_g and $C_e$such that,

$$\begin{aligned} \dot{C}_g &= -\frac{i}{\hbar}\gamma \cos\omega t e^{-i\omega_0 t} C_e, \\ \dot{C}_e &= -\frac{i}{\hbar}\gamma \cos\omega t e^{i\omega_0 t} C_g, \end{aligned} \tag{2.52}$$

where $\gamma = \langle e|\hat{V}_0|g\rangle = -\boldsymbol{d}_{eg}.\boldsymbol{E}_0$, which is taken to be real. As an example we consider the initial condition where all the population to be in the ground state: $C_g(0) = 1$ and $C_e(0) = 0$. Neglecting energy non conserving terms i.e., using rotating wave approximation(RWA) and retaining terms oscillating at the frequency $\omega_0 - \omega$ the (2.52) becomes

$$\dot{C}_g = -\frac{i}{2\hbar}\gamma \exp\left[i(\omega - \omega_0)t\right] C_e, \tag{2.53}$$

$$\dot{C}_e = -\frac{i}{\hbar}\gamma \exp\left[-i(\omega - \omega_0)t\right] C_g. \tag{2.54}$$

Differentiating Eq. (2.54) and eliminating C_g using Eq. (2.54) we get

$$\ddot{C}_e + i(\omega - \omega_0)\dot{C}_e + \frac{1}{4}\frac{\gamma^2}{\hbar^2}C_e = 0. \tag{2.55}$$

Now we try a solution of Eq. (2.55) of the form

$$C_e(t) = e^{i\lambda t}, \tag{2.56}$$

and substituting Eq. (2.56) in Eq. (2.55) we obtain the possible value of λ as

$$\lambda_\pm = \frac{1}{2}\left\{\Delta \pm \left[\Delta^2 + \gamma^2/\hbar^2\right]^{1/2}\right\}, \tag{2.57}$$

where $\Delta = \omega - \omega_0$ is the detuning of the atomic transition frequency and the electromagnetic field. Now the general solution can be written as

$$C_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t} \tag{2.58}$$

By applying the initial conditions we get,

$$A_\pm = \pm\frac{1}{2\hbar}\gamma\left[\Delta^2 + \gamma^2/\hbar^2\right]^{-1/2}. \tag{2.59}$$

and the corresponding solution is

$$C_e(t) = i\frac{\gamma}{\Omega_R\hbar}e^{i\Delta t/2}\sin(\Omega_R t/2), \tag{2.60}$$

$$C_g(t) = e^{i\Delta t/2}\left[\cos(\Omega_R t/2) - i\frac{\Delta}{\Omega_R}\sin(\Omega_R t/2)\right], \tag{2.61}$$

where

$$\Omega_R = \left[\Delta^2 + \gamma^2/\hbar^2\right]^{1/2} \tag{2.62}$$

is the so called Rabi frequency. The probability that the atom is in state $|e\rangle$ is

$$P_e(t) = |C_e(t)|^2 = \frac{\gamma^2}{\Omega_R^2\hbar^2}\sin^2(\Omega_R t/2). \tag{2.63}$$

For exact resonance cases, $\Delta = 0$, we have

$$P_e(t) = \sin^2\left(\frac{\gamma t}{2\hbar}\right), \tag{2.64}$$

and at the time $t = \pi\hbar/\gamma$, $P_e(t) = 1$ *i.e,* all the atomic population has been transferred to the excited state.
It is convenient to consider the quantity known as the atomic inversion $W(t)$, also known as population inversion, defined as the difference in the excited and ground state populations. i.e.,

$$W(t) = P_e(t) - P_g(t). \tag{2.65}$$

Population inversion for the resonant case and with the atom initially in the ground state is

$$W(t) = \sin^2\left(\frac{\gamma t}{2\hbar}\right) - \cos^2\left(\frac{\gamma t}{2\hbar}\right) = -\cos\left(\gamma t/\hbar\right). \tag{2.66}$$

Note that for $\Delta = 0$ the Rabi frequency is just $\Omega_R = \gamma/\hbar$, the oscillation frequency of the atomic inversion. Again, for $t = \pi\hbar/\gamma$ all the population is transferred to the excited state: $W(\pi\hbar/\gamma) = 1$. Also when $t = \pi\hbar/2\gamma$, then $W(\pi\hbar/2\gamma) = 0$ *i.e,* and the population is shared coherently between the excited and ground states with

$$C_e(\pi\hbar/2\gamma) = \frac{i}{\sqrt{2}}, \tag{2.67}$$

$$C_g(\pi\hbar/2\gamma) = \frac{1}{\sqrt{2}}, \tag{2.68}$$

so that,

$$|\psi(t = \pi\hbar/2\gamma)\rangle = \frac{1}{\sqrt{2}}((|g\rangle + i|e\rangle). \tag{2.69}$$

2.7 The Jaynes-Cummings model

Recent developments in cavity electrodynamics show that, it is possible to manufacture environments where the density of electromagnetic

field modes is significantly different than in free space. For example small micro cavities or optical cavities are capable of supporting only a single mode radiation field. And we may say that a single mode field atom interaction is realizable in a laboratory set up. Jaynes and Cummings introduced a model to describe the interaction of a two level atom and single mode electromagnetic field in 1963 [12] and it is capable of exhibiting purely quantum mechanical phenomena.
Here also we consider an atom with two levels $|g\rangle$ and $|e\rangle$ interacting with a single mode cavity with electric field,

$$\hat{\boldsymbol{E}} = \boldsymbol{e}\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\left(\hat{a}+\hat{a}^\dagger\right)\sin(kz), \tag{2.70}$$

where $\boldsymbol{e}$ is an arbitrary oriented polarization vector. The corresponding interaction Hamiltonian in the operator form is

$$\hat{H}^{(I)} = -\hat{\boldsymbol{d}}.\hat{\boldsymbol{E}} = \hat{d}g\left(\hat{a}+\hat{a}^\dagger\right), \tag{2.71}$$

where

$$g = -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\sin(kz). \tag{2.72}$$

and $\hat{d} = \hat{\boldsymbol{d}}.\boldsymbol{e}$. At this point it is convenient to introduce the so-called atomic transition operators

$$\hat{\sigma}_+ = |e\rangle\langle g|, \quad \hat{\sigma}_- = |g\rangle\langle e| = \sigma_+^\dagger \tag{2.73}$$

and the atomic inversion operator

$$\hat{\sigma}_z = |e\rangle\langle e| - |g\rangle\langle g|. \tag{2.74}$$

These operators obey the Pauli spin algebra

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z \tag{2.75}$$

$$[\hat{\sigma}_z, \hat{\sigma}_\pm] = 2\hat{\sigma}_\pm \tag{2.76}$$

From parity considerations, the diagonal matrix elements of the dipole moment operator is zero and only the off diagonal matrix elements are non zero i.e., $\langle e|\hat{d}|g\rangle \neq 0$. We can now write

$$\hat{d} = d|g\rangle\langle e| + d|e\rangle\langle g| \tag{2.77}$$

$$= d\hat{\sigma}_- + d\hat{\sigma}_+ = d\left(\hat{\sigma}_+ + \hat{\sigma}_-\right) \tag{2.78}$$

where we have set $\langle e|\hat{d}|g\rangle = d$ and is taken to be real. Thus the interaction Hamiltonian is

$$\hat{H}^{(I)} = \hbar\lambda\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right) \tag{2.79}$$

where $\lambda = dg/\hbar$. Now we define the level of zero energy is at midway between the states $|g\rangle$ and $|e\rangle$ as shown in Fig. 2.4.

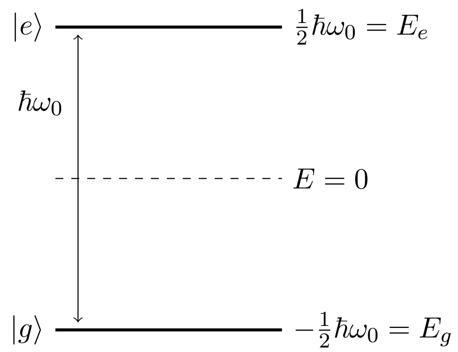


Figure 2.4: Atomic energy level diagram. $E = 0$ level is taken half way between the two levels $|g\rangle$ and $|e\rangle$

Then the free atomic Hamiltonian can be written as

$$\hat{H}_A = \frac{1}{2}\left(E_e - E_g\right)\hat{\sigma}_z = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z. \tag{2.80}$$

The free field Hamiltonian after dropping the zero point energy term is

$$\hat{H}_F = \hbar\omega\hat{a}^\dagger\hat{a}. \tag{2.81}$$

Now the total Hamiltonian is

$$\begin{aligned} \hat{H} &= \hat{H}_A + \hat{H}_F + \hat{H}^{(I)} \\ &= \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)\left(\hat{a} + \hat{a}^\dagger\right) \end{aligned} \quad (2.82)$$

The description of a system using the total Hamiltonian given in Eq. (2.82) is known as Jaynes-Cummings Model(JCM).
In the free field case we have the time dependence of field operators as

$$\hat{a}(t) = \hat{a}(0)e^{-i\omega t}, \qquad \hat{a}^\dagger(t) = \hat{a}^\dagger(0)e^{i\omega t}. \quad (2.83)$$

Similarly for the free atomic cases

$$\hat{\sigma}_\pm(t) = \hat{\sigma}_\pm(0)e^{\pm i\omega_0 t}. \quad (2.84)$$

Using Eqs. (2.83) and (2.84) the time dependence of the following factors in the total Hamiltonian in Eq. (2.82) is expected as

$$\begin{aligned} \hat{\sigma}_+\hat{a} &\sim e^{i(\omega_0-\omega)t}, \\ \hat{\sigma}_-\hat{a}^\dagger &\sim e^{-i(\omega_0-\omega)t}, \\ \hat{\sigma}_+\hat{a}^\dagger &\sim e^{i(\omega_0+\omega)t}, \\ \hat{\sigma}_-\hat{a} &\sim e^{-i(\omega_0+\omega)t}. \end{aligned} \quad (2.85)$$

For $\omega \approx \omega_0$, the last two terms vary much more rapidly than the first two. Further more the last two terms does not conserve energy in comparison with the first two. The term $\hat{\sigma}_+\hat{a}^\dagger$ corresponds to the emission of a photon as the atom goes from the ground to the excited state. Similarly $\hat{\sigma}_-\hat{a}$ corresponds to the absorption of photon when an atom is de-excited from upper to lower. These two processes do not conserve energy and we are dropping these two terms from the total Hamiltonian, making the Rotating Wave Approximation(RWA).
Now the Hamiltonian in the RWA is

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger\right). \quad (2.86)$$

Let us now consider the example of resonant interaction, *i.e.*, $\Delta = 0$. We assume that the atom is initially in the excited state $|e\rangle$ and the field initially in the number state $|n\rangle$. Now the initial state of the atom field system is $|i\rangle = |e\rangle|n\rangle$. The energy of such a state is $E_i = \frac{1}{2}\hbar\omega + n\hbar\omega$. The only final state of the system is $|f\rangle = |g\rangle|n+1\rangle$ with energy $E_i = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega$, *i.e.*, $E_i = E_f$. The general state of the atom field system at any time t is

$$|\psi\rangle(t) = C_i(t)|i\rangle + C_f(t)|f\rangle. \tag{2.87}$$

Substituting Eq. (2.86) in the time dependent Schrödinger equation we get

$$\begin{aligned} \dot{C}_i &= -i\lambda\sqrt{n+1}\, C_f, \\ \dot{C}_f &= -i\lambda\sqrt{n+1}\, C_i. \end{aligned} \tag{2.88}$$

Eliminating C_f we obtain

$$\ddot{C}_i + \lambda^2(n+1)C_i = 0. \tag{2.89}$$

The solution, using the initial condition, is

$$C_i(t) = \cos\left(\lambda t\sqrt{n+1}\right). \tag{2.90}$$

From Eq. (2.88) we get

$$C_f(t) = -i\sin\left(\lambda t\sqrt{n+1}\right) \tag{2.91}$$

Thus the state of the atom field system at any time t is

$$|\psi\rangle = \cos\left(\lambda t\sqrt{n+1}\right)|e\rangle|n\rangle - i\sin\left(\lambda t\sqrt{n+1}\right)|g\rangle|n+1\rangle. \tag{2.92}$$

The probability of the system remains in the initial state is

$$P_i(t) = |C_i(t)|^2 = \cos^2\left(\lambda t\sqrt{n+1}\right) \tag{2.93}$$

and the probability that it makes a transition to the final state $|f\rangle$ is

$$P_f(t) = |C_f(t)|^2 = \sin^2\left(\lambda t\sqrt{n+1}\right). \tag{2.94}$$

The atomic inversion or population inversion is given by

$$\begin{aligned} W(t) &= \langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle \\ &= P_i(t) - P_f(t) \\ &= \cos\left(2\lambda t\sqrt{n+1}\right) \end{aligned} \tag{2.95}$$

We may define a quantum electrodynamic Rabi frequency $\Omega(n) = 2\lambda\sqrt{n+1}$ so that

$$W(t) = \cos\left[\Omega(n)t\right] \tag{2.96}$$

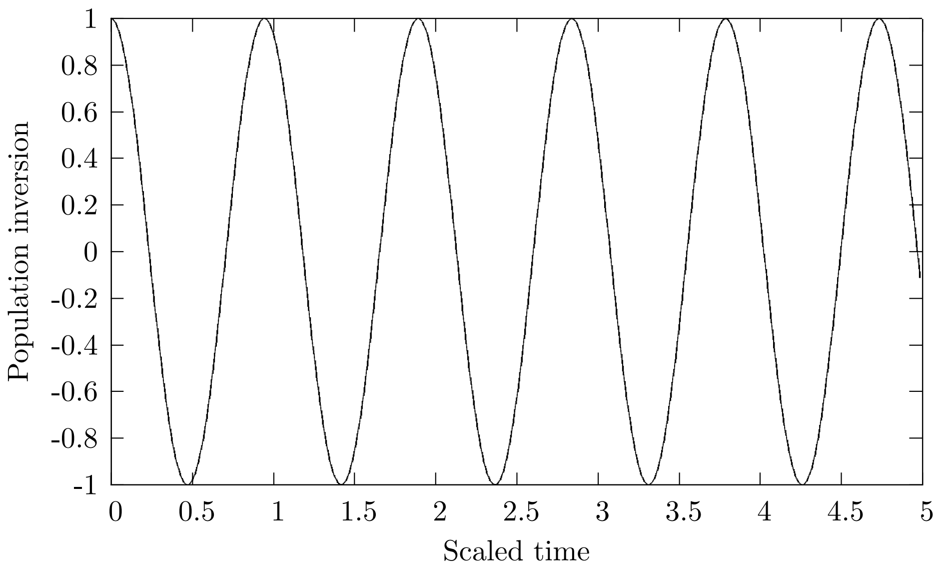


Figure 2.5: Atomic population inversion plotted against scaled time (λt). Field is initially in a Fock state with number of photons equal to 10.

Here in the quantum mechanical case it is clear from Eq. (2.95) that there are Rabi oscillations even when the number of photons is

zero($n = 0$). These are vacuum field Rabi oscillation which does not have a classical counter part. These vacuum Rabi oscillations are the result of the spontaneous emission of photon from the atom. Vacuum Rabi oscillations are visible in high Q cavities. Apart from this, the behaviour of atomic inversion for a definite number of photons is very much similar to the semiclassical Rabi oscillations.
As a more general case let us consider the atom initially in a superposition of states $|g\rangle$ and $|e\rangle$; i.e.,

$$|\psi(0)\rangle_{\text{atom}} = C_g|g\rangle + C_e|e\rangle \tag{2.97}$$

and initially the field is in a state

$$|\psi(0)\rangle_{\text{field}} = \sum_{n=0}^{\infty} C_n|n\rangle. \tag{2.98}$$

So the initial atom-field state is

$$|\psi(0)\rangle = |\psi(0)\rangle_{\text{atom}} \otimes |\psi(0)\rangle_{\text{field}} \tag{2.99}$$

The solution of the Schrödinger equation now becomes

$$\begin{aligned} |\psi(t)\rangle &= \sum_{n=0}^{\infty} \left[C_e C_n \cos\left(\lambda t\sqrt{n+1}\right) - iC_g C_{n+1} \sin\left(\lambda t\sqrt{n+1}\right)\right]|e\rangle \\ &+ \left[-iC_e C_{n-1} \sin\left(\lambda t\sqrt{n}\right) + C_g C_n \cos\left(\lambda t\sqrt{n}\right)\right]|g\rangle|n\rangle. \end{aligned} \tag{2.100}$$

If the atom is initially in the excited state, where $C_e = 1$ and $C_g = 0$, the solution can be rewritten as

$$|\psi(t)\rangle = |\psi_g(t)\rangle|g\rangle + |\psi_e(t)\rangle|e\rangle. \tag{2.101}$$

The field components $|\psi_g(t)\rangle$ and $|\psi_e(t)\rangle$ respectively when atom in the ground and excited state are

$$
\begin{aligned}
|\psi_g(t)\rangle &= -i\sum_{n=0}^{\infty} C_n \sin\left(\lambda t\sqrt{n+1}\right)|n+1\rangle, \\
|\psi_e(t)\rangle &= \sum_{n=0}^{\infty} C_n \cos\left(\lambda t\sqrt{n+1}\right)|n\rangle. \qquad (2.102)
\end{aligned}
$$

The corresponding atomic population inversion is

$$
\begin{aligned}
W(t) &= \langle\psi(t)|\hat{\sigma}_z|\psi(t)\rangle \\
&= \langle\psi_e(t)|\psi_e(t)\rangle - \langle\psi_g(t)|\psi_g(t)\rangle \\
&= \sum_{n=0}^{\infty} |C_n|^2 \cos\left(2\lambda t\sqrt{n+1}\right). \qquad (2.103)
\end{aligned}
$$

The population inversion is just the sum of n-photon inversions of Eq. (2.95) weighted with the photon number distribution of the initial field state. If the initial field is coherent, i.e.,

$$
C_n = e^{|\alpha^2|/2}\frac{\alpha^n}{\sqrt{n!}}, \qquad (2.104)
$$

then the population inversion is

$$
W(t) = e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\cos\left(2\lambda t\sqrt{n+1}\right). \qquad (2.105)
$$

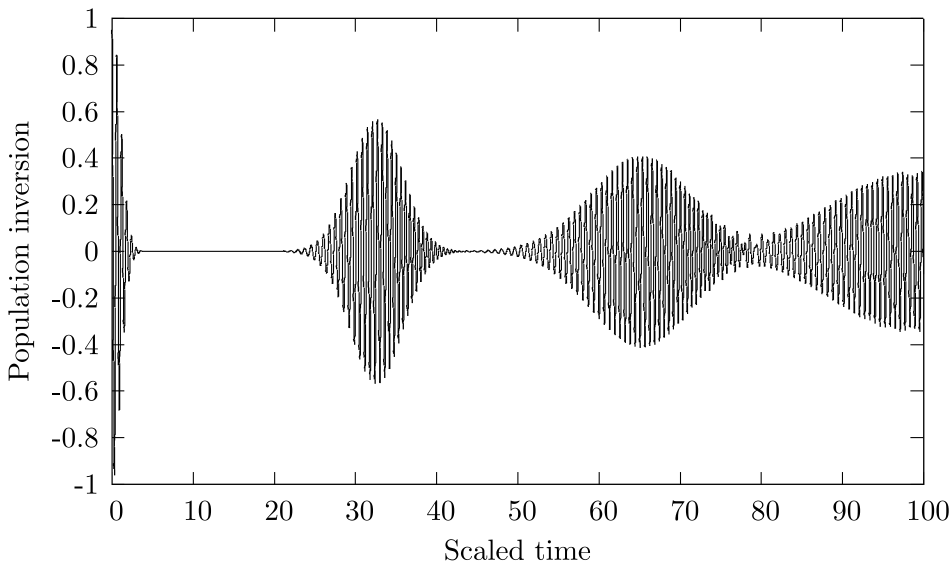


Figure 2.6: Atomic population inversion plotted against scaled time (λt). Field is initially in coherent state with average number of photons, $\bar{n} = 25$.

A plot of atomic population inversion, $W(t)$, versus the scaled time(λt) is shown in Fig. 2.6. The initial Rabi oscillation die out or collapse as time elapses and after a period of quiescence following the collapse, the Rabi oscillations starts to revive, although not completely. At longer times one can find a series of collapses and revivals, the revivals becoming less distinct as time increases. This collapse and revival behaviour of population inversion is a fully quantum mechanical effect and is strikingly different than in the semi classical case. These collapses and revivals are due to the spread of probabilities about $\bar{n}$ for photon numbers in the range $\bar{n} \pm \Delta n$. Due to this spread in probability there will be many Rabi frequencies other than the dominant one, corresponding to the average photon number $\bar{n}$. The collapse time t_c may be estimated approximately from the time-frequency uncertainty relation and $t_c \simeq 1/(2\lambda)$.

2.8 Motivation

2.8.1 Application of an atom-field system as qubit QIP

Due to the recent developments in the cavity quantum electrodynamics experiments, what was earlier considered as 'toy models' are today realized in laboratories. A controlled isolated coherent evolution of one or a few atoms coupled to a single mode electromagnetic field inside a cavity can be accomplished experimentally[13–16]. These atom field systems are well suited for studying purely quantum mechanical effects and applicable for performing quantum logic operations in a quantum computer. The Jaynes-Cummings model has served as a theoretical description of this interacting system. As the experimental techniques are improved, for example, atom cooling, the use of multi-level atoms or multi-modes and driving of atoms or fields by external lasers, there are many extensions to the original Jaynes-Cummings model.
Jayness cumming model and its later extensions to this have made many theoretical predictions in cavity QED, for example collapse-revivals [17], atomic disentanglement [18–21], field squeezing [22–24], non demolition measurements [25, 26], state reconstruction [27, 28], single photon Fock states [29], superposition of large amplitude coherent states (Schrödinger cats) [30, 31], decoherence [32, 33] etc.. Many of these and other predictions have been verified in experiments [34–41, 41, 41–48]. At about the same time as the realization of cavity QED experiments has began, a new sub-field of quantum mechanics; quantum information has been developed explosively [49, 50]. The basic unit of information in quantum information is a "qubit" and it

can be decoded in the two level atom or in the field. Quantum states of an atom can be used for information storage and a photon coupled to the atom can act as information carriers. Control over the interaction between atom and field will help in managing quantum data in quantum information process(QIP). The applications of atom-photon interactions towards quantum information and computing have been studied and also experimentally tested, see for example [37–42, 50]. After the realisation that atom field states and its interaction dynamics are possible candidates for data storage and processing in quantum informatics, there are many works enquiring about the various possibilities of atom field interaction dynamics suitable for applications in quantum information processes [51–55]. In our work we seek various possibilities for controlling and manipulating state evolution in the interaction between a two level atom and electromagnetic field. We also examine the entanglement entropy evolution of an atom field system during the interaction. A controllable or fine tuned atom field interaction can improve the data handling ability of it in quantum computers.

2.8.2 Coupled cavity system for data transmission in QIP

In the current setting, information transmission is one of the main challenges facing in the realization of quantum computers. Many systems have been proposed in this area for effectively transmitting and manipulating data. For short length quantum communication, linear spin chain channels are introduced [56, 57]. Another concept, which can serve the purpose of information transmission from one part to other is a coupled cavity array, which can be modelled as an effective and controllable many body system [58], has been the subject of discussion in many recent studies [59, 60]. The solid state analogue of coupled cavity systems are realized by coupling superconducting qubits to stripline resonators [61–63]. There are many theoretical and experimental studies devoted to this area. This includes the developments of nanocavities in photonic crystals and the Josephson junction arrays, which is found to be useful in many significant applications in quantum information processing. Remarkable development also has been accomplished recently by considering cold atoms trapped in optical lattices which can effectively be described by a Bose-Hubbard Hamiltonian, unveils its potential applications as quantum optical simulators. In the following chapters we work with coupled cavity systems and analyse its interaction dynamics. We investigate the state transfer between two adjacent cavities and possibilities of controlling the data transfer between them.

3

Interaction of two level atom and electromagnetic field with time varying frequency

3.1 Introduction

A two level atom field system can be used for data storage and for their operations in quantum information processes(QIP). But a controllable atom-field interaction is essential for using it in QIP. There are many models suggesting various methods to control the atom-field interactions [64–67]. After the invention of frequency chirped lasers there are tremendous progresses in the research on atom field interactions where the frequency of the laser field is time dependent. Recent studies show that time dependent field frequency considerably modifies the dynamical evolution of an atom-field system[68–70]. In this chapter we discuss methods to control or manipulate the atom field probability amplitudes during the interaction between a two level atom and quantized electromagnetic field using time dependent field frequency fluctuations. It is to be noted that the sinusoidal field frequency fluctuation can be used for controlling the dynamics of atom field system. A detailed study of the dependence of population inversion on the applied frequency modulation parameters by varying the initial photon distributions are presented in this chapter. As a continuation, the dynamics of interaction between a two level atom and electromagnetic field with phase shifted sinusoidally varying frequency is also included in this chapter. In the case of phase shifted frequency fluctuations the population inversion behaves as in the case of Fock state atom interaction.

