SOME PROPERTIES OF RECIPROCAL COORDINATE SUBTANGENT IN THE CONTEXT OF STOCHASTIC MODELLING

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By

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CERTIFICATE

Certified that the thesis entitled "Some Properties of Reciprocal Coordinate Subtangent in the Context of Stochastic Modelling", is a bonafide record of work done by Mr. Sreejith T. B. under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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Certified that all the relevant corrections and modifications suggested by the audience during pre-synopsis seminar and recommended by the Doctoral committee of the candidate has been incorporated in the thesis.

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Cochin 682 022 *Sreejith T. B.*

July 2017

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Introduction and review of literature

1.1 Introduction

The concept of coordinate subtangent is not only of use in geometry but also of importance as a statistical measure. The reciprocal coordinate subtangent (RCST) has been used in the statistical literature as a useful tool to describe the behaviour of a density curve. As the strong unimodal densities have an increasing RCST function, RCST is considered as a measure for strongly unimodal property (see Hajek and Sidak (1967)). By definition RCST to a curve $y = f(x)$ is given by

$$\eta(x) = -\frac{f'(x)}{f(x)}, \tag{1.1}$$

provided $f'(x)$ exist.

Let X be an absolutely continuous random variable (rv) with probability density function (pdf) $f(\cdot)$ such that $f'(\cdot)$ exists. Then we say that the RCST of the density curve of X exists and is defined by (1.1). Equivalently,

$$\eta(x) = -\frac{d}{dx} \log f(x). \tag{1.2}$$

Since the application of RCST is closely related to reliability modelling, in following sections we give a brief review on basic reliability concepts and other related concepts useful in the present study, followed by more applications of RCST are discussed.

1.2 Some basic concepts in reliability theory - Univariate case

The term reliability of a device, component, material or structure, is to denote the probability of performing its intended function satisfactorily, for a given period of time when operating under normal environmental conditions.

1.2.1 Reliability function

Let $a = \inf\{x|F(x) > 0\}$ and $b = \sup\{x|F(x) < 1\}$ be such that (a, b), $-\infty \leq a < b \leq \infty$ is the interval support of rv X. The reliability function or survival function (sf) of a rv X, denoted by $\bar{F}(\cdot)$ is defined as

$$\bar{F}(x) = P(X > x) = 1 - F(x),$$

where $F(\cdot)$ is the distribution function (df) of rv X. It gives the probability of failure free operation for a time period greater than x.

1.2.2 Hazard rate

The hazard rate (or failure rate) of a rv X, denoted by $h(\cdot)$, is defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P[x \leq X < x + \Delta x | X > x]}{\Delta x}. \tag{1.3}$$

The failure rate $h(x)$, measures the instantaneous rate of failure at time x, given that the component survives at least up to time x. $h(x)\Delta x$ represents the approximate probability of failure in the interval $[x, x+\Delta x)$, given the component survived up to time x, provided Δx is very small. Kotz and Shanbhag (1980) defined failure rate as the Radon Nikodym derivative with respect to Lebesgue measure on $\{x : F(x) < 1\}$, of the hazard measure $H(B) = \int_B \frac{dF(x)}{[1-F(x)]}$ for every Borel set B of the form $(-\infty, L)$, where $L = \inf\{x : F(x) = 1\}$. If $f(\cdot)$ is the pdf of X, (1.3) can be equivalently written as

$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d}{dx} \log \bar{F}(x).$$

$h(\cdot)$ uniquely determines the sf $\bar{F}(\cdot)$ through the relationship

$$\bar{F}(x) = \exp\left(-\int_0^x h(u)du\right) = \exp(-H(x)),$$

where $H(x) = \int_0^x h(u)du$ is known as cumulative hazard rate.

The concept of hazard rate is widely used for characterizing lifetime distributions. For example, constancy of hazard rate is a characteristic property of exponential distribution (Galambos and Kotz (1978)). A large volume of literature is available on characterizations and other properties of hazard rate function (see, for example, Barlow et al. (1963), Nanda and Shaked (2001), Nair and Asha (2004), Nanda (2010), Noughabi et al. (2013) and references therein).

1.2.3 Reversed hazard rate

Barlow et al. (1963) proposed reversed hazard rate function for a rv X, denoted by $\bar{h}(\cdot)$ and is defined as

$$\bar{h}(x) = \lim_{\Delta x \to 0} \frac{P[x - \Delta x < X \leq x | X \leq x]}{\Delta x}.$$

$\bar{h}(x)$ measures the instantaneous rate of failure of a unit at time x, given that it failed before time x. Thus, $\bar{h}(x)\Delta x$ gives the probability that the unit failed in an infinitesimal interval $(x - \Delta x, x]$, given that it failed before x. If the pdf $f(\cdot)$ exists, the above equation can be expressed as

$$\bar{h}(x) = \frac{f(x)}{F(x)} = \frac{d}{dx} \log F(x).$$

Keilson and Sumita (1982) shown that $\bar{h}(\cdot)$ determines the df through the relationship

$$F(x) = \exp\left(- \int_x^b \bar{h}(u) du \right) = \exp(-\bar{H}(x)),$$

where $\bar{H}(x) = \int_x^b \bar{h}(u) du$ denotes the cumulative reversed hazard rate.

Finkelstein (2002) established the relationship between $\bar{h}(\cdot)$ and $h(\cdot)$ as

$$\bar{h}(x) = \frac{h(x)}{\exp\left(\int_0^x h(u) du\right) - 1}.$$

For more details on reversed hazard rate one can refer to Gupta and Nanda (2001), Nanda and Shaked (2001), Nair and Asha (2004), Bartoszewicz and Skolimowska (2004), Chandra and Roy (2005), Nair et al. (2005), Sunoj and Maya (2006), Sankaran et al. (2007) and Kundu and Ghosh (2017) .

1.2.4 Mean residual life function

For a rv X with $E(X) < \infty$, the mean residual life function (MRLF) denoted by $r(\cdot)$, defined by Swartz (1973) as

$$r(x) = E(X - x | X > x). \tag{1.4}$$

$r(x)$ measures the average residual life of a component which has survived a time x. If the df $F(\cdot)$ is continuous with respect to Lebesgue measure, (1.4) becomes

$$r(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(u) du.$$

$r(\cdot)$ uniquely determines the underlying distribution through the relationship

$$\bar{F}(x) = \frac{r(0)}{r(x)} \exp\left[-\int_0^x \frac{1}{r(u)} du\right].$$

Model identification can be done easily by knowing the functional form of $r(\cdot)$. For example, characterization of distribution using the linear form of $r(\cdot)$ is available in Hall and Wellner (1981). MRLF is related to the failure rate by the equation

$$h(x) = \frac{1 + r'(x)}{r(x)}.$$

Bryson and Siddiqui (1969) proved that increasing hazard rate of a component implies decreasing MRLF of that component.

For more properties on $r(\cdot)$, one could refer to Hall and Wellner (1981), Mukherjee and Roy (1986), Nanda (2010), Gupta (2016) and references therein.

1.2.5 Reversed mean residual life function

The reversed mean residual life function is an analogous concept of MRLF but defined for the past lifetime $(x - X|X \leq x)$, given by

$$\bar{r}(x) = E(x - X|X \leq x).$$

It measures the average past lifetime of a rv which failed at time x. It is also known as mean inactivity time or mean past lifetime in reliability. If the df $F(\cdot)$ is continuous with respect to Lebesgue measure, $\bar{r}(\cdot)$ can be written as

$$\bar{r}(x) = \frac{1}{F(x)} \int_0^x F(u)du.$$

The reversed mean residual life time is related to reversed hazard rate through the relationship,

$$\bar{h}(x) = \frac{1 - \bar{r}'(x)}{\bar{r}(x)}.$$

Like $r(\cdot)$, $\bar{r}(\cdot)$ also uniquely determines the underlying df by the relationship (Chandra and Roy (2001)),

$$F(x) = \exp\left(-\int_x^\infty \frac{1 - \bar{r}'(u)}{\bar{r}(u)}du\right).$$

For more details on reversed mean residual life functions we refer to Kayid and Ahmad (2004), Ahmad and Kayid (2005), Gandotra et al. (2011), Kayid and Izadkhah (2014), Kundu and Ghosh (2017) and references therein.

1.2.6 Vitality function

Kupka and Loo (1989) introduced the concept of vitality function as a Borel-measurable function on the real line as

$$m(x) = E\left(X \,|X > x\right) = \frac{1}{\bar{F}(x)} \int_x^\infty u f(u) du. \tag{1.5}$$

Clearly, (1.5) measures the expected life of a component, when it has survived x units of time. The vitality function is closely related to MRLF by the relationship

$$m(x) = x + r(x)$$

and

$$m'(x) = r(x)h(x),$$

where $m'(x)$ is the derivative of $m(x)$. Due to the one-to-one relationship between $m(\cdot)$ and $r(\cdot)$, the vitality function uniquely determines the underlying distribution.

1.3 Bivariate case

Let (X_1, X_2) be a random vector defined on $\mathbb{R}_2 = (-\infty, \infty) \times (-\infty, \infty)$. Then joint (bivariate) df of (X_1, X_2) is defined as $F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)$. It satisfies the following properties:

1) $\lim\limits_{x_1 \to -\infty} \lim\limits_{x_2 \to -\infty} F(x_1, x_2) = \lim\limits_{x_1 \to -\infty} F(x_1, x_2) = \lim\limits_{x_2 \to -\infty} F(x_1, x_2) = 0,$

2) $\lim\limits_{x_1 \to \infty} \lim\limits_{x_2 \to \infty} F(x_1, x_2) = 1,$

3) If $a < b$ and $c < d$, then $F(a, c) < F(b, d),$

4) If $a > x_1$ and $b > x_2$, then $F(a, b) - F(a, x_2) - F(x_1, b) + F(x_1, x_2) \geq 0.$

The bivariate sf of (X_1, X_2) is defined as $\bar{F}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$. $\bar{F}(x_1, x_2)$ is related to $F(x_1, x_2)$ by the equation

$$\bar{F}(x_1, x_2) = 1 - \lim_{x_2 \to \infty} F(x_1, x_2) - \lim_{x_1 \to \infty} F(x_1, x_2) + F(x_1, x_2).$$

If $F(x_1, x_2)$ is absolutely continuous and if the second order derivative exists then the joint density function $f(x_1, x_2)$ can be defined as

$$f(x_1, x_2) = \frac{\partial^2 \bar{F}(x_1, x_2)}{\partial x_1 \partial x_2} = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}.$$

1.3.1 Bivariate hazard rate

A straightforward extension of hazard rate (or failure rate) in univariate case to the bivariate case is due to Basu (1971), defined as a scalar failure rate,

$$k(x_1, x_2) = \frac{f(x_1, x_2)}{\bar{F}(x_1, x_2)}.$$

Puri and Rubin (1974) characterized a mixture of exponential distributions by the constancy $k(x_1, x_2) = c$ for $x_1 > 0$ and $x_2 > 0$. However, in general $k(x_1, x_2)$ does not determine a bivariate distribution uniquely. For more properties, see Yang and Nachlas (2001), Finkelstein (2003) and Finkelstein and Esaulova (2005).

An alternative and a more popular definition on bivariate hazard rate is due to Johnson and Kotz (1975) who proposed a vector-valued bivariate failure rate,

$$h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)),$$

where

$$h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \log \bar{F}(x_1, x_2), \quad i = 1, 2,$$

is the instantaneous failure rate of X_i at time x_i given that X_i was alive at time x_i and that X_{3-i} survived beyond time x_{3-i}, $i = 1, 2$. Unlike $k(x_1, x_2)$, $h(x_1, x_2)$ uniquely determines the df (see Marshall and Olkin (1979) and Shanbhag and Kotz (1987)) through the expression

$$\bar{F}(x_1, x_2) = \exp\left[-\int_0^{x_1} h_1(u, 0)du - \int_0^{x_2} h_2(x_1, v)dv\right]$$

or

$$\bar{F}(x_1, x_2) = \exp\left[-\int_0^{x_1} h_1(u, x_2)du - \int_0^{x_2} h_2(0, v)dv\right].$$

Some characterizations of probability models based on $h(x_1, x_2)$ can be found in Navarro and Ruiz (2004), Kotz et al. (2007) and Navarro et al. (2007).

Some other versions of failure rate in bivariate set up are also available in literature, for example Cox (1972), Marshall (1975), Shaked and Shanthikumar (1987), Basu and Sun (1997), Finkelstein (2003) and references therein.

1.3.2 Bivariate reversed hazard rate

Motivated with the wide applicability of bivariate failure rate due to Johnson and Kotz (1975), Roy (2002a) proposed a vector-valued reversed hazard rate. Let (X_1, X_2) be a random vector with joint df $F(x_1, x_2)$ and $F_i(\cdot)$ denotes the marginal df of $X_i, i = 1, 2$. The support of (X_1, X_2) be $D = [0, b_1] \times [0, b_2]$ where (b_1, b_2) is such that $F(b_1, b_2) < 1$ then the bivariate reversed failure rate is defined as

$$\bar{h}(x_1, x_2) = (\bar{h}_i(x_1, x_2), \bar{h}_i(x_1, x_2)),$$

where

$$\begin{aligned}\bar{h}_i(x_1, x_2) &= \lim_{\Delta x_i \to 0} \frac{P\left(x_i - \Delta x_i \leq X_i \leq x_i | X_1 \leq x_1, X_2 \leq x_2\right)}{\Delta x_i} \\ &= \frac{\partial}{\partial x_i} \log F(x_1, x_2), i = 1, 2.\end{aligned}$$

Here $\bar{h}_1(x_1, x_2)\Delta x_1$, represents the probability of failure of the first component in the interval $(x_1 - \Delta x_1, x_1]$ given that it has failed before x_1 and the second has failed before x_2. The interpretation for $\bar{h}_2(x_1, x_2)$ is similar.

$\bar{h}(x_1, x_2)$ uniquely determine $F(x_1, x_2)$ by the relationships

$$F(x_1, x_2) = \exp\left[-\int_{x_1}^{b_1} \bar{h}_1(u, b_2)du - \int_{x_2}^{b_2} \bar{h}_2(x_1, v)dv\right]$$

or

$$F(x_1, x_2) = \exp\left[-\int_{x_1}^{b_1} \bar{h}_1(u, x_2)du - \int_{x_2}^{b_2} \bar{h}_2(b_1, v)dv\right].$$

For more details on bivariate reversed hazard rate we refer to Sankaran and Gleeja (2006), Asha and Rejeesh (2007), Sankaran and Gleeja (2008), Asha and Rejeesh (2009), Domma (2011) and Kundu and Kundu (2017).

1.3.3 Bivariate mean residual life function

Buchanan and Singpurwalla (1977) introduced a bivariate MRLF as

$$e(x_1, x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_0^\infty \int_0^\infty P[X_1 > x_1 + t_1, X_2 > x_2 + t_2] dt_1 dt_2, \ x_i > 0, i = 1, 2.$$

Even if $e(x_1, x_2)$ is a direct extension of univariate MRLF, it does not uniquely determine the underlying distribution.

An alternative definition to bivariate MRLF is provided by Shanbhag and Kotz (1987) and Arnold and Zahedi (1988) as follows. Let (X_1, X_2) be a random vector on $\mathbb{R}_2^+ = \{(x_1, x_2)|x_i > 0, i = 1, 2\}$ with joint df $F(x_1, x_2)$ and let (L_1, L_2) be the vector of extended real numbers such that $L_i = \inf\{x|F_i(x_i) = 1\}$ where $F_i(\cdot)$ is the df of X_i. Further let $E(X_i) < \infty$, for $i = 1, 2$. The vector-valued Borel-measurable function $r(x_1, x_2)$ on $\mathbb{R}_2^+$ is given by

$$\begin{aligned} r(x_1, x_2) &= (r_1(x_1, x_2), r_2(x_1, x_2)) \\ &= (E(X_1 - x_1|X_1 > x_1, X_2 > x_2), E(X_2 - x_2|X_1 > x_1, X_2 > x_2)), \end{aligned}$$

for all $(X_1, X_2) \in \mathbb{R}_2^+$, $x_i < L_i$, $i = 1, 2$, is called the bivariate mean residual life function. When (X_1, X_2) is continuous and nonnegative, the components of bivariate MRLF are given by

$$r_1(x_1, x_2) = E(X_1 - x_1|X_1 > x_1, X_2 > x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_{x_1}^\infty \bar{F}(u, x_2) du$$

and

$$r_2(x_1, x_2) = E(X_2 - x_2 | X_1 > x_1, X_2 > x_2) = \frac{1}{\bar{F}(x_1, x_2)} \int_{x_2}^{\infty} \bar{F}(x_1, v) dv.$$

Unlike $e(x_1, x_2)$, the bivariate MRLF $r(x_1, x_2)$ uniquely determines the distribution through the identities (Nair and Nair (1988))

$$\bar{F}(x_1, x_2) = \frac{r_1(0,0) r_2(x_1, 0)}{r_1(x_1, 0) r_2(x_1, x_2)} \exp\left[-\int_0^{x_1} \frac{du}{r_1(u, 0)} - \int_0^{x_2} \frac{dv}{r_2(x_1, v)} \right]$$

or

$$\bar{F}(x_1, x_2) = \frac{r_1(0, x_2) r_2(0, 0)}{r_1(x_1, x_2) r_2(0, x_2)} \exp\left[-\int_0^{x_2} \frac{dv}{r_2(0, v)} - \int_0^{x_1} \frac{du}{r_1(u, x_2)} \right].$$

Similar to the relationship between failure rate and MRLF in the univariate case, the bivariate MRLF is related to bivariate failure rate by

$$h_i(x_1, x_2) = \frac{1 + \frac{\partial}{\partial x_i} r_i(x_1, x_2)}{r_i(x_1, x_2)}, \quad i = 1, 2.$$

For more applications of bivariate mean residual life function we refer to Sankaran and Nair (1993a), Roy (2002b), Nair et al. (2004) and Sunoj and Vipin (2017).