3.2 Model and Hamiltonian

We have the Jayness-Cumming Hamiltonian [12], discussed in the previous chapter, corresponding to the interaction between a two level atom and quantized single mode electromagnetic field. Setting $\hbar = 1$ the Hamiltonian is

$$\hat{H} = \nu_0 \hat{a}^\dagger \hat{a} + \frac{1}{2}\omega\hat{\sigma}_z + g\left(\hat{\sigma}_+\hat{a} + \hat{\sigma}\hat{a}^\dagger\right), \tag{3.1}$$

where ν_0 is the frequency of the field and ω is the transition frequency of the two level atom. The general state of the atom-field system derived using the time dependent Schrödinger equation and the initial condition such that atom is in the excited state at $t = 0$ is,

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} C_n \cos\left(\lambda t\sqrt{n+1}\right)|n\rangle - i\sum_{n=0}^{\infty} C_n \sin\left(\lambda t\sqrt{n+1}\right)|n+1\rangle, \tag{3.2}$$

where C_n is obtained from the initial photon distribution. These results have been discussed in the previous chapter for the photons initially in Fock state and coherent state. In all these cases the frequency of the interacting field, ν_0, is kept constant; does not have any time dependence. Now let us consider the case with the frequency of the field is also varying with time. In such cases the Hamiltonian and the entire system evolution changes. The fluctuating field frequency can be represented as the combination of a time independent mean frequency, ν_0, and a time dependent function $f(t)$ such that

$$\nu(t) = \nu_0 + f(t). \tag{3.3}$$

Because of the fluctuations in field frequency, all the parameters involved in the interaction which has a dependency on field frequency also changes with time. From JCM, it is known that the atom field coupling strength g is proportional to the field frequency and inversely proportional to the quantization volume(V) of the cavity, i.e.,

$$g \propto -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\left(\frac{d}{\hbar}\right) \tag{3.4}$$

According to the quantization of electromagnetic field, V is always inversely proportional to the field frequency such that

$$V(t) = \frac{V_0}{1+f(t)/\nu_0} \tag{3.5}$$

and now g reads

$$g = g_0\left(1+f(t)/\nu_0\right), \tag{3.6}$$

where g_0 is the coupling strength corresponds to the mean frequency ν_0. The time dependent total Hamiltonian after considering the changes in coupling strength due to the field frequency fluctuation is,

$$\hat{H}(t) = \nu_0\hat{a}^\dagger\hat{a} + \frac{1}{2}\omega\hat{\sigma}_z + \left(1+f(t)/\nu_0\right)\left[\hat{\sigma}_+\hat{a}+\hat{\sigma}_-\hat{a}^\dagger\right]. \tag{3.7}$$

Now the general state of the two level atom-field system at any arbitrary time t can be represented as

$$|\psi\rangle = \sum_n \left[C_{e,n}(t)|e,n\rangle + C_{g,n}(t)|g,n\rangle\right]. \tag{3.8}$$

Here $|e,n\rangle$($|g,n\rangle$) represents the atom in excited(ground) state with n photons and $C_{e,n}(t)$(or $C_{g,n}(t)$) are the coefficients corresponding to their probabilities. Substituting the Hamiltonian in Eq. (3.7) and the general state given by Eq. (3.8) in the time dependent Schrödinger equation we get the following infinite set of equations for the evolution of probability amplitudes, $C_{e,n}(t)$ and $C_{g,n+1}(t)$:

$$\begin{aligned}\frac{d}{dt}C_{e,n}(t) &= -in\left[\nu_0 + f(t)\right]C_{e,n}(t) - \frac{i}{2}\omega C_{1,n} \\ &\quad -ig_0\left[1 + f(t)/\nu_0\right]\sqrt{n+1}C_{g,n+1}(t), \qquad (3.9)\\ \frac{d}{dt}C_{g,n+1} &= -i(n+1)\left[\nu_0 + f(t)\right]C_{g,n+1} + \frac{i}{2}\omega C_{g,n+1} \\ &\quad -ig_0\left[1 + f(t)/\nu_0\right]\sqrt{n+1}C_{e,n}(t). \qquad (3.10)\end{aligned}$$

In order to make the equations simple and eventually to solve them we define another set of coefficients $M_{e,n}(t)$ and $M_{g,n+1}(t)$ such that

$$\begin{aligned}C_{e,n}(t) &= \exp\left[-i\left(n\nu_0 + \omega/2\right)t\right] \times \\ &\quad \exp\left[-in\int_0^t f(t')dt'\right]M_{e,n}(t) \qquad (3.11)\\ C_{g,n+1}(t) &= \exp\left[-i\left[(n+1)\nu_0 - \omega/2\right]t\right] \times \\ &\quad \exp\left[-i(n+1)\int_0^t f(t')dt'\right]M_{g,n+1}(t). \qquad (3.12)\end{aligned}$$

It is important to note that,

$$|M_{e,n}(t)|^2 = |C_{e,n}(t)|^2 \ ; \ |M_{g,n}(t)|^2 = |C_{g,n}(t)|^2. \qquad (3.13)$$

Substituting Eqs. (3.11) and (3.12) in Eqs. (3.9) and (3.10) we obtain

$$\begin{aligned}\frac{d}{dt}M_{e,n}(t) &= -ig\sqrt{n+1}\exp[-i(\nu_o - \omega)t] \times \\ &\quad \exp\left(-i\int_0^t f(t')dt'\right)M_{g,n+1} \qquad (3.14)\\ \frac{d}{dt}M_{g,n+1}(t) &= -ig\sqrt{n+1}\exp\left[i(\nu_0 - \omega)t\right] \times \\ &\quad \exp\left(i\int_0^t f(t')dt'\right)M_{1,n} \qquad (3.15)\end{aligned}$$

The above set of equations Eqs. (3.14) and (3.15) can be solved to find the nature of evolution of the atom field system. In the previous chapter we have defined the atomic population inversion as the difference in the probability of finding the atom in excited state to ground state; i.e.,

$$W(t) = \sum_n \left[|C_{e,n}(t)|^2 - |C_{g,n+1}|^2 \right]. \quad (3.16)$$

Using Eqs. (3.11) and (3.12) and Eq. (3.13), it can be written in the form:

$$W(t) = \sum_n \left[|M_{e,n}(t)|^2 - |M_{g,n+1}|^2 \right]. \quad (3.17)$$

In the case where the frequency is constant *i.e.* $f(t) = 0$, the set of Eqs. (3.14) and (3.15) can be re written as

$$\frac{dM_{e,n}(t)}{dt} = -ig_0\sqrt{n+1}\; e^{-i\Delta t}\; M_{g,n+1}(t), \quad (3.18)$$

$$\frac{dM_{g,n+1}(t)}{dt} = -ig_0\sqrt{n+1}\; e^{+i\Delta t}\; M_{e,n}(t), \quad (3.19)$$

where $\Delta = \nu_0 - \omega$. We can solve Eqs.(3.18) and (3.19) analytically using the standard techniques of solving coupled differential equations as follows.
Differentiating Eq. (3.18) and substituting Eq. (3.19) gives

$$\frac{\partial^2}{\partial t^2} M_{g,n+1}(t) - i\Delta \frac{\partial}{\partial t} M_{g,n+1}(t) + g_0^2(n+1) M_{g,n+1}(t) = 0 \quad (3.20)$$

Assume a solution of the form $M_{g,n+1}(t) = Ae^{i\theta t}$ and substituting it in Eq. (3.20) we get

$$\theta^2 - \Delta\theta - g_0^2(n+1) = 0 \quad (3.21)$$

i.e.,

$$\theta = \frac{\Delta \pm \sqrt{\Delta^2 + 4g_0^2(n+1)}}{2} \quad (3.22)$$

Taking $\Omega_n^2 = \Delta^2 + 4g_0^2(n+1)$ the general solution is

$$M_{g,n+1}(t) = A_+ e^{\frac{i}{2}(\Delta+\Omega_n)t} + A_- e^{\frac{i}{2}(\Delta-\Omega_n)t} \quad (3.23)$$

We have the initial condition such that at time $t = 0$, $M_{g,n+1}(0) = 0$. i.e., $M_{g,n+1}(0) = 0 = A_+ + A_- \quad \Rightarrow \; A_+ = -A_- = A(\text{say})$. Then we get

$$\begin{aligned} M_{g,n+1}(t) &= A\left\{e^{\frac{i}{2}(\Delta+\Omega_n)t} - e^{\frac{i}{2}(\Delta-\Omega_n)t}\right\} \\ &= Ae^{\frac{i\Delta t}{2}}\left\{e^{\frac{i}{2}\Omega_n t} - e^{-\frac{i}{2}\Omega_n t}\right\} \\ \text{i.e.,} \qquad M_{g,n+1}(t) &= Ae^{\frac{i\Delta t}{2}} 2i\sin(\Omega_n t/2). \end{aligned} \tag{3.24}$$

Substituting it in Eq. (3.19) we obtain

$$M_{e,n}(t) = (iA/g_0)\, e^{-i\Delta t/2}\left[\frac{\Omega_n}{2}\cos(\Omega_n t/2) + \frac{i\Delta}{2}\sin(\Omega_n t/2)\right] \tag{3.25}$$

Using the Eqs. (3.24) and (3.25), the atomic population inversion $W(t)$ can now be written as,

$$W(t) = \sum_{n=0}^{\infty}\rho_{nn}(0)\left[\frac{\Delta^2}{\Omega_n^2} + \frac{4g_0(n+1)}{\Omega_n^2}\cos\left(\Omega_n t\right)\right], \tag{3.26}$$

where $\rho_{nn}(0)$ is obtainable from the initial photon distribution.

3.3 Sinusoidally varying field frequency

We now consider the case with the frequency of the electromagnetic radiation varies sinusoidally with time as given below

$$\nu(t) = \nu_0 + \Delta\nu\sin(\beta t), \tag{3.27}$$

where ν_0 is the initial mean frequency and $\Delta\nu\sin(\beta t)$ is the fluctuation with an amplitude $\Delta\nu$ and a periodicity $1/\beta$. Now the coupling strength g becomes

$$g = g_0\left[1 + \frac{\Delta\nu\sin(\beta t)}{\nu_0}\right]. \tag{3.28}$$

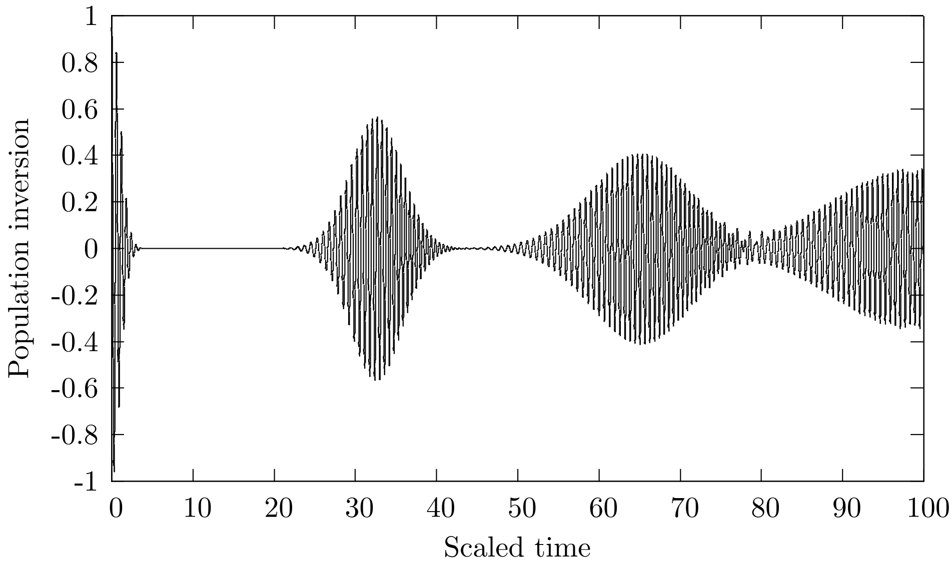
With the oscillating field frequency the evolution equations for the probability amplitudes takes the form

$$\begin{aligned}\frac{dM_{e,n}(t)}{dt} &= -ig_0\sqrt{n+1}\; e^{\Delta\nu/\beta[cos\beta t-1]} \\ &\qquad \left[1+\frac{\Delta\nu\sin(\beta t)}{\nu_0}\right] M_{g,n+1}(t) \qquad (3.29)\\ \frac{dM_{g,n+1}(t)}{dt} &= -ig_0\sqrt{n+1}\; e^{\Delta\nu/\beta[cos\beta t-1]} \\ &\qquad \left[1+\frac{\Delta\nu\sin(\beta t)}{\nu_0}\right] M_{e,n}(t) \qquad (3.30)\end{aligned}$$

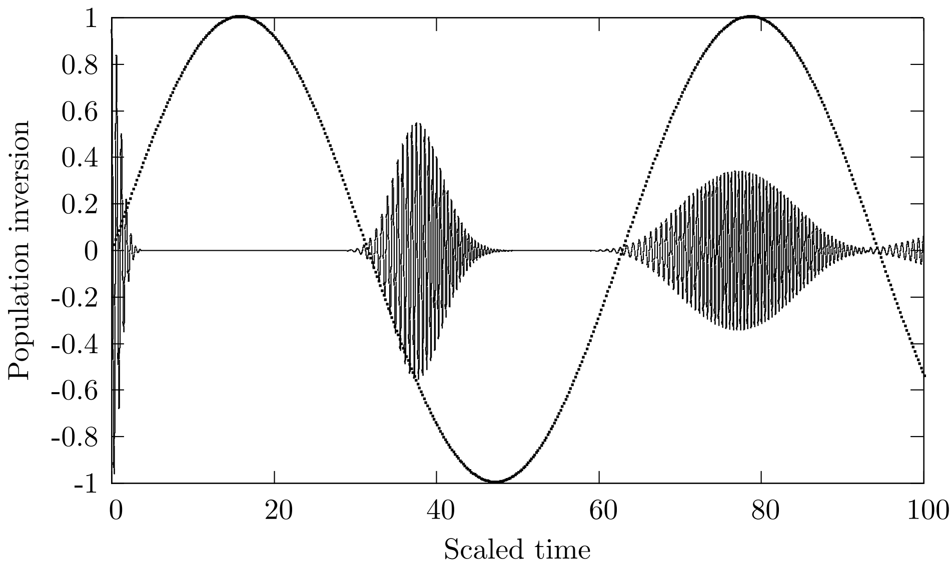
The above set of equations Eqs. (3.29) and (3.30) are solved numerically using the fourth order Runge-Kutta method for the given initial photon distribution. In the following sections we discuss the interaction for coherent and squeezed field.

3.3.1 Interaction with coherent field

Consider the case in which a two level atom interacting with an initial coherent field with time varying frequency. We notice a weak influence of frequency fluctuation in the evolution of population inversion when the amplitude of frequency modulation $\Delta\nu$ and angular frequency β is small. For example we choose $\Delta\nu \approx 0.001\nu_0$, and $\beta \approx 0.1g_0$ and the corresponding evolution of atomic population inversion is plotted in Fig. 3.1 as the function of scaled time $\tau = gt$



(a): $f(t) = 0$



(b): $f(t) = \Delta\nu \sin(\beta t)$

Figure 3.1: Atomic population inversion against scaled time. Initial coherent field with $\bar{n} = 25$. Field frequency ν_0 is taken to be $10000g_0$, $\Delta\nu = 0.001\nu_0$ and $\beta = 0.1g_0$. Dotted line shows the field frequency fluctuation.

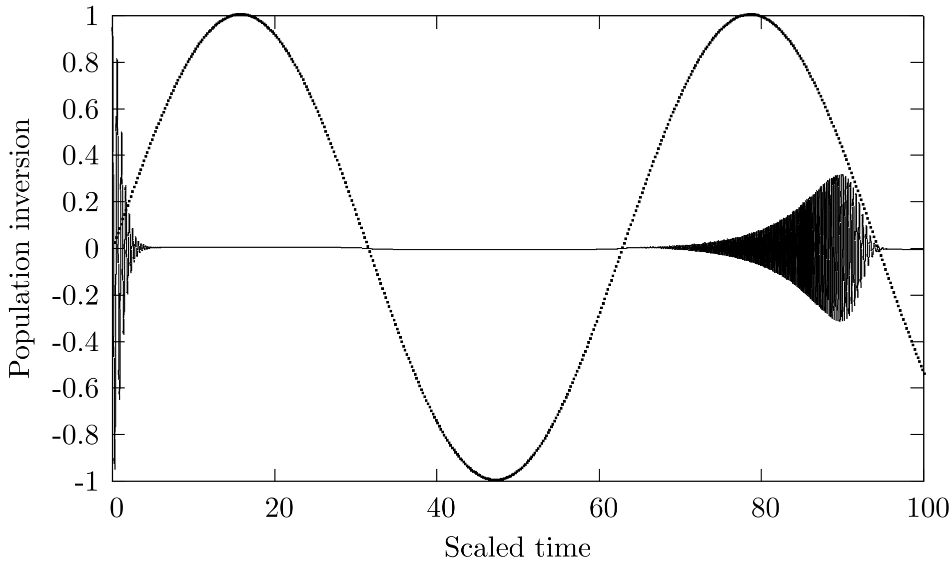
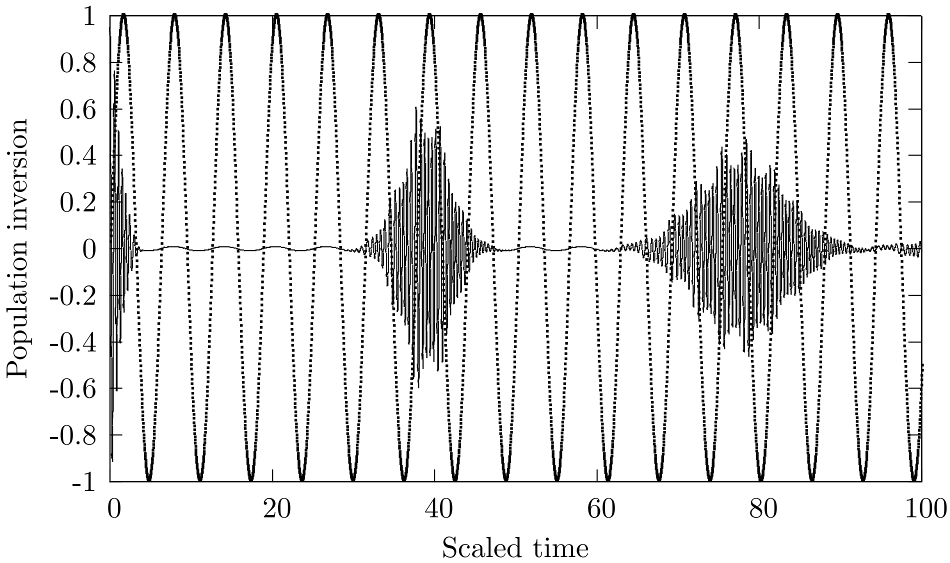
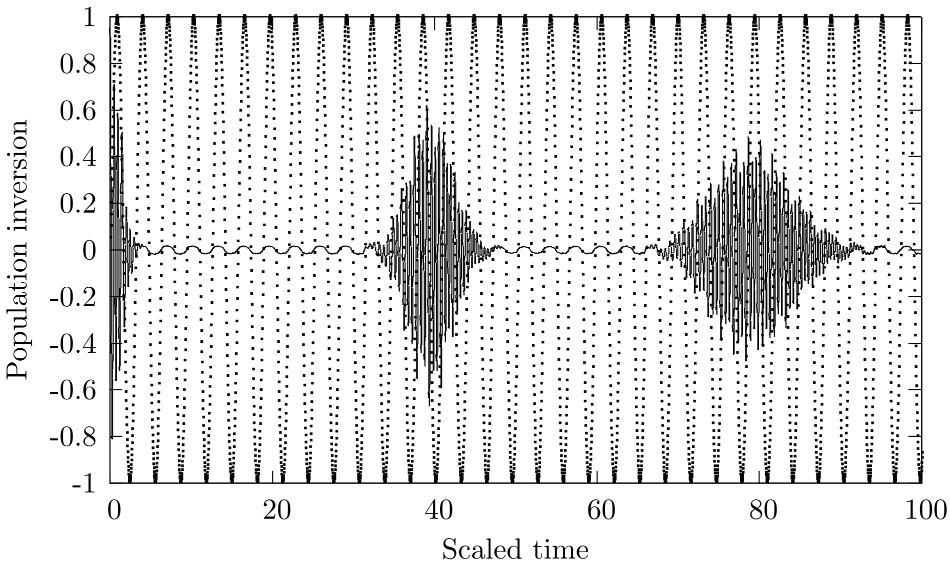


Figure 3.2: Atomic population inversion against scaled time. Initial coherent field with $\bar{n} = 25$. $\nu_0 = 1000 g_0$, $\Delta\nu = 0.005\nu_0$, $\beta = 0.1 g_0$. Dotted line shows the field frequency fluctuation.

Here the evolution is similar to the constant frequency case with normal collapses and revivals. But when the amplitude $\Delta\nu$ increases, there are visible changes in the nature of population inversion. In Fig. 3.1 population inversion versus time for both, with frequency fluctuation and without frequency fluctuation, cases are plotted. When there is a frequency variation with a considerable amplitude, we can see from Fig. 3.2 that the revivals are shifted towards to the right. We may say that the system spend more time in the collapse region with a coherent sharing of probability amplitudes between atom and field or in other words, with a zero atomic inversion. In Fig. 3.3 we have plotted the atomic inversion by varying the periodicity of the field frequency fluctuations. Here we observed the quasi periodic oscillations in population inversion with time. It is also noted that, during the collapse period the atomic inversion is not exactly collapsing to zero but oscillating with a small amplitude. The period of these oscillations are exactly equal to the period of field frequency fluctuations.



(a): $\beta = g_0$



(b): $\beta = 2g_0$

Figure 3.3: Population Inversion versus time. Initial coherent field with $\bar{n} = 25$. Field frequency $\nu_0 = 10000g_0$, $\Delta\nu = 10g_0$. Field frequency variation is shown by the dotted line.

3.3.2 Interaction with squeezed field

Quadrature squeezed electromagnetic field has been discussed in the proceeding section 1.4. Squeezed light is the light with minimum uncertainty but the value of uncertainty in different quadratures are not the same. One way in which this can be achieved is to squeeze the uncertainty circle of the vacuum or the coherent state into an ellipse of the same area. The squeezed states for a single mode radiation field may be generated from the vacuum $|0\rangle$ by

$$|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle. \tag{3.31}$$

Here $\hat{S}(\xi)$ and $\hat{D}(\alpha)$ are the squeezing and coherent displacement operators respectively and are given by

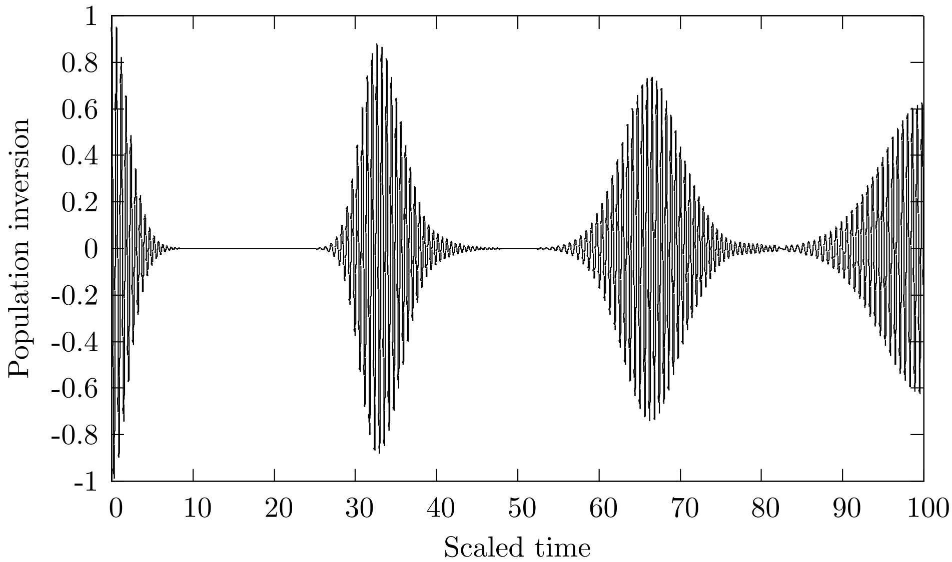
$$\hat{S}(\xi) = \exp[\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger^2})], \tag{3.32}$$

$$\hat{D}(\alpha) = \exp(\alpha \hat{a^\dagger} - \alpha^* \hat{a}), \tag{3.33}$$

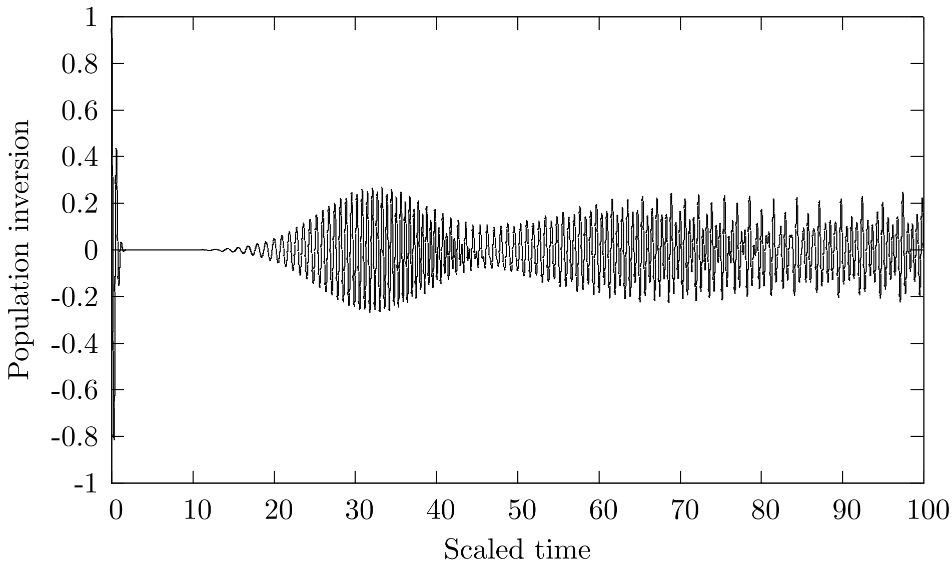
where $\alpha = |\alpha| e^{i\psi}$, $\xi = re^{i\theta}$, r is known as the squeezing parameter and $0 \leq r < \infty$ and $0 \leq \theta < 2\pi$. When $\xi = 0$ we obtain the coherent state. The photon number distribution for a squeezed field is

$$\begin{aligned} P_n &= \frac{(\frac{1}{2}\tanh r)^n}{n! \cosh r} \exp\left[-|\alpha|^2 - \frac{1}{2}\left(\alpha^2 e^{i\theta} + \alpha^{*^2} e^{-i\theta}\right)\tanh r\right] \times \\ & \left| H_n \left[\gamma (e^{i\theta}\sinh 2r)^{\frac{1}{2}})\right]\right|^2, \end{aligned} \tag{3.34}$$

with $\gamma = \alpha \cosh r + \alpha^* \sinh r$ and H_n's are the Hermite polynomials. Now consider the interaction of quadrature squeezed light with a two level atom. In the interaction the effects of squeezing is maximum when the value θ is π and the effects are minimum for $\theta = 0$. The behaviour of population inversion with time for $\theta = 0$ is almost similar to that of a coherent field atom interaction. There are series of clear collapses and revivals in population inversion with time. But when the value of $\theta = \pi$ only first two revivals are prominent and clear. The collapse and revival phenomenon is not retained for long time and after the first few there occurs random oscillations in population inversion as shown in Fig. 3.4. When the value of r is increased the random oscillation of population inversion starts early in time; immediately after the first revival itself. These randomness in the time evolution of atomic inversion is due to the squeezing of the interacting field.



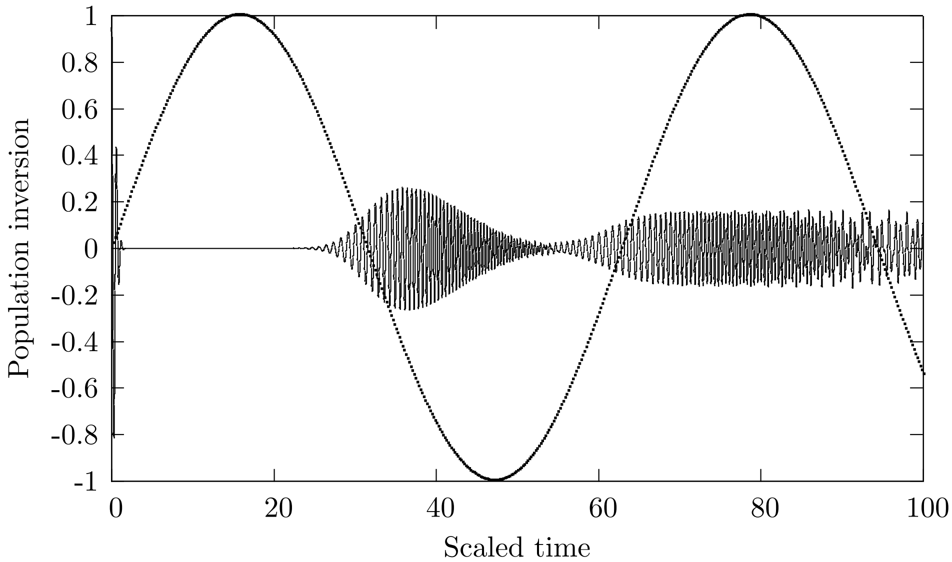
(a): Squeezed field with $\theta = 0$



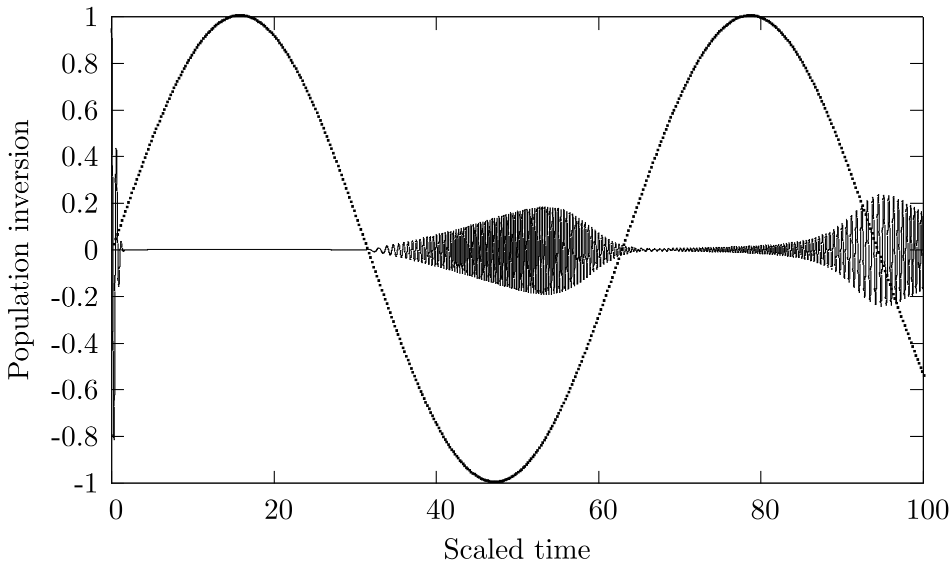
(b): Squeezed field with $\theta = \pi$

Figure 3.4: Population inversion versus time. Initial squeezed field with $\bar{n} = 25$. Squeezing parameter $r = 0.8$.

Now we take the case of interaction of atom and squeezed light with time varying frequency. We are considering only the case with $\theta = \pi$ where the effect of squeezing is maximum. The frequency variation is $f(t) = \Delta\nu \sin(\beta t)$ with $\Delta\nu << \nu_0$ and β is taken to be very small. For example we choose $\beta = 0.1g_0$ and $\Delta\nu = 0.001\nu_0$. The corresponding evolution of population inversion is shown in Fig. 3.5. In the case of interactions in which field frequency is a constant, we have already noticed that the variation of population inversion is random after the first revival. From Figs. 3.5 and 3.6 it is clear that the randomness in the population inversion is reduced by applying a sinusoidal frequency variation for the squeezed field. Now the collapses and revivals are clear and distinct. Here the variation in population inversion is controlled and collapses and revivals are retained. Also, similar to the behaviour noticed in coherent field case, the occurrence of revivals shift towards the right when the amplitude of fluctuation increases. Thus the revival periodicity has a noticeable dependence on the amplitude($\Delta\nu$) of the field frequency modulation.

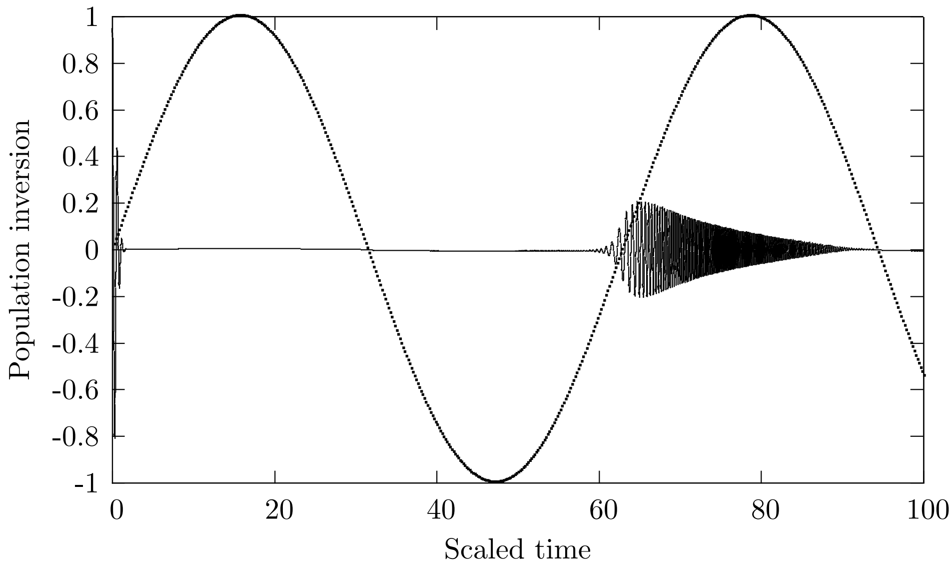


(a): $\Delta\nu = 10g_0$



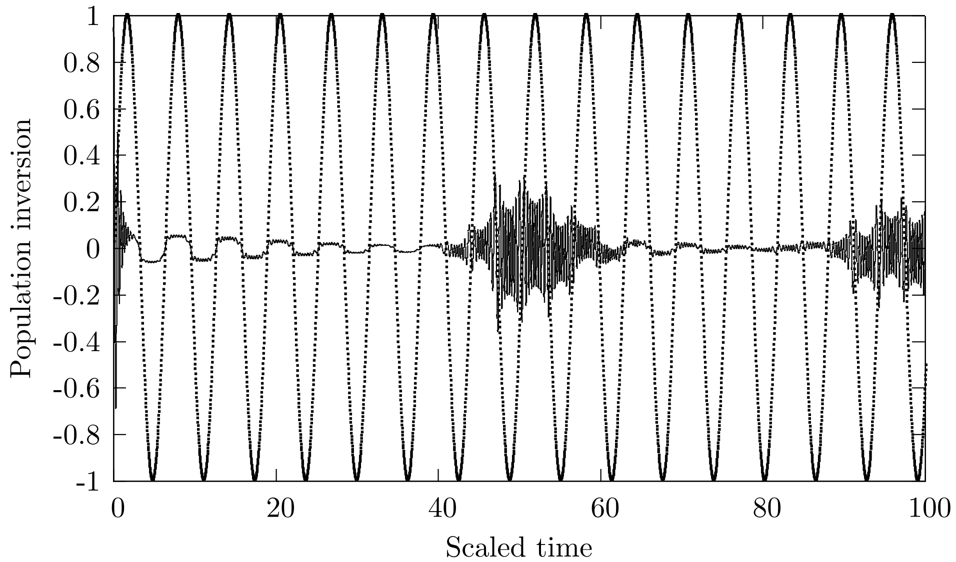
(b): $\Delta\nu = 20g_0$

Figure 3.5: Population inversion versus time for two level atom and squeezed field interaction with field frequency fluctuations for $\bar{n} = 25$, $r = 0.8, \theta = \pi$, $\beta = 0.1g_0$. Dotted line shows the field frequency fluctuation.

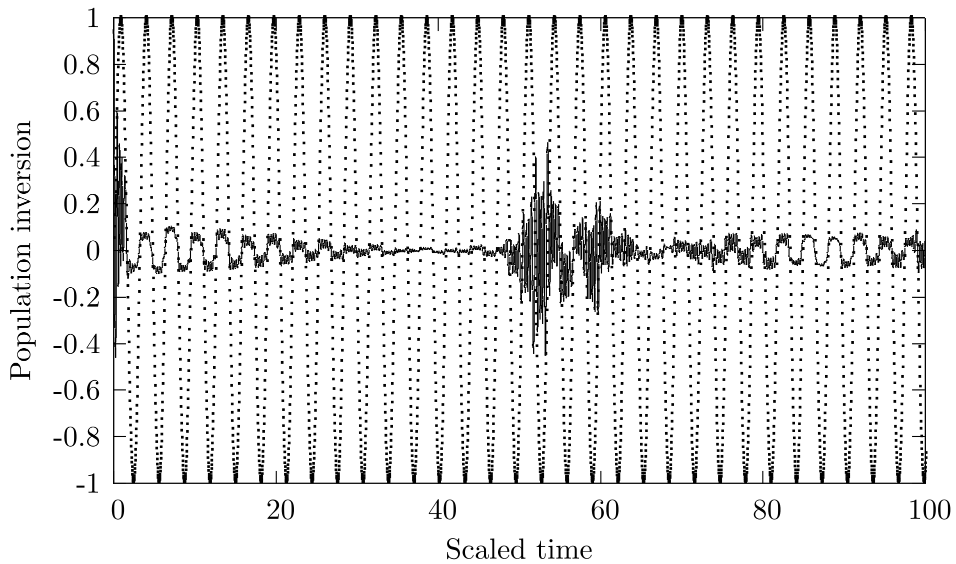


$\Delta\nu = 40g_0$

Figure 3.6: Population inversion versus time for two level atom and squeezed field interaction with field frequency fluctuations. $\bar{n} = 25$, $r = 0.8, \theta = \pi$, $\beta = 0.1g_0$. Field frequency variation is shown by the dotted line.



(a): $\beta = 1g_0$



(b): $\beta = 2g_0$

Figure 3.7: Population inversion versus time for two level atom and squeezed field interaction with field frequency fluctuations. Field parameters $\bar{n} = 25$, $r = 0.8, \theta = \pi$, $\nu_0 = 10000g_0$, $\Delta\nu = 20g_0$ (a) $f(t) = 20g_0 \sin(1g_0t)$ and in (b) $f(t) = 20g_0 \sin(2g_0t)$. Dotted line shows the field frequency fluctuations.

When we vary the periodicity for frequency fluctuation for a constant amplitude, for example $\beta = 1g_0, 2g_0$ etc., the population inversion executes a quasi periodic oscillation. Also the collapses are not exactly at zeros and we can see a small amplitude oscillations in the collapse region. The periodicity of these small amplitude oscillations are equal to that of the applied field frequency fluctuations as seen in the Fig. 3.7. We can conclude that the field frequency fluctuation can be used for controlling or manipulating atom field probability amplitudes in an atom field system and this method can be used for realising a controllable atom field interaction in QIP.

3.4 Phase shifted frequency modulation

The effect of field frequency variation in the interaction of two level atom in Kerr medium has been studied in a recent publication by Li Wang *et. al.*[70]. Their results shows that the coupling between the atom and photon is enhanced in the Kerr medium and even the atomic population inversion for an initial coherent field behaves like that of an initial Fock state. In this section we suggest another model in linear medium for which the population inversion behaves in a similar manner. We considered the interaction between a two level atom and electromagnetic field in a linear medium where the field frequency has a phase shifted sinusoidal fluctuation. The system has been studied for both coherent and squeezed field and it is noted that the population inversion oscillates sinusoidally just as in the Fock state atom interaction; without collapses and revivals. Phase shifted frequency fluctuations improves the coupling between the two level atom and field.

3.4.1 Sinusoidal frequency variation with a phase difference

Here we consider a field with sinusoidally varying frequency which has a phase difference with the mean frequency ν_0. In such cases, the field frequency at any time t can be written in the form,

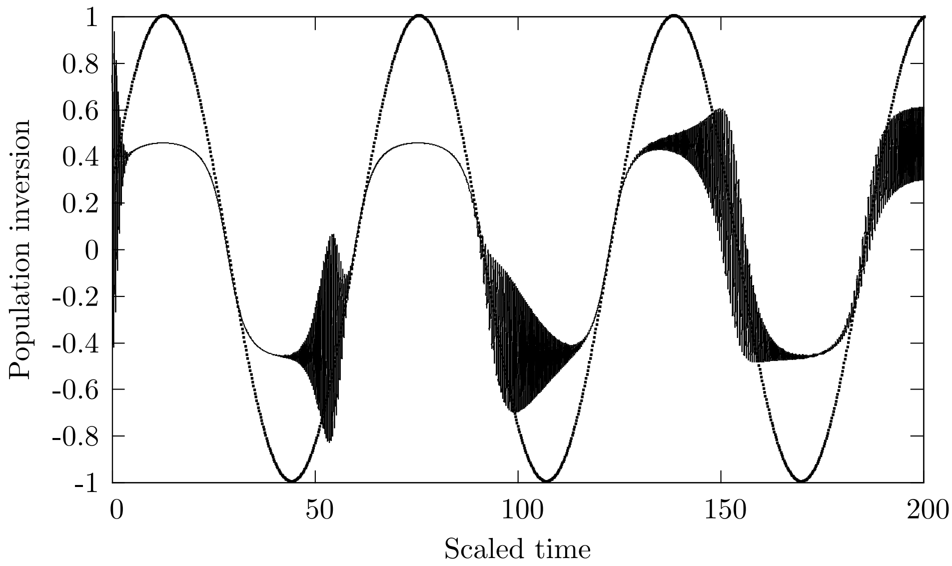
$$\nu(t) = \nu_0 + \Delta\nu \sin(\beta t + \phi). \tag{3.35}$$

We have the time evolution equations, Eqs. (3.14) and (3.15) for the coefficients corresponding to the probability amplitudes in section 3.2. Substituting Eq. (3.35) in Eqs. (3.14) and (3.14) the time evolution equations for the probability amplitudes becomes

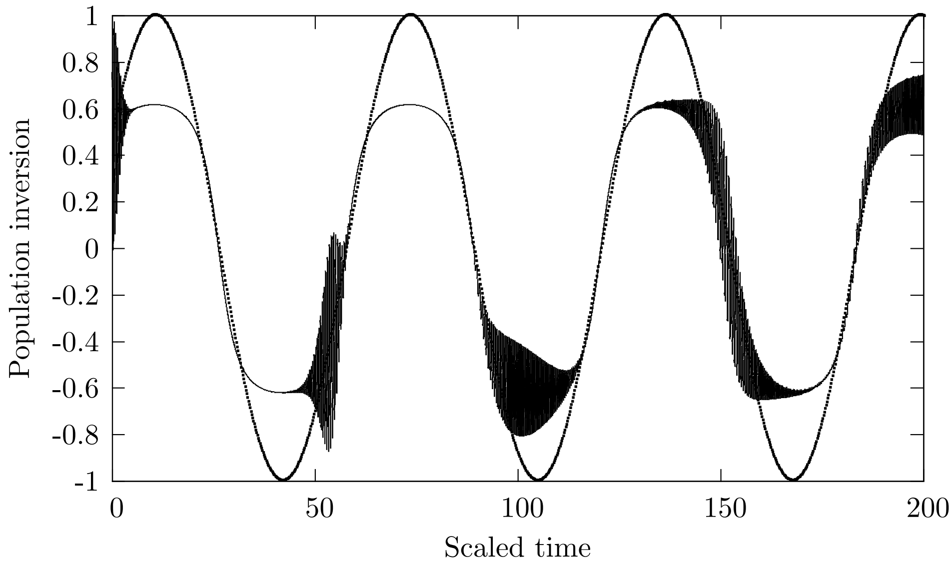
$$\begin{aligned} \frac{d}{dt}M_{1,n} &= -ig_0\left(1+\frac{\Delta\nu\sin(\beta t+\phi)}{\nu_0}\right)\sqrt{n+1} \\ &\quad e^{-i(\nu_0-\omega)t}e^{-i\int_0^t \Delta\nu\sin(\beta t'+\phi)dt'}M_{0,n+1} \qquad (3.36) \\ \frac{d}{dt}M_{0,n+1} &= -ig_0\left(1+\frac{\Delta\nu\sin(\beta t+\phi)}{\nu_0}\right)\sqrt{n+1} \\ &\quad e^{i(\nu_0-\omega)t}e^{-i\int_0^t \Delta\nu\sin(\beta t'+\phi)dt'}M_{1,n} \qquad (3.37) \end{aligned}$$

The time evolution of the system is now investigated by numerically solving the Eqs. (3.36) and (3.37).

3.4.2 Initial coherent field



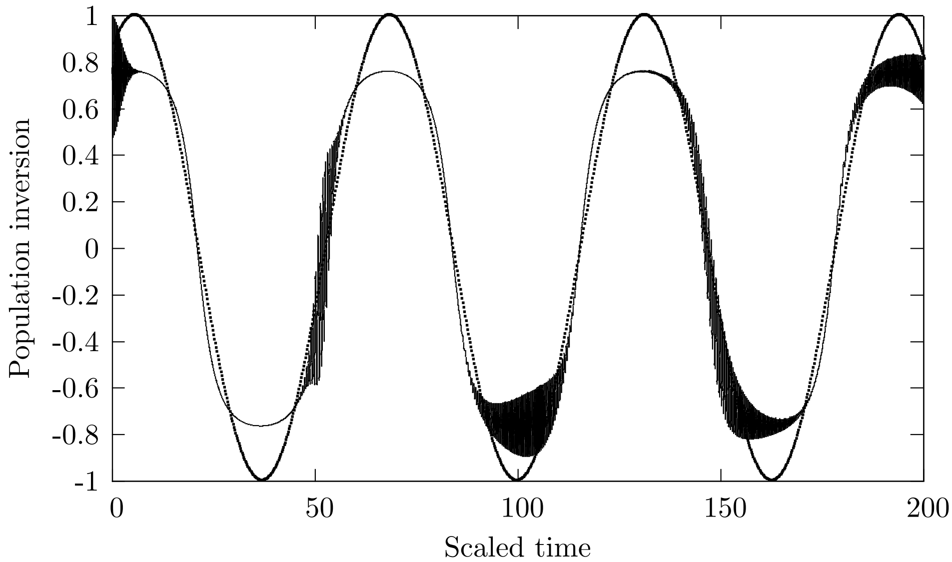
(a): $\phi = \pi/10$



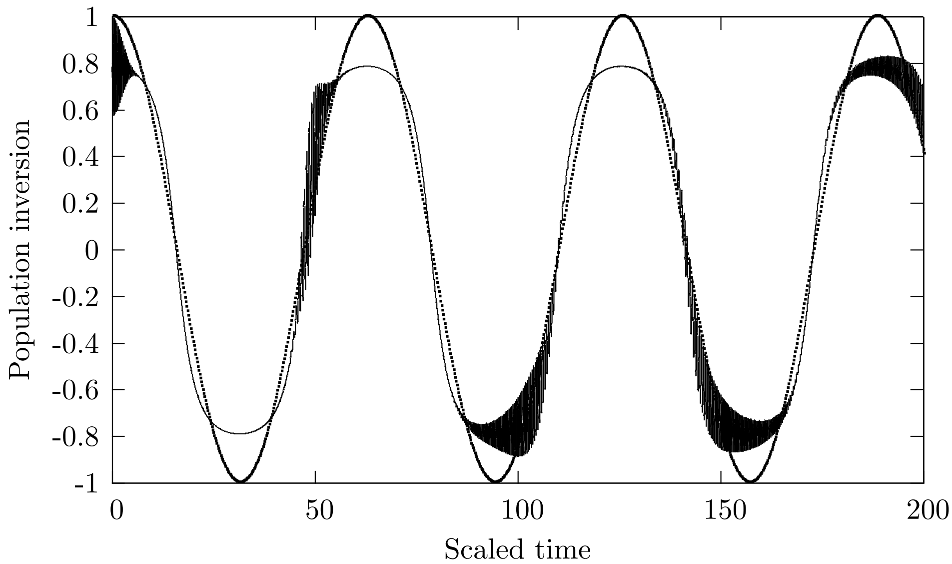
(b): $\phi = \pi/6$

Figure 3.8: Population inversion against scaled time. Initial coherent field with $\bar{n} = 25$. Field frequency has phase shifted sinusoidal fluctuations with $\alpha = 20g_0$, $\beta = 0.1g_0$. Dotted line shows the field frequency fluctuations.

In Figs. 3.8 and 3.9 the population inversion is plotted against time for the interaction of two level atom with initial coherent field with a phase shifted sinusoidal frequency modulation. It is to be noted that there are no exact collapses and revivals in population inversion but it oscillates sinusoidally with time. When the value of ϕ is equal to $\pi/2$ the evolution of population inversion is identical to the case of Fock field - atom interaction; a sinusoidal oscillation.



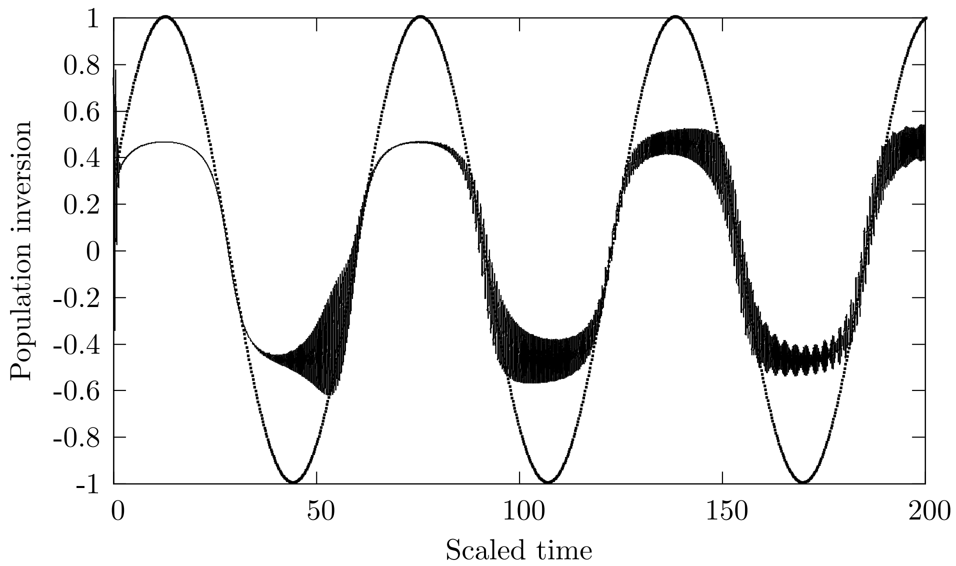
(a): $\phi = \pi/3$



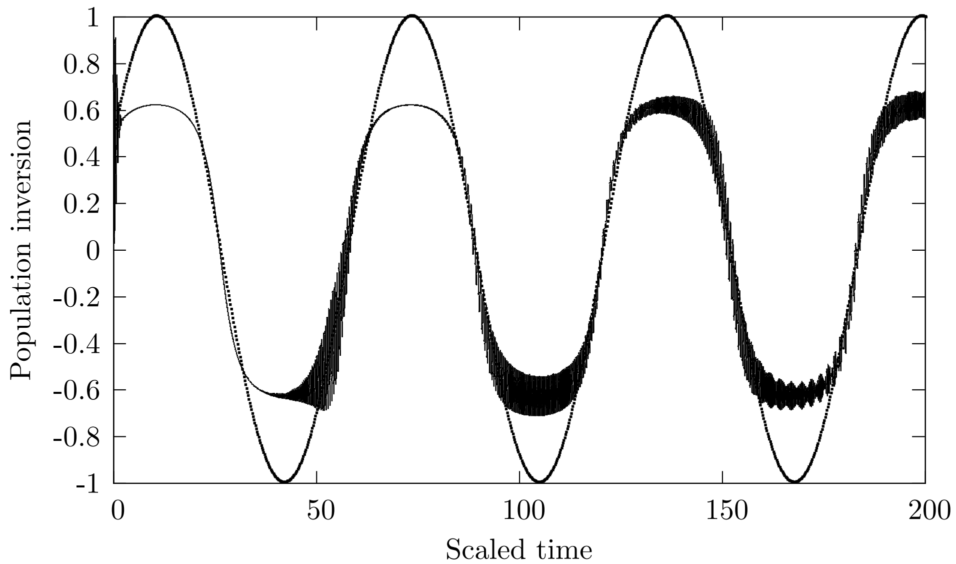
(b): $\phi = \pi/2$

Figure 3.9: Population inversion against scaled time. Initial coherent field with $\bar{n} = 25$. Field frequency has phase shifted sinusoidal fluctuations for $\alpha = 20g_0$, $\beta = 0.1g_0$. Dotted line shows the field frequency fluctuations.

3.4.3 Initial squeezed field



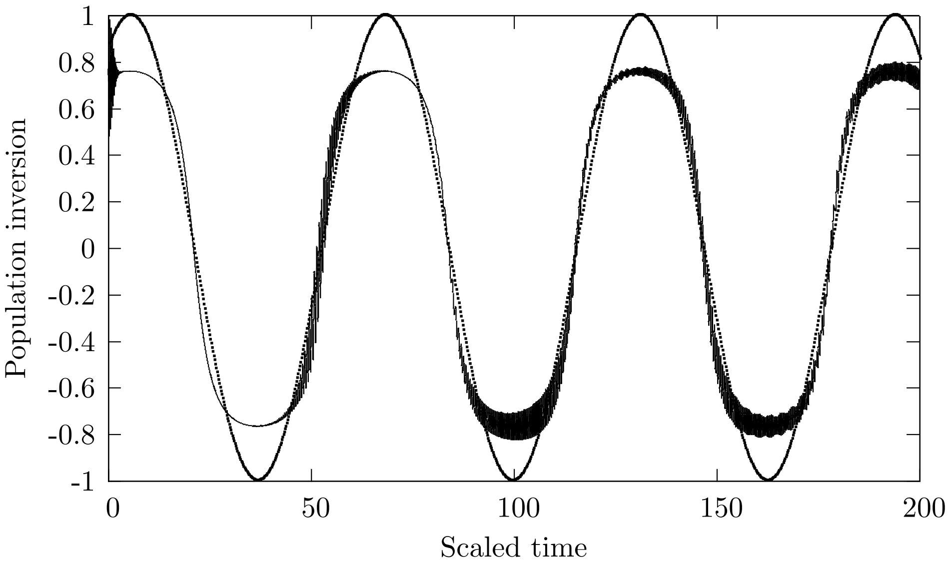
(a): $\phi = \pi/10$



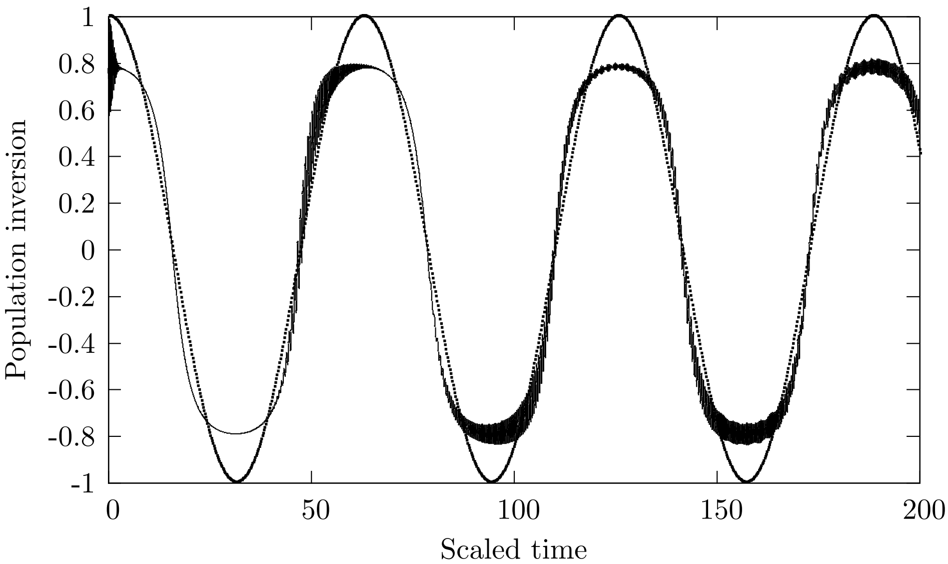
(b): $\phi = \pi/6$

Figure 3.10: Population inversion against scaled time. Initial squeezed field with $\bar{n} = 25$, $\theta = \pi$ and $r = 0.8$. Field frequency has phase shifted sinusoidal fluctuations for $\alpha = 20g_0$, $\beta = 0.1g_0$. Field frequency fluctuation is shown by the dotted lines.