1.3.4 Bivariate reversed mean residual life function

A vector-valued bivariate reversed mean residual life function is proposed by Nair and Asha (2008). Let (X_1, X_2) be a random vector defined on $\mathbb{R}_2$ with joint df $F(x_1, x_2)$ and marginal df $F_i(\cdot)$, $i = 1, 2$, $E(X_1, X_2) < \infty$ and let (a_1, a_2) and (b_1, b_2) be vectors of real numbers such that $a_i = \inf\{x | F_i(x) > 0\}$ and $b_i = \sup\{x | F_i(x) < 1\}$ then bivariate reversed mean residual life function is defined as a Borel-measurable function

$$\bar{r}(x_1, x_2) = (\bar{r}_1(x_1, x_2), \bar{r}_2(x_1, x_2)),$$

where

$$\bar{r}_1(x_1, x_2) = E(x_1 - X_1 | X_1 \leq x_1, X_2 \leq x_2) = \frac{1}{F(x_1, x_2)} \int_{a_1}^{x_1} F(u, x_2) du$$

and

$$\bar{r}_2(x_1, x_2) = E(x_2 - X_2 | X_1 \leq x_1, X_2 \leq x_2) = \frac{1}{F(x_1, x_2)} \int_{a_2}^{x_2} F(x_1, v) dv.$$

The bivariate reversed mean residual life function uniquely determines the underlying distribution through the relationships

$$F(x_1, x_2) = \frac{\bar{r}_1(b_1, b_2)\bar{r}_2(x_1, b_2)}{\bar{r}_1(x_1, b_2)\bar{r}_2(x_1, x_2)} \exp\left(- \int_{x_1}^{b_1} \frac{du}{\bar{r}_1(u, b_2)} - \int_{x_2}^{b_2} \frac{dv}{\bar{r}_2(x_1, v)} \right)$$

and

$$F(x_1, x_2) = \frac{\bar{r}_1(b_1, x_2)\bar{r}_2(b_1, b_2)}{\bar{r}_1(x_1, x_2)\bar{r}_2(b_1, x_2)} \exp\left(- \int_{x_1}^{b_1} \frac{du}{\bar{r}_1(u, x_2)} - \int_{x_2}^{b_2} \frac{dv}{\bar{r}_2(b_1, v)} \right).$$

Further, bivariate reversed mean residual life function is related to bivariate reversed hazard rate by

$$\bar{h}_i(x_1, x_2) = \frac{1 - \frac{\partial}{\partial x_i}\bar{r}_i(x_1, x_2)}{\bar{r}_i(x_1, x_2)}, \quad i = 1, 2.$$

For more properties and results based on bivariate reversed mean residual life function, we refer to Kayid (2006), Asha and Rejeesh (2009) and Ghosh and Kundu (2017).

1.4 Weighted distributions

The concept of weighted distributions can be traced from the studies by Fisher (1934) on how methods of ascertainment can influence the form of distribution of recorded observations. However, Rao (1965) identified the need for a unifying the concept of weighted distributions and studied various sampling situations that can be modeled by weighted distributions. These situations happen when the recorded observations cannot be considered as a random sample from the original distributions, such as non observability of some events or damage occurred to the original observation resulting in reduced value, or the adoption of a sampling mechanism which gives unequal chances to the units in the original.

A mathematical definition of a weighted distribution is obtained by considering a probability space $(\Omega, \Im, P)$ and a rv $X : \Omega \to H$, where $H = (a, b)$ is an interval on the real line with $a > 0$ and $b(> a)$ can be finite or infinite. When the df $F(\cdot)$ of X is absolutely continuous with pdf $f(\cdot)$ and $w(\cdot)$, a nonnegative function satisfying $\mu^w = E(w(X)) < \infty$, then the rv X^w with pdf

$$f^w(x) = \frac{w(x)}{\mu^w} f(x), \quad a < x < b,$$

is said to have weighted distribution, corresponding to the distribution of X. The definition in the discrete case is similar.

Depending on the selection of weight function $w(\cdot)$, we have different weighted distributions. For example, when $w(x) = x$, then X^w is called the length-biased rv X^l with pdf,

$$f^l(x) = \frac{x}{\mu} f(x), \quad a < x < b,$$

where $\mu = E(X) < \infty$. Length-biased sampling is usually adopted when a suitable sampling frame is absent. In length-biased sampling items are selected at a rate proportional to their length, so that larger values of the quantity being measured are sampled with higher probabilities. In such situations, the possible bias due to the nature of data collection process can be utilized to connect the population parameters to that of the sampling distribution. That is, if we know the choice mechanism behind the biased sample, then the process of inference on population parameters is easier. Length-biased sampling has wide variety of applications on various topics such as reliability theory, survival analysis, population studies and clinical trials. For a more details on various aspects of length-biased sampling one can refer to Fisher (1934), Rao (1965), Neel and Schull (1966), Eberhardt (1968), Zelen (1971), Cook and Martin (1974), Patil and Rao (1977, 1978), Eberhardt (1978), Sankaran and Nair (1993b), Sen and Khattree (1996), Oluyede (1999, 2000), Van et al. (2000), Sunoj (2004), Bar-Lev and Schouten (2004), Kersey and Oluyede (2013) and Das and Kundu (2016).

When the weight is inversely proportional to length of unit of interest, we use $w(x) = \frac{1}{x}$, called inversed length-biased distribution (see Barmi and Simonoff (2000)). Barmi and Simonoff (2000) proposed a transformation-based technique for the density estimation of weighted distributions and used length-biased and inverse length-biased sampling for the study.

Some of the known and important distributions in statistics and applied probability can be expressed as weighted distributions. Equilibrium distributions, residual-life distributions, distribution of order statistics, proportional hazards models (see Gupta and Kirmani (1990), Bartoszewicz and Skolimowska (2004)) are some of the examples. Some of the special cases of weighted distributions are given Table 1.1. Thus the theory of weighted distributions is appropriate whenever these distributions are applied. For more details of applications and recent works of weighted distributions, we refer to Gupta and Kirmani (1990), Jones (1991), Navarro et al. (2001), Sunoj and Maya (2006), Di Crescenzo and Longobardi (2006), Maya and Sunoj (2008), Navarro et al. (2014), Jarrahiferiz et al. (2016) and Sunoj and Vipin (2017).

Table 1.1: Special cases of weighted distributions

$w(x)$	**Distribution**	**pdf**
$\frac{1}{h(x)}$	Equilibrium distribution	$\frac{\bar{F}(x)}{E(X)}$
$\left[\bar{F}(x)\right]^{\theta-1}, \theta > 0$	Proportional hazards model	$\theta \left[\bar{F}(x)\right]^{\theta-1} f(x)$
$\left[F(x)\right]^{\theta-1}, \theta > 0$	Proportional reversed hazards model	$\theta \left[F(x)\right]^{\theta-1} f(x)$
$\frac{f(x+t)}{f(x)}$	Residual life distribution	$\frac{f(x+t)}{\bar{F}(t)}$
$\frac{f(t-x)}{f(x)}, t > x$	Reversed residual life distribution	$\frac{f(t-x)}{F(t)}$
$\left[F(x)\right]^{j-1} \left[\bar{F}(x)\right]^{n-j}$, $j = 1, 2, ..., n$	Distribution of jth order statistics	$\frac{n!}{(j-1)!(n-j)!} \left[F(x)\right]^{j-1} \left[\bar{F}(x)\right]^{n-j} f(x)$
$\left[-\log \bar{F}(x)\right]^{n-1}$	Distribution of upper record value	$\frac{\left[-\log \bar{F}(x)\right]^{n-1}}{(n-1)!} f(x)$
$\left[-\log F(x)\right]^{n-1}$	Distribution of lower record value	$\frac{\left[-\log F(x)\right]^{n-1}}{(n-1)!} f(x)$

1.4.1 Bivariate weighted distributions

The wide applicability of weighted distributions in the univariate case has motivated many researchers to extend the concept of weighted distribution to higher dimensions. Let $(X_1, X_2)'$ be a bivariate random vector in the support of $(a_1, b_1) \times (a_2, b_2)$,

$b_i > a_i$, $i = 1, 2$ where (a_i, b_i) is an interval on the real line with absolutely continuous df $F(x_1, x_2)$, and pdf $f(x_1, x_2)$. By defining $w(x_1, x_2)$ as a nonnegative weight function satisfying $E(w(X_1, X_2)) < \infty$, Mahfoud and Patil (1982) defined bivariate weighted distribution as the distribution of the random vector $(X_1^w, X_2^w)'$ with pdf

$$f^w(x_1, x_2) = \frac{w(x_1, x_2)}{E(w(X_1, X_2))} f(x_1, x_2), \quad a_i < x_i < b_i,\ i = 1, 2. \tag{1.6}$$

For more properties of bivariate weighted distributions one can refer to Nair and Sunoj (2003), Sunoj and Sankaran (2005), Navarro et al. (2006), Arnold et al. (2016), Alavi (2017), Kayal and Sunoj (2017) and references therein.

Jain and Nanda (1995) extended the definition to the p - variate case. Let $\boldsymbol{X} = (X_1, X_2, ..., X_P)'$ be a p - dimensional nonnegative random vector with pdf $f(\boldsymbol{x})$ and $\boldsymbol{X}^w = (X_1^w, X_2^w, ..., X_p^w)'$ be the multivariate weighted version of $\boldsymbol{X}$ such that the weight function $w(\boldsymbol{x})[w : \boldsymbol{X} \to A \subseteq R^+$, where R^+ denotes the positive real line] is nonnegative with finite and nonzero expectation. Then the multivariate weighted density corresponding to $f(\boldsymbol{x})$ is given by

$$f^w(\boldsymbol{x}) \;=\; \frac{w(\boldsymbol{x}) f(\boldsymbol{x})}{E(w(\boldsymbol{X}))}. \tag{1.7}$$

For more applications of multivariate weighted distributions, see Navarro et al. (2006), Kim (2008) and Kim (2010a,b).

1.5 Characterization

In modelling statistical data, an important problem one encounters is the identification of an appropriate model that is supposed to generate the observations. In such a situation one can start with a general system of distributions and then select an appropriate member from the system that fits the data. The difficulty here is that most of the models used can have different right tail behaviour and the sample size may not be large enough to observe such differences. A standard practice used in such situations is to ascertain the physical properties of the process generating the observations, express them by means of equations or inequalities and then solve them to obtain the model. The only exact method of finding a probability distribution is to use a characterization theorem, which in general terms say that under certain conditions a family of distributions $\boldsymbol{F}$ is the only one possessing a designated property $\boldsymbol{P}$. For instance, in reliability theory, the failure rate or mean residual life are uniquely determines the underlying distribution.

1.6 Applications of reciprocal coordinate subtangent (RCST)

RCST plays a very important role in reliability analysis, however, used it rather unknowingly. For example, failure rate or hazard rate is the RCST measured on the curve $y = \bar{F}(x)$. Also, since many of the failure rate functions have complex expressions, Glaser (1980) identified (1.1) (but not called as 'RCST') as an easy statistical tool to determine the shape of the failure rate function. $\eta(\cdot)$ in (1.1) can also be represented in terms of the failure rate $h(\cdot)$ by

$$\eta(x) = h(x) - \frac{h'(x)}{h(x)}, \tag{1.8}$$

provided $h'(x)$ exist. According to Glaser (1980), the nature of failure rate function can be determined in the following way.

(i). If $\eta(\cdot) \in I$, then $h(\cdot) \in I$ (increasing failure rate, IFR)

(ii). If $\eta(\cdot) \in D$, then $h(\cdot) \in D$ (decreasing failure rate, DFR)

(iii). If $\eta(\cdot) \in BS$, then $h(\cdot)$ is either BS (bathtub shaped failure rate) or D

(iv). If $\eta(\cdot) \in UBS$, then $h(\cdot)$ is either UBS (upside-down bathtub shaped failure rate) or I.

Glaser (1980) and Gupta and Warren (2001) used $\eta(\cdot)$ to determine the shape of the failure rate of the mixture of two gamma densities. Navarro and Hernandez (2004) have considered the shape of the failure rate of the mixture of two positively truncated normal distributions by using $\eta(\cdot)$.

However, the Theorem 3.1 given in Mukherjee and Roy (1989) established RCST as a measure to characterize various models by a unique determination of $f(\cdot)$ from $\eta(\cdot)$ by

$$f(x) = k \ \exp\left[-\int_0^x \eta(u)du\right], \tag{1.9}$$

where k is a normalizing constant. Mukherjee and Roy (1989) also studied some properties and applications of $\eta(\cdot)$ and proved characterization results to certain important life distributions. For more applications of $\eta(\cdot)$ in reliability theory, we refer to Gupta (2001), Gupta and Warren (2001), Block et al. (2002), Ghitany (2004), Mi (2004), Lai and Xie (2006) and Navarro (2008). Recently, Roy and Roy (2009) further extended the concept of RCST in the multivariate setup and proved some characterization theorems useful in reliability modelling.

1.7 Present Study

The thesis is organized into seven chapters. The present work is fully devoted to study RCST and its various properties. In continuation of the present chapter, in Chapter 2, we further explore the concept of RCST in the context of weighted models. We prove characterizations to some important distributions such as gamma and Rayleigh, equilibrium, residual lifetime (reversed residual lifetime) and proportional hazards model. We derive an identity for weighted distribution when RCST takes the form of a general class of distributions which contains many important moment relationships, and a generalization of the result due to Nair and Sankaran (2008).

In Chapter 3, we study the monotone properties of weighted random variable based on RCST and illustrated it using some examples. Roy and Roy (2013) introduced Mean RCST (MRCST) as a counterpart of mean residual life function in the density domain. We study the relationship between RCST and MRCST for characterizing distributions. The different stochastic orderings of two random variables based on RCST and MRCST are also studied. We also prove certain characterizations of probability models based on RCST of record values. We conclude the chapter on the study of some properties of RCST in the context of circular distributions.

In Chapter 4, RCST is studied in context of bivariate and conditionally specified models. We have obtained characterization results for a general bivariate model proposed by Navarro and Sarabia (2013), Sarmanov family, Farlie-Gumbel-Morgenstern (FGM) family and Ali-Mikhail-Haq family proposed by Ali et al. (1978). We define RCST for conditionally specified distributions and proved characterization results based on it. We also obtain a relationship between local dependence function of

Holland and Wang (1987) and RCST.

Chapter 5 is focused on finding the properties of RCST in discrete time. Gupta et al. (1997) introduced a discrete measure that can be useful for measuring the shape of a failure rate function. We study the usefulness of that measure (discrete analogue of RCST) in modelling different discrete distributions/families. A new definition for discrete analogue of RCST is also introduced that possesses certain new properties than the measure introduced by Gupta et al. (1997). A discrete proportional hazards model is characterized based on this definition.

In Chapter 6, we propose a nonparametric estimator for RCST under right censored dependent case. We have examined the asymptotic property of the estimator. A simulation study is carried out to illustrate the performance of the estimator. Finally, in Chapter 7, we have given a conclusion of the thesis and a brief outline about the future research.