As in the case of coherent field atom interaction discussed in the previous section, for the squeezed field also, the population inversion varies like a sinusoidal function with time where the collapses and revivals are insignificant. For the phase $\phi = \pi/2$ the population inversion oscillation is exactly similar to the population inversion variation that occurs in the Fock field atom interactions.



(a): $\phi = \pi/3$



(b): $\phi = \pi/2$

Figure 3.11: Population inversion against scaled time. Initial squeezed field with $\bar{n} = 25$, $\theta = \pi$ and $r = 0.8$. Field frequency has phase shifted sinusoidal fluctuations for $\alpha = 20g_0$, $\beta = 0.1g_0$. Dotted line shows the field frequency fluctuations.

3.5 Conclusion

We have studied the interaction of a two level atom and squeezed field with time varying frequency. When the quadrature squeezed field interacts with a two level atom, the population inversion oscillates in random with time for squeezing parameter $r > 0.5$ and $\theta = \pi$ without collapses and periodic revivals. However, by applying a sinusoidal variation in the frequency of the field, the randomness in population inversion is reduced and the collapses and periodic revivals are regained. Thus the field frequency modulation manipulates the population inversion in the case of squeezed light atom interaction. Also, the periodicity of revival depends on the amplitude of applied frequency modulation. By varying the periodicity of the applied frequency fluctuation the dynamics of population inversion with time can be manipulated. Two level atom field interaction has an important role in the field of quantum computation. Our results suggest a new method to control and manipulate the population of states in two level atom radiation interaction, which is very essential for quantum information processing.
We have also studied the variation of atomic population inversion with time, when a two level atom interacts with light field, where the light field has a sinusoidal frequency variation with a constant phase. In both coherent field and squeezed field cases, the population inversion variation is completely different from the phase zero frequency modulation case. Variation of phases from 0 to π have been considered. It is observed that in the presence of a non zero phase ϕ, the population inversion oscillates sinusoidally. Also the collapses and revivals gradually disappears when ϕ increases from 0 to $\pi/2$. When $\phi = \pi/2$ the evolution of population inversion is identical to the case when two level atom interacts with a Fock state. This is

the same behaviour of population inversion when two level atom and a frequency varying light interacts in Kerr medium discussed by Li Wang *et al*[70]. Thus, by applying a phase shifted frequency modulation one can induce sinusoidal oscillations of atomic inversion in linear medium, those normally observed in Kerr medium. We propose this as a method to control the atom field state probability amplitudes in an atom field system. In the field of quantum computation various quantum states can be used for the data storage and this method can be utilized for efficiently handling the data in quantum computation.

4

Interaction of two level atom and electromagnetic field in Kerr medium

4.1 Introduction

In this chapter we investigate the evolution of population inversion and von Neumann entropy in a system consisting of a two level atom in Kerr medium interacting with photon field, which is initially prepared in a quadrature squeezed state. The frequency of the field is set to be varying sinusoidally with time and also has a phase difference from the initial frequency. By following steps that used in the previous chapter we derived an analytical expressions for the probability amplitudes for constant frequency case. Again the time dependent field frequency case has to be investigated using the numerical techniques. Population inversion is examined for various nolinear strengths for both time dependent and time independent frequencies. The effects of frequency modulation on the evolution of the system for phase zero and phase non-zero cases have been analysed. It is observed that in Kerr medium also frequency fluctuation modifies the interaction between squeezed field and two level atom. The presence of phase factor in the frequency fluctuation enhances the modifications. The von Neumann entropy of the system which is a direct measure of the entanglement between the two subsystems; the atom and the field is also analysed for different damping and susceptibility. The effects of frequency fluctuation on the time evolution of von Neumann entropy and the corresponding entanglement is examined. It is also noticed that some interesting behaviour of the evolution of a two level atom in nonlinear medium can also be induced in linear medium itself by adding a phase factor in the field frequency modulations.

4.2 Model and Hamiltonian

Consider the system consists of a single two level atom interacting with single mode electromagnetic field in an infinite Q-cavity containing Kerr medium. Let ω_a be the frequency equivalent corresponding to the energy difference between the ground and exited states of the two level atom and ω_f is the single mode photon frequency. Kerr nonlinearity of the medium can be modelled by an anharmonic oscillator with frequency ω_k. Let $\hat{b}(\hat{b}^\dagger)$ be the annihilation(creation) operator corresponding to the medium and $\hat{a}(\hat{a}^\dagger)$ be that of the photon. The Hamiltonian of the system is derived using the Jayness Cumming model where the rotating wave approximation is applied. The total Hamiltonian of the system can be written as($\hbar = 1$):

$$\begin{aligned} \hat{H} &= \omega_f \hat{a}^\dagger \hat{a} + \omega \hat{\sigma}_z + g\left(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right) + \omega_k \hat{b}^\dagger \hat{b} + q\hat{b^\dagger}^2 \hat{b}^2 \\ &\quad + p(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) \end{aligned} \tag{4.1}$$

Here $\hat{\sigma}$'s are the atomic operators satisfying $[\hat{\sigma}_+, \hat{\sigma}_-] = i\hat{\sigma}_z$, q is the anharmonicity parameter, p is the field-medium coupling strength and g is the atom field coupling strength. If the response time of the nonlinear medium is so short that the medium follows the field in an adiabatic manner, the total Hamiltonian can be transformed to an effective Hamiltonian involving only the photon and the atomic operators. In the adiabatic limit the field frequency and anharmonic frequency are assumed to be very far from each other(i.e., $\omega_k \ll \omega_f$). In such cases one can introduce the third order susceptibility factor in the Hamiltonian[71, 72] and it can be rewritten as,

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_z + g\left(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right) + \chi \hat{a}^{\dagger^2} \hat{a}^2 \qquad (4.2)$$

where the new frequency ω and the coupling constant χ are related to q and p by, $\chi = qp^4/\delta^4$, $\omega = \omega_f - p^2/\delta$, $\delta = \omega_f - \omega_k$. The coupling constant χ is the dispersive part of the third-order nonlinearity of the Kerr-like medium.

We now consider the case where the field frequency depends on time, such that

$$\omega(t) = \omega_0 + \lambda \sin(\beta t + \phi), \qquad (4.3)$$

where the amplitude of fluctuation $\lambda \ll \omega_0$ and ϕ is the phase of the field frequency modulation. The modifications in the atom field coupling strength g is obtained from Eq. (3.6) of section 3.2 and is given by

$$g = g_0\left[1 + \lambda \sin(\beta t + \phi)\right]. \qquad (4.4)$$

The susceptibility factor χ also changes due to the frequency fluctuation and takes the form

$$\chi = \chi_0 + \epsilon\lambda \sin(\beta t + \phi), \qquad (4.5)$$

where $\epsilon << \chi_0$ such that the changes in χ due to the frequency fluctuation is always small. Now the total Hamiltonian of the system in the frequency fluctuating case becomes:

$$\begin{aligned}\hat{H} &= \left[\omega_0 + \lambda \sin(\beta t + \phi)\right] \hat{a}^\dagger \hat{a} + \omega \hat{\sigma}_z + \left[\chi_0 + \epsilon\lambda \sin(\beta t + \phi)\right] \hat{a}^{\dagger^2} \hat{a}^2 \\ &+ g_0\left(1 + \frac{\lambda \sin(\beta t + \phi)}{\nu_0}\right)\left(\hat{a}^\dagger \hat{\sigma}_- + \hat{a}\hat{\sigma}_+\right) \qquad (4.6)\end{aligned}$$

The general state of the atom field system at any time, t can be taken as

$$|\psi(t)\rangle = \sum_n C_n^e(t)|e\rangle|n\rangle + \sum_n C_{n+1}^g(t)|g\rangle|n+1\rangle. \tag{4.7}$$

As we have done in the preceding sections substitute Eq. (4.6) and Eq. (4.7) in the time dependent Schrödinger equation and obtain

$$\begin{aligned} \frac{d}{dt}\, C_n^e(t) &= -ig\sqrt{n+1}\left(1+\frac{\lambda\sin(\beta t+\phi)}{\omega_f}\right)C_{n+1}^g(t) \\ &\quad -i\left[n\omega+\frac{\omega_a}{2}+n(n-1)\chi_0\right]C_n^e(t) \\ &\quad -i\left[\lambda\sin(\beta t+\phi)\right]\left[1+\epsilon n(n-1)\right]C_n^e(t) \end{aligned} \tag{4.8}$$

$$\begin{aligned} \frac{d}{dt}\, C_{n+1}^g(t) &= -ig\sqrt{n+1}\left(1+\frac{\lambda\sin(\beta t+\phi)}{\omega_f}\right)C_n^e(t) \\ &\quad -i\left[n\omega-\frac{\omega_a}{2}+n(n+1)\chi_0\right]C_{n+1}^g(t) \\ &\quad -i\left[\lambda\sin(\beta t+\phi)\right]\left[1+\epsilon n(n+1)\right]C_{n+1}^g(t). \end{aligned} \tag{4.9}$$

By solving Eqs. (4.8) and (4.9) the time evolution of the system can be obtained. Variation of population inversion and entanglement entropy are examined from the obtained solutions.

4.3 Evolution of probabilities

If the field frequency does not have any time dependence i.e., $\lambda = 0$, we can solve the Eqs. (4.8) and (4.9) analytically by following the steps that we have used in the previous chapter. When $\lambda = 0$ Eqs.

(4.8) and (4.9) becomes

$$\frac{d}{dt}\, C_n^e(t) = -ig\sqrt{n+1}\, C_{n+1}^g(t) - i\,(n\omega + \omega_a/2)\, C_n^e(t) -i\; n(n-1)\chi_0\; C_n^e(t) \tag{4.10}$$

$$\frac{d}{dt}\, C_{n+1}^g(t) = -ig\sqrt{n+1}\, C_n^e(t) - i\,(n\omega - \omega_a/2)\, C_{n+1}^g(t) -i\; n(n+1)\chi_0\; C_{n+1}^g(t). \tag{4.11}$$

Now define $H_{e,n}$ and $H_{g,n+1}$ such that

$$C_{e,n}(t) = \exp -i\left[n\omega + \frac{1}{2}\omega_a + \chi_0 n(n+1)\right] H_{e,n}(t) \tag{4.12}$$

$$C_{g,n+1}(t) = \exp -i\left[(n+1)\omega - \frac{1}{2}\omega_a + \chi_0(n+1)n\right] H_{g,n+1}(t). \tag{4.13}$$

It is useful to note that

$$|C_{e,n}(t)|^2 = |H_{e,n}(t)|^2, \tag{4.14}$$

$$|C_{g,n+1}(t)|^2 = |H_{g,n+1}(t)|^2. \tag{4.15}$$

Substituting Eqs. (4.12) and (4.13) in Eqs. (4.10) and (4.11) we get the following equations

$$i\frac{d}{dt}H_{e,n} = \lambda\sqrt{n+1}\; H_{g,n+1}(t)\; e^{-ik_n t}, \tag{4.16}$$

$$i\frac{d}{dt}H_{g,n+1} = \lambda\sqrt{n+1}\; H_{e,n}(t)\; e^{+ik_n t}, \tag{4.17}$$

where

$$k_n = \omega - \omega_0 + 2\chi n \tag{4.18}$$

It is assumed that initially the atom is in excited state, i.e.,

$$C_{n+1}^g(0) = H_{n+1}^g(0) = 0 \tag{4.19}$$

Using this initial condition Eq. (4.19), Eqs. (4.16) and (4.17) are solved and the solutions are

$$H_{g,n+1}(t) = \frac{\lambda\sqrt{(n+1)}}{\Omega_n} e^{(ik_n t/2)}\ 2i\ \sin(\Omega_n t/2)\,, \tag{4.20}$$

$$H_{e,n}(t) = \frac{1}{2\Omega_n} e^{(-ik_n t/2)}\left\{ik \sin(\Omega_n t/2) + \Omega_n \cos(\Omega_n t/2)\right\}, \tag{4.21}$$

with $\Omega_n^2 = k_n^2 + 4g^2(n+1)$. Using Eqs. (4.12) and (4.13) in Eqs. (4.20) and (4.21) we get

$$\begin{aligned} C_{e,n}(t) &= (1/\Omega_n)\left[ik_n \sin(\Omega_n t/2) + \Omega_n \cos(\Omega_n t/2)\right] \\ &\quad \times\ \exp\left\{\left[-i\left(n\omega + \omega_a/2 + \chi_0 n(n+1) + k_n t/2\right)\right]\right\} \end{aligned} \tag{4.22}$$

$$\begin{aligned} C_{g,n+1}(t) &= \left(2\lambda\sqrt{(n+1)}/\Omega_n\right) 2i \sin(\Omega_n t/2) \\ &\quad \times\ \exp\left\{-i\left[(n+1)\omega - \omega_a/2 + \chi_0(n+1)n - k_n t/2\right]\right\} \end{aligned} \tag{4.23}$$

From Eqs. (4.22) and (4.23) the population inversion of the system is

$$\begin{aligned} W(t) = \quad & -\frac{1}{\Omega_n^2}\left[-k_n^2 \sin^2(\Omega_n t/2) + \Omega_n^2 \cos^2(\Omega_n t/2) - \right. \\ & \left. k_n \Omega_n \sin(\Omega_n t/2)\cos(\Omega_n t/2)\right] - \\ & \frac{\lambda^2(n+1)}{\Omega_n^2} \sin^2(\Omega_n t/2) \end{aligned} \tag{4.24}$$

In the case of time dependent field frequencies we use numerical techniques to solve the Eqs. (4.8) and (4.9) to find the population inversion.

4.4 Entanglement entropy

In this section we look at the quantum entropy properties of the atom field system. It is well known that the interaction between a two level atom and photon field leads to entanglement between the two sub systems, the atom and the field. Many methods are there to study entanglement dynamics between the atom and field and we calculated the von Neumann entropy which gives a direct measure of the entanglement. The reduced density operator of the atom(ρ_a) in the bare basis is obtained by taking the partial trace over all the field states and is given by,

$$\rho_a(t) = \begin{bmatrix} \sum_{n=0}^{\infty} C_n^e(t) C_n^{e*}(t) & \sum_{n=0}^{\infty} C_n^e(t) C_{n+1}^{g\ *}(t) \\ \\ \sum_{n=0}^{\infty} C_{n+1}^g(t) C_n^{e*}(t) & \sum_0^{\infty} C_{n+1}^g(t) C_{n+1}^{g\ *}(t) \end{bmatrix}. \quad (4.25)$$

It can be shown that the reduced density operator for field(ρ_f) is also the same. The components of Bloch vectors are written in terms of the elements in the reduced density matrix as,

$$\begin{aligned} s_1(t) &= \sum_{n=0}^{\infty} \left[C_n^e(t) C_{n+1}^{g\ *}(t) + C_{n+1}^g(t) C_n^{e*}(t) \right] \\ s_2(t) &= \sum_{n=0}^{\infty} \left[C_n^e(t) C_{n+1}^{g\ *}(t) - C_{n+1}^g(t) C_n^{e*}(t) \right] \\ s_3(t) &= \sum_{n=0}^{\infty} \left[|C_n^e(t)|^2 - |C_{n+1}^g(t)|^2 \right] \end{aligned} \quad (4.26)$$

and the von Neumann entropy, $S(\rho_a)$, is

$$S(\rho_a) = -g_1(t) \ln \{g_1(t)\} - g_2 \ln \{g_2(t)\}, \quad (4.27)$$

where

$$\begin{aligned} g_1(t) &= 1 + \sqrt{|s_1(t)|^2 + |s_2(t)|^2 + |s_3(t)|^2}, \\ g_2(t) &= 1 - \sqrt{|s_1(t)|^2 + |s_2(t)|^2 + |s_3(t)|^2}. \end{aligned} \quad (4.28)$$

The effects of frequency modulation on the entropy evolution in a system of two level atom and field in a linear medium has been already studied by Jia Fei *et al.*[73]. According to their results, in the case of interaction between two level atom and coherent field, when the decay coefficient is small, the system is in the entangled state all the time except the initial time; when the decay coefficient increases, the entanglement between the atom and the field decays to zero as the time increases. Here we focus on the entropy evolution of the system in Kerr medium and also the role of phase shifted frequency fluctuation on the entropy evolution. The initial photon field is chosen to be quadrature squeezed.

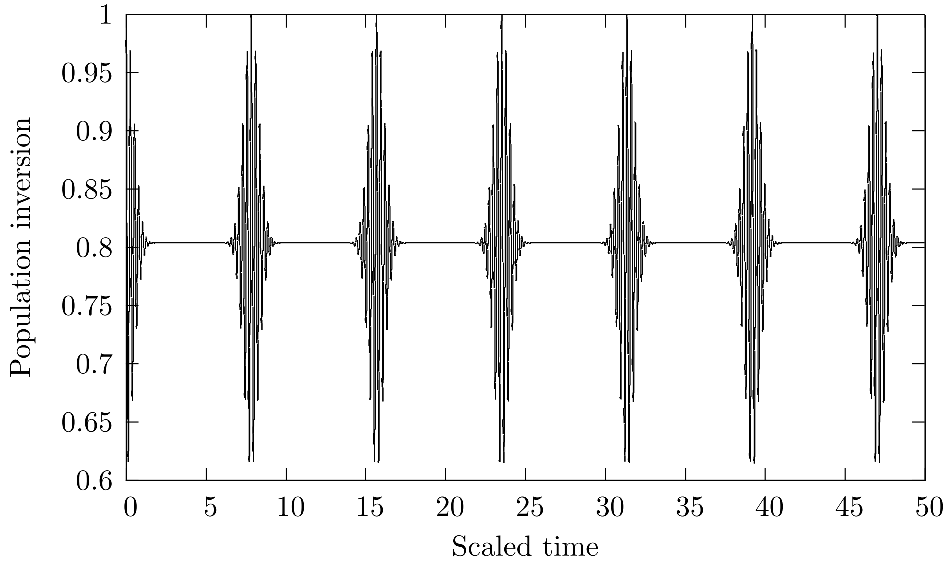
4.5 Evolution of population inversion

4.5.1 No field frequency fluctuations

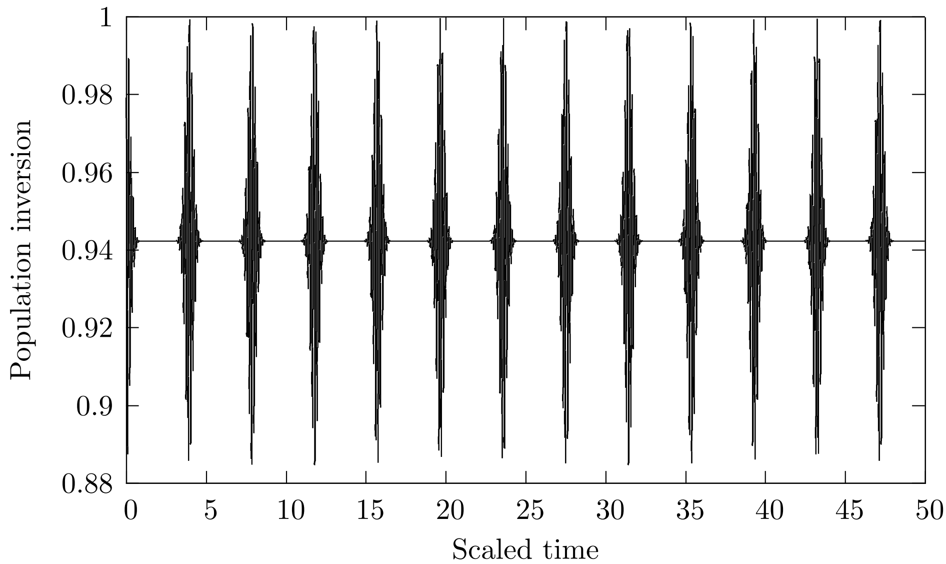
We have the analytical solutions for this case given by Eqs. (4.22) and (4.23) and using it, the analytical expression for the population inversion is obtained in Eq. (4.24). In the Kerr medium population inversion oscillates with periodic collapses and revivals as shown in Fig. 4.1, similar to the case of linear medium. But the population inversion never collapses to zero, rather it settles to a positive value during the collapse region, which indicates that the probability of atom being in the initial excited state is large comparing with that of the atom being in ground state. When the value of nonlinear susceptibility χ_0 is high, the probability of the atom being in the excited state itself is very near to 1 and the ground state probability is close to zero. One can say that the high nonlinearity of the medium suppresses the probability of deexcitation of the two level atom. It is also noted that the number of collapses and revivals in the population inversion during a particular time interval increases with the nonlinearity(χ_0) of the medium.

4.5.2 With field frequency fluctuations

For frequency fluctuating cases we use numerical techniques to solve the time evolution equation given in Eqs. (4.8) and (4.9). In our discussion the parameters of frequency variation such as amplitude and frequency are chosen to be $\lambda = 30g_0$ and $\beta = 1g_0$. In the phase shifted case we take $\phi = \pi/2$ for which the effects are maximum, which we have already seen in the previous chapter.

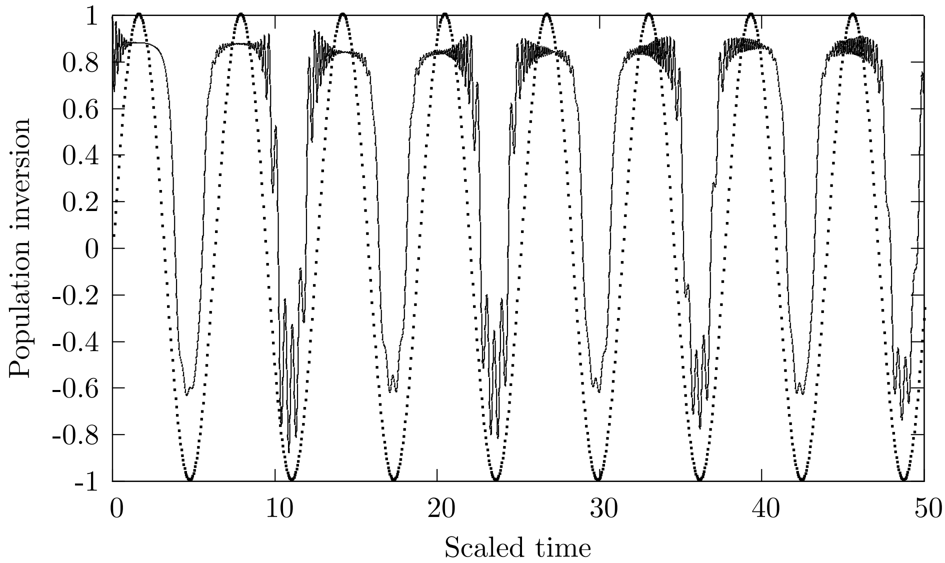


(a): Susceptibility $\chi_0 = 0.4g_0$

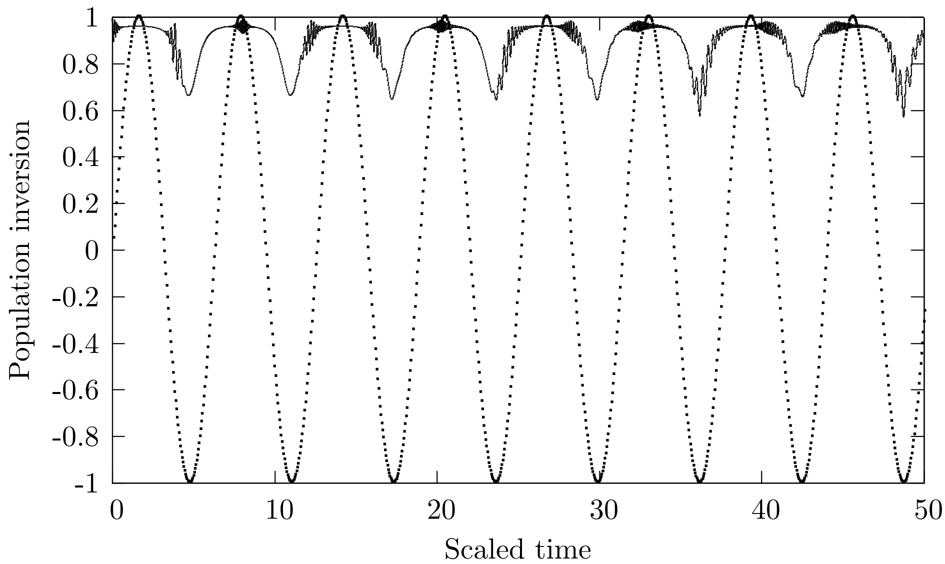


(b): Susceptibility $\chi_0 = 0.8g_0$

Figure 4.1: Population inversion versus time for constant field frequency. The initial field is squeezed with parameters $\bar{n} = 25$, $r = 0.8$, $\theta = \pi$. Damping $\gamma = 0$.

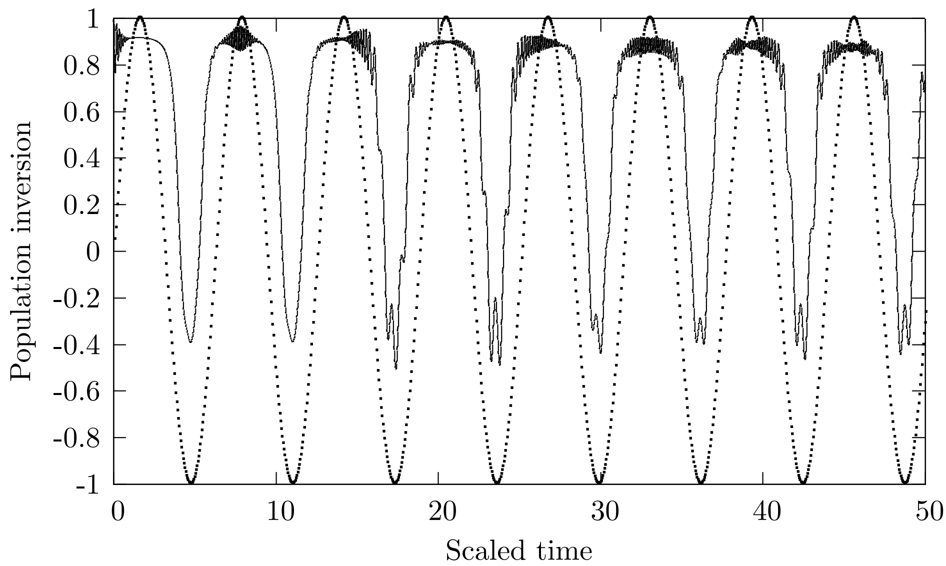


(a): Susceptibility $\chi_0 = 0.4g_0$

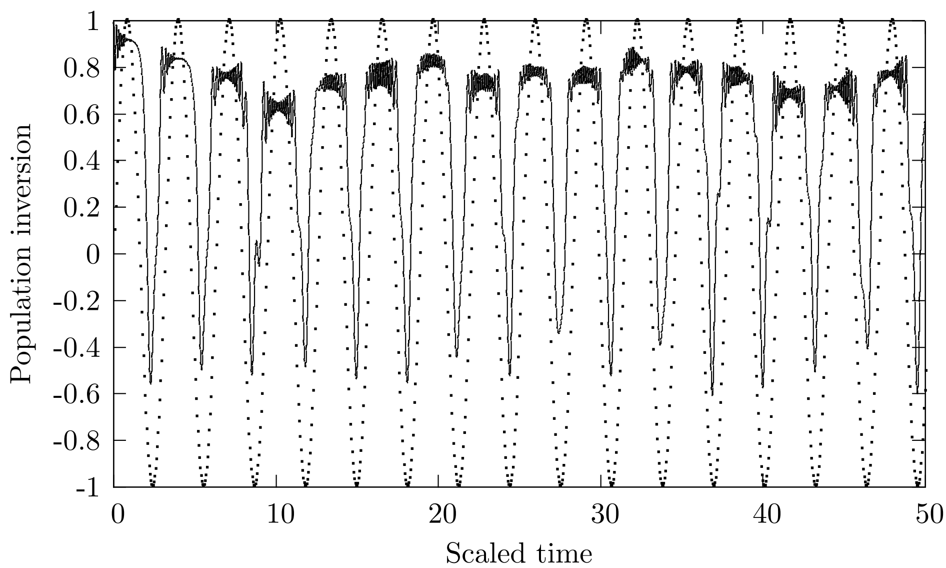


(b): Susceptibility $\chi_0 = 0.8g_0$

Figure 4.2: Population inversion versus time for fluctuating field frequency with $c = 30g_0$ and $\beta = 1g_0$. Squeezed field parameters are $\bar{n} = 25$, $r = 0.8$, Damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.



(a): $c = 30g_0$, $\beta = g_0$



(b): $c = 30g_0$, $\beta = 2g_0$

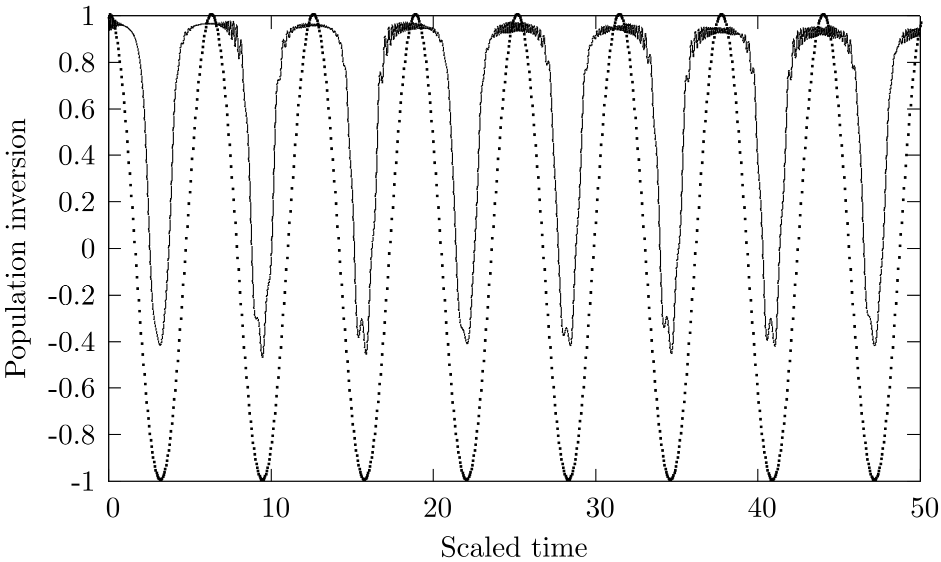
Figure 4.3: Population inversion versus time for fluctuating field frequency. Initial squeezed field parameters are $\bar{n} = 25$, $r = 0.8$, $\theta = 0$. Susceptibility $\chi_0 = 0.5g_0$ and damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.

Case:1 Phase $\phi = 0$.

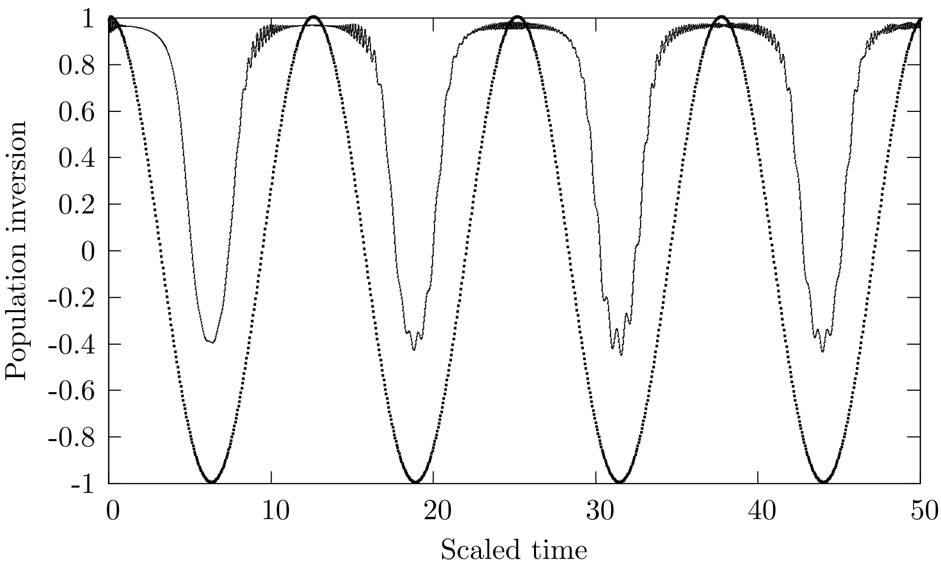
As in the case of coherent field atom interaction in Kerr medium, discussed by L Wang *et al.*[70], for the case of initial squeezed field also, collapses and revivals in population inversion are unclear. Population inversion or the probabilities correspond to the atom in excited and ground states executes oscillations with time and these oscillations are very close to a sinusoidal wave, which is shown in Figs. 4.2 and 4.3. The period of oscillation of the population inversion is same as that of the field frequency modulation.

Case:2 Phase $\phi \neq 0$.

When there is a phase factor in the frequency modulation the oscillations in population inversion are more sinusoidal compared to the zero phase case, which is clear from the Fig. 4.4. The oscillation in population inversion starts earlier than the zero phase case, depending on ϕ. In the figures shown we set $\phi = \pi/2$, for which the effect of phase factor on the evolution is most visible. The same behaviour of the population inversion is observed in linear medium also when there is a phase factor in the frequency modulation. In the Fig. 4.5, where the population inversion is plotted for linear medium i.e., $\chi_0 = 0$ with $\phi = \pi/2$, the evolution of population inversion is very similar to the behaviour of population inversion when a two level atom interacts with frequency modulated field in Kerr medium plotted in Fig. 4.4. The dynamics of a two level atom-photon system in nonlinear medium is induced in a linear medium due to the phase factor in the field frequency fluctuations. It is suggested that the effect of nonlinearity in the medium in a two level atom-time varying frequency field interaction can be induced in linear medium also by introducing a phase factor in the frequency modulation.

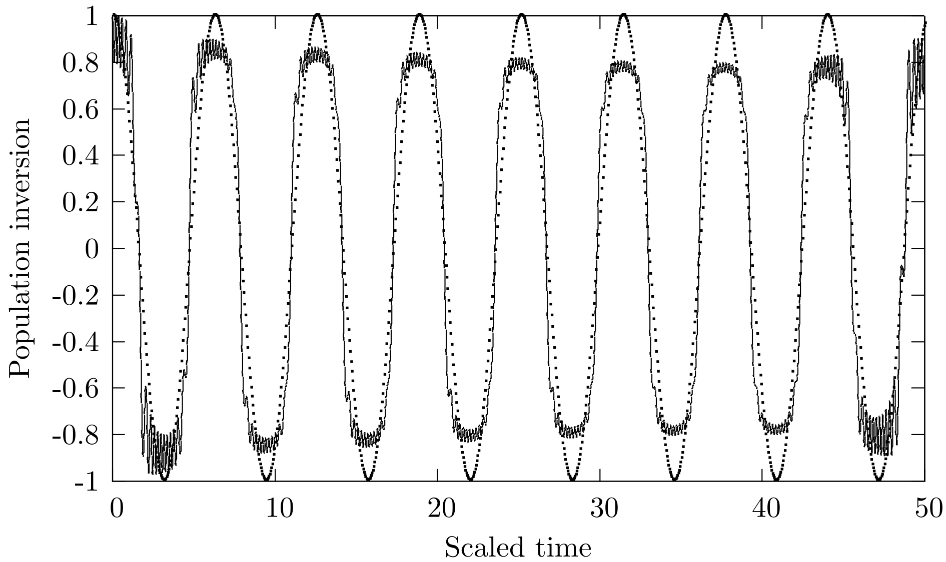


(a): $\lambda = 30g_0$, $\phi = \pi/2$, $\beta = g_0$

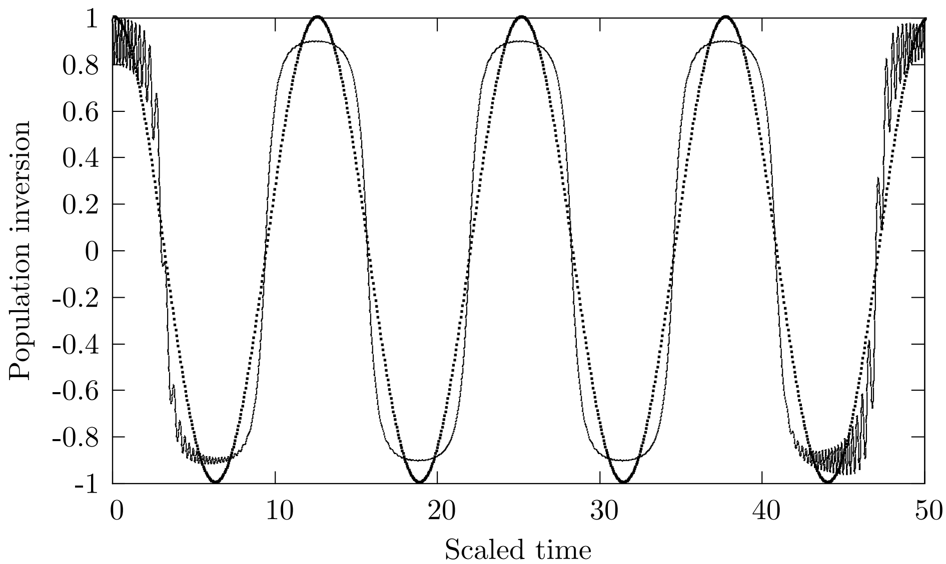


(b): $\lambda = 30g_0$, $\phi = \pi/2$, $\beta = 0.5g_0$

Figure 4.4: Population inversion versus time where the field has a phase shifted frequency fluctuation. Initial squeezed field parameters are $\bar{n} = 25$, $r = 0.8$ and $\theta = 0$. Susceptibility $\chi_0 = 0.5g_0$ and damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.



(a): $\beta = g_0$



(b): $\beta = 0.5g_0$

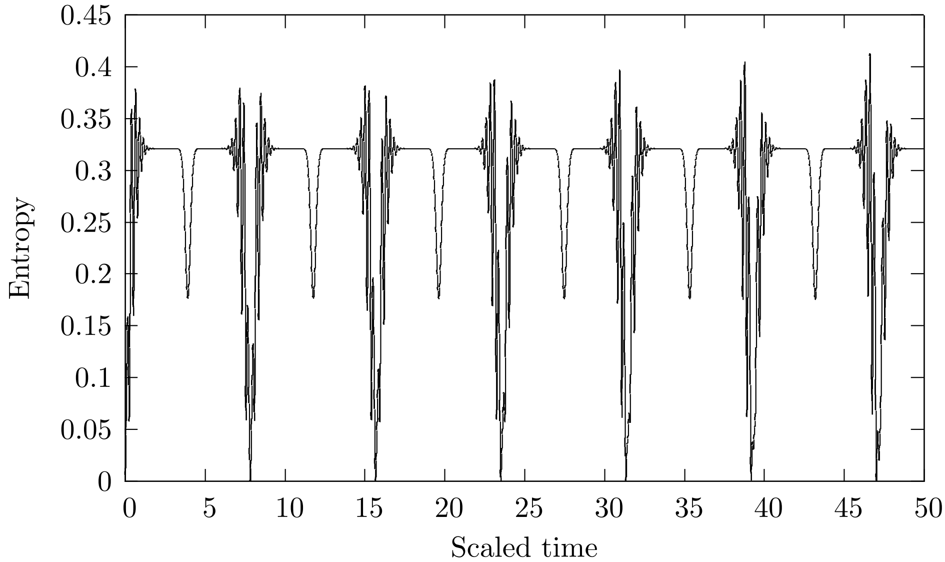
Figure 4.5: Population inversion versus time in linear medium where field has a phase shifted frequency fluctuation. Initial squeezed field parameters are $r = 0.8$, $\theta = \pi$, $\bar{n} = 25$, $\chi_0 = 0$, $\gamma = 0$ and $\phi = \pi/2$. Dotted curve shows the field frequency fluctuation.

4.6 Evolution of entanglement entropy

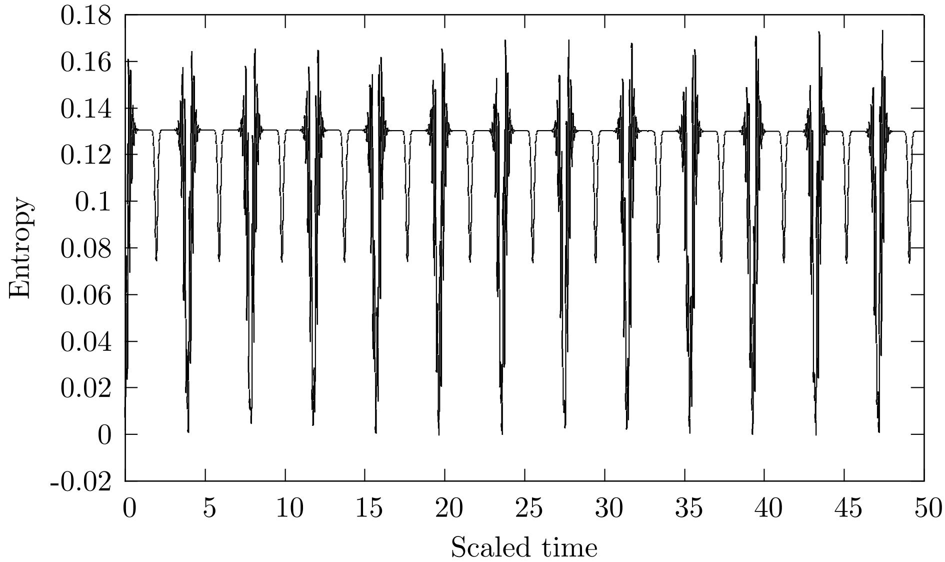
The von Neumann entropy of the system can be calculated by using the atom field evolution coefficients in Eqs. (4.25), (4.26) and (4.27). The evolution of entropy is plotted in Figs. 4.6 - 4.9, by varying various parameters.

Case: 1 No frequency fluctuation

Evolution of entropy for no field frequency fluctuation($\lambda = 0$) is shown in Fig. 4.6.



(a): Susceptibility $\chi = 0.4g_0$



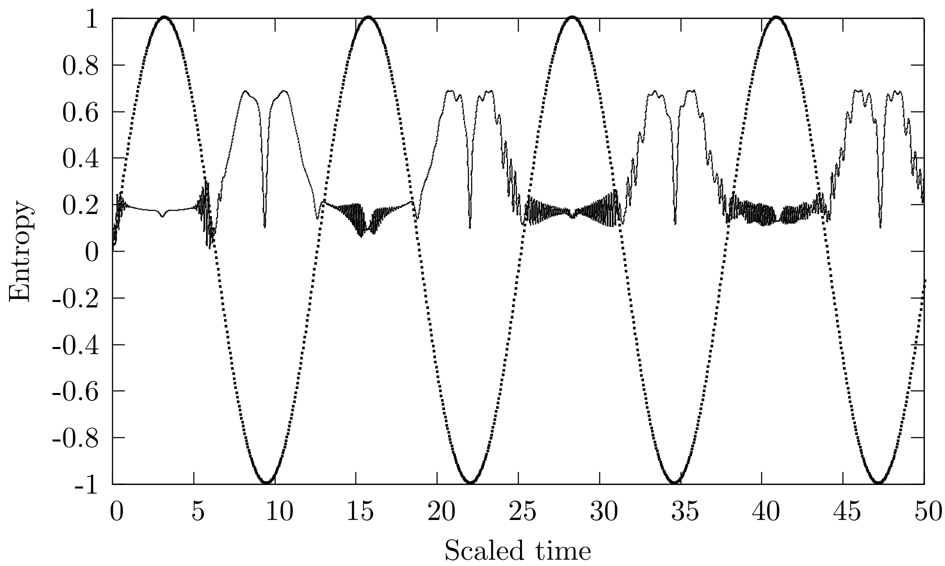
(b): Susceptibility $\chi = 0.8g_0$

Figure 4.6: Evolution of entropy with time in Kerr medium with initial squeezed field with $\bar{n} = 25$, $r = 0.8$ and $\theta = 0$.

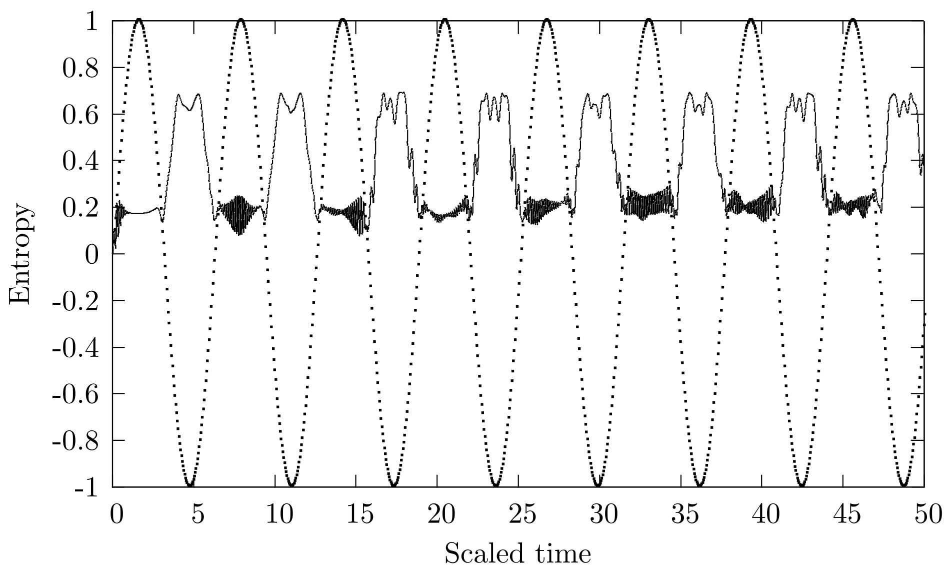
It is clear that the entropy remains a constant value with frequent periodic drops to some minimum value, where few of the minimum points are very close to zero. Interesting thing is that these drops occurs at the same time when revivals occurs in the population inversion. The number of falls to minimum increases with increase in the susceptibility. Thus the average entropy is small for higher susceptibility values. We can infer that during each revival period the atom and field becomes minimally entangled and the nonlinearity reduces the entanglement between atom and field. If the damping is present, entropy increases to a maximum and then it decreases and damp out to zero.

Case: 2 With frequency fluctuation

Now consider the effect of frequency fluctuations with a zero phase, shown in Figs. 4.7 and 4.8. Here the entropy oscillates in the shape of a quasi sine wave, which is in sync with the applied field frequency fluctuation. We can say that the frequency fluctuation make the entropy more ordered and controllable. The additional phase factor does not make any noticeable modifications in the entropy evolution. As seen from Fig. 4.9, where the entropy evolution is plotted for linear medium with a phase shifted($\phi = \pi/2$) field frequency, quasi sinusoidal oscillation in entropy is present in this case also. Thus for the entropy evolution also the effects of non linearity can be induced by the phase shifted frequency modulation in linear medium itself. One can understand this by comparing Fig. 4.9 with Fig.4.7.

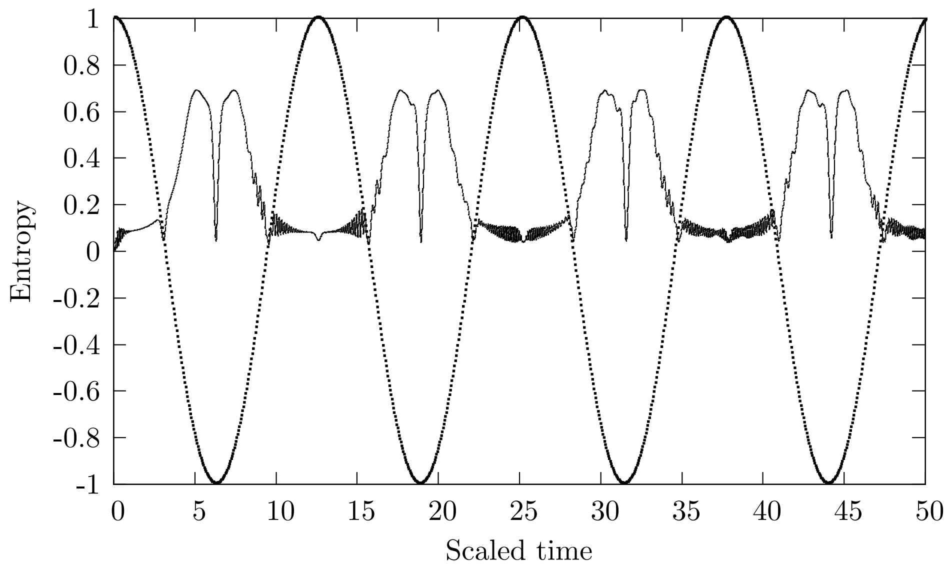


(a): $\chi_0 = 0.5g_0$, $\beta = 0.5g_0$

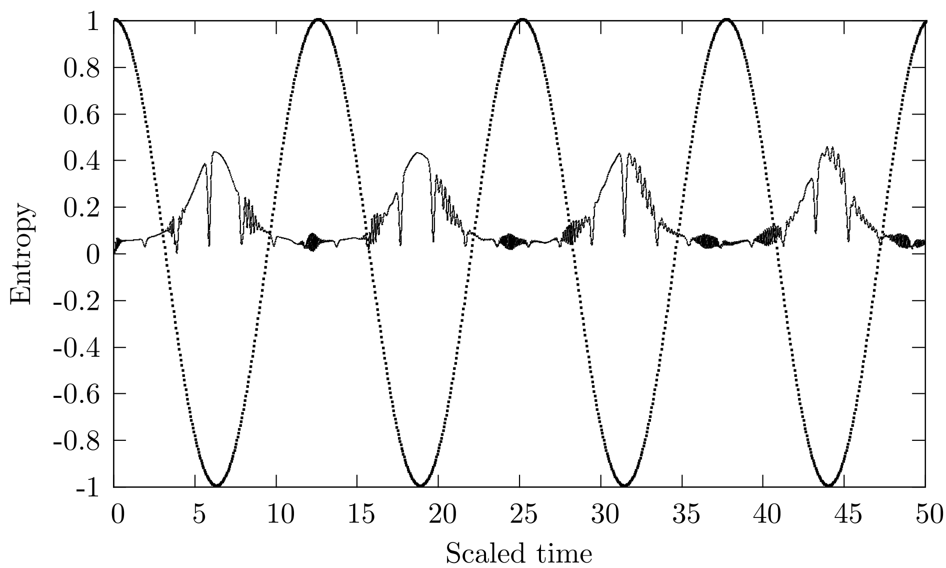


(b): $\chi_0 = 0.5g_0$, $\beta = g_0$

Figure 4.7: Evolution of entropy with time in Kerr medium for initial squeezed field with $\bar{n} = 25$, $r = 0.8$ and $\theta = 0$. Field frequency is sinusoidally fluctuating. Damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.

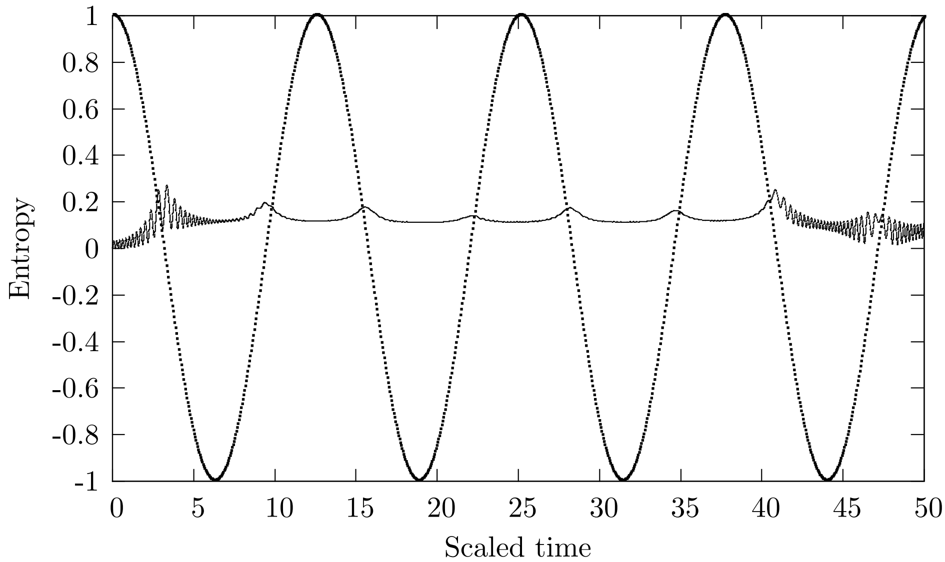


(a): $\chi_0 = 0.5g_0$

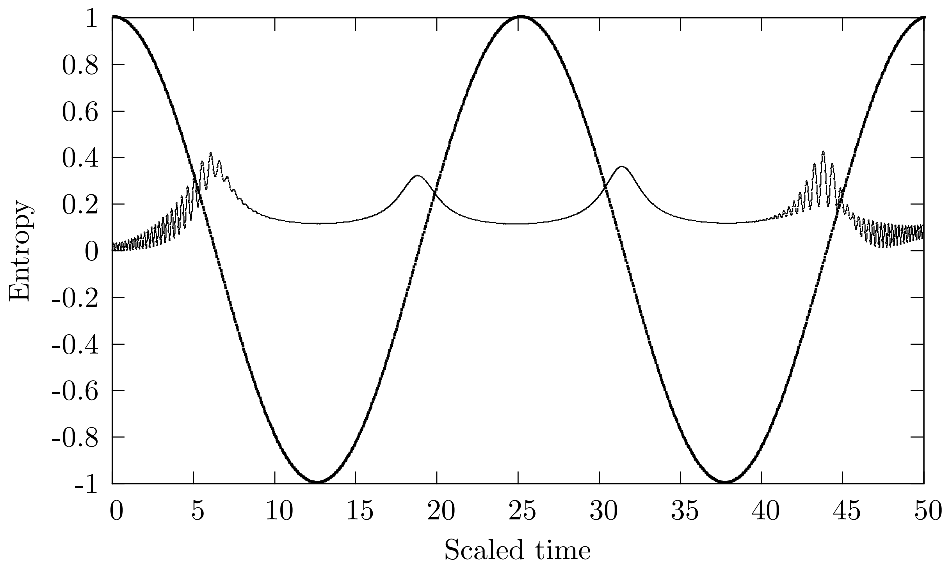


(b): $\chi_0 = 0.8g_0$

Figure 4.8: Evolution of entropy in Kerr medium with phase shifted field frequency variation. Initial squeezed field parameters are $\bar{n}$, $r = 0.8$ and $\theta = \pi$. Damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.