Chapter 2

Some results on reciprocal coordinate subtangent in the context of weighted distributions[1]

2.1 Introduction

In most cases of analysing lifetime data, a fundamental problem is the identification of an appropriate model that is supposed to generate the observations. Generally, it is not easy to segregate all the physical properties that individually or collectively contributing to the life mechanism and to mathematically account for each and therefore the task of identifying the appropriate model representing the data is challenging. A standard practice used in such a modelling situation is to ascertain the physical properties of the data generating mechanism, express them in terms of equations or inequalities and then solve them to obtain the best model. For exam-

[1]Contents of this chapter have been published as entitled "Some results on reciprocal subtangent in the context of weighted models", *Communications in Statistics–Theory and Methods*, 41(8),1397–1410 (see Sunoj and Sreejith (2012)).

ple, in reliability modelling the basic concepts such as hazard rate, mean residual life function, vitality function, etc. are used to describe the physical characteristics of the life mechanism and therefore these concepts form the basis of specifying a probability distribution of lifetimes. If one can translate the characteristics of the life mechanism in terms of the reliability properties such as failure rate, mean residual life function or any other reliability related concepts and if there exists a probability distribution characterised by such a property or concept, the problem of model identification is satisfactorily resolved.

In this chapter, we study the concept of RCST in the context of weighted models. The chapter is organized as follows. In Section 2.2, we introduce RCST for weighted models and prove some univariate characterizations to distributions such as gamma and Rayleigh, under the inversed length-biased model. We also introduce characterizations to equilibrium, residual lifetime (reversed residual lifetime) and proportional hazards model in the context of weighted distributions. An identity for weighted distribution is also obtained when the RCST takes the form in terms of a general class of distributions. In Sections 2.3 and 2.4, we further extend RCST of weighted distributions to bivariate and multivariate setup and examine some characterization theorems arising out of it.

2.2 Univariate RCST for weighted models

By virtue of the definition of RCST in (1.1), the RCST for the weighted rv X^w is given by

$$\eta^w(x) \;=\; -\,\frac{f^{w\prime}(x)}{f^w(x)},$$

where $f^{w\prime}(\cdot)$ is the derivative of $f^w(\cdot)$. Equivalently,

$$\eta^w(x) \;=\; \eta(x) - \frac{w'(x)}{w(x)}, \tag{2.1}$$

provided $w'(\cdot)$ exist.

Remark 2.2.1. If $w(\cdot)$ is monotonically increasing (decreasing) and $\eta(\cdot)$ is monotonically increasing or decreasing, then $\eta^w(x) \leq (\geq)\, \eta(x)$.

The following theorem uniquely determines the weighted distribution by using the RCST of weighted distributions.

Theorem 2.2.2. *For a nonnegative rv X, the RCST function of X^w, $\eta^w(\cdot)$ uniquely determines the pdf $f^w(\cdot)$ by*

$$f^w(x) = C\ \exp\left[-\int_0^x \eta^w(u)du\right], \tag{2.2}$$

where C is a constant to be determined from the identity $\int_0^\infty f^w(x)dx = 1$.

Proof. The proof follows from Theorem 3.1 of Mukherjee and Roy (1989). □

If we know the functional form of RCST for weighted distribution $\eta^w(\cdot)$, we can also find the original distribution, which is given in the following corollary.

Corollary 2.2.3. *For a nonnegative rv X, the RCST function of X^w, $\eta^w(\cdot)$ uniquely determines the pdf $f(\cdot)$ by*

$$f(x) = \frac{K}{w(x)}\ \exp\left[-\int_0^x \eta^w(u)du\right], \tag{2.3}$$

where K is a constant to be determined from the identity $\int_0^\infty f(x)dx = 1$.

Now we prove some characterization theorems to certain well-known univariate models viz. gamma, Rayleigh and some applied models such as equilibrium, proportional hazards and residual lifetime models using weighted RCST function $\eta^w(\cdot)$.

Theorem 2.2.4. *For a nonnegative rv X and weight function $w(x) = \frac{1}{x}$, then $\eta^w(x) = cx + d$ if and only if X follows a gamma distribution with pdf*

$$f(x) = a^2 x e^{-a\,x} \ \ , \ x > 0, \ a > 0 \tag{2.4}$$

according as $c = 0$ and $d > 0$, and Rayleigh distribution with pdf

$$f(x) = 2a\ x e^{-a\,x^2} \ \ , \ x > 0, \ a > 0 \tag{2.5}$$

according as $c > 0$ and $d = 0$.

Proof. Suppose X follows a gamma distribution with pdf (2.4), then

$$\eta(x) = \frac{ax - 1}{x},$$

and hence

$$\eta^w(x) = d,$$

which is of the form $\eta^w(x) = cx + d$ where $c = 0$ and $d > 0$.
On the other hand, when X follows a Rayleigh distribution with pdf (2.5), then

$$\eta(x) = \frac{2ax^2 - 1}{x},$$

and therefore

$$\eta^w(x) = 2ax,$$

which is of the form $\eta^w(x) = cx + d$ with $c > 0$ and $d = 0$.

Conversely, assume that $\eta^w(x) = d$, a constant, then from (2.3) we have

$$\begin{aligned} f(x) =& K\ xe^{-\int_0^x d\ du} \\ =& K\ xe^{-dx}, \end{aligned}$$

where K is a constant. Now using the identity $\int_0^\infty f(x)dx = 1$, we get $K = d^2$ and hence

$$f(x) = d^2 xe^{-dx}, x > 0, d > 0,$$

the gamma model (2.4). Similarly, if we assume that $\eta^w(x) = cx$, then from (2.3), we get

$$\begin{aligned} f(x) =& K\ xe^{-\int_0^x cu\ du} \\ =& K\ xe^{-\frac{c}{2}x^2}. \end{aligned}$$

Using the identity $\int_0^\infty f(x)dx = 1$, we have $K = c$ and therefore

$$f(x) = c\ xe^{-\frac{c}{2}x^2}, x > 0, c > 0,$$

which is the Rayleigh distribution (2.5). □

Theorem 2.2.5. *For a nonnegative rv X, the relationship $\eta^w(x) = h(x)$ holds if and only if X^w follows an equilibrium distribution.*

Proof. Suppose X^w follows an equilibrium distribution. *i.e.*, $w(x) = \frac{1}{h(x)}$ (see Gupta and Kirmani (1990)), then $\frac{w'(x)}{w(x)} = -\frac{h'(x)}{h(x)}$. Using (1.8) and (2.1), we get $\eta^w(x) = h(x)$.

Conversely, suppose that $\eta^w(x) = h(x)$, then from (2.2) we get

$$f^w(x) = C\ e^{-\int_0^x h(u)\ du}.$$

Equivalently,

$$f^w(x) = C\ \bar{F}(x). \tag{2.6}$$

Now from $\int_0^\infty f^w(x)dx = 1$, we have $C = \frac{1}{E(X)}$, where $E(X) = \int_0^\infty \bar{F}(x)\ dx$ and therefore (2.6) becomes

$$f^w(x) = \frac{\bar{F}(x)}{E(X)}.$$

i.e., X^w follows an equilibrium distribution. □

Theorem 2.2.6. *For a nonnegative rv X, then*

$$\eta^w(x) = \theta\ h(x) - \frac{h'(x)}{h(x)}\ \ ,\theta > 0$$

if and only if X^w follows proportional hazards model.

Proof. Suppose X^w follows proportional hazards model. *i.e.*, $w(x) = \left[\bar{F}(x)\right]^{\theta-1}, \theta > 0$ (see Bartoszewicz and Skolimowska (2004)), then $\frac{w'(x)}{w(x)} = (1-\theta)\, h(x)$. Using (1.8) and (2.1) we get, $\eta^w(x) = \theta\ h(x) - \frac{h'(x)}{h(x)}$.
Conversely, assume that $\eta^w(x) = \theta\ h(x) - \frac{h'(x)}{h(x)}$ holds, then from (2.2)

$$\begin{aligned}
f^w(x) &= C\ e^{-\int_0^x \left[\theta\ h(u) - \frac{h'(u)}{h(u)}\right]\ du}, \\
&= C\ e^{\theta \int_0^x \left[\ \frac{d}{du} \log \bar{F}(u)\right]\ du + \int_0^x \left[\ \frac{d}{du} \log h(u)\right]\ du}, \\
&= A\ \left[\bar{F}(x)\right]^{\theta} h(x), \\
&= A\ \left[\bar{F}(x)\right]^{\theta-1} f(x).
\end{aligned} \tag{2.7}$$

Using $\int_0^\infty f^w(x)dx = 1$, we get $A = \theta$ and therefore (2.7) becomes

$$f^w(x) = \theta \,\left[\bar{F}(x)\right]^{\theta-1} f(x).$$

i.e., X^w follows proportional hazards model. $\square$

Theorem 2.2.7. *For a nonnegative rv X, $\eta^w(x) = \eta(x+t)$ if and only if X^w follows a residual life distribution.*

Proof. Suppose X^w follows a residual life distribution. *i.e.*, $w(x) = \frac{f(x+t)}{f(x)}$ (see Gupta and Kirmani (1990)), then $\frac{w'(x)}{w(x)} = \eta(x) - \eta(x+t)$. From (2.1), we have $\eta^w(x) = \eta(x+t)$.

To prove the converse, assume that $\eta^w(x) = \eta(x+t)$ holds. Now from (2.2),

$$\begin{aligned} f^w(x) &= C\ e^{-\int_0^x \eta(u+t)\ du}, \\ &= C\ e^{\int_0^x \frac{d}{du} \log f(u+t)\ du}, \\ &= A(t) f(x+t). \end{aligned} \tag{2.8}$$

Using the identity $\int_0^\infty f^w(x)dx = 1$, we get $A(t) = \frac{1}{\bar{F}(t)}$ and therefore (2.8) becomes $f^w(x) = \frac{f(x+t)}{\bar{F}(t)}$, which is the residual life distribution. $\square$

Corollary 2.2.8. $\eta^w(x) = \eta(t-x)$ *if and only if X^w follows a reversed residual life distribution. (where $w(x) = \frac{f(t-x)}{f(x)}$).*

Theorem 2.2.9. *Let $\eta^w(\cdot)$ be the RCST function of X^w and let $\eta(\cdot)$, $h(\cdot)$ and $\bar{h}(\cdot)$ respectively be the RCST function, failure rate and reversed failure rate of X. Then*

$$\eta^w(x) = \eta(x) + (1-j)\,\bar{h}(x) + (n-j)\,h(x),\ j = 1, 2, ..., n.$$

if X^w follows the distribution of a j^{th} order statistics.

Proof. Let X^w follows the distribution of a j^{th} order statistics. *i.e.*, $w_j(x) = [F(x)]^{j-1}\left[\bar{F}(x)\right]^{n-j}$ (see Bartoszewicz and Skolimowska (2004)), then

$$\frac{w_j'(x)}{w_j(x)} = -\left[(1-j)\,\bar{h}(x) + (n-j)\,h(x)\right].$$

From (2.1), we have

$$\eta^w(x) = \eta(x) + (1-j)\,\bar{h}(x) + (n-j)\,h(x),\ j = 1, 2, ..., n.$$

□

Corollary 2.2.10. *If X^w follows the distribution of a first order statistics, then $\eta^w(x) = \eta(x) + (n-1)\,h(x)$.*

Corollary 2.2.11. *If X^w follows the distribution of a n^{th} order statistics, then $\eta^w(x) = \eta(x) + (1-n)\,\bar{h}(x)$.*

In the next theorem, we consider a general class of distributions by defining a rv X in the support of (a, b), a subset of the real line, $-\infty \leq a < x < b \leq \infty$ with $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$. We say that the rv X belongs to the general class of distributions if the RCST $\eta(\cdot)$ is of the form

$$\eta(x) = -\frac{k - B(x) - g'(x)}{g(x)}, \tag{2.9}$$

where k is a constant, $B(\cdot)$ is a suitably chosen function of X, $g(\cdot)$ is a real function defined on (a, b) and the derivative of $g(\cdot)$ exist. The identity (2.9) is equivalent to

$$E\left[B(X)\,|X > x\right] = k + g(x)h(x) \tag{2.10}$$

and if $\lim\limits_{x\to a}[g(x)h(x)] = 0$, then (2.10) reduces to $E\left[B(X)\,|X > x\right] = \mu + g(x)h(x)$, where $\mu = E\left[B(X)\right]$ (see Nair and Sankaran (2008)). If $B(x) = x$ then the identity (2.10) becomes

$$E\left[X\left|X > x\right.\right] = k + g(x)h(x)$$

or by using (1.5), we have

$$m(x) = k + g(x)h(x),$$

which gives the relationship connecting the vitality function and the hazard rate function for a general class of a distribution given in (2.9).

Now the following theorem gives the generalization of the identity (2.10) in the context of weighted distributions.

Theorem 2.2.12. *A rv X belongs to the general class of distributions* (2.9) *if and only if it satisfies the weighted identity corresponding to* (2.10) *as*

$$E\left[B(X)\, w(X)\left|X > x\right.\right] = k\; E\left[w(X)\left|X > x\right.\right] + w(x)g(x)h(x) + E\left[w'(X)g(X)\left|X > x\right.\right] \tag{2.11}$$

under the regularity condition $\lim\limits_{x \to b}\left[w(x)g(x)f(x)\right] = 0.$

Proof. When the rv X belongs to the general class of distributions (2.9), then from (2.1), we have

$$\eta^{w}(x) = -\frac{k - B(x) - g'(x)}{g(x)} - \frac{w'(x)}{w(x)}.$$

Equivalently,

$$\frac{f^{w\prime}(x)}{f^{w}(x)} - \frac{k - B(x) - g'(x)}{g(x)} = \frac{w'(x)}{w(x)},$$

$$g(x)f^{w\prime}(x) - k\; f^{w}(x) + B(x)\; f^{w}(x) + g'(x)f^{w}(x) = \frac{w'(x)}{w(x)}g(x)f^{w}(x),$$

which gives

$$B(x)f^w(x) = k\ f^w(x) - \frac{d}{dx}\left[g(x)f^w(x)\right] + \frac{w'(x)}{w(x)}g(x)f^w(x),$$

or

$$B(x)w(x)f(x) = k\ w(x)f(x) - \frac{d}{dx}\left[w(x)g(x)f(x)\right] + w'(x)g(x)f(x). \tag{2.12}$$

Integrating (2.12) and applying regularity condition, we get

$$\int_x^b B(u)w(u)f(u)du = k\int_x^b w(u)f(u)du + w(x)g(x)f(x) + \int_x^b w'(u)g(u)f(u)du. \tag{2.13}$$

Dividing (2.13) by $\bar{F}(x)$, we get (2.11). The converse part can be proved by retracing the above steps. □

As a special case of the above theorem, we have the following corollaries. Here we consider the length-biased distribution *i.e.,* $w(x) = x$ and for $B(x) = x$.

Corollary 2.2.13. *If* $B(x) = x$ *and* $w(x) = x$, (2.11) *reduces to*

$$E(X^2\,|X > x\,) = k\ E(X\,|X > x\,) + x\ g(x)h(x) + E(g(X)\,|X > x\,) \tag{2.14}$$

Using (2.10), (2.14) *can be written as*

$$E(X^2\,|X > x\,) = k^2 + g(x)h(x)l(x) + E(g(X)\,|X > x\,), \tag{2.15}$$

where $l(x)$ *is a linear function in* x *and if* $\lim_{x\to a}(g(x)h(x)l(x)) = 0$, *then* $k^2 = E(X^2) - E(g(X))$.

Corollary 2.2.14. *If the random variable X follows Pearson family, i.e., $g(x) = a_0 + a_1x + a_2x^2$ and $B(x) = x$, $w(x) = x$, then*

$$E(X^2\,|X > x) = l_1(x)E(X\,|X > x) + l_2(x), \tag{2.16}$$

where $l_1(x)$ and $l_2(x)$ are linear functions in x, the form same as given in Glänzel (1991). Using (2.10), (2.16) *can also be expressed in terms of failure rate $h(x)$ as*

$$E(X^2\,|X > x) = A + (A_0 + A_1x + A_2x^2)h(x)l_3(x),$$

where A is a constant, $A_i = \frac{a_i}{1-a_2}$, $a_2 \neq 1$, $i = 0, 1, 2.$ and $l_3(x)$ is a linear function in x. If $\lim\limits_{x\to a}(g(x)h(x)l_3(x)) = 0$, then $A = E(X^2)$.

Example 2.2.15 (Exponential)**.** $k = 0\,, g(x) = \frac{ax+1}{a^2}\,, f(x) = a\,e^{-a\,x},\ x > 0,\ a > 0,$ then $E(X) = \frac{1}{a}\,,\ \ V(X) = \frac{1}{a^2}$. In this case (2.15) becomes

$$E(X^2\,|X > x) = V(X) + h(x)q_1(x),$$

where $q_1(x)$ is a quadratic function in x.