(a): $\beta = 0.5g_0$



(b): $\beta = 0.25g_0$

Figure 4.9: Evolution of entropy in linear medium with phase shifted frequency fluctuation. Initial squeezed field parameters are $r = 0.8$, $\theta = \pi$, $\bar{n} = 25$. Susceptibility $\chi_0 = 0$, damping $\gamma = 0$ and phase $\phi = \pi/2$. Dotted curve shows the field frequency fluctuation.

4.7 Conclusion

In this work we have considered the system consisting of a two level atom in Kerr medium interacting with quadrature squeezed photon field in the adiabatic limit. The frequency of the field is set to be fluctuating and phase shifted. Evolution of population inversion and entanglement entropy of the system is analysed by varying parameters. It is observed that in the nonlinear medium also sinusoidal frequency fluctuation modifies the time evolution of population inversion. These modifications are enhanced in the presence of a phase in the frequency fluctuation. The entanglement entropy of the system also has a close dependence on the field frequency fluctuations. It becomes more ordered and controllable when the frequency is sinusoidally fluctuating. We noticed that the entanglement between the atom and field can be controlled by varying the period of the field frequency fluctuations. As we have discussed in the previous chapter, many interesting behaviour in the evolution of a two level atom in Kerr medium can also be produced in linear medium by including phase factor in the frequency modulation. State evolution during the interaction of an isolated atom-photon system is an interesting area of research nowadays, which contribute much to the developments of controllable quantum computation and QIP. The understanding of the entanglement between the atom and field can be applied in generating optimal methods for the qubit operations. Thus the results produced in this work may help to progress the research in QIP and speed up the realisation of a quantum computer.

5

Dynamics of coupled cavity system

5.1 Quantum state transfer in a coupled cavity system

5.1.1 Introduction

Two level atom and electromagnetic field interaction is an area which has been studied extensively in the field of quantum optics[74–79] and it has attained considerable attention for its applications in quantum information processes. Atom-field states and its interaction dynamics are possible candidates for data storage and processing in quantum informatics and following to this many works have been reported about the various possibilities of atom field interaction dynamics suitable for applications in quantum information processes[52, 54, 80–82]. In the current setting, information transmission is one of the main challenges facing in the realization of quantum computers. Many systems have been proposed in this area for effectively transmitting and manipulating the data. For example in the case of short length quantum communication, linear spin chain channels are introduced [56, 57]. Another concept, which can serve the purpose of information transmission from one part to other is a coupled cavity array, which can be modelled as an effective and controllable many body system[58], has been the subject of discussion in many recent studies[59, 60]. The solid state analogue of coupled cavity systems are realized by coupling superconducting qubits to stripline resonators[62, 63, 83]. There are many theoretical and experimental studies devoted to this area. This includes the developments of nanocavities in photonic crystals and the Josephson junction arrays, which is found to be useful in many significant applications in quantum information processing. Remarkable development also has been accomplished recently by considering cold atoms trapped in optical lattices which can effectively be described by a Bose Hubbard Hamiltonian, unveils its potential applications as quantum optical simulators.

Quantum state transfer between cavities[84–88] as well as the dissipation properties and emission characteristics of coupled cavity arrays[89, 90] corresponding to different experimental situations are the main subjects of study in recent publications. This chapter concentrate on the study of quantum state transfer in a coupled cavity system with two level atom inside each cavity. Jaynes Cummings model (JCM) with rotating wave approximation is used to account for the atom field interaction Hamiltonian. Cavity-cavity tunnelling is modelled by photon hopping between the neighbouring cavities where the coupling is due to the overlap of the fading fields in the region between the two cavities[91]. The evolution of such a system is studied analytically and the dynamics of the probability amplitudes of each possible states are analyzed. The model and Hamiltonian of the system are discussed in the next section and the evolution of the system in single excitation subspace is addressed in the subsequent sections.

5.1.2 Model and Hamiltonian

Consider the coupled cavity system in which two cavities are coupled together and both of them contains a two level atom. Each of the cavity fields interact with the two level atoms inside and this interaction Hamiltonian is described by JCM. The cavity-cavity coupling is such that photon can hop between them. A schematic diagram of the coupled cavity system is shown in Fig. 5.1.

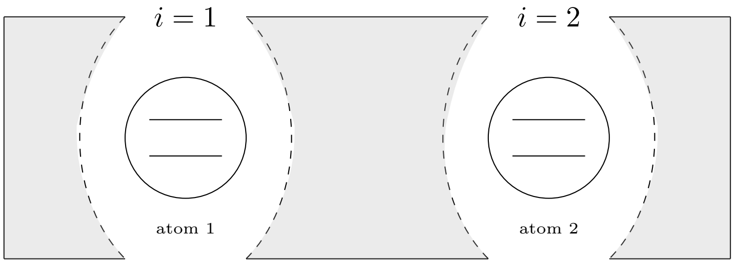


Figure 5.1: Schematic diagram of a coupled cavity array where two cavities are coupled and each of the cavities contains a two level atom inside.

Hamiltonian of the individual field and atom in ith cavity are respectively given by

$$\hat{H}_{Ci} = \hbar\omega_{fi}\hat{a}_i^\dagger\hat{a}_i \tag{5.1}$$

and

$$\hat{H}_{ai} = \frac{\hbar\omega_{ai}}{2}\hat{\sigma}_{iz}. \tag{5.2}$$

Here $\hat{a}_i^\dagger(\hat{a}_i)$ is the photon creation(annihilation) operator for the field in the ith cavity with $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}\hat{I}$. The raising(lowering) operator for the atom in ith cavity is $\hat{\sigma}_{i+}(\hat{\sigma}_{i-})$, where $[\hat{\sigma}_{i+}, \hat{\sigma}_{i-}] = \hat{\sigma}_{iz}$ and field and atomic frequencies are ω_{fi} and ω_{ai} respectively. According to the JCM, employing RWA, the interaction Hamiltonian between atom and field in ith cavity is

$$\hat{H}_{aCi} = \hbar g_i\left(\hat{a}_i\hat{\sigma}_{i+} + \hat{a}_i^\dagger\hat{\sigma}_{i-}\right). \tag{5.3}$$

where g_i is the atom field coupling strength inside the ith cavity. The cavity-cavity coupling Hamiltonian can be deduced phenomenologically, conceived from the work of Zoubi *et al.*[91]. The single mode vector potential operator of cavity $i = 1$ is given by

$$\vec{A}_1(r) = \vec{u}_1(\boldsymbol{r} - \boldsymbol{r}_1)\hat{a}_1 + \vec{u}_1^*(\boldsymbol{r} - \boldsymbol{r}_1)\hat{a}_1^\dagger, \tag{5.4}$$

where $u_1(\boldsymbol{r} - \boldsymbol{r}_1)$ is the vector field distribution of the mode, $\boldsymbol{r}$ is an arbitrary position vector in the cavity and $\boldsymbol{r}_1$ is a reference position within the cavity C, say its geometric center. For a single mode we can take $u(\boldsymbol{r})$ to be real. Similarly, we write for cavity 2

$$\vec{A}_2(r) = \vec{u}_2(\boldsymbol{r} - \boldsymbol{r}_2)\hat{a}_2 + \vec{u}_2^*(\boldsymbol{r} - \boldsymbol{r}_2)\hat{a}_2^\dagger \tag{5.5}$$

Now we take the cases where the coupling is not too strong. For example, dielectric cavities are well separated in space. In these framework the coupling between the two cavities can taken to be proportional to the overlap integral between the fields of the two modes. For such a system the cavity cavity coupling can be written as

$$\hat{H}_{CC} = A\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1 + \hat{a}_1\hat{a}_2 + \hat{a}_1^\dagger\hat{a}_2^\dagger\right) \tag{5.6}$$

where the coupling parameter A, with dimensions of frequency, is proportional to the overlap integral

$$A \propto \int d\boldsymbol{r}\boldsymbol{u}_1(\boldsymbol{r} - \boldsymbol{r}_1).\boldsymbol{u}_2(\boldsymbol{r} - \boldsymbol{r}_2). \tag{5.7}$$

It depends, among other things, on the distance between the cavities. The last two terms in Eq. (5.6) can be precluded by applying RWA and the cavity-cavity coupling Hamiltonian now reduces to

$$\hat{H}_{CC} = A\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right). \tag{5.8}$$

The total Hamiltonian of the coupled cavity system is obtained by adding all the individual Hamiltonians in Eqs. (5.1)-(5.3) and (5.8). i.e.,

$$\hat{H} = \sum_{i=1,2} (H_{Ci} + H_{ai} + H_{aCi}) + H_{CC}, \tag{5.9}$$

taking $\hbar = 1$ we can write it as

$$\begin{aligned} \hat{H} &= \sum_{i=1,2} \left[\omega_i \hat{a}_i^\dagger \hat{a}_i + \frac{\omega_{ai}}{2} \left(|e\rangle_i \langle e| - |g\rangle_i \langle g| \right) + g(\hat{a}_i \hat{\sigma}_{i+} + \hat{a}^\dagger \hat{\sigma}_{i-}) \right] \\ &\quad + A \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \right). \end{aligned} \tag{5.10}$$

When $A = 0$, the Hamiltonian in Eq. (5.10) reduces to the Jaynes Cummings Hamiltonian of two uncoupled two level atom field system. The general state of the system is a superposition of product states of all possible field and atomic states in each cavity, which can be written as

$$\begin{aligned} |\psi(t)\rangle &= \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \Big[C_{l_1,l_2}^{g,g}(t)\ |l_1\rangle\ |l_2\rangle\ |g_1\rangle\ |g_2\rangle \\ &\quad + C_{l_1,l_2}^{g,e}(t)\ |l_1\rangle\ |l_2\rangle\ |g_1\rangle\ |e_2\rangle + \qquad (5.11) \\ &\quad C_{l_1,l_2}^{e,g}(t)\ |l_1\rangle\ |l_2\rangle\ |e_1\rangle\ |g_2\rangle + C_{l_1,l_2}^{e,e}(t)\ |l_1\rangle\ |l_2\rangle\ |e_1\rangle\ |e_2\rangle \Big] \qquad (5.12) \end{aligned}$$

where $|g_i\rangle$ and $|e_i\rangle$ are the possible atomic states in ith cavity. The field states inside ith cavity is $|l_i\rangle$. Since we have obtained the Hamiltonian in Eq. (5.10) and general state of the system in Eq. (5.11), these can be substituted in the time dependent Schrödinger equation to examine the dynamics of the coupled cavity system.

5.1.3 Evolution of the system in single excitation subspace

Single excitation subspace is a space where the total number of excited state in the system is restricted to 1. That is to say either any of the atom is in the excited state($|e\rangle$) or any of the cavity field is in one photon state $|1\rangle$. The other atom must be in ground state($|g\rangle$) and field must be vacuum state($|0\rangle$). Thus the general state of the coupled cavity system in a single excitation subspace is

$$\begin{aligned} |\psi(t)\rangle &= C_{00}^{eg}(t)\ |e\rangle_1|g\rangle_2|0\rangle_1|0\rangle_2 + C_{00}^{ge}(t)\ |g\rangle_1|e\rangle_2|0\rangle_1|0\rangle_2 + \\ & C_{10}^{gg}(t)\ |g\rangle_1|g\rangle_2|1\rangle_1|0\rangle_2 + C_{01}^{gg}(t)\ |g\rangle_1|g\rangle_2|0\rangle_1|1\rangle_2, \end{aligned} \tag{5.13}$$

where $|C_{00}^{eg}(t)|^2$, $|C_{00}^{ge}(t)|^2$, $|C_{10}^{gg}(t)|^2$ and $|C_{01}^{gg}(t)|^2$ gives the probabilities of finding the system in the respective states. In the following discussions the coupled cavity system is confined to a single excitation subspace.

We also assume that the field and atoms in both cavities have the same frequencies, i.e.,

$$\omega_{f1} = \omega_{f2} = \omega_f \tag{5.14}$$

and

$$\omega_{a1} = \omega_{a2} = \omega_a \tag{5.15}$$

so that $g_1 = g_2 = g$. Substituting the general state of the coupled cavity system in a single excitation subspace given by Eq.(5.13) in the time dependent Schrödinger equation we get the following set of coupled differential equations for the amplitudes:

$$i\frac{d}{dt}\ C_{10}^{gg}(t) = (\omega_f - \omega_a)\ C_{10}^{gg}(t) + g\ C_{00}^{eg} + AC_{01}^{gg}(t), \tag{5.16}$$

$$i\frac{d}{dt}\ C_{00}^{eg}(t) = g\ C_{10}^{gg}(t), \tag{5.17}$$

$$i\frac{d}{dt}\ C_{01}^{gg}(t) = (\omega_f - \omega_a)\ C_{01}^{gg}(t) + g\ C_{00}^{ge}(t) + AC_{10}^{gg}(t), \tag{5.18}$$

$$i\frac{d}{dt}\ C_{00}^{ge}(t) = g\ C_{01}^{gg}(t). \tag{5.19}$$

The above set of coupled equations, Eqs. (5.16) - (5.19) can be solved analytically and these solutions give the evolution of atom field state probability with time in each cavity.

5.1.3.1 Analytical expressions for the probability amplitudes of states

We can use the typical techniques of solving coupled differential equations to solve Eqs. (5.16) - (5.19). For convenience let us define

$$\alpha_1(t) = C_{10}^{gg}(t) + C_{01}^{gg}(t) \tag{5.20}$$

$$\alpha_2(t) = C_{10}^{gg}(t) - C_{01}^{gg}(t) \tag{5.21}$$

and

$$\beta_1(t) = C_{00}^{eg}(t) + C_{00}^{ge}(t) \tag{5.22}$$

$$\beta_2(t) = C_{00}^{eg}(t) - C_{00}^{ge}(t) \tag{5.23}$$

Now by adding Eq. (5.16) and Eq. (5.18), subtracting Eq. (5.18) from Eq. (5.16) we obtain the following equations

$$i\frac{d}{dt}\alpha_1 = (\omega - \omega_a + A)\,\alpha_1 + g\beta_1 \tag{5.24}$$

$$i\frac{d}{dt}\alpha_2 = (\omega - \omega_a - A)\,\alpha_2 + g\beta_2. \tag{5.25}$$

Similarly by adding Eq. (5.17) and Eq. (5.19), subtracting Eq. (5.17) from Eq. (5.19) we get

$$i\hbar\frac{d}{dt}\beta_1 = g\alpha_1 \tag{5.26}$$

$$i\hbar\frac{d}{dt}\beta_2 = g\alpha_2 \tag{5.27}$$

By solving the Eqs. (5.24) - (5.27) we get the expression for the amplitudes of states as below:

$$\begin{aligned}
C_{10}^{gg} &= \left(\frac{-1}{2g}\right) e^{\left(\frac{-i\Delta_1 t}{2}\right)} \left[P_1(\Omega_1-\Delta_1)e^{i\Omega_1 t/2} - P_2(\Omega_1+\Delta_1)e^{-i\Omega_1 t/2}\right] + \\
&\quad \left(\frac{-1}{2g}\right) e^{\left(\frac{-i\Delta_2 t}{2}\right)} \left[M_1(\Omega_2-\Delta_2)e^{i\Omega_2 t/2} - M_2(\Omega_2+\Delta_2)e^{-i\Omega_2 t/2}\right], \\
C_{10}^{gg} &= \left(\frac{-1}{2g}\right) e^{\left(\frac{-i\Delta_1 t}{2}\right)} \left[P_1(\Omega_1-\Delta_1)e^{i\Omega_1 t/2} - P_2(\Omega_1+\Delta_1)e^{-i\Omega_1 t/2}\right] - \\
&\quad \left(\frac{-1}{2g}\right) e^{\left(\frac{-i\Delta_2 t}{2}\right)} \left[M_1(\Omega_2-\Delta_2)e^{i\Omega_2 t/2} - M_2(\Omega_2+\Delta_2)e^{-i\Omega_2 t/2}\right], \\
C_{00}^{eg} &= e^{\left(\frac{-i\Delta_1 t}{2}\right)} \left[P_1 e^{i\Omega_1 t} + P_2 e^{-i\Omega_1 t}\right] + e^{\left(\frac{-i\Delta_2 t}{2}\right)} \left[M_1 e^{i\Omega_2 t} + M_2 e^{-i\Omega_2 t}\right], \\
C_{00}^{ge} &= e^{\left(\frac{-i\Delta_1 t}{2}\right)} \left[P_1 e^{i\Omega_1 t} + P_2 e^{-i\Omega_1 t}\right] + e^{\left(\frac{-i\Delta_2 t}{2}\right)} \left[M_1 e^{i\Omega_2 t} + M_2 e^{-i\Omega_2 t}\right].
\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}
&\Delta_1 = \omega_f - \omega_a + A \quad , \quad \Delta_2 = \omega_f - \omega_a - A \\
\text{and}\quad &\Omega_1 = \sqrt{\Delta_1^2 + 4\ g^2} \quad , \quad \Omega_2 = \sqrt{\Delta_2^2 + 4\ g^2}.
\end{aligned} \tag{5.29}$$

The constants P_1, P_2, M_1 and M_2 are determined by the initial conditions and are given by

$$\begin{aligned}
P_1 &= \frac{1}{\Omega_1}\left\{(\Omega_1+\Delta_1)\left[\frac{C_{00}^{eg}(0)+C_{00}^{eg}(0)}{2}\right] - 2\ g\ \left[\frac{C_{10}^{gg}(0)+C_{01}^{gg}(0)}{2}\right]\right\}, \\
P_2 &= \frac{1}{\Omega_1}\left\{(\Omega_1-\Delta_1)\left[\frac{C_{00}^{eg}(0)+C_{00}^{eg}(0)}{2}\right] + 2\ g\ \left[\frac{C_{10}^{gg}(0)+C_{01}^{gg}(0)}{2}\right]\right\}, \\
M_1 &= \frac{1}{\Omega_2}\left\{(\Omega_2+\Delta_2)\left[\frac{C_{00}^{eg}(0)-C_{00}^{eg}(0)}{2}\right] - 2\ g\ \left[\frac{C_{10}^{gg}(0)-C_{01}^{gg}(0)}{2}\right]\right\}, \\
M_2 &= \frac{1}{\Omega_2}\left\{(\Omega_2-\Delta_2)\left[\frac{C_{00}^{eg}(0)-C_{00}^{eg}(0)}{2}\right] + 2\ g\ \left[\frac{C_{10}^{gg}(0)-C_{01}^{gg}(0)}{2}\right]\right\}.
\end{aligned} \tag{5.30}$$

Using the set of Eqs. (5.28) - (5.30) the dynamics of the coupled cavity system for any initial conditions with different atom field coupling strength(g) and cavity-cavity coupling strength(A) can be studied.

5.1.3.2 Transfer of atomic excitation between cavities

Here we focus our attention on the transfer of atomic excitation probability between cavities 1 and 2. The initial conditions are set in such a way that at time $t = 0$ atom 1 is in the excited state, atom 2 in ground state and both the field states are vacuum states, i.e.,

$$|C_{00}^{ge}(0)|^2 = |C_{10}^{gg}(0)|^2 = |C_{01}^{gg}|^2 = 0, \tag{5.31}$$

$$|C_{00}^{eg}(0)|^2 = 1. \tag{5.32}$$

With these initial conditions the constants given in Eq.(5.30) becomes,

$$\begin{aligned} P_1 &= \frac{1}{2\ \Omega_1}(\Omega_1 + \Delta_1) \quad , \quad P_2 = \frac{1}{2\ \Omega_1}(\Omega_1 - \Delta_1) \\ \text{and} \quad M_1 &= \frac{1}{2\ \Omega_2}(\Omega_2 + \Delta_2) \quad , \quad M_1 = \frac{1}{2\ \Omega_2}(\Omega_2 - \Delta_2). \end{aligned} \tag{5.33}$$

Utilizing the Eq. (5.33) in Eq. (5.28), we get

$$\begin{aligned}
|C_{10}^{gg}(t)|^2 &= (1/4\ g)^2 \{[k_1 \sin(\Delta_1\ t/2) \sin(\Omega_1\ t/2) + \\
&\quad k_2 \sin(\Delta_2\ t/2) \sin(\Omega_2\ t/2)]^2 \\
&\quad + [k_1 \cos(\Delta_1\ t/2) \sin(\Omega_1\ t/2) + \\
&\quad k_2 \cos(\Delta_2\ t/2) \sin(\Omega_2\ t/2)]^2 \Big\}, \qquad (5.34) \\
|C_{01}^{gg}(t)|^2 &= \left(\frac{-1}{4\ g}\right)^2 \{[k_1 \sin(\Delta_1\ t/2) \sin(\Omega_1\ t/2) - \\
&\quad k_2 \sin(\Delta_2\ t/2) \sin(\Omega_2\ t/2)]^2 \\
&\quad + [k_1 \cos(\Delta_1\ t/2) \sin(\Omega_1\ t/2) - \\
&\quad k_2 \cos(\Delta_2\ t/2) \sin(\Omega_2\ t/2)]^2 \Big\}, \qquad (5.35) \\
|C_{00}^{eg}(t)|^2 &= \{\sum_{j=1,2} \frac{1}{2\Omega_j} [\Omega_j \cos(\Omega_j t/2) \cos(\Delta_j t/2)) + \\
&\quad \Delta_j \sin(\Omega_j t/2) \sin(\Delta_j\ t/2)]\}^2 + \\
&\quad \{\sum_{j=1,2} \frac{1}{2\ \Delta_j}\ [\Omega_j \sin(\Omega_j\ t/2) \cos(\Delta_j\ t/2) - \\
&\quad \Omega_j \cos(\Omega_j t/2) \sin(\Delta_j\ t/2)]\}^2, \qquad (5.36) \\
|C_{00}^{ge}(t)|^2 &= \{\sum_{j=1,2} (-1)^{j+1} \frac{1}{2\Omega_j} [\Omega_j \cos(\Omega_j t/2) \cos(\Delta_j t/2) + \\
&\quad \Delta_j \sin(\Omega_j t/2) \sin(\Delta_j t/2)]\}^2 + \\
&\quad \{\sum_{j=1,2} (-1)^{j+1} \frac{1}{2\Delta_j} [\Delta_j \sin(\Omega_j t/2) \cos(\Delta_j t/2) - \\
&\quad \Omega_j \cos(\Omega_j t/2) \sin(\Delta_j t/2)]\}^2. \qquad (5.37)
\end{aligned}$$

where $k_i = \left(\Omega_i^2 - \Delta_i^2\right)/\Omega_i^2,\ i = 1, 2$. Eqs. (5.34)-(5.37) give the evolution of probabilities of various achievable states in the coupled cavity system in a single excitation subspace.

Now if the atom and field are at resonance in each cavity i.e., $\omega_f = \omega_a$, we have $\Delta_1 = A$, $\Delta_2 = -A$ and $\Omega_1 = \Omega_2$. In this case Eqs. (5.34)-(5.37) simplifies to

$$|C_{10}^{gg}(t)|^2 = \left\{\frac{2g}{\Omega} \cos\left(\frac{A}{2}\,t\right) \sin\left(\frac{\Omega}{2}\,t\right)\right\}^2, \tag{5.38}$$

$$|C_{01}^{gg}(t)|^2 = \left\{\frac{2g}{\Omega} \sin\left(\frac{A}{2}\,t\right) \sin\left(\frac{\Omega}{2}\,t\right)\right\}^2, \tag{5.39}$$

$$|C_{00}^{eg}(t)|^2 = \left\{\frac{1}{\Omega}\left[\Omega\cos\left(\frac{\Omega t}{2}\right)\cos\left(\frac{At}{2}\right) + A\sin\left(\frac{\Omega t}{2}\right)\sin(\frac{At}{2})\right]\right\}^2, \tag{5.40}$$

$$|C_{00}^{ge}(t)|^2 = \left\{\frac{1}{\Omega}\left[A\sin\left(\frac{\Omega t}{2}\right)\cos\left(\frac{At}{2}\right) - \Omega\cos\left(\frac{\Omega t}{2}\right)\sin(\frac{At}{2})\right]\right\}^2. \tag{5.41}$$

The evolution of probabilities in resonance case; expressed in Eqs. (5.38) - (5.41) can be written as the harmonic superposition in the following way:

$$|C_{10}^{gg}(t)|^2 = (g/\Omega)^2 \{1 - (1/2)\cos[(A-\Omega)t] \\ -(1/2)\cos[(A+\Omega)t] + \cos(At) - \cos(\Omega t)\}, \tag{5.42}$$

$$|C_{01}^{gg}(t)|^2 = (g/\Omega)^2 [1 + (1/2)\cos[(A-\Omega)t] + \\ (1/2)\cos[(A+\Omega)t] - \cos(At) - \cos(\Omega t)], \tag{5.43}$$

$$|C_{00}^{eg}(t)|^2 = (1/2\Omega)^2 \left\{\Omega^2 + A^2 + \left[(\Omega+A)^2/2\right]\cos[(\Omega-A)t] \\ + \left[(\Omega-A)^2/2\right]\cos[(\Omega+A)t] \\ - +(\Omega^2 - A^2)\left[\cos(At) + \cos(\Omega t)\right]\right\}, \tag{5.44}$$

$$|C_{00}^{eg}(t)|^2 = (1/2\Omega)^2 \left\{\Omega^2 + A^2 - \left[(\Omega-A)^2/2\right]\cos[(\Omega+A)t] \\ - \left[(\Omega+A)^2/2\right]\cos[(\Omega-A)t] - \\ - (A^2 - \Omega^2)\left[\cos(At) - \cos(\Omega t)\right]\right\}. \tag{5.45}$$

From the above set of equations (5.42) - (5.45), it is clear that each state execute different modes of oscillations with different amplitudes. These amplitudes and frequencies of each term depends on the atom field coupling strengths g inside a cavity and the cavity-cavity coupling strength A.

5.1.4 Results and discussion

5.1.4.1 Case 1

We first consider the case in which the atom field coupling strength, g is small compared to the cavity-cavity coupling strength, A (i.e., $g \ll A$). In such cases $\Omega^2 \approx A^2 \gg g^2$. In this limit the values of field excitation probabilities $|C_{10}^{gg}(t)|^2$ and $|C_{01}^{gg}(t)|^2$ are negligible compared to the values of atomic excitation probabilities $|C_{00}^{eg}(t)|^2$ and $|C_{00}^{ge}(t)|^2$ as shown in the Figs. 5.2 and 5.3.