Example 2.2.16 (Gamma)**.** $k = \mu\,,\ g(x) = \frac{x}{a},\ f(x) = \frac{a^{a\mu}}{\Gamma a\mu}x^{a\mu\,-1}e^{\,-a\,x}\,,\ x > 0,\ a > 0,\ \ \mu > 0$, then $E(X) = \mu\,,\ \ V(X) = \frac{\mu}{a}\,,\ \ E(X^2) = \frac{\mu}{a} + \mu^2$. Here (2.15) reduces to

$$E(X^2\,|X > x) = E(X^2) + h(x)q_2(x),$$

where $q_2(x)$ is a quadratic function in x.

Example 2.2.17 (Beta)**.** $k = \mu\,,\ g(x) = \frac{x(1-x)}{a+b},\ f(x) = \frac{1}{B(a,b)}x^{a\,-1}(1-x)^{b-1},\ 0 < x < 1\,,\ a > 0,\ b > 0$, then $E(X) = \frac{a}{a+b}$, $V(X) = \frac{ab}{(a+b)^2(a+b+1)}$,$E(X^2) = \frac{a(1+a)}{(a+b)(a+b+1)}$.

Therefore (2.15) becomes

$$E(X^2\,|X > x\,) = E(X^2) + h(x)c_1(x),$$

where $c_1\,(x)$ is a cubic function in x.

Example 2.2.18 (Pareto)**.** $k = \mu\,,\;\; g(x) = \frac{x(x-a)}{c-1},\;\; c > 1,\; f(x) = \frac{c}{a}\left(\frac{x}{a}\right)^{-c-1},\;\; a < x < \infty$, $a > 0$, then $E(X) = \frac{ac}{c-1}\,,\;\; V(X) = \frac{a^2c}{(c-1)^2(c-2)}\,,\;\; E(X^2) = \frac{a^2c^3-2a^2c^2+a^2c}{(c-1)^2(c-2)}$. The identity (2.15),

$$E(X^2\,|X > x\,) = E(X^2) + h(x)c_2(x),$$

where $c_2(x)$ is a cubic function in x.

Example 2.2.19 (Normal)**.** $k = \mu\,,\;\; g(x) = \sigma^2, f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},\;\; -\infty < x < \infty, -\infty < \mu < \infty$, $\sigma > 0$, then $E(X) = \mu\;,\;\; V(X) = \sigma^2\;,\;\; E(X^2) = \sigma^2 + \mu^2$. Then (2.15) becomes

$$E(X^2\,|X > x\,) = E(X^2) + h(x)l_4(x),$$

where $l_4(x)$ is a linear function in x.

Example 2.2.20 (Student - t)**.** $k = 0\,,\; g(x) = \frac{n+x^2}{n-1},\; f(x) = \frac{1}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)}\frac{1}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$, $-\infty < x < \infty$, then $V(X) = \frac{n}{n-2}\,,\;\; n > 2$. (2.15) becomes

$$E(X^2\,|X > x\,) = V(X) + h(x)c_3(x),$$

where $c_3(x)$ is a cubic function in x.

2.3 Bivariate RCST for weighted models

For a nonnegative vector variable $\boldsymbol{X} = (X_1, X_2)'$ with pdf $f(x_1, x_2)$, the vector-valued bivariate RCST (see Roy and Roy (2009)) is given by

$$\eta_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \log f(x_1, x_2),\ i = 1, 2.$$

Denoting $\boldsymbol{X}^w = (X_1^w, X_2^w)'$ be the bivariate weighted version of $\boldsymbol{X}$, then the vector-valued bivariate RCST for $\boldsymbol{X}^w$ is defined as

$$\eta_i^w(x_1, x_2) = -\frac{\partial}{\partial x_i} \log f^w(x_1, x_2),\ i = 1, 2.$$

Using (1.6), $\eta_i^w(x_1, x_2)$ can be written as

$$\eta_i^w(x_1, x_2) \ = \ \eta_i(x_1, x_2) - w_i(x_1, x_2), \tag{2.17}$$

where $w_i(x_1, x_2) = \frac{\partial}{\partial x_i} \log w(x_1, x_2)$.

The following theorem uniquely determines the bivariate weighted distribution by using the bivariate RCST for weighted models.

Theorem 2.3.1. *For a bivariate setup, if the i^{th} RCST of $\boldsymbol{X}^w$ is $\eta_i^w(x_1, x_2), i = 1, 2$ and is continuous, then the weighted density curve can be uniquely determined in terms of the following two alternative forms:*

$$f^w(x_1, x_2) = C \exp\left[-\int_0^{x_1} \eta_1^w(u, 0)du - \int_0^{x_2} \eta_2^w(x_1, v)dv\right] \tag{2.18}$$

$$f^w(x_1, x_2) = C \exp\left[-\int_0^{x_2} \eta_2^w(0, v)dv - \int_0^{x_1} \eta_1^w(u, x_2)du\right]. \tag{2.19}$$

We can also uniquely determines the bivariate original distribution by using the bivariate RCST for weighted models.

Corollary 2.3.2. *For the bivariate setup, if the* i^{th} *RCST of* $\boldsymbol{X}^w$ *is* $\eta_i^w(x_1, x_2), i = 1, 2$ *and is continuous, then the density curve can be uniquely determined in terms of the following two alternative forms:*

$$f(x_1, x_2) = \frac{K}{w(x_1, x_2)} \exp\left[-\int_0^{x_1} \eta_1^w(u, 0)du - \int_0^{x_2} \eta_2^w(x_1, v)dv\right]$$

$$f(x_1, x_2) = \frac{K}{w(x_1, x_2)} \exp\left[-\int_0^{x_2} \eta_2^w(0, v)dv - \int_0^{x_1} \eta_1^w(u, x_2)du\right].$$

Analogues to the univariate case we have the following characterization theorems for the special cases of weighted distributions.

Theorem 2.3.3. *Let* $h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$ *be the vector-valued bivariate failure rate of a nonnegative random vector* $\boldsymbol{X} = (X_1, X_2)'$. *Then*

$$\eta_i^w(x_1, x_2) \;=\; h_i(x_1, x_2)\ , i = 1, 2,$$

where $h_i(x_1, x_2) = -\frac{\partial}{\partial x_i} \log \bar{F}(x_1, x_2)$ *if and only if* $\boldsymbol{X}^w$ *follows a bivariate equilibrium distribution.*

Proof. If $\boldsymbol{X}^w$ follows a bivariate equilibrium distribution, then $w(x_1, x_2) = \frac{1}{k(x_1,x_2)}$ (see Navarro et al. (2006)), where $k(x_1, x_2) = \frac{f(x_1,x_2)}{\bar{F}(x_1,x_2)}$. It follows that $w_i(x_1, x_2) = \eta_i(x_1, x_2) - h_i(x_1, x_2), i = 1, 2$ and therefore (2.17) becomes $\eta_i^w(x_1, x_2) = h_i(x_1, x_2), i = 1, 2$.

Conversely, assume that $\eta_i^w(x_1, x_2) = h_i(x_1, x_2), i = 1, 2$ holds. Now using (2.18) and (2.19), we have

$$f^w(x_1,x_2) = C\exp\left[-\int_0^{x_1} h_1(u,0)du - \int_0^{x_2} h_2(x_1,v)dv\right] \tag{2.20}$$

$$f^w(x_1,x_2) = C\exp\left[-\int_0^{x_2} h_2(0,v)dv - \int_0^{x_1} h_1(u,x_2)du\right]. \tag{2.21}$$

Equations (2.20) and (2.21) together implies

$$f^w(x_1,x_2) = C\bar{F}(x_1,x_2). \tag{2.22}$$

Applying the condition of total probability, we obtain $C = \frac{1}{E(X_1X_2)}$, (2.22) reduces to $f^w(x_1,x_2) = \frac{\bar{F}(x_1,x_2)}{E(X_1X_2)}$, the bivariate equilibrium distribution. □

Theorem 2.3.4. *For a nonnegative random vector $\boldsymbol{X} = (X_1,X_2)'$, the relationship*

$$\eta_i^w(x_1,x_2) = \eta_i(x_1+t_1,x_2+t_2), i = 1,2$$

satisfies if and only if $\boldsymbol{X}^w$ follows a bivariate residual life distribution.

Proof. If $\boldsymbol{X}^w$ follows a bivariate residual life distribution, then $w(x_1,x_2) = \frac{f(x_1+t_1,x_2+t_2)}{f(x_1,x_2)}$, $w_i(x_1,x_2) = \eta_i(x_1,x_2) - \eta_i(x_1+t_1,x_2+t_2), i = 1,2$ and therefore $\eta_i^w(x_1,x_2) = \eta_i(x_1+t_1,x_2+t_2), i = 1,2$.

Conversely, assume that $\eta_i^w(x_1,x_2) = \eta_i(x_1+t_1,x_2+t_2), i = 1,2$ holds, then using (2.18) and (2.19), we get

$$f^w(x_1,x_2) = C\exp\left[-\int_0^{x_1} \eta_1(u+t_1,t_2)du - \int_0^{x_2} \eta_2(x_1+t_1,v+t_2)dv\right] \tag{2.23}$$

$$f^w(x_1,x_2) = C\exp\left[-\int_0^{x_2} \eta_2(t_1,v+t_2)dv - \int_0^{x_1} \eta_1(u+t_1,x_2+t_2)du\right]. \tag{2.24}$$

Equations (2.23) and (2.24) can be rewritten as

$$f^w(x_1,x_2) = C\exp\left[\int_0^{x_1}\left[\frac{\partial}{\partial u}\log f(u+t_1,t_2)\right]du + \int_0^{x_2}\left[\frac{\partial}{\partial v}\log f(x_1+t_1,v+t_2)\right]dv\right] \tag{2.25}$$

$$f^w(x_1,x_2) = C\exp\left[\int_0^{x_2}\left[\frac{\partial}{\partial v}\log f(t_1,v+t_2)\right]dv + \int_0^{x_1}\left[\frac{\partial}{\partial u}\log f(u+t_1,x_2+t_2)\right]du\right]. \tag{2.26}$$

Equations (2.25) and (2.26) together give

$$f^w(x_1,x_2) = A(t_1,t_2)f(x_1+t_1,x_2+t_2).$$

Applying the condition of total probability, yield $A(t_1,t_2) = \frac{1}{\bar{F}(t_1,t_2)}$ and therefore $f^w(x_1,x_2) = \frac{f(x_1+t_1,x_2+t_2)}{\bar{F}(t_1,t_2)}$, proves the result. □

Corollary 2.3.5. *$\eta_i^w(x_1,x_2) = \eta_i(t_1-x_1,t_2-x_2), i=1,2$ if and only if $\boldsymbol{X}^w$ follows a reversed residual life distribution. (where $w(x_1,x_2) = \frac{f(t_1-x_1,t_2-x_2)}{f(x_1,x_2)}$).*

Theorem 2.3.6. *Let X_1 and X_2 be independent and identically distributed nonnegative random variables (rv's) and let $\bar{h}_1(\cdot)$ be the reversed failure rate of X_1, $h_2(\cdot)$ be the failure rate of X_2 and $a_1(x_1,x_2)$, $a_2(x_1,x_2)$ be the generalized failure rates, where $a_i(x_1,x_2) = \frac{f(x_i)}{F(x_2)-F(x_1)}, i=1,2$ (see Navarro and Ruiz (1996)), then the identities*

$$\eta_1^w(x_1,x_2) = \eta_1(x_1,x_2) + (1-j)\,\bar{h}_1(x_1) + (k-j-1)\,a_1(x_1,x_2)$$

and

$$\eta_2^w(x_1,x_2) = \eta_2(x_1,x_2) + (n-k)\,h_2(x_2) - (k-j-1)\,a_2(x_1,x_2)$$

holds if $\boldsymbol{X}^w$ follows the joint pdf of j^{th} and k^{th} order statistics, $1 \le j < k \le n$.

Proof. Let $\boldsymbol{X}^w$ follows the joint pdf of j^{th} and k^{th} order statistics. *i.e.*, $w_{jk}(x_1,x_2) =$

$[F(x_1)]^{j-1}\left[F(x_2)-F(x_1)\right]^{k-j-1}\left[\bar{F}(x_2)\right]^{n-k}$, then

$$w_1(x_1,x_2)=\frac{\partial}{\partial x_1}\log w_{jk}(x_1,x_2)=-\left[(1-j)\,\bar{h}_1(x_1)+(k-j-1)\,a_1(x_1,x_2)\right]$$

and

$$w_2(x_1,x_2)=\frac{\partial}{\partial x_2}\log w_{jk}(x_1,x_2)=-\left[(n-k)\,h_2(x_2)-(k-j-1)\,a_2(x_1,x_2)\right].$$

Clearly, (2.17) becomes

$$\eta_1^w(x_1,x_2)=\eta_1(x_1,x_2)+(1-j)\,\bar{h}_1(x_1)+(k-j-1)\,a_1(x_1,x_2)$$

and

$$\eta_2^w(x_1,x_2)=\eta_2(x_1,x_2)+(n-k)\,h_2(x_2)-(k-j-1)\,a_2(x_1,x_2),1\le j<k\le n.$$

□

Corollary 2.3.7. *If $\boldsymbol{X}^w$ follows the joint pdf of j^{th} and $(j+1)^{th}$ order statistics then $\eta_1^w(x_1,x_2)=\eta_1(x_1,x_2)+(1-j)\,\bar{h}_1(x_1)$ and $\eta_2^w(x_1,x_2)=\eta_2(x_1,x_2)+(n-j-1)\,h_2(x_2)$.*

Corollary 2.3.8. *If $\boldsymbol{X}^w$ follows the joint pdf of first and second order statistics then $\eta_1^w(x_1,x_2)=\eta_1(x_1,x_2)$ and $\eta_2^w(x_1,x_2)=\eta_2(x_1,x_2)+(n-2)\,h_2(x_2)$.*

Corollary 2.3.9. *If $\boldsymbol{X}^w$ follows the joint pdf of $(n-1)^{th}$ and n^{th} order statistics then $\eta_1^w(x_1,x_2)=\eta_1(x_1,x_2)-(n-2)\,\bar{h}_1(x_1)$ and $\eta_2^w(x_1,x_2)=\eta_2(x_1,x_2)$.*

Corollary 2.3.10. *If $\boldsymbol{X}^w$ follows the joint pdf of first and n^{th} order statistics then $\eta_1^w(x_1,x_2)=\eta_1(x_1,x_2)+(n-2)\,a_1(x_1,x_2)$ and $\eta_2^w(x_1,x_2)=\eta_2(x_1,x_2)-(n-2)\,a_2(x_1,x_2)$.*

Example 2.3.11 (Exponential)**.** $f(x) = b\, e^{-\,bx},\ 0 < x < \infty,\ b > 0$ and $f(x_1, x_2) = b^2\, e^{-\,b(x_1+x_2)}$, then

$$\eta_1^w(x_1, x_2) = b + (1-j)b\frac{1}{e^{b\,x_1}-1} + (k-j-1)b\frac{1}{1-e^{b(x_1-x_2)}}$$

and

$$\eta_2^w(x_1, x_2) = b + (n-k)b + (k-j-1)b\frac{1}{1-e^{b(x_2-x_1)}}.$$

Example 2.3.12 (Power)**.** $f(x) = cx^{c-1},\ 0 \le x \le 1,\ c > 0$ and $f(x_1, x_2) = c^2\, x_1^{c-1} x_2^{c-1}$, then

$$\eta_1^w(x_1, x_2) = (1-j)\frac{1}{x_1} + (k-j-1)\frac{c\, x_1^{c-1}}{x_2^c - x_1^c}$$

and

$$\eta_2^w(x_1, x_2) = \frac{1-c}{x_2} + (n-k)\frac{c\, x_2^{c-1}}{1-x_2^c} - (k-j-1)\frac{c\, x_2^{c-1}}{x_2^c - x_1^c}.$$

Example 2.3.13 (Pareto)**.** $f(x) = \frac{c}{a}\left(\frac{x}{a}\right)^{-c-1},\ a \le x < \infty,\ a > 0,\ c > 0$ and $f(x_1, x_2) = \left(\frac{c}{a}\right)^2 \left(\frac{x_1}{a}\right)^{-c-1} \left(\frac{x_2}{a}\right)^{-c-1}$, then

$$\eta_1^w(x_1, x_2) = \frac{c+1}{x_1} + (1-j)\frac{c\, a^c}{x_1(x_1^c - a^c)} + (k-j-1)\frac{c\, x_2^c}{x_1(x_2^c - x_1^c)}$$

and

$$\eta_2^w(x_1, x_2) = \frac{1}{x_2} + (n-k+1)\frac{c}{x_2} - (k-j-1)\frac{c\, x_1^c}{x_2(x_2^c - x_1^c)}.$$

2.4 Multivariate RCST for weighted models

The vector-valued multivariate RCST of $\boldsymbol{X}$ is given by $\eta(\boldsymbol{x}) = (\eta_1(\boldsymbol{x}), \eta_2(\boldsymbol{x}), ..., \eta_p(\boldsymbol{x}))^{'}$, where $\eta_i(\boldsymbol{x}) = -\frac{\partial}{\partial x_i}\log f(\boldsymbol{x}), i = 1, 2, ..., p$. The corresponding vector-valued multivariate RCST of $\boldsymbol{X}^w$ is given by

$$\eta^w(\boldsymbol{x}) = \left(\eta_1^w(\boldsymbol{x}), \eta_2^w(\boldsymbol{x}), ..., \eta_p^w(\boldsymbol{x})\right)',$$

where $\eta_i^w(\boldsymbol{x}) = -\frac{\partial}{\partial x_i} \log f^w(\boldsymbol{x}), i = 1, 2, ..., p$. Using (1.7), $\eta_i^w(\boldsymbol{x})$ can be written as

$$\eta_i^w(\boldsymbol{x}) = \eta_i(\boldsymbol{x}) - w_i(\boldsymbol{x}),$$

where $w_i(\boldsymbol{x}) = \frac{\partial}{\partial x_i} \log w(\boldsymbol{x})$.