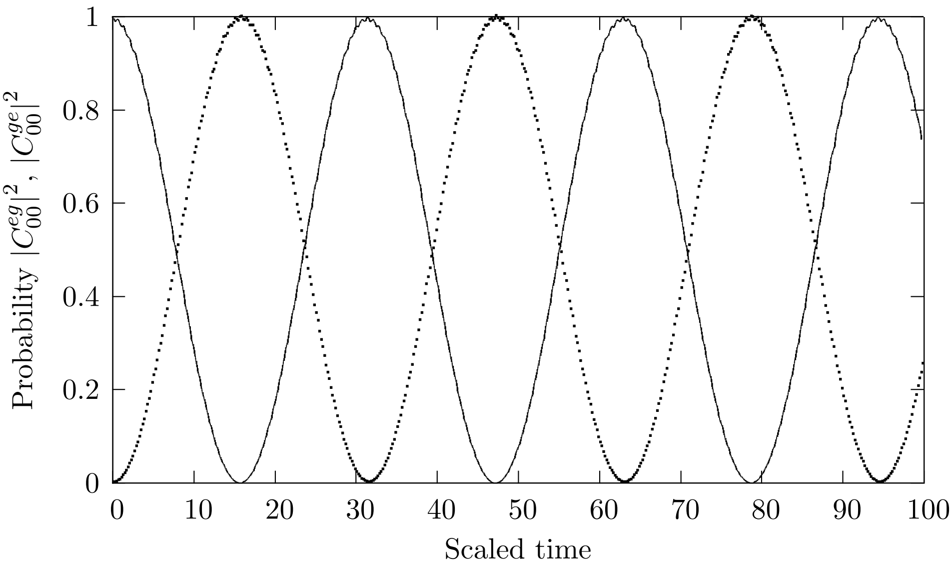


Figure 5.2: Evolution of atomic excitation probability of atom 1 (solid line) and atom 2 (dashed line) for the resonant interaction($\omega_f = \omega_a$) inside each cavity. The coupling constants are $g = 2$ and $A = 20g$.

Here the atomic excitation probability of atom 1 diminishes to a minimum with a corresponding increase in probability for atom 2. As time elapses we can observe the repetition of these cycles . This indicates the existence of atomic excitation probability transfer between cavities, while keeping the field excitation probabilities small.

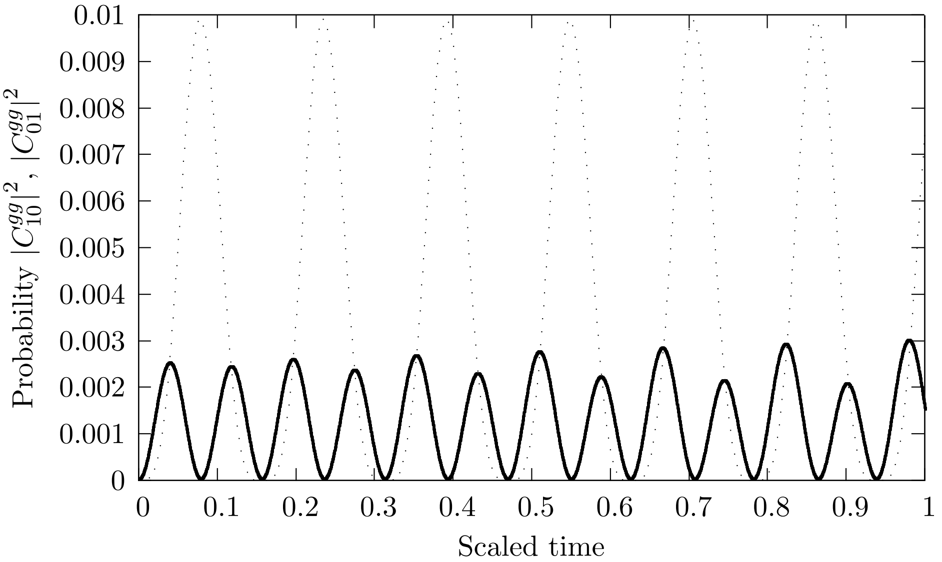


Figure 5.3: Evolution of field excitation probability of cavity 1 (dotted line) and cavity 2 (solid line) for resonant interaction($\omega_f = \omega_a$) inside each cavity. The coupling constants are $g = 2$ and $A = 20g$. Field excitation probabilities are very small compared to the atomic excitation probability shown in figure 5.2.

Now we apply this approximation in the expressions for the probabilities given by Eqs. (5.42). Since $g^2 \ll \Omega^2$ both the field excitation probabilities will be very small. But for the atomic excitation probabilities the prevailing terms are

$$|C_{00}^{eg}(t)|^2 \propto \frac{(\Omega + A)^2}{2} \cos\left[(\Omega - A)t\right], \tag{5.46}$$

and

$$|C_{00}^{ge}(t)|^2 \propto \frac{(\Omega + A)^2}{2} \cos\left[(\Omega - A)t\right]. \tag{5.47}$$

From Eqs. (5.46) and (5.47) it is clear that the probability oscillates in time and the frequency of oscillation is given by $\Omega - A$ and the time period for complete excitation probability transfer between atom 1 and atom 2 is

$$T = [\pi/(\Omega - A)] \, . \tag{5.48}$$

Thus at $t = \pi/(\Omega - A)$ the probability for the atom 2 excited is maximum, where as that of atom 1 is approximately zero.

5.1.4.2 Case 2

Now consider the case where the atom field coupling strength is large compared to the cavity-cavity coupling strength(i.e., $g \gg A$). In such case we get $\Omega^2 \approx 4g^2 \gg A^2$. Initial condition is taken to be the same that only atom 1 is in the excited state. Here all the probability amplitudes are comparable and they oscillate as shown in Figs. 5.4 and 5.5. It is also observed that there is a fast exchange of probability amplitudes between the atom and field in each cavity in the course of time. Atomic probabilities exchanges between cavity 1 and cavity 2 slowly and a similar exchange of field probability between cavities also observed. In other words the probability transfer between cavities is slow compared with the atom field probability exchange inside each cavity.

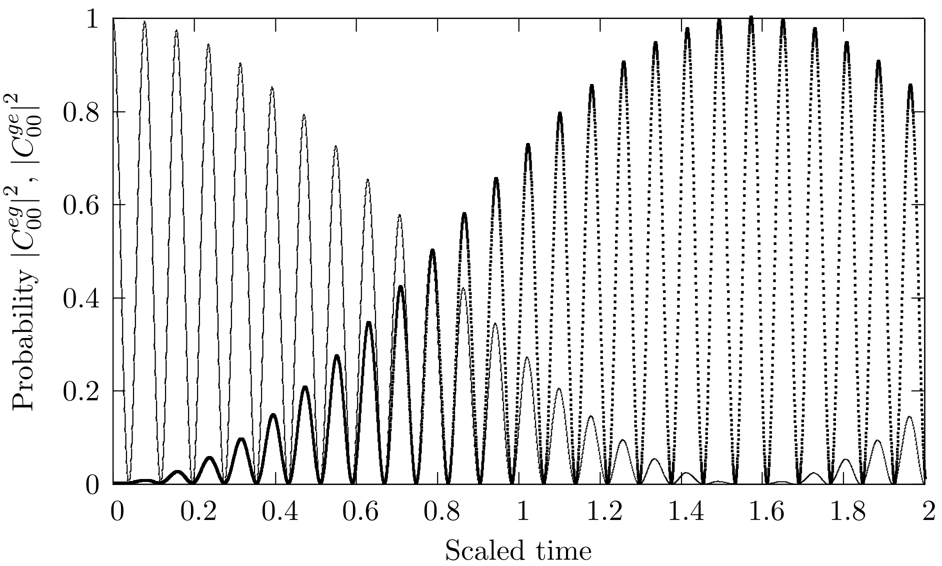


Figure 5.4: Evolution of atomic excitation probability of atom 1 (solid line) and atom 2 (dashed line). Resonant interaction($\omega_f = \omega_a$) inside each cavity and the coupling strengths are $g = 40$, $A = 0.05g$.

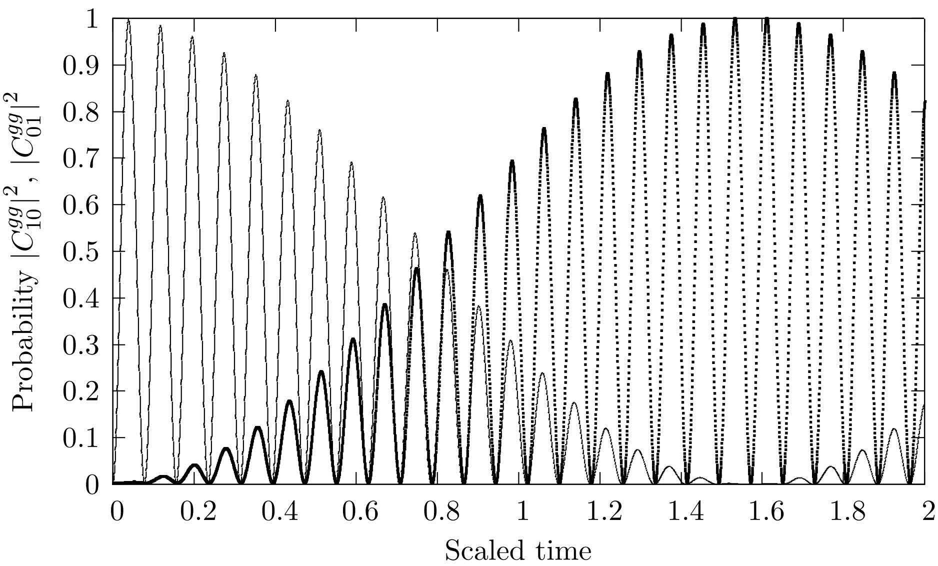


Figure 5.5: Evolution of field excitation probability of cavity 1 (solid line) and cavity 2 (dashed line) for resonant interaction($\omega_f = \omega_a$) inside each cavity. The coupling strengths are $g = 40$, $A = 0.05g$.

5.1.4.3 Case 3

When the atom field coupling strength and cavity cavity coupling strength are comparable(i.e., $g \approx A$) the probabilities of atomic excitation and field excitation oscillates with comparable amplitude and is periodic, which is shown in Figs. 5.6 and 5.7.

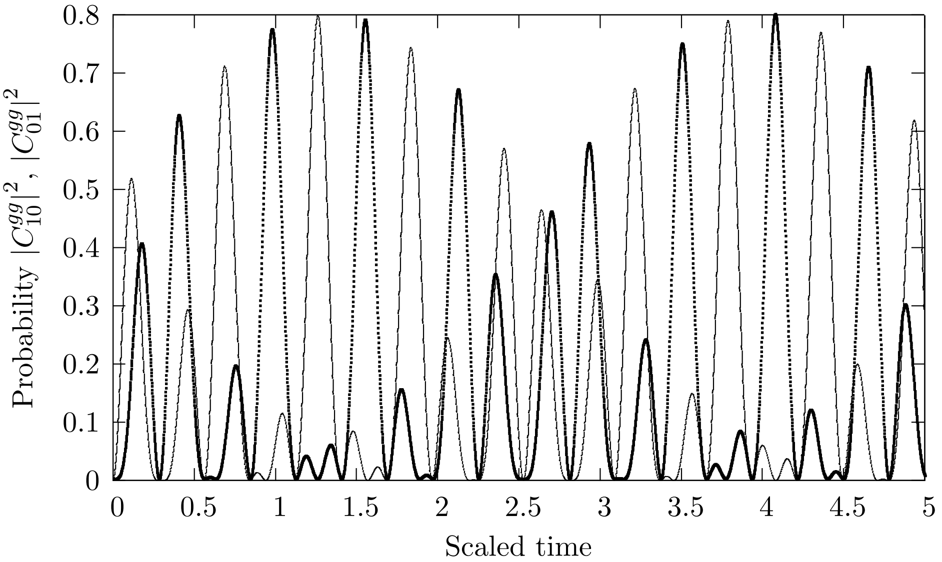


Figure 5.6: Evolution of field excitation probability of cavity 1 (solid line) and cavity 2 (dashed line) for resonant interaction($\omega_f = \omega_a$) inside each cavity. The coupling strengths are $g = 10$ and $A = g$.

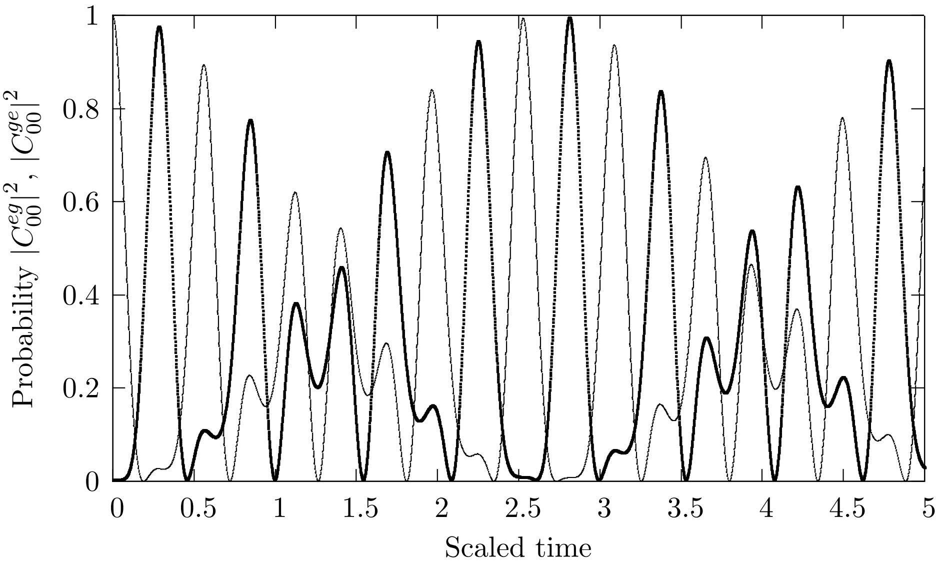


Figure 5.7: Evolution of atomic excitation probability of atom 1 (solid line) and atom 2 (dashed line) for resonant ($\omega_f = \omega_a$) interaction parameters are $g = 10$ and $A = g$.

5.1.4.4 Population inversion

In the case of a coupled cavity system in single excitation subspace we define the population inversion $W(t)$ as

$$W(t) = |C_{00}^{eg}(t)|^2 + |C_{00}^{ge}(t)|^2 - |C_{10}^{gg}(t)|^2 - |C_{01}^{gg}(t)|^2, \qquad (5.49)$$

which is the difference in probabilities between the states with any of the atom excited and both the atoms in ground state. Substituting the set of Eqs. (5.42) in the expression for population inversion in Eq. (5.49) we get

$$W(t) = \frac{1}{\Omega^2}\left[A^2 + 4g^2\cos(\Omega t)\right]. \qquad (5.50)$$

It is clear that the population inversion execute sinusoidal oscillations with time as shown in Fig. 5.8 with a frequency equal to Ω and amplitude in the range between 1 and $\left(\dfrac{A^2 - 4g^2}{\Omega^2}\right)$.

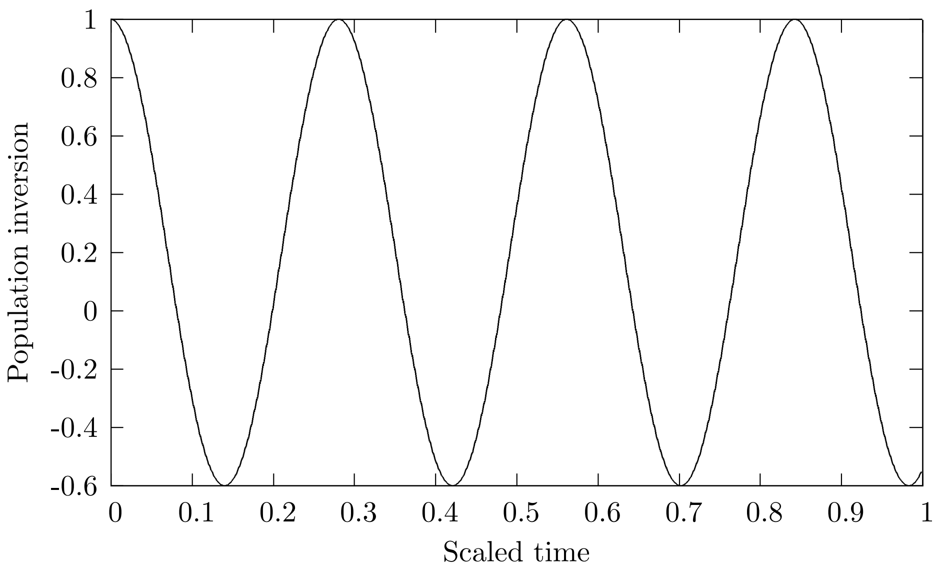


Figure 5.8: Population inversion is plotted against time for resonant ($\omega_f = \omega_a$) interaction with parameters $g = 10$ and $A = g$.

5.1.5 Conclusion

We have analysed the evolution of atom field state probability in a coupled cavity system. Analytical formulation for the time variation of atomic and field probability is done in a single excitation subspace. Atomic excitation transfer between cavities for different limits of coupling strengths g and A are investigated. It is observed that periodic transfer of excitation probability between cavities exists. The time period for complete excitation transfer between cavities for various limiting cases of coupling strength are predicted. An analytical expression is obtained for the population inversion of the system which evolves sinusoidally with time. These analytical expressions for the time evolution of various atom field states in the coupled cavity system can be used to study the system with possible experimentally realizable initial states and coupling strengths. These findings will help in the realization of an effective quantum data transfer device which is essential for the of quantum computers.

5.2 Coupled cavity system with Kerr nonlinear medium

5.2.1 Introduction

In the previous section we have analysed a coupled cavity system with a linear medium in each cavity. Interesting results are noticed for various coupling strengths. State transfer between two cavities occurs and the periodicity of probability exchange has a close connection with the coupling strength of the system. In this section we consider the same system with two cavities, each containing a two level atom, are coupled *via.* photon hopping and the cavity is filled with Kerr nonlinear medium.

5.2.2 Model and Hamiltonian

In the present study, we considered a system with two cavities coupled together via inter cavity photon hopping. Both cavities are filled with non-linear Kerr medium and contains a two level atom in it. Each cavity field interacts with the two level atom inside, and this interaction Hamiltonian is taken according to Jaynes Cummings model(JCM) and cavity-cavity coupling is modelled *via.* photon hopping. Here the non-linear medium Hamiltonian is assumed to be equivalent to that of an anharmonic oscillator[71, 72]. Now the total Hamiltonian of such a system in the adiabatic limit, where anharmonic frequency and the field frequency are far from each other, can be written as

$$\begin{aligned}\hat{H} &= \sum_{i=1,2}\left[\omega_{fi}\hat{a}_i^\dagger\hat{a}_i + \chi_0\hat{a}_i^{\dagger^2}\hat{a}_i^2 + \frac{\omega_{ai}}{2}\,\hat{\sigma}_{iz} + g(\hat{a}_i\hat{\sigma}_{i+} + \hat{a}^\dagger\hat{\sigma}_{i-})\right] \\ &\quad + A\left(\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1\right) \end{aligned} \tag{5.51}$$

Here it is assumed that the changes in the cavity cavity coupling strength due to the nonlinearity of the medium is negligible and the coupling Hamiltonian does not contains any higher order terms of the field operator. Hamiltonian in Eq.(5.51) is identical to the the Hamiltonian given in (5.10) except the presence of a nonlinear part viz., $\chi_0\hat{a}_i^{\dagger^2}\hat{a}_i^2$. It arises from the nonlinear Kerr medium with third order susceptibility χ_0 [92]. As in the previous section, the general state of a coupled cavity system, with two cavities can be written as the combination of all the possible field and atomic states in each cavity such that

$$\begin{aligned}|\psi(t)\rangle &= \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\Big[C_{l_1,l_2}^{g,g}\;|l_1\rangle\;|l_2\rangle\;|g_1\rangle\;|g_2\rangle + \\ &\quad C_{l_1,l_2}^{g,e}\;|l_1\rangle\;|l_2\rangle\;|g_1\rangle\;|e_2\rangle + C_{l_1,l_2}^{e,g}\;|l_1\rangle\;|l_2\rangle\;|e_1\rangle\;|g_2\rangle + \\ &\quad C_{l_1,l_2}^{e,e}\;|l_1\rangle\;|l_2\rangle\;|e_1\rangle\;|e_2\rangle\Big]\,. \end{aligned} \tag{5.52}$$

Substituting the Hamiltonian in Eq. (5.51) and general state given by Eq. (5.52) in time dependent Schrödinger equation we get

$$
\begin{aligned}
i\dot{C}^{gg}_{l_1,l_2} &= \left[\left(\omega_{f1}l_1+\omega_{f2}l_2\right)-\frac{(\omega_{a1}+\omega_{a2})}{2}\right]C^{gg}_{l_1,l_2}+\\
&[\chi_0 l_1(l_1-1)+\chi_0 l_2(l_2-1)]\,C^{gg}_{l_1,l_2}\\
&+g\left[\sqrt{l_2}\,C^{ge}_{l_1,l_2-1}+\sqrt{l_1}\,C^{eg}_{l_1-1,l_2}\right]\\
&+A\left[\sqrt{l_1(l_2+1)}\,C^{gg}_{l_1-1,l_2+1}+\sqrt{(l_1+1)l_2}\,C^{gg}_{l_1+1,l_2-1}\right]\\
i\dot{C}^{ge}_{l_1,l_2} &= \left[\left(\omega_{f1}l_1+\omega_{f2}l_2\right)+\frac{(\omega_{a2}-\omega_{a1})}{2}\right]C^{ge}_{l_1,l_2}+\\
&[\chi_0 l_1(l_1-1)+\chi_0 l_2(l_2-1)]\,C^{ge}_{l_1,l_2}\\
&+g\left[\sqrt{l_2+1}\,C^{gg}_{l_1,l_2+1}+\sqrt{l_1}\,C^{ee}_{l_1-1,l_2}\right]\\
&+A\left[\sqrt{l_1(l_2+1)}\,C^{ge}_{l_1-1,l_2+1}+\sqrt{(l_1+1)l_2}\,C^{ge}_{l_1+1,l_2-1}\right]\\
i\dot{C}^{eg}_{l_1,l_2} &= \left[\left(\omega_{f1}l_1+\omega_{f2}l_2\right)+\frac{(\omega_{a1}-\omega_{a2})}{2}\right]C^{eg}_{l_1,l_2}+\\
&[\chi_0 l_1(l_1-1)+\chi_0 l_2(l_2-1)]\,C^{eg}_{l_1,l_2}\\
&+g\left[\sqrt{l_1+1}\,C^{gg}_{l_1+1,l_2}+\sqrt{l_2}\,C^{ee}_{l_1,l_2-1}\right]\\
&+A\left[\sqrt{l_1(l_2+1)}\,C^{eg}_{l_1-1,l_2+1}+\sqrt{(l_1+1)l_2}\,C^{eg}_{l_1+1,l_2-1}\right]\\
i\dot{C}^{ee}_{l_1,l_2} &= \left[\left(\omega_{f1}l_1+\omega_{f2}l_2\right)+\frac{(\omega_{a1}+\omega_{a2})}{2}\right]C^{ee}_{l_1,l_2}+\\
&[\chi_0 l_1(l_1-1)+\chi_0 l_2(l_2-1)]\,C^{ee}_{l_1,l_2}\\
&+g\left[\sqrt{l_1+1}\,C^{ge}_{l_1+1,l_2}+\sqrt{l_2+1}\,C^{eg}_{l_1,l_2+1}\right]\\
&+A\left[\sqrt{l_1(l_2+1)}\,C^{ee}_{l_1-1,l_2+1}+\sqrt{(l_1+1)l_2}\,C^{gg}_{l_1+1,l_2-1}\right]
\end{aligned}
\tag{5.53}
$$

which represent the evolution of the amplitudes in a general two cavity system with any number of photons.

5.2.3 Evolution of the System in two excitation subspace

Since the effect of nonlinearity is not visible in single excitation subspace we are considering the evolution of the system in two excitation subspace. In a single excitation subspace the eigenvalues of the nonlinear part of Hamiltonian vanish and we can not consider it for studying the effect of nonlinearity. In two excitation subspace the total number of excitation is limited to two and the general state of the coupled cavity system in such a subspace is

$$\begin{aligned}|\psi(t)\rangle &= C^{gg}_{20}|g_1\rangle|g_2\rangle|2\rangle|0\rangle + C^{gg}_{02}|g_1\rangle|g_2\rangle|0\rangle|2\rangle + \\ & C^{gg}_{11}|g_1\rangle|g_2\rangle|1\rangle|1\rangle + C^{eg}_{10}|e_1\rangle|g_2\rangle|1\rangle|0\rangle + \\ & C^{eg}_{01}|e_1\rangle|g_2\rangle|0\rangle|1\rangle + C^{ge}_{10}|g_1\rangle|e_2\rangle|1\rangle|0\rangle + \\ & C^{ge}_{01}|g_1\rangle|e_2\rangle|0\rangle|1\rangle + C^{ee}_{00}|e_1\rangle|e_2\rangle|0\rangle|0\rangle. \qquad (5.54)\end{aligned}$$

Assume that the field (atom) frequency in both cavities are the same i.e., $\omega_{f1} = \omega_{f2} = \omega_f$ and $\omega_{a1} = \omega_{a2} = \omega_a$ and using the general Eq. (5.53) we get the following set of coupled differential equations corresponding to the evolution of system in the two excitation subspace:

$$i\,\frac{d}{dt}C_{10}^{ge} = \omega_f\, C_{10}^{ge} + g\,(C_{11}^{gg} + C_{00}^{ee}) + A\, C_{ge}^{01} \tag{5.55}$$

$$i\,\frac{d}{dt}C_{01}^{ge} = \omega_f\, C_{01}^{ge} + \sqrt{2}\, g\, C_{02}^{ee} + A\, C_{10}^{ge} \tag{5.56}$$

$$i\,\frac{d}{dt}C_{10}^{eg} = \omega_f\, C_{10}^{eg} + \sqrt{2}\, g\, C_{20}^{ee} + A\, C_{01}^{eg} \tag{5.57}$$

$$i\,\frac{d}{dt}C_{01}^{eg} = \omega_f\, C_{01}^{eg} + g\,(C_{11}^{gg} + C_{00}^{ee}) + A\, C_{eg}^{10} \tag{5.58}$$

$$i\,\frac{d}{dt}C_{00}^{ee} = \omega_a\, C_{00}^{ee} + g(C_{01}^{eg} + C_{10}^{ge}) \tag{5.59}$$

$$i\,\frac{d}{dt}C_{20}^{gg} = (2\omega_f - \omega_a)C_{20}^{gg} + \sqrt{2}\, g\, C_{10}^{eg} + A\, C_{11}^{gg} + 2\chi C_{20}^{gg} \tag{5.60}$$

$$i\,\frac{d}{dt}C_{02}^{gg} = (2\omega_f - \omega_a)C_{02}^{gg} + \sqrt{2}\, g\, C_{01}^{ge} + A\, C_{11}^{gg} + 2\chi C_{02}^{gg} \tag{5.61}$$

$$i\frac{d}{dt}C_{11}^{gg} = (2\omega_f - \omega_a)C_{11}^{gg} + g(C_{01}^{eg} + C_{10}^{ge}) + \sqrt{2}A(C_{20}^{gg} + C_{20}^{gg}). \tag{5.62}$$

The above Eqs. (5.55) - (5.62) determine the evolution of amplitudes correspond to the various possible states of the coupled cavity system in the two excitation subspace. These set of equations are solved numerically using the fourth order Range Kutta method and the evolution of various atom-field probabilities are analysed in the succeeding sections.