Remark 2.4.1. For $p = 1$ and $p = 2$ the above definition reduces to the corresponding univariate and bivariate RCST of weighted distributions respectively.

Remark 2.4.2. If $w(\boldsymbol{x})$ is monotonically increasing (decreasing) and $\eta_i(\boldsymbol{x})$ is monotonically increasing or decreasing, then $\eta_i^w(\boldsymbol{x}) \leq (\geq)\, \eta_i(\boldsymbol{x}), i = 1, 2, .., p$.

Theorem 2.4.3. *For the multivariate setup if the i^{th} RCST of $\boldsymbol{X}^w$ is $\eta_i^w(x_1, x_2, ..., x_p)$, $i = 1, 2, ..., p$ and is continuous, then the weighted density curve can be uniquely determined as*

$$f^w(\boldsymbol{x}) = C\ \exp\left[-\int_{\Gamma} \eta^w(\boldsymbol{u}) d\boldsymbol{u}\right],$$

where the integration is a line integration with respect to $\boldsymbol{u} = (u_1, u_2, ..., u_p)$ over a piecewise smooth curve Γ, joining the points $(0, 0, ..., 0)$ and $\boldsymbol{x} = (x_1, x_2, ..., x_p)$ and where C is the normalizing constant such that the total probability is one (see Roy and Roy (2009)).

Corollary 2.4.4. *For a multivariate setup if the i^{th} RCST of $\boldsymbol{X}^w$ is $\eta_i^w(x_1, x_2, ..., x_p)$, $i = 1, 2, ..., p$ and is continuous, then the density curve can be uniquely determined by*

$$f(\boldsymbol{x}) = \frac{K}{w(\boldsymbol{x})}\ \exp\left[-\int_{\Gamma} \eta^w(\boldsymbol{u}) d\boldsymbol{u}\right],$$

where K is the normalizing constant such that the total probability is one.

Similar to Roy and Roy (2009), we have the following theorem for weighted random variables.

Theorem 2.4.5. *If $X_1^w, X_2^w, ..., X_p^w$ are independent rv's then*

$$\eta^w(\boldsymbol{x}) = \left(\eta_1^w(x_1), \eta_2^w(x_2), ..., \eta_p^w(x_p)\right)^{'},$$

where $\eta_i^w(x_i) = \eta_i(x_i) - w_i(x_i)$ is the univariate RCST of $X_i^w, i = 1, 2, ..., p$.

We can define strictly constant vector-valued multivariate RCST of $\boldsymbol{X}^w$ as $\eta^w(\boldsymbol{x}) = (a_1, a_2, ..., a_p)^{'}$, where $\boldsymbol{a} = (a_1, a_2, ..., a_p)^{'}$ is an absolute constant with respect to all the variables.

Theorem 2.4.6. *If $w(\boldsymbol{x}) = \prod_{i=1}^{p} \frac{1}{x_i}$ then the vector-valued multivariate RCST of $\boldsymbol{X}^w$ is an absolute constant if and only if the underlying distribution is a joint collection of independent univariate gamma distribution with pdf*

$$f(x) = a^2 x e^{-a\,x} \;\;, \; x > 0, \; a > 0. \tag{2.27}$$

Proof. Suppose X_i's are follows independent univariate gamma distribution with pdf given by (2.27). Under the weight function $w(\boldsymbol{x}) = \prod_{i=1}^{p} \frac{1}{x_i}$, X_i^w's are independent (see Arnold and Nagaraja (1991)) for the given pdf (2.27) with $w_i(x_i) = -\frac{1}{x_i}$, $i = 1, 2, ..., p$ and therefore $\eta^w(\boldsymbol{x}) = (a_1, a_2, ..., a_p)^{'}$.

Conversely, suppose $\eta^w(\boldsymbol{x})$ is constant.

i.e.,

$$\eta_i^w(x_i) = a_i \;, i = 1, 2, ..., p,$$

or

$$\frac{\partial}{\partial x_i} \log f^w(x) = -a_i.$$

Equivalently,

$$\frac{\partial}{\partial x_i} \log f(x) + \frac{\partial}{\partial x_i} \log w(x) = -a_i.$$

Integrating both sides with respect to x_i, we get

$$f(\boldsymbol{x})w(\boldsymbol{x}) = \exp(-a_i x_i) g_i(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_p), i = 1, 2, ..., p$$

or

$$f(\boldsymbol{x}) = \frac{1}{w(\boldsymbol{x})} \exp(-a_i x_i) g_i(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_p), i = 1, 2, ..., p.$$

Combining $f(\boldsymbol{x})$, $i = 1, 2, ..., p$, we have

$$f(\boldsymbol{x}) \propto \left[\prod_{i=1}^{p} x_i\right] \left[\exp\left(-\sum_{i=1}^{p} a_i x_i\right)\right].$$

Applying the condition of total probability, we obtain $f(\boldsymbol{x}) = \prod_{i=1}^{p} a_i^2 x_i \exp[-a_i x_i]$, which proves the result. $\square$

Chapter 3

Some properties of reciprocal coordinate subtangents in the context stochastic modelling[2]

3.1 Introduction

The study of lifetime of organisms, devices, components, materials, etc., is of major importance in the reliability and survival analysis. A major part of such studies is devoted to the mathematical description of the lifetime by a failure distribution. In reliability studies a useful tool for identifying the failure behaviour of an item is by the study on the shape and monotonicity of its failure rate function because it explain some of the characteristics of the mechanism leading to the identification of its underlying distribution. Since the monotone properties of lifetime of device or component plays a very important role in reliability modelling, we study the

[2]Some of the results in this Chapter have been accepted as entitled "Some properties of reciprocal coordinate subtangents in the context stochastic modelling", in the *Journal of the Indian Statistical Association* (see Sunoj and Sreejith (2017)).

monotone properties of weighted rv based on RCST. As given in (1.2) the RCST to a curve $y = f(x)$ of the rv X is given by

$$\eta(x) = -\frac{d}{dx}\log f(x). \tag{3.1}$$

Recalling the RCST for the weighted rv X^w given in (2.1),

$$\eta^w(x) = \eta(x) - \frac{d}{dx}\log w(x). \tag{3.2}$$

We make use of (3.1) and (3.2) to study the monotone properties weighted distribution in comparison with the original distribution.

Mixture distributions have an important role in statistical analysis as they represent heterogeneity in the distribution of a rv. It describes the random variables that are drawn from more than one parent population. A finite mixture model is a convex combination of two or more probability density functions. By combining the properties of individual probability density functions, finite mixture models are a powerful and flexible tool for modelling complex data. Roy and Roy (2013) introduced Mean RCST (MRCST) as the counterpart of mean residual life function in the density domain. For a nonnegative rv X, MRCST denoted by $M(x)$ is defined by

$$M(x) = \int_0^\infty \frac{f(x+t)}{f(x)}dt,$$

Equivalently,

$$M(x) = \frac{1}{f(x)}\int_x^\infty f(u)du. \tag{3.3}$$

From (3.3) it is clear that $M(x)$ is an inverse of failure rate, and an increasing $M(x)$

will lead to an increasing mean residual life function. $\eta(\cdot)$ can be expressed in terms of $M(\cdot)$ by

$$\eta(x) = \frac{1 + M'(x)}{M(x)}.$$

Hence we look into how the RCST and MRCST helpful to characterize finite mixtures of lifetime distributions.

The classical theory of records is an important area considered by many researchers in past. The theory of records and order statistics are closely related. Various studies on the properties and characterizations related to order statistics and record values are available in Arnold et al. (1992), Arnold et al. (2011), Wu and Lee (2001), Balakrishnan and Stepanov (2004), Su et al. (2008), Kundu and Nanda (2010), Kumar (2015) and the references therein. Motivated with these, we obtain characterizations of probability models based on RCST of record values.

The present Chapter is organized as follows. In Section 3.2, we prove that monotone failure properties of probability models are invariant under nonsingular transformation. We further extend the concept to study the failure property of weighted rv with respect to its original rv. The stochastic comparison of two rvs based on RCST and MRCST are also studied. In Section 3.3, characterization result is obtained for mixtures of exponential, Lomax and beta distributions using a relationship between RCST and MRCST. We also prove characterization results based on RCST of record values in Section 3.4 and in Section 3.5 we study some properties of RCST in the context of circular distributions.

3.2 Properties of RCST

In this section, we study the monotone properties of probability models and comparison of two random variables X and Y based on RCST.

3.2.1 Ageing properties of RCST

In the following theorem, we prove that increasing RCST (IRCST) and decreasing RCST (DRCST) classes are invariant under nonsingular transformations.

Theorem 3.2.1. *(i). If X is IRCST and ϕ' is nonnegative and log-convex, then ϕ is also IRCST.*

(ii). If X is DRCST and ϕ' is nonnegative and log-concave, then ϕ is also DRCST.

Proof. If $f_Y(\cdot)$ is the pdf of $Y = \phi(X)$, then $f_Y(y) = \frac{f_X(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$. Now using (3.1), we get

$$\eta_Y(y) = \eta_X(\phi^{-1}(y)) + \frac{d}{dy} \log \phi'(\phi^{-1}(y)),$$

where $\eta_Y(\cdot)$ is the RCST of Y. Now if X is IRCST and ϕ' is nonnegative and log-convex, then ϕ is also IRCST. The proof of (ii) is similar. $\square$

Corollary 3.2.2. *If X is IRCST (DRCST) then the distribution of location-scale-shape family $\left(i.e., Y = \left(\frac{X-\theta}{\beta}\right)^{\alpha},\ X > \theta, \beta > 0, \alpha > 0\right)$ is IRCST (DRCST) if $0 < \alpha < 1$ $(\alpha > 1)$.*

Proof. Here $\phi(x) = \left(\frac{x-\theta}{\beta}\right)^{\alpha},\ x > \theta, \beta > 0, \alpha > 0$ which implies $\phi'(x) = \frac{\alpha}{\beta}\left(\frac{x-\theta}{\beta}\right)^{\alpha-1}$ and is log-convex (log-concave) if $0 < \alpha < 1$ $(\alpha > 1)$. Therefore from Theorem 3.2.1, the distribution of $Y = \left(\frac{X-\theta}{\beta}\right)^{\alpha}$ is IRCST (DRCST) if $0 < \alpha < 1$ $(\alpha > 1)$. $\square$

Example 3.2.3. Assume that X follows an exponential distribution with pdf $f_X(x) = \lambda e^{-\lambda x}, x > 0, \lambda > 0$. Let $Y = \phi(X) = X^{\frac{1}{\alpha}}, \alpha > 0$, then Y has the Weibull distribution with pdf $f_Y(y) = \alpha\lambda y^{\alpha-1}e^{-\lambda y^{\alpha}}$. Clearly $\phi'(x) = \frac{1}{\alpha}x^{\frac{1}{\alpha}-1}$ is log-convex (log-concave) if $\alpha > 1$ $(0 < \alpha < 1)$. Then from Theorem 3.2.1, we can conclude that Weibull distribution is IRCST (DRCST).

In the next theorem we prove the preservation properties of weighted distribution and illustrate with examples.

Theorem 3.2.4. *(i). If X is DRCST and $w(\cdot)$ is log-convex, then X^w is DRCST.*

(ii). If X is IRCST and $w(\cdot)$ is log-concave, then X^w is IRCST.

(iii). If X is BS and $w(\cdot)$ is log-convex, then X^w is DRCST. The change point when X^w become BS will be greater than the change point when X is BS.

(iv). If X is UBS and $w(\cdot)$ is log-concave, then X^w is IRCST. The change point when X^w become UBS will be greater than the change point when X is UBS.

Proof. (i) and (ii) follows from (3.2). For (iii), suppose that X is BS, then from (3.2) we have,

$$\frac{d}{dx}\eta(x) = \frac{d}{dx}\eta^w(x) + \frac{d}{dx}\log w(x) = 0$$

for some $x = x_0$ which means that

$$\frac{d}{dx}\eta^w(x) = -\left(\frac{d}{dx}\log w(x)\right)$$

at x_0. Since $w(x)$ is log-convex, we have that $\eta^w(x)$ is still nonincreasing and hence X^w is DRCST. It is clear that, X^w is DRCST even at the change point of X is BS. Therefore, if X^w is BS, the change point will be greater than the change point of X is BS (see Nair and Sankaran (2012)). The proof of (iv) when X is UBS and $w(x)$ is log-concave is similar. $\square$

Example 3.2.5. Let $X \sim Gamma(p, \lambda)$ and the corresponding RCST is $\eta(x) = \lambda - \frac{p-1}{x}$. Then the gamma distribution is DRCST (IRCST) if $0 < p < 1$ ($p > 1$). Suppose $w(x) = \frac{1}{x}$ and is log-convex then from Theorem 3.2.4, the inverse length-biased gamma distribution $X^w \sim Gamma(p-1, \lambda)$ is also DRCST for $0 < p < 1$ (Figure 3.1.1) and for a log-concave weight function $w(x) = x$, the length-biased gamma distribution $X^w \sim Gamma(p+1, \lambda)$ is IRCST for $p > 1$ (Figure 3.1.2).

Example 3.2.6. Let X follows a modified Weibull distribution proposed by Lai et al. (2003) with pdf $f(x) = a(\alpha + \lambda x)x^{\alpha-1}e^{\lambda x}\exp(-ax^{\alpha}e^{\lambda x}), x > 0, a > 0, \alpha > 0$ and $\lambda > 0$, which is an extension of two parameter Weibull distribution, and has a BS when $0 < \alpha < 1$ (Lai and Xie (2006)). The corresponding RCST is $\eta(x) = \frac{(\alpha+\lambda x)}{x}\left(\frac{\alpha}{(\alpha+\lambda x)^2} + ax^{\alpha}e^{\lambda x} - 1\right)$, which is plotted in Figure 3.1.3 along with $\eta^w(x)$ for a log-convex weight function $w(x) = \frac{1}{x}$.

Example 3.2.7. A Burr XII distribution with pdf $f(x) = \frac{kcx^{c-1}}{(1+x^c)^{k+1}}, x > 0, k > 0, c > 0$ has a UBS if $c > 2$ (Lai and Xie (2006)) and $\eta(x) = \frac{1-c}{x} + (k+1)\frac{cx^{c-1}}{(1+x^c)}$ is the corresponding RCST, which is plotted in Figure 3.1.4 along with $\eta^w(x)$ for a log-concave weight function $w(x) = x$.