5.2.4 Results and discussion

Dynamics of the coupled cavity system with Kerr medium is represented in Eqs. (5.55) - (5.62). The system is examined by varying the atom field coupling strengths(g), cavity-cavity coupling strength(A) and the susceptibility χ_0 for different initial states. For simplicity we use the term “atom 1” or “field 1” to denote the atom or field in first cavity and likewise the terms “atom 2” or “field 2” for cavity 2.

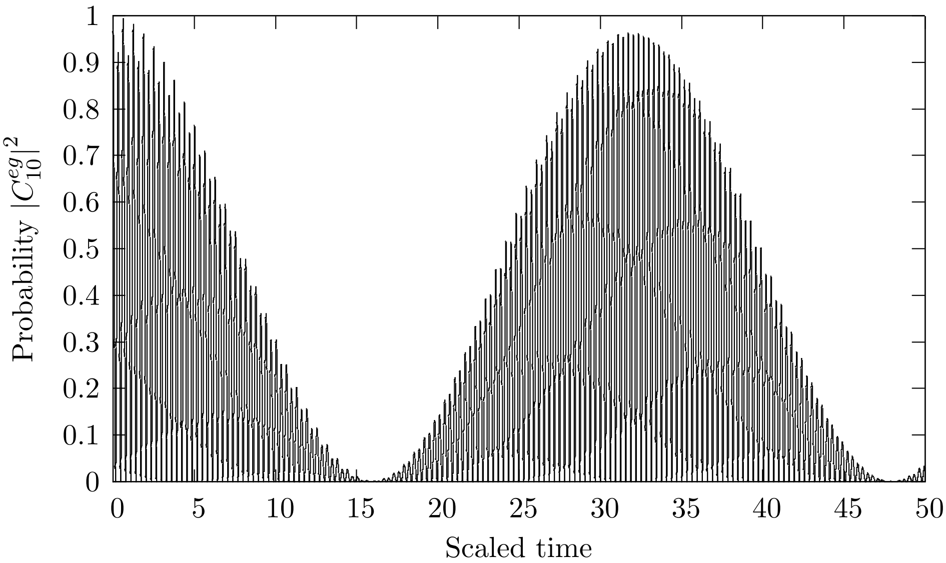
Also for convenience the value of the susceptibility is limited between 0 and 0.9 in our treatment. It is observed that the susceptibility of the cavity has a significant role in the evolution of the coupled cavity system and its effect is different for various coupling strengths and different initial state of the system. We examine the system with different initial states for which the susceptibility has a substantial role in the dynamics.

Case 1 : $|C_{10}^{eg}(0)|^2 = 1$

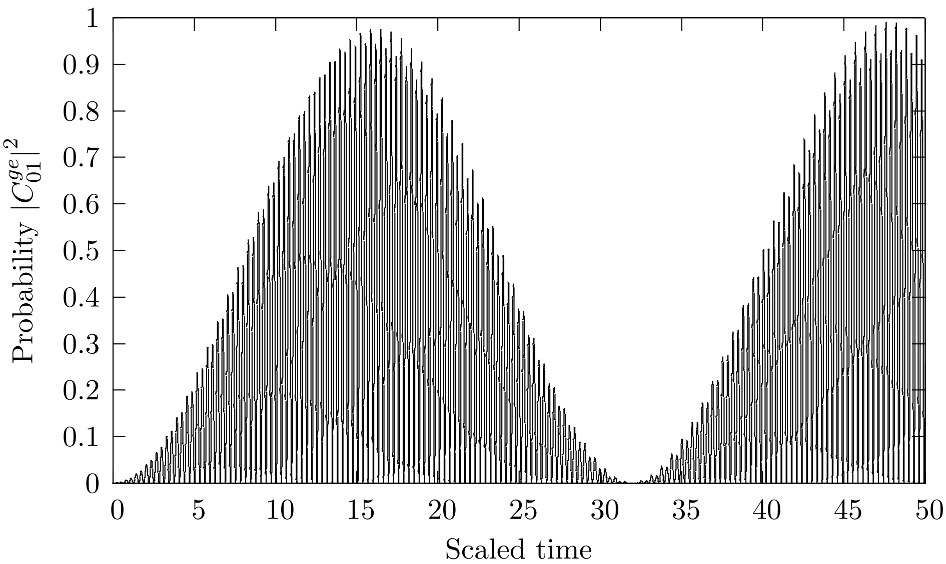
Here we look at the case with initially atom 1 is excited and field 1 is a one photon state. i.e.,

$$|\psi(0)\rangle = |e\rangle|g\rangle|1\rangle|0\rangle \tag{5.63}$$

In such cases, for strong cavity-cavity coupling strength compared to the atom field coupling strength, the probabilities $|C_{10}^{eg}(t)|^2$ and $|C_{01}^{eg}(t)|^2$ oscillate with time in the shape of an amplitude modulated wave as shown in Fig. 5.9. Similar behaviour is shown by the probabilities $|C_{10}^{ge}(t)|^2$ and $|C_{01}^{ge}(t)|^2$, which is displayed in Fig. 5.10. It is to be noted that there is a periodic exchange of probabilities between two states $|e\rangle|g\rangle|1\rangle|0\rangle$ and $|g\rangle|e\rangle|0\rangle|1\rangle$, which is clear from Figs. 5.9 and 5.10. Interesting observation is that the periodicity of these probability exchange do not have any noticeable dependence on the susceptibility of the medium. Precisely the same behaviour is shown by the probabilities $|C_{01}^{eg}(t)|^2$ and $|C_{10}^{ge}(t)|^2$. We may conclude in this case that the probability amplitude of the states are exchanged between cavity 1 and cavity 2 periodically and the period depends on the coupling strengths and independent of the susceptibility of the medium. Probabilities of all the other possible states show oscillation with time but they are very small.



(a)



(b)

Figure 5.9: Probability amplitude versus time is plotted, with parameters $A = 10g$, $\chi = 0$ and $\omega_f = \omega_a$. (a): Atom 1 excited and field 1 excited. (b): Atom 2 excited and field 2 excited

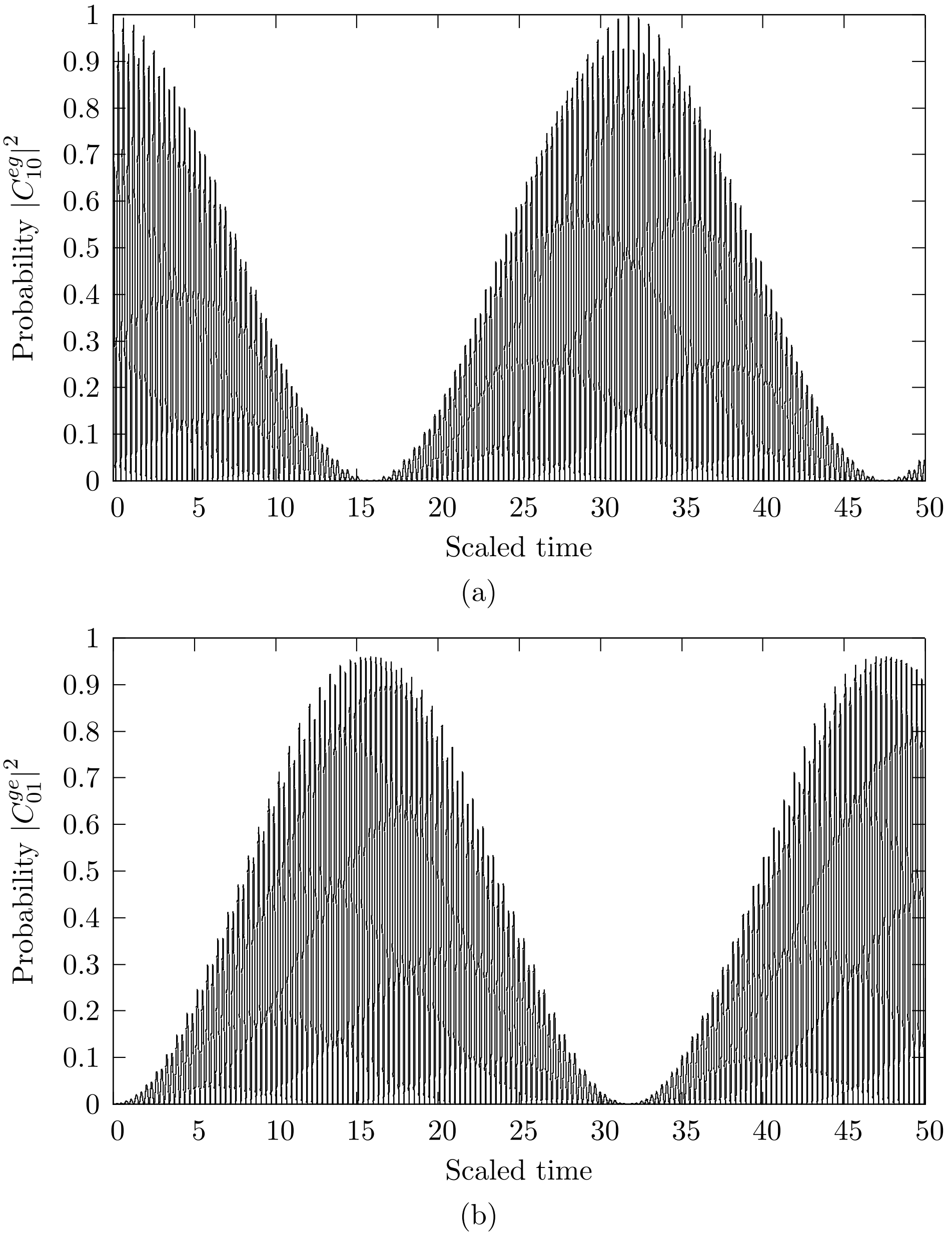


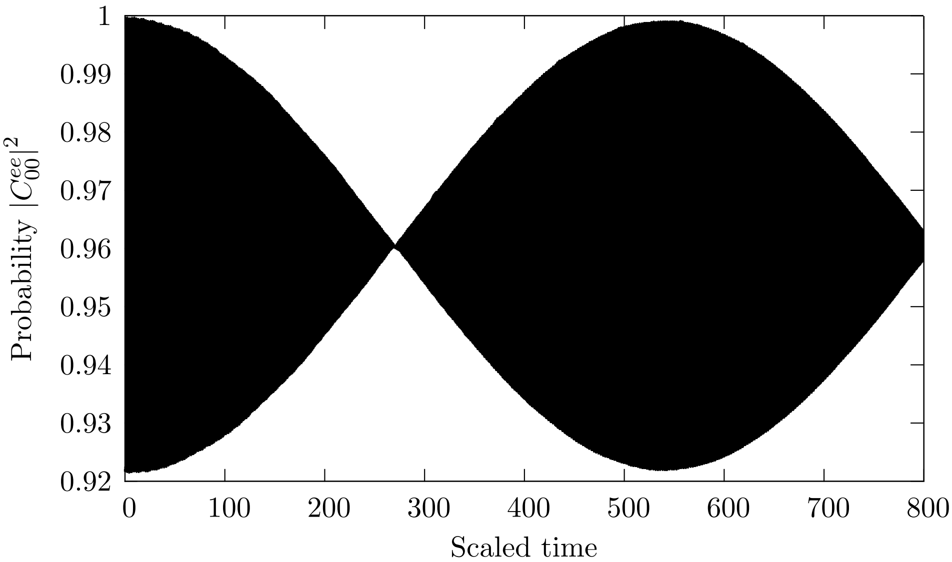
Figure 5.10: Probability amplitude versus time is plotted, with parameters $A = 10g$ $\chi = 0.9$ and $\omega_f = \omega_a$. (a): Atom 1 excited and field 1 excited. (b): Atom 2 excited and field 2 excited. Variations in probability amplitudes are same as that shown in Fig. 5.9

Case 2 : $|C^{ee}_{00}(0)|^2 = 1$

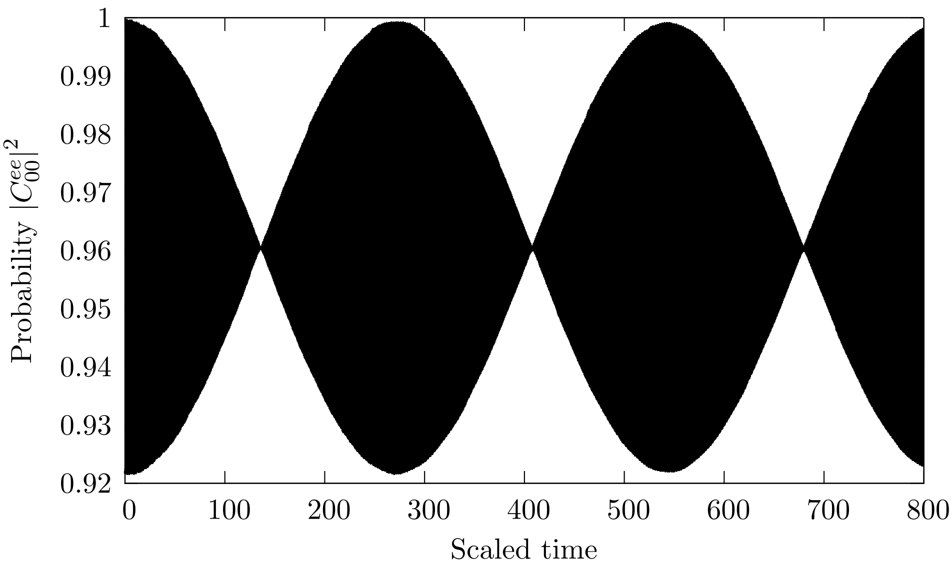
In this case the initial condition is such that both the atoms are excited and both the fields are vacuum states. i.e.,

$$|\psi(0)\rangle = |e\rangle|e\rangle|0\rangle|0\rangle \tag{5.64}$$

With this initial condition, for higher hopping strengths the probability of the system to exist in the same state as the initial state is high. Thus the probability of the state $|e\rangle|e\rangle|0\rangle|0\rangle$ maintains a value very close to 1 all the time and probabilities for rest of the states are always near to zero. When the value of susceptibility is zero, we find sinusoidal oscillations in the probability of the state with an amplitude range in between 0.9 and 1.0. But when the value of susceptibility is nonzero there is a change in the nature of oscillation of the probability. It is in the shape of an amplitude modulated wave, keeping the amplitude in the same range as the linear medium case, which is shown in Figs. 5.11 and 5.12. By varying respective coupling strengths and the susceptibility of the system we observed that the periodicity of the envelope wave is determined by the cavity-cavity coupling and the susceptibility of the medium. For fixed coupling strengths the frequency of the envelop is proportional to the susceptibility of the medium. From Figs. 5.11 and 5.12 one can visualize the dependence of periodicity on the susceptibility factor. It is also noted that when susceptibility is kept constant the number of envelops decreases with the increase of A.



(a)



(b)

Figure 5.11: Probability amplitude of the state with both atoms excited is plotted against time. (a): With parameters $\chi = 0.6$, $A = 10g$ and $\omega_a = \omega_f$. (b): With $\chi = 0.9$, $A = 10g$ and $\omega_a = \omega_f$.

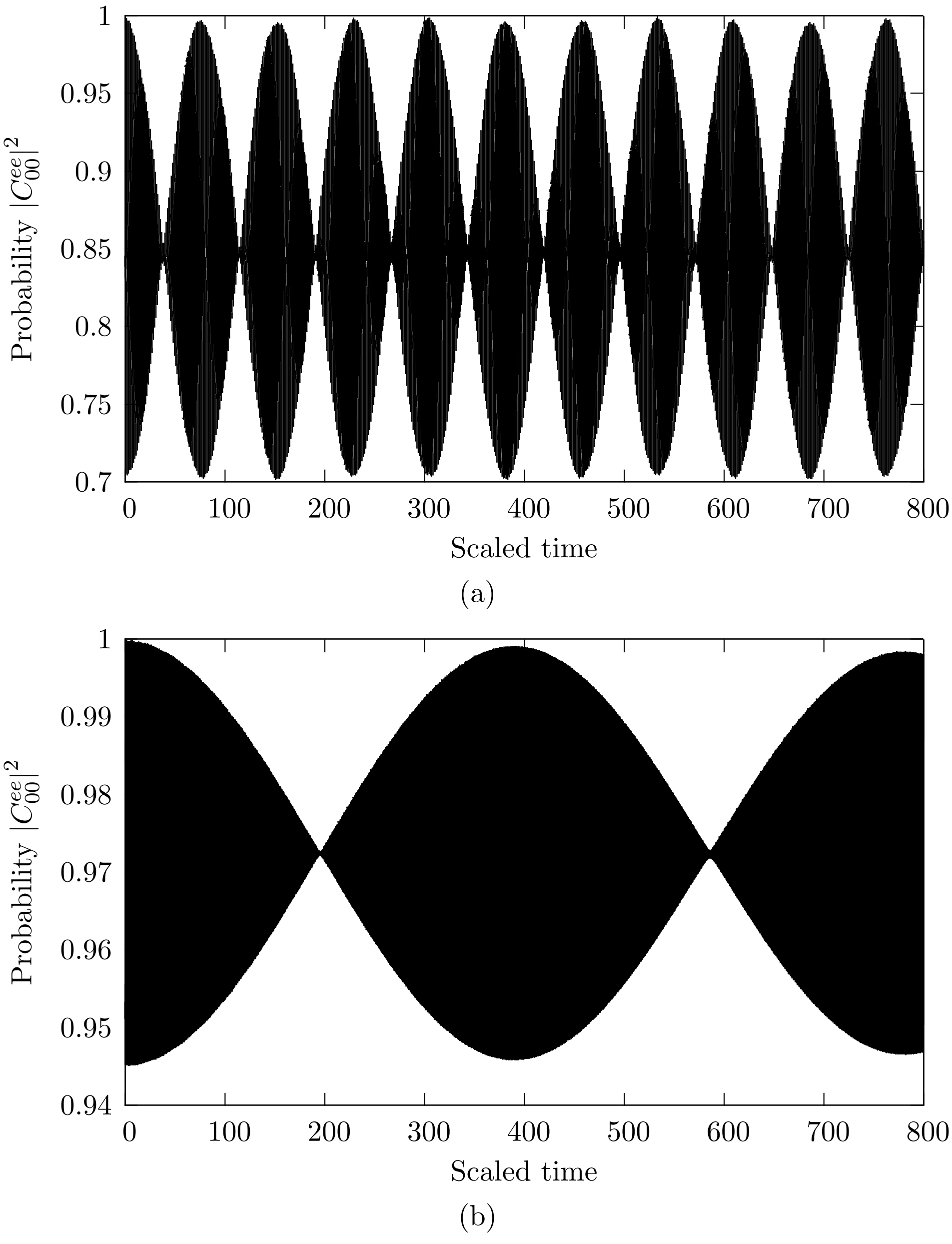


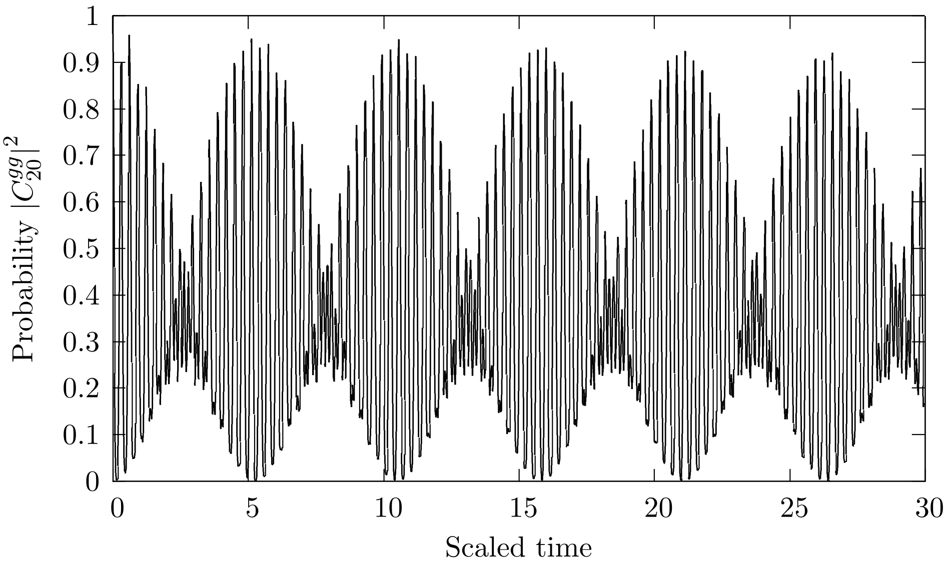
Figure 5.12: Probability amplitude of the state with both atoms excited is plotted against time. (a): With parameters $\chi = 0.9$, $A = 5g$ and $\omega_a = \omega_f$. (b): With $\chi = 0.9$, $A = 12g$ and $\omega_a = \omega_f$

5.2.4.1 **Case** 3 : $|C_{20}^{gg}(0)|^2 = 1$

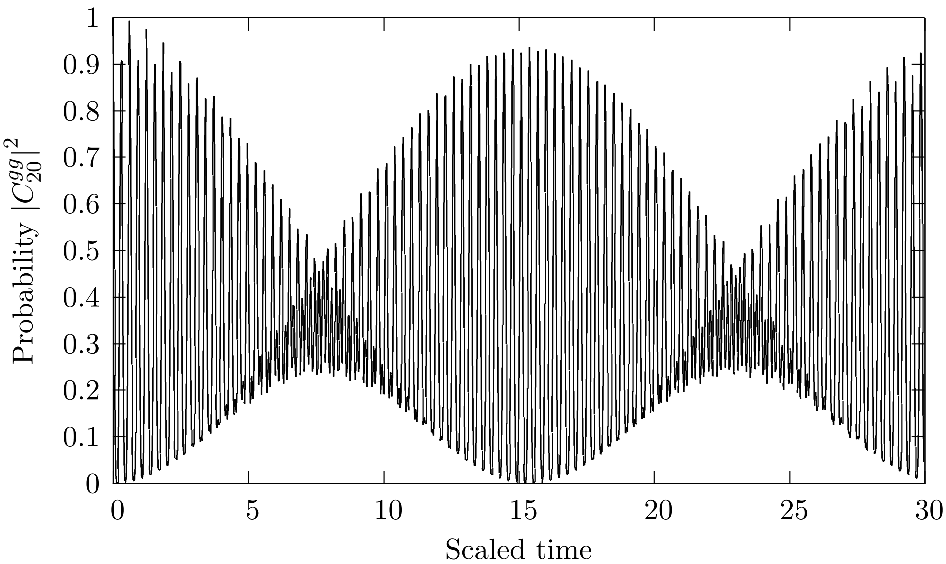
Now consider a system with an initial condition such that both atoms are in ground state and both photons are in the cavity 1, i.e.,

$$|\psi(0)\rangle = |g\rangle|g\rangle|2\rangle|0\rangle. \tag{5.65}$$

In this case the probability of the state $|g\rangle|g\rangle|2\rangle|0\rangle$ oscillates sinusoidally for strong cavity coupling and with $\chi = 0$. A different nature of evolution is shown when the value of χ_0 is nonzero. The oscillation is in the form of an amplitude modulated wave as shown in Fig. 5.13. As in the previous case, the frequency of envelop is proportional to susceptibility of the medium for a constant cavity cavity coupling. Identical nature of oscillation is shown by the probability of state $|g\rangle|g\rangle|0\rangle|2\rangle$.



(a)



(b)

Figure 5.13: Probability amplitude of the state with both the field excitation in cavity 1 is plotted against time. (a): $A = 10g$, $\chi = 0.9$ and $\omega_a = \omega_f$. (b): $A = 10g$, $\chi = 0.5$ and $\omega_a = \omega_f$.

5.2.5 Conclusion

In this work we have investigated the evolution of various atom field state probability amplitudes in a coupled cavity system, where cavities are filled with Kerr medium. The system has been solved numerically and the behaviour of it for different initial conditions and different susceptibility values are analysed. It is observed that, for weak cavity coupling the effect of susceptibility is minimal. In cases of strong cavity coupling, susceptibility factor modifies the nature in which the probability oscillates with time. Effect of susceptibility on probability of states is closely related to the initial state of the system. Dependence on the third order susceptibility of the medium open up a way to control or tune up the evolution of probabilities of states in a coupled cavity system. A controlled state evolution is essential for the data transfer processes in quantum computation.

6
Summary and Conclusion

We have studied the interaction of a two level atom and squeezed field with time varying frequency. By applying a sinusoidal variation in the frequency of the field, the randomness in population inversion is reduced and the collapses and periodic revivals are regained. Thus the field frequency modulation manipulates the population inversion in the case of squeezed light atom interaction. Also, the periodicity of revival depends on the amplitude of applied frequency modulation. By varying the periodicity of the applied frequency fluctuation the dynamics of population inversion with time can be manipulated. Two level atom field interaction has an important role in the field of quantum computation. Our results suggest a new method to control and manipulate the population of states in two level atom radiation interaction, which is very essential for quantum information processing. We have also studied the variation of atomic population inversion with time, when a two level atom interacts with light field, where the light field has a sinusoidal frequency variation with a constant phase. In both coherent field and squeezed field cases, the population inversion variation is completely different from the phase zero frequency modulation case. It is observed that in the presence of a non zero phase ϕ, the population inversion oscillates sinusoidally. Also the collapses and revivals gradually disappears when ϕ increases from 0 to $\pi/2$. When $\phi = \pi/2$ the evolution of population inversion is identical to the case when a two level atom interacts with a

Fock state. Thus, by applying a phase shifted frequency modulation one can induce sinusoidal oscillations of atomic inversion in linear medium, those normally observed in Kerr medium. We have considered the system consisting of a two level atom in Kerr medium interacting with quadrature squeezed photon field in the adiabatic limit. The frequency of the field is set to be fluctuating and phase shifted. Evolution of population inversion and entanglement entropy of the system is analysed by varying parameters. It is observed that in the nonlinear medium also sinusoidal frequency fluctuation modifies the time evolution of population inversion. These modifications are enhanced in the presence of a phase in the frequency fluctuation. The entanglement entropy of the system also has a close dependence on the field frequency fluctuations. It becomes more ordered and controllable when the frequency is sinusoidally fluctuating. We noticed that the entanglement between the atom and field can be controlled by varying the period of the field frequency fluctuations. Many interesting behaviour in the evolution of a two level atom in Kerr medium can also be produced in linear medium by including phase factor in the frequency modulation. We have analysed the evolution of atom field state probability in a coupled cavity system. Analytical formulation for the time variation of atomic and field probability is done in a single excitation subspace. Atomic excitation transfer between cavities for different limits of atom-field coupling strength, g and cavity-cavity coupling strength, A are investigated. It is observed that periodic transfer of excitation probability between cavities exists. The time period for complete excitation transfer between cavities for various limiting cases of coupling strength are predicted. An analytical expression is obtained for the population inversion of the system which evolves sinusoidally with time. We have also investigated the evolution of various atom field state probability amplitudes in a coupled cavity system, where cavities are filled with Kerr medium. The system has been solved numerically and the behaviour of it for different initial conditions and different susceptibility values are analysed. It is observed that, for weak cavity coupling the effect of susceptibility is minimal. In cases of strong cavity coupling, susceptibility factor modifies the nature in which the probability oscillates with time. Effect of susceptibility on probability of states is closely related to the initial state of the system.

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