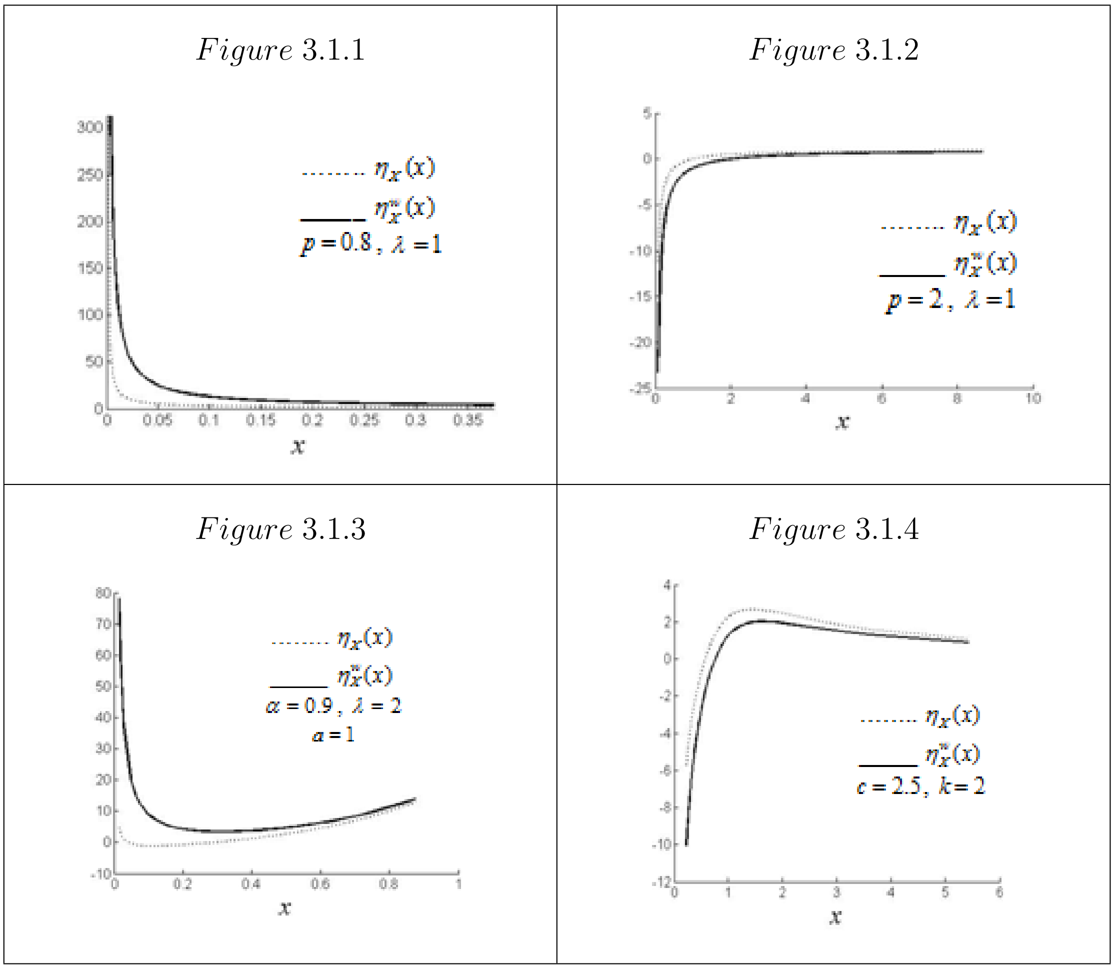
On the other hand, if we know the monotonicity property of weighted distribution, then the following theorem establishes the nature of monotonicity of original distribution.

Theorem 3.2.8. *(i). If X^w is DRCST and $w(\cdot)$ is log-concave, then X is DRCST.*

(ii). If X^w is IRCST and $w(\cdot)$ is log-convex, then X is IRCST.

(iii). If X^w is BS and $w(\cdot)$ is log-concave, then X is DRCST. The change point when X become BS will be greater than the change point when X^w is BS.

Figure 3.1: RCST for original and weighted distributions



(iv). If X^w is UBS and $w(\cdot)$ is log-convex, then X is IRCST. The change point when X become UBS will be greater than the change point when X^w is UBS.

Proof. The proof is similar to Theorem 3.2.4. □

3.2.2 Some ordering results

In this section we compare two random variable X and Y based on $\eta(\cdot)$ and $M(\cdot)$ by using the stochastic ordering.

Stochastic orders have been in use over the past few decades, at an accelerated rate, in many diverse areas of probability and statistics such as reliability theory, survival analysis, economics, actuarial science, operations research, etc. There are several ways in which one can assert that a rv X is 'greater than' (or 'less than') another rv Y (see Marshall and Olkin (1979), Ross (1983) and Shaked and Shanthikumar (2007)). Likelihood ratio ordering and failure rate ordering are among the various notions of ordering between rv's.

Definition 3.2.9. If X and Y are two random variables with respective density functions $f_X(\cdot)$ and $f_Y(\cdot)$, X is said to be less than Y in likelihood ratio ordering $(X \underset{LR}{\leq} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ is decreasing in $x \geq 0$.

Definition 3.2.10. If X and Y are two random variables with respective failure rate functions $h_X(\cdot)$ and $h_Y(\cdot)$, X is said to be less than Y in failure rate ordering $(X \underset{FR}{\leq} Y)$ if $h_Y(x) \leq h_X(x)$.

Navarro (2008) proved that if $X \underset{LR}{\leq} Y$ if and only if $\eta_X(x) \geq \eta_Y(x)$ and in the context of weighted distributions, Kochar and Gupta (1987) proved that if the weight function $w(\cdot)$ is monotonically increasing, then $X \underset{LR}{\leq} X^w$. Now we have following results and the proofs are omitted as they are directly obtained.

Theorem 3.2.11. *For an increasing (decreasing) weight function $w(\cdot)$, $X \underset{LR}{\leq} X^w$ if and only if $\eta_X(x) \geq (\leq) \eta_{X^w}(x)$.*

Corollary 3.2.12. *For an increasing (decreasing) weight function $w(x) = \frac{1}{h_X(x)}$ (equilibrium distribution), $X \underset{LR}{\leq} X^w$ if and only if $\eta_X(x) \geq (\leq) h_X(x)$.*

Corollary 3.2.13. *For an increasing (decreasing) weight function $w(x) = \left[\bar{F}_X(x)\right]^{\theta-1}$ (proportional hazards distribution), where $\theta > 0$ and $\bar{F}_X(x)$ is the reliability function, $X \underset{LR}{\leq} X^w$ if and only if $\eta_X(x) \geq (\leq) \theta h_X(x) - \frac{h_X'(x)}{h_X(x)}$ or $\theta \leq (\geq) 1$, by using the fact that $\eta_X(x) = h_X(x) - \frac{h'_X(x)}{h_X(x)}$.*

Corollary 3.2.14. *For an increasing (decreasing) weight function* $w(x) = \frac{f_X(x+t)}{f_X(x)}$ *(residual life distribution),* $X \underset{LR}{\leq} X^w$ *if and only if* $\eta_X(x) \geq (\leq)\eta_X(x+t)$.

Corollary 3.2.15. *For an increasing (decreasing) weight function* $w(x) = \frac{f_X(t-x)}{f_X(x)}, t > x$ *(reversed residual life distribution),* $X \underset{LR}{\leq} X^w$ *if and only if* $\eta_X(x) \geq (\leq)\eta_X(t-x)$.

Theorem 3.2.16. $X \underset{FR}{\leq} Y$ *if* $M_X(x) \geq M_Y(x)$.

Theorem 3.2.17. *For an increasing (decreasing) weight function* $w(\cdot)$, $X \underset{FR}{\leq} X^w$ *if* $M_{X^w}(x) \geq (\leq)M_X(x)$.

Corollary 3.2.18. *For an increasing (decreasing) weight function* $w(x) = \frac{1}{h_X(x)}$ *(equilibrium distribution),* $X \underset{FR}{\leq} X^w$ *if* $r_X(x) \geq (\leq)M_X(x)$, *where* $r_X(x)$*is the mean residual life function of the rv* X.

Corollary 3.2.19. *For an increasing (decreasing) weight function* $w(x) = \left[\bar{F}_X(x)\right]^{\theta-1}$ *(proportional hazards distribution), where* $\theta > 0$ *and* $\bar{F}_X(x)$ *is the reliability function,* $X \underset{FR}{\leq} X^w$ *if* $\frac{1}{\theta h_X(x)} \geq (\leq)M_X(x)$ *or* $\theta \leq (\geq)1$.

Corollary 3.2.20. *For an increasing (decreasing) weight function* $w(x) = \frac{f_X(x+t)}{f_X(x)}$ *(residual life distribution),* $X \underset{FR}{\leq} X^w$ *if* $\frac{1}{h_X(x+t)} \geq (\leq)M_X(x)$.

Corollary 3.2.21. *For an increasing (decreasing) weight function* $w(x) = \frac{f_X(t-x)}{f_X(x)}, t > x$ *(reversed residual life distribution),* $X \underset{FR}{\leq} X^w$ *if* $\frac{1}{h_X(t-x)} \geq (\leq)M_X(x)$.

3.3 RCST of finite mixture models

The mixture of distributions are common models in reliability since they represent populations with different kinds of units. Even if the shapes of the failure rate functions of the members in the mixture are known, it is not easy to determine the shape of the failure rate of the mixture. Glaser (1980) and Gupta and Warren

(2001) used (3.1) to determine the shape of the failure rate of the mixture of two gamma densities. Navarro and Hernandez (2004) have considered the shape of the failure rate of the mixture of two positively truncated normal distributions by using (3.1). Navarro (2008) defined $\eta(\cdot)$ as the Glasers function associated to the mixture density $f_P(x) = pf_1(x) + (1-p)f_2(x)$ by

$$\eta_P(x) = -\frac{pf_1'(x) + (1-p)f_2'(x)}{pf_1(x) + (1-p)f_2(x)}, \tag{3.4}$$

or

$$\eta_P(x) = \varphi(x)\eta_1(x) + (1-\varphi(x))\eta_2(x),$$

where $f_1(\cdot)$, $f_2(\cdot)$ denotes the pdf's of the random variables X_1 and X_2 respectively, and $\varphi(x) = \frac{pf_1(x)}{pf_1(x)+(1-p)f_2(x)}$.

Example 3.3.1. Consider two random variables X_1 and X_2 satisfy proportional hazards model such that $\bar{F}_2(x) = \left(\bar{F}_1(x)\right)^{\theta}$, then

$$\eta_P(x) = \left(\theta(1-\varphi(x)) + \varphi(x)\right) h_1(x) - \tfrac{h_1'(x)}{h_1(x)},$$

where $\varphi(x) = \frac{p}{p+(1-p)\theta\left(\bar{F}_1(x)\right)^{\theta-1}}$.

Example 3.3.2. Let X_1 and X_2 have weighted distributions corresponding to exponential distributions with parameters α and β, respectively, and with weight function $\psi(x)$. Their densities are then given by, $f_1(x) = \frac{\psi(x)e^{-\alpha x}}{\mu(\alpha)}$, $f_2(x) = \frac{\psi(x)e^{-\beta x}}{\mu(\beta)}, x > 0$. Then

$$\eta_P(x) = (\alpha-\beta)\varphi(x) + \beta - \tfrac{\psi'(x)}{\psi(x)},$$

where $\varphi(x) = \frac{\mu(\beta)pe^{-\alpha x}}{\mu(\beta)pe^{-\alpha x}+\mu(\alpha)(1-p)e^{-\beta x}}$.

Example 3.3.3. Let X_1 and X_2 have equilibrium distributions with means μ_1 and

μ_2, respectively, and with densities given by, $f_1(x) = \frac{\bar{F}_1(x)}{\mu_1}$, $f_2(x) = \frac{\bar{F}_2(x)}{\mu_2}, x > 0$. Then

$$\eta_P(x) = \varphi(x)h_1(x) + (1 - \varphi(x))\, h_2(x).$$

The following theorem gives an identity connecting RCST and MRCST that characterizes mixtures of exponential, Lomax and beta distributions.

Theorem 3.3.4. *The relationship*

$$\eta_P(x) = (\theta_1 + \theta_2 + a)(1 + ax)^{-1} - \theta_1\theta_2(1 + ax)^{-2} M_P(x), \tag{3.5}$$

where $M_P(\cdot)$ is the MRCST for mixture density holds, if and only if i^{th} population X_i, $i = 1, 2$ follows exponential distribution with pdf

$$f_i(x) = \lambda_i e^{-\lambda_i x},\ \lambda_i > 0, x > 0\ for\ a = 0, \tag{3.6}$$

Lomax distribution with pdf

$$f_i(x) = \alpha_i(1 + x)^{-(\alpha_i+1)},\ \alpha_i > 0, x > 0\ for\ a = 1, \tag{3.7}$$

and, beta density with pdf

$$f_i(x) = \beta_i(1 - x)^{\beta_i - 1},\ \beta_i > 0, 0 < x < 1\ for\ a = -1. \tag{3.8}$$

Proof. To prove the 'if' part, assume that X_i follows exponential distribution with pdf (3.6), then using (3.4) we get

$$\begin{aligned}
\eta_P(x) &= \frac{p\lambda_1^2 e^{-\lambda_1 x} + (1-p)\lambda_2^2 e^{-\lambda_2 x}}{p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}} \\
&= \frac{(\lambda_1+\lambda_2)\left(p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}\right) - \lambda_1\lambda_2\left(pe^{-\lambda_1 x} + (1-p)e^{-\lambda_2 x}\right)}{p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2 e^{-\lambda_2 x}} \\
&= (\lambda_1+\lambda_2) - \lambda_1\lambda_2 \frac{\bar{F}_P(x)}{f_P(x)}, \qquad (3.9)
\end{aligned}$$

where $\bar{F}_P(x) = pe^{-\lambda_1 x} + (1-p)e^{-\lambda_2 x}$, reduces (3.9) to the required identity (3.5) with $a = 0$. When X_i follows Lomax distribution in (3.7), then (3.4) becomes

$$\begin{aligned}
\eta_P(x) &= \frac{p\alpha_1(\alpha_1+1)(1+x)^{-(\alpha_1+2)} + (1-p)\alpha_2(\alpha_2+1)(1+x)^{-(\alpha_2+2)}}{p\alpha_1(1+x)^{-(\alpha_1+1)} + (1-p)\alpha_2(1+x)^{-(\alpha_2+1)}} \\
&= \frac{(\alpha_1+\alpha_2+1)\left(p\alpha_1(1+x)^{-(\alpha_1+2)} + (1-p)\alpha_2(1+x)^{-(\alpha_2+2)}\right)}{p\alpha_1(1+x)^{-(\alpha_1+1)} + (1-p)\alpha_2(1+x)^{-(\alpha_2+1)}} \\
&\quad - \frac{\alpha_1\alpha_2\left(p(1+x)^{-(\alpha_1+2)} + (1-p)(1+x)^{-(\alpha_2+2)}\right)}{p\alpha_1(1+x)^{-(\alpha_1+1)} + (1-p)\alpha_2(1+x)^{-(\alpha_2+1)}} \\
&= (\alpha_1+\alpha_2+1)(1+x)^{-1} - \alpha_1\alpha_2(1+x)^{-2}M_P(x). \qquad (3.10)
\end{aligned}$$

Now (3.10) gives the identity (3.5) with $a = 1$. In a similar way we can prove (3.5) for (3.8) with $a = -1$.

To prove the 'only if' part, first we assume that $a = 0$ in (3.5) and which can be written as

$$f'_P(x) + (\theta_1+\theta_2)f_P(x) - \theta_1\theta_2\bar{F}_P(x) = 0$$

Differentiating both sides with respect to x, we obtain

$$f''_P(x) + (\theta_1+\theta_2)f'_P(x) + \theta_1\theta_2 f_P(x) = 0$$

or

$$\frac{1}{\theta_1}\frac{1}{\theta_2}f''_P(x) + (\frac{1}{\theta_1} + \frac{1}{\theta_2})f'_P(x) + f_P(x) = 0,$$

which is a second order differential equation whose solution is

$$f_P(x) = A\theta_1 e^{-\theta_1 x} + B\theta_2 e^{-\theta_2 x}.$$

Since $\int_0^\infty f_P(x)dx = 1$, we obtain $B = 1 - A$ and hence $f_P(x)$ is the pdf of a mixture of two exponentials with means $\frac{1}{\theta_1}$ and $\frac{1}{\theta_2}$.

Now consider the case $a \neq 0$ in (3.5), then

$$-\frac{f_P'(x)}{f_P(x)} = (\theta_1 + \theta_2 + a)(1 + ax)^{-1} - \theta_1\theta_2(1 + ax)^{-2}\frac{\bar{F}_P(x)}{f_P(x)}$$

or equivalently,

$$(1 + ax)^2 f_P'(x) + (\theta_1 + \theta_2 + a)(1 + ax)f_P(x) - \theta_1\theta_2\bar{F}_P(x) = 0. \tag{3.11}$$

Differentiating (3.11) with respect to x, we get

$$(1 + ax)^2 f_P''(x) + (\theta_1 + \theta_2 + 3a)(1 + ax)f_P'(x) + ((\theta_1 + \theta_2 + a)a + \theta_1\theta_2)\, f_P(x) = 0. \tag{3.12}$$

To solve the differential equation (3.12), we set $e^z = 1 + ax$ and $y = f(x)$, then (3.12) becomes

$$a^2\frac{d^2y}{dz^2} + (\theta_1 + \theta_2 + 2a)\, a\frac{dy}{dz} + ((\theta_1 + \theta_2 + a)a + \theta_1\theta_2)\, y = 0,$$

which is a homogeneous differential equation with constant coefficients. The auxiliary equation is

$$m^2 + \left(\frac{\theta_1 + \theta_2}{a} + 2\right)m + \left(\frac{\theta_1 + \theta_2}{a} + \frac{\theta_1\theta_2}{a^2} + 1\right) = 0,$$

and the roots are $m_1 = 1+\theta_1 a^{-1}$ and $m_2 = 1+\theta_2 a^{-1}$. The solution of (3.12) is then

$$y = Ae^{-(1+\theta_1 a^{-1})z} + Be^{-(1+\theta_2 a^{-1})z},$$

and correspondingly we obtain

$$f(x) = A(1+ax)^{-(1+\theta_1 a^{-1})} + B(1+ax)^{-(1+\theta_2 a^{-1})}. \tag{3.13}$$

Now choose $A = p\theta_1$ and $B = q\theta_2$, (3.13) reduces to

$$f(x) = p\theta_1(1+ax)^{-(1+\theta_1 a^{-1})} + q\theta_2(1+ax)^{-(1+\theta_2 a^{-1})}. \tag{3.14}$$

For $a = 1$, (3.14) provides

$$f(x) = p\theta_1(1+x)^{-(\theta_1+1)} + q\theta_2(1+x)^{-(\theta_2+1)}.$$

Now applying the condition of total probability and $f(x) \geq 0$, we get $\theta_i > 0$ and $q = 1-p$ hence $f_P(\cdot)$ is the pdf of a mixture of two Lomax distributions (3.7). Similarly for $a = -1$, (3.14) gives $f_P(\cdot)$, the pdf of a mixture of two beta distributions (3.8). □

Theorem 3.3.5. *The relationship*

$$\eta_P^w(x) = (\lambda_1 + \lambda_2) - \lambda_1\lambda_2 M_P^w(x),$$

where $M_P^w(\cdot)$ is the MRCST for mixture weighted density holds, if and only if for $i = 1,2,\ f_i(x) = \lambda_i^2 x e^{-\lambda_i x},\ \lambda_i > 0, x > 0$ with $w_i(x) = \frac{1}{x}$.

Proof. For $i = 1,2,\ f_i(x) = \lambda_i^2 x e^{-\lambda_i x}$ with $w_i(x) = \frac{1}{x}$, we have$f_{X_i^w}(x) = \lambda_i e^{-\lambda_i x}$ and the remaining part of the theorem follows from Theorem 3.3.4 for the exponential distribution case. □

3.4 RCST of record values

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed continuous rvs with cumulative distribution function (cdf) $F(.)$ and pdf $f(.)$. The random variable X_n is called an upper (lower) record value of this sequence if $X_n > X_i(X_n < X_i)$ for all $i = 1, 2, ..., n-1$. By convention X_1 is a record value. For more details about record values we refer to Arnold et al. (1992). We denote the n^{th} upper (lower) record values by $U_n(L_n)$. The pdf of U_n is given by

$$f^{U_n}(x) = \frac{[-\log \bar{F}(x)]^{n-1}}{(n-1)!} f(x), \quad -\infty < x < \infty, \tag{3.15}$$

where $\bar{F}(x) = 1 - F(x)$ and the pdf of L_n is given by

$$f^{L_n}(x) = \frac{[-\log F(x)]^{n-1}}{(n-1)!} f(x), \quad -\infty < x < \infty. \tag{3.16}$$

By applying (3.15) in (3.1) we get,

$$\eta^{U_n}(x) = \eta(x) - (n-1)\frac{d}{dx}\log[-\log \bar{F}(x)] \tag{3.17}$$

or

$$\frac{\eta(x) - \eta^{U_n}(x)}{(n-1)} = \frac{d}{dx}\log[-\log \bar{F}(x)]. \tag{3.18}$$

Similarly by applying (3.16) in (3.1) we get,

$$\frac{\eta(x) - \eta^{L_n}(x)}{(n-1)} = \frac{d}{dx}\log[-\log F(x)].$$

Theorem 3.4.1. *For a rv X, $\frac{\eta(x)-\eta^{U_n}(x)}{(n-1)} = \frac{p}{x}$ if and only if X follows a Weibull distribution with $\bar{F}(x) = e^{-(\lambda x)^p}$, $\lambda > 0, p > 0,\ x > 0$.*

Proof. Suppose $\frac{\eta(x)-\eta^{U_n}(x)}{(n-1)} = \frac{p}{x}$, then from (3.18) we have

$$\frac{d}{dx}\log[-\log \bar{F}(x)] = \frac{p}{x}.$$

Integrating the above equation on both sides with respect to x, we get

$$\log[-\log \bar{F}(x)] = p\left(\log x + \log c\right), \tag{3.19}$$

where $\log c$ is a constant of integration. (3.19) can be written as $\bar{F}(x) = e^{-(cx)^p}$, reliability function of Weibull distribution. The converse part is straight forward. □

Corollary 3.4.2. *For a rv X, $\frac{\eta(x)-\eta^{U_n}(x)}{(n-1)} = \frac{1}{x}$ if and only if X follows an exponential distribution with $\bar{F}(x) = e^{-\lambda x}$, $\lambda > 0, x > 0$.*

Remark 3.4.3. For a Weibull distribution with $f(x) = \lambda p(\lambda x)^{p-1}e^{-(\lambda x)^p}, \lambda > 0, p > 0,\ 0 < x < \infty$, we have $\eta^{U_n}(x) = \frac{1-np}{x} + \lambda^p p x^{p-1}$, then $\eta^{U_n}(x)$ is increasing for $p = 1, n > 1$ and $p > 1, n > \frac{1}{p}$ and $\eta^{U_n}(x)$ is decreasing for $p < 1, n < \frac{1}{p}$.

Theorem 3.4.4. *For a rv X, $\frac{\eta(x)-\eta^{U_n}(x)}{(n-1)} = \frac{\lambda}{(1+\lambda x)\log(1+\lambda x)}$ if and only if X follows a Lomax distribution with $\bar{F}(x) = (1+\lambda x)^{-p}$, $\lambda > 0,\ p > 0,\ x > 0$.*

Proof. The proof is similar to Theorem 3.4.1. □

Theorem 3.4.5. *For a rv X, $\frac{\eta(x)-\eta^{U_n}(x)}{(n-1)} = \frac{\lambda}{(1-\lambda x)\log(1-\lambda x)}$ if and only if X follows a beta distribution with $\bar{F}(x) = (1-\lambda x)^{p}$, $\lambda > 0,\ p > 0,\ 0 < x < \frac{1}{\lambda}$.*

Proof. The proof is similar to Theorem 3.4.1. □

Similar characterizations results can be obtained in the case of lower record values also (Table 3.1).

Table 3.1: $\frac{\eta(x)-\eta^{L_n}(x)}{(n-1)}$ for different distributions

Distribution	$f(x)$	$\frac{\eta(x)-\eta^{L_n}(x)}{(n-1)}$
Exponential	$\lambda e^{-\lambda x},\ \lambda>0, x>0$	$\frac{\lambda e^{-\lambda x}}{(1-e^{-\lambda x})\log(1-e^{-\lambda x})}$
Pareto I	$px^{-(p+1)}, p>0, x>1$	$\frac{px^{-(p+1)}}{(1-x^{-p})\log(1-x^{-p})}$
Pareto II (Lomax)	$\lambda p(1+\lambda x)^{-(p+1)}, \lambda>0, p>0, x>0$	$\frac{\lambda p(1+\lambda x)^{-(p+1)}}{(1-(1+\lambda x)^{-p}\log(1-(1+\lambda x)^{-p}}$
Beta	$\lambda p(1-\lambda x)^{p-1}, \lambda>0, p>0, 0<x<\frac{1}{\lambda}$	$\frac{\lambda p(1-\lambda x)^{p-1}}{(1-(1-\lambda x)^{p}\log(1-(1-\lambda x)^{p}}$
Weibull	$\lambda p(\lambda x)^{p-1}e^{-(\lambda x)^p}, \lambda>0, p>0, x>0$	$\frac{\lambda p(\lambda x)^{p-1}e^{-(\lambda x)^p}}{\left(1-e^{-(\lambda x)^p}\right)\log\left(1-e^{-(\lambda x)^p}\right)}$

Theorem 3.4.6. *For a rv Y, the n^{th} upper record value,*

$$\eta_Y^{U_n}(x)=\eta_X^{U_n}(x)+(\theta-1)h_X(x) \tag{3.20}$$

if and only if Y follows proportional hazards model with $\bar{F}_Y(x)=\left[\bar{F}_X(x)\right]^{\theta}, \theta>0$.

Proof. For the proportional hazards model we have $\bar{F}_Y(x)=\left[\bar{F}_X(x)\right]^{\theta}$, and $f_Y(x)=\theta\left[\bar{F}_X(x)\right]^{\theta-1}f_X(x)$, then (3.15) becomes

$$\begin{aligned}
f_Y^{U_n}(x) &= \frac{\left[-\log\bar{F}_Y(x)\right]^{n-1}}{(n-1)!}f_Y(x),\\
&= \frac{\left[-\theta\log\bar{F}_X(x)\right]^{n-1}}{(n-1)!}\theta\left[\bar{F}_X(x)\right]^{\theta-1}f_X(x),\\
&= \frac{\left[-\log\bar{F}_X(x)\right]^{n-1}}{(n-1)!}\theta^n\left[\bar{F}_X(x)\right]^{\theta-1}f_X(x),\\
&= f_X^{U_n}(x)\theta^n\left[\bar{F}_X(x)\right]^{\theta-1}.
\end{aligned}$$

Now taking logarithm on both sides, we get

$$\log f_Y^{U_n}(x) = \log f_X^{U_n}(x) + n\log\theta + (\theta-1)\log\left[\bar{F}_X(x)\right].$$

Differentiating with respect to x, gives

$$-\frac{d}{dx}\log f_Y^{U_n}(x) = -\frac{d}{dx}\log f_X^{U_n}(x) - (\theta-1)\frac{d}{dx}\log\left[\bar{F}_X(x)\right],$$

which can be written as (3.20).

Conversely, suppose that (3.20) holds, then by substituting (3.17) in (3.20) we have,

$$\eta_Y^{U_n}(x) = \eta_X(x) - (n-1)\frac{d}{dx}\log\left[-\log\bar{F}_X(x)\right] + (\theta-1)h_X(x).$$

Now from (1.9), we have

$$\begin{aligned}
f_Y^{U_n}(x) &= k\ \exp\left[-\int_0^x \left(\eta_X(u) - (n-1)\frac{d}{du}\log\left[-\log\bar{F}_X(u)\right] + (\theta-1)h_X(u)\right)du\right] \\
&= k\ \exp\left[-\int_0^x \eta_X(u)du\right]\exp\left[(n-1)\int_0^x \frac{d}{du}\log\left[-\log\bar{F}_X(u)\right]du\right] \\
&\qquad \times\exp\left[-(\theta-1)\int_0^x h_X(u)du\right] \\
&= k\ f_X(x)\exp\left[(n-1)\log\left[-\log\bar{F}_X(x)\right]\right]\exp\left[(\theta-1)\log\bar{F}_X(x)\right] \\
&= k\ f_X(x)\left[-\log\bar{F}_X(x)\right]^{(n-1)}\left[\bar{F}_X(x)\right]^{(\theta-1)},
\end{aligned}$$

which is the pdf of the n^{th} upper record value of proportional hazards model. □

3.5 RCST of circular models

Circular distributions play an important role in modelling directional data which occurs in wide variety of fields such as biology, medicine, physics, oceanography and geology. For more details on circular distributions we refer to Mardia and Jupp (2000) and Jammalamadaka and Sengupta (2001).

The wrapped (around the circle) or a circular rv $\Theta = X(\text{mod} 2\pi)$, has the density

$$f_\Theta(\theta) = \sum_{k=-\infty}^{\infty} f_X(\theta + 2k\pi), \quad \theta \in [0, 2\pi),$$

then the corresponding RCST can be defined as

$$\eta_\Theta(\theta) = -\frac{f'_\Theta(\theta)}{f_\Theta(\theta)}.$$

Example 3.5.1 (Wrapped symmetric Laplace (see Jammalamadaka and Kozubowski (2003))). If $f_\Theta(\theta) = \frac{\lambda}{2}\left(\frac{e^{\lambda(2\pi-\theta)}+e^{\lambda\theta}}{e^{2\pi\lambda}-1}\right)$, $\lambda > 0, 0 \le \theta < 2\pi$, then

$$\eta_\Theta(\theta) = \lambda\left(\frac{e^{2\lambda(\pi-\theta)} - 1}{e^{2\lambda(\pi-\theta)} + 1}\right).$$

Example 3.5.2 (Wrapped asymmetric Laplace (see Jammalamadaka and Kozubowski (2003))). If $f_\Theta(\theta) = \frac{\lambda k}{1+k^2}\left(\frac{e^{-\lambda k\theta}}{1-e^{-2\pi\lambda k}} + \frac{e^{(\lambda/k)\theta}}{e^{2\pi\lambda/k}-1}\right)$, $\lambda > 0, k > 0, 0 \le \theta < 2\pi$, then

$$\eta_\Theta(\theta) = \frac{\frac{\lambda k e^{-\lambda k\theta}}{1-e^{-2\pi\lambda k}} - \frac{\frac{\lambda}{k}e^{(\lambda/k)\theta}}{e^{2\pi\lambda/k}-1}}{\frac{e^{-\lambda k\theta}}{1-e^{-2\pi\lambda k}} + \frac{e^{(\lambda/k)\theta}}{e^{2\pi\lambda/k}-1}}.$$

Example 3.5.3 (Wrapped weighted exponential (see Roy and Adnan (2012))).

If $f_\Theta(\theta) = \frac{\alpha+1}{\alpha}\lambda e^{-\lambda\theta}\sum_{m=o}^{\infty} e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)}), \alpha > 0, \lambda > 0, 0 \leq \theta < 2\pi$, then

$$\eta_\Theta(\theta) = \lambda - \frac{\alpha\lambda\sum_{m=o}^{\infty} e^{-2m\pi\lambda}e^{-\alpha\lambda(\theta+2m\pi)})}{\sum_{m=o}^{\infty} e^{-2m\pi\lambda}(1-e^{-\alpha\lambda(\theta+2m\pi)})}.$$

Example 3.5.4 (Wrapped skew-normal (see Pewsey (2000))).

If $f_\Theta(\theta) = \frac{2}{\alpha}\sum_{m=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi m-\mu}{\alpha}\right)\Phi\left(\lambda\left(\frac{\theta+2\pi m-\mu}{\alpha}\right)\right)$, $\alpha > 0, -\infty < \lambda < \infty$, $-\infty < \mu < \infty, 0 \leq \theta < 2\pi$, then

$$\eta_\Theta(\theta) = \frac{\sum\limits_{k=-\infty}^{\infty}\left(\phi\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\Phi'\left(\lambda\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\right)\frac{\lambda}{\alpha} + \Phi\left(\lambda\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\right)\phi'\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\frac{1}{\alpha}\right)}{\sum\limits_{k=-\infty}^{\infty}\phi\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\Phi\left(\lambda\left(\frac{\theta+2\pi k-\mu}{\alpha}\right)\right)}.$$

Theorem 3.5.5. *For a circular rv Θ,the RCST function $\eta_\Theta(\theta)$ uniquely determines the circular distribution $f_\Theta(\theta)$ by*

$$f_\Theta(\theta) = C\exp\left[-\int_0^\theta \eta_\Theta(u)du\right], \tag{3.21}$$

where C is a constant to be determined by $\int_0^{2\pi} f_\Theta(\theta)d\theta = 1$.

Theorem 3.5.6. *For a rv Θ, $\eta_\Theta(\theta) = \lambda$ if and only if Θ follows a wrapped exponential distribution given by Jammalamadaka and Kozubowski (2001)*

$$f_\Theta(\theta) = \frac{\lambda e^{-\lambda\theta}}{1-e^{-2\pi\lambda}},\ \lambda > 0, 0 \leq \theta < 2\pi.$$

Proof. Suppose that $\eta_\Theta(\theta) = \lambda$, then from (3.21) we have

$$f_\Theta(\theta) = C\exp\left[-\lambda\theta\right].$$

Now using the identity $\int_0^{2\pi} f_\Theta(\theta)d\theta = 1$, e get $C = \frac{\lambda}{1-e^{-2\pi\lambda}}$ and hence

$$f_\Theta(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-2\pi\lambda}}.$$

The other part is quite straightforward. $\square$

Chapter 4

Characterizations of some bivariate models using reciprocal coordinate subtangents[3]

4.1 Introduction

Conditional densities are always easier to visualise as compared to the marginal or joint densities. For example, it is reasonable to visualize that in some human population, the distribution of heights for a given weight with the mode of the conditional distribution changing monotonically with weight. In similar fashion a unimodal distribution of weights for a given height can be easier to visualize with the mode changing monotonically with the height. However, it is not so easy to visualise the appropriate joint distributions without certain assertion. One can characterize the joint distribution by using generating functions such as joint characteristic function, joint moment generating function, etc. or by using reliability concepts like vector-

[3]Contents of this chapter have been published as entitled "Characterizations of some bivariate models using reciprocal coordinate subtangents", *Statistica*, 74(2):153–170 (see Sunoj et al. (2014)).

valued hazard function, mean residual life function etc. They are well defined and uniquely determine the joint distribution.

To determine the joint df, the knowledge of the marginals is inadequate. But if we introduce a conditional specification instead of a marginal specification or together with a marginal specification then the situation brightens. The study of reliability properties in conditionally specified models is quite recent. Arnold (1995, 1996) and Arnold and Kim (1996) have studied several classes of conditional survival models. The identification of the joint distribution of (X_1, X_2) when conditional distributions of $(X_1|X_2 = x_2)$ and $(X_2|X_1 = x_1)$ are known has been an important problem studied by many researchers in the past. This approach of identifying a bivariate density using the conditionals is called the conditional specification of the joint distribution (see Arnold et al. (1999)). These conditional models are often useful in many two component reliability systems where the operational status of one component is known in advance. Another important problem closely associated to this is the identification of the joint distribution of (X_1, X_2) when the conditional distribution or corresponding reliability measures of the rv's $(X_1|X_2 > x_2)$ and $(X_2|X_1 > x_1)$ are known. That is, instead of conditioning on a component failing (down) at a specified time, we study the system when the survival time of one of component is known. For a recent study of these models, we refer to Navarro et al. (2011), Navarro and Sarabia (2013) and the references therein.

The Chapter is organized as follows. In Section 4.2, we prove characterization results for a general bivariate model where conditional distributions are proportional hazard rate models, Sarmanov family and Ali-Mikhail-Haq family of bivariate distributions and establish a relationship between local dependence function and RCST.

In Section 4.3 and 4.4, we define RCST for conditionally specified distributions and some characterization results are proved.

4.2 Bivariate RCST

For a nonnegative vector random variable (X_1, X_2) with pdf $f(x_1, x_2)$, the vector-valued bivariate RCST (see Roy and Roy (2009)) is given by

$$\eta_1(x_1, x_2) = -\frac{\partial}{\partial x_1} \log f(x_1, x_2) \tag{4.1}$$

and

$$\eta_2(x_1, x_2) = -\frac{\partial}{\partial x_2} \log f(x_1, x_2). \tag{4.2}$$

If the bivariate RCST $(\eta_1(x_1, x_2), \eta_2(x_1, x_2))$ is continuous, then the density curve can be uniquely determined in terms of the following two alternative forms:

$$f(x_1, x_2) = C \exp\left[-\int_0^{x_1} \eta_1(u, 0)du - \int_0^{x_2} \eta_2(x_1, v)dv\right] \tag{4.3}$$

and

$$f(x_1, x_2) = C \exp\left[-\int_0^{x_2} \eta_2(0, v)dv - \int_0^{x_1} \eta_1(u, x_2)du\right]. \tag{4.4}$$

In the following subsections, we consider some bivariate models that are important in terms of richness in members and usefulness.

4.2.1 Bivariate model with conditional distributions as proportional hazards models

Recently, Navarro and Sarabia (2013) studied the reliability properties in two classes of bivariate continuous distributions based on the specification of conditional hazard functions. These classes were constructed by conditioning on two types of events viz. events of type $\{X_1 = x_1\}$ and type $\{X_2 = x_2\}$ and events of type $\{X_1 > x_1\}$ and type $\{X_2 > x_2\}$ respectively, that has been used in Arnold and Kim (1996). In survival studies the most widely used semi-parametric regression model is the Cox proportional hazards rate (PHR) model. The univariate Cox PHR model is a class of modelling distributions with pdf and sf given by

$$f(x) = \alpha\lambda(x)\exp\{-\alpha\Lambda(x)\}, \quad x \geq 0, \tag{4.5}$$

and

$$\bar{F}(x) = \exp\{-\alpha\Lambda(x)\}, \quad x \geq 0,$$

where $\alpha > 0$, $\lambda(x)$ is the baseline hazard rate function and $\Lambda(x) = \int_0^x \lambda(u)du$ is the baseline cumulative hazard function, where both $\lambda(x)$ and $\Lambda(x)$ might involve parameter θ, besides the parameter α. The hazard (or failure) rate function of $f(x)$ is $h(x) = \alpha\lambda(x)$. A rv with the pdf (4.5) can be denoted by $X \sim PHR(\alpha; \Lambda(x))$. Special cases of $f(x)$ in (4.5) include exponential, Burr, Pareto and Weibull. Navarro and Sarabia (2013) obtained a general form of a bivariate pdf with conditional distributions satisfying $(X_1|X_2 = x_2) \sim PHR(\alpha_1(x_2); \Lambda_1(x_1))$ and $(X_2|X_1 = x_1) \sim PHR(\alpha_2(x_1); \Lambda_2(x_2))$, given by

$$f(x_1, x_2) = c(\phi)a_1a_2\lambda_1(x_1)\lambda_2(x_2)\exp\left[-a_1\Lambda_1(x_1) - a_2\Lambda_2(x_2) - \phi a_1a_2\Lambda_1(x_1)\Lambda_2(x_2)\right], \tag{4.6}$$

for $x_1, x_2 \geq 0$, where $a_1, a_2 > 0$ and $\phi \geq 0$. The model given in (4.6) is a reparametrization of the bivariate conditional proportional hazard model due to Arnold and Kim (1996). The case when $\phi = 0$ corresponds to the case of independence. In particular, if $\Lambda_1(x_1) = x_1$ and $\Lambda_2(x_2) = x_2$, we obtain the class of bivariate distributions with exponential conditionals considered by Arnold and Strauss (1988). Navarro and Sarabia (2013) also obtained a bivariate pdf with conditional distributions satisfying $(X_1|X_2 > x_2) \sim PHR(\alpha_1(x_2); \Lambda_1(x_1))$ and $(X_2|X_1 > x_1) \sim PHR(\alpha_2(x_1); \Lambda_2(x_2))$, with joint pdf is given by

$$\begin{aligned} f(x_1, x_2) &= a_1 a_2 \lambda_1(x_1)\lambda_2(x_2)\left(\frac{\alpha_1(x_2)\alpha_2(x_1)}{a_1 a_2} - \phi\right) \\ &\quad \times \exp\left[-a_1\Lambda_1(x_1) - a_2\Lambda_2(x_2) - \phi a_1 a_2 \Lambda_1(x_1)\Lambda_2(x_2)\right], \end{aligned} \tag{4.7}$$

where in both cases $\alpha_1(x_2) = a_1[1 + \phi a_2\Lambda_2(x_2)]$ and $\alpha_2(x_1) = a_2[1 + \phi a_1\Lambda_1(x_1)]$, Λ_1 and Λ_2 are two cumulative hazard functions and λ_1 and λ_2 are their respective hazard rate functions.

The following two theorems provide characterizations to the vector-valued bivariate RCST using the baseline hazard functions.

Theorem 4.2.1. *For a nonnegative random vector (X_1, X_2), the relationships*

$$\eta_1(x_1, x_2) = a_1\left[1 + \phi a_2\Lambda_2(x_2)\right]\lambda_1(x_1) - \frac{\lambda_1'(x_1)}{\lambda_1(x_1)} \tag{4.8}$$

and

$$\eta_2(x_1, x_2) = a_2\left[1 + \phi a_1\Lambda_1(x_1)\right]\lambda_2(x_2) - \frac{\lambda_2'(x_2)}{\lambda_2(x_2)}, \tag{4.9}$$

holds if and only if $f(x_1, x_2)$ is of the form (4.6).

Proof. Assume that equations (4.8) and (4.9) holds, then using (4.3) we obtain $f(x_1, x_2)$

$$= C\exp\left[-a_1\int_0^{x_1}\lambda_1(u)du + \int_0^{x_1}\frac{\lambda_1'(u)}{\lambda_1(u)}du - a_2(1 + \phi a_1\Lambda_1(x_1))\int_0^{x_2}\lambda_2(v)dv + \int_0^{x_2}\frac{\lambda_2'(v)}{\lambda_2(v)}dv\right],$$

and thus

$$f(x_1, x_2) = C \exp\left[-a_1 \Lambda_1(x_1) + \log \lambda_1(x_1) - a_2 \Lambda_2(x_2)\left(1 + \phi a_1 \Lambda_1(x_1)\right) + \log \lambda_2(x_2)\right]$$

and we have the model (4.6). The other part is quite straightforward. □

Example 4.2.2. Bivariate exponential (*i.e.*, $\Lambda_1(x_1) = x_1$ and $\Lambda_2(x_2) = x_2$) with joint pdf

$$f(x_1, x_2) = c(\phi) a_1 a_2 \exp\left[-a_1 x_1 - a_2 x_2 - \phi a_1 a_2 x_1 x_2\right],$$

obtains characterizing relationships

$$\eta_1(x_1, x_2) = a_1 + \phi a_1 a_2 x_2$$

and

$$\eta_2(x_1, x_2) = a_2 + \phi a_1 a_2 x_1.$$

Example 4.2.3. Bivariate Weibull (*i.e.*, $\Lambda_1(x_1) = {x_1}^{\gamma_1}$ and $\Lambda_2(x_2) = {x_2}^{\gamma_2}$) with joint pdf

$$f(x_1, x_2) = c(\phi) a_1 a_2 \gamma_1 {x_1}^{\gamma_1 - 1} \gamma_2 {x_2}^{\gamma_2 - 1} \exp\left[-a_1 {x_1}^{\gamma_1} - a_2 {x_2}^{\gamma_2} - \phi a_1 a_2 {x_1}^{\gamma_1} {x_2}^{\gamma_2}\right],$$

the relationships are

$$\eta_1(x_1, x_2) = \gamma_1 {x_1}^{\gamma_1 - 1}\left(a_1 + \phi a_1 a_2 {x_2}^{\gamma_2}\right) - \frac{\gamma_1 - 1}{x_1}$$

and

$$\eta_2(x_1, x_2) = \gamma_2 {x_2}^{\gamma_2 - 1}\left(a_2 + \phi a_1 a_2 {x_1}^{\gamma_1}\right) - \frac{\gamma_2 - 1}{x_2}.$$

Example 4.2.4. Bivariate Pareto (*i.e.*, $\Lambda_1(x_1) = \log \frac{\beta_1 + x_1}{\beta_1}$ and $\Lambda_2(x_2) = \log \frac{\beta_2 + x_2}{\beta_2}$)

with joint pdf

$$\begin{aligned} f(x_1,x_2) &= c(\phi)a_1a_2\beta_1^{a_1}\beta_2^{a_2}\left(\frac{1}{\beta_1+x_1}\right)^{a_1+1}\left(\frac{1}{\beta_2+x_2}\right)^{a_2+1} \\ &\quad\times\exp\left[-\phi a_1a_2\log\frac{\beta_1+x_1}{\beta_1}\log\frac{\beta_2+x_2}{\beta_2}\right], \end{aligned}$$

characterizes

$$\eta_1(x_1,x_2)=\frac{1}{\beta_1+x_1}\left(1+a_1+\phi a_1a_2\log\frac{\beta_2+x_2}{\beta_2}\right)$$

and

$$\eta_2(x_1,x_2)=\frac{1}{\beta_2+x_2}\left(1+a_2+\phi a_1a_2\log\frac{\beta_1+x_1}{\beta_1}\right).$$

Example 4.2.5. Bivariate Burr (*i.e.*, $\Lambda_1(x_1)=\log\frac{\beta_1+x_1{}^{\gamma_1}}{\beta_1}$ and $\Lambda_2(x_2)=\log\frac{\beta_2+x_2{}^{\gamma_2}}{\beta_2}$) with joint pdf

$$f(x_1,x_2)=c(\phi)a_1a_2\gamma_1\gamma_2\beta_1^{a_1}\beta_2^{a_2}x_1{}^{\gamma_1-1}x_2{}^{\gamma_2-1}\left(\frac{1}{\beta_1+x_1{}^{\gamma_1}}\right)^{a_1+1}\left(\frac{1}{\beta_2+x_2{}^{\gamma_2}}\right)^{a_2+1}$$

$$\times\exp\left[-\phi a_1a_2\log\frac{\beta_1+x_1{}^{\gamma_1}}{\beta_1}\log\frac{\beta_2+x_2{}^{\gamma_2}}{\beta_2}\right],$$

we have

$$\eta_1(x_1,x_2)=\frac{\gamma_1x_1{}^{\gamma_1-1}}{\beta_1+x_1{}^{\gamma_1}}\left(1+a_1+\phi a_1a_2\log\frac{\beta_2+x_2{}^{\gamma_2}}{\beta_2}\right)-\frac{\gamma_1-1}{x_1}$$

and

$$\eta_2(x_1,x_2)=\frac{\gamma_2x_2{}^{\gamma_2-1}}{\beta_2+x_2{}^{\gamma_2}}\left(1+a_2+\phi a_1a_2\log\frac{\beta_1+x_1{}^{\gamma_1}}{\beta_1}\right)-\frac{\gamma_2-1}{x_2}.$$

Theorem 4.2.6. *For a nonnegative random vector, the relationships*

$$\eta_1(x_1,x_2) = \alpha_1(x_2)\lambda_1(x_1)\left(1 - \frac{\phi a_1 a_2}{\alpha_1(x_2)\alpha_2(x_1) - \phi a_1 a_2}\right) - \frac{\lambda_1'(x_1)}{\lambda_1(x_1)} \tag{4.10}$$

and

$$\eta_2(x_1,x_2) = \alpha_2(x_1)\lambda_2(x_2)\left(1 - \frac{\phi a_1 a_2}{\alpha_1(x_2)\alpha_2(x_1) - \phi a_1 a_2}\right) - \frac{\lambda_2'(x_2)}{\lambda_2(x_2)} \tag{4.11}$$

hold if and only if $f(x_1,x_2)$ *is* (4.7)*.*

Proof. Assume that equations (4.10) and (4.11) holds, then using (4.3) we obtain

$$\begin{aligned} f(x_1,x_2) =& C\exp\left[-\int_0^{x_1}\left(a_1\lambda_1(u)\left(1 - \frac{\phi a_1 a_2}{a_1\alpha_2(u) - \phi a_1 a_2}\right) - \frac{\lambda_1'(u)}{\lambda_1(u)}\right)du\right] \\ &\times \exp\left[-\int_0^{x_2}\left(\alpha_2(x_1)\lambda_2(v)\left(1 - \frac{\phi a_1 a_2}{\alpha_1(v)\alpha_2(x_1) - \phi a_1 a_2}\right) - \frac{\lambda_2'(v)}{\lambda_2(v)}\right)dv\right], \end{aligned}$$

Equivalently, we have

$$\begin{aligned} f(x_1,x_2) =& C\exp\left[-\int_0^{x_1}\left(a_1\lambda_1(u) - \frac{\phi a_1\lambda_1(u)}{[1+\phi a_1\Lambda_1(u)] - \phi} - \frac{\lambda_1'(u)}{\lambda_1(u)}\right)du\right] \\ &\times \exp\left[-\int_0^{x_2}\left(\alpha_2(x_1)\lambda_2(v) - \frac{\phi a_2\alpha_2(x_1)\lambda_2(v)}{[1+\phi a_2\Lambda_2(v)]\alpha_2(x_1) - \phi a_2} - \frac{\lambda_2'(v)}{\lambda_2(v)}\right)dv\right] \end{aligned}$$

which on further simplification yield

$$\begin{aligned} f(x_1,x_2) =& C^*\lambda_1(x_1)\lambda_2(x_2)\left([1+\phi a_1\Lambda_1(x_1)][1+\phi a_2\Lambda_2(x_2)] - \phi\right) \\ &\times \exp\left[-a_1\Lambda_1(x_1) - a_2\Lambda_2(x_2) - \phi a_1 a_2\Lambda_1(x_1)\Lambda_2(x_2)\right] \end{aligned}$$

reduces to the model (4.7). The other part is direct. □

4.2.2 Sarmanov family of bivariate distributions

Assume that $f_1(\cdot)$ and $f_2(\cdot)$ are univariate pdfs with supports defined on $A_1 \subseteq R$ and $A_2 \subseteq R$. Let $\phi_1(x_1)$ and $\phi_2(x_2)$ be bounded nonconstant functions such that

$$\int_{-\infty}^{\infty} \phi_1(u) f_1(u) du = 0$$

and

$$\int_{-\infty}^{\infty} \phi_2(v) f_2(v) dv = 0.$$

Then the function defined by

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) \left[1 + \omega \phi_1(x_1) \phi_2(x_2)\right] \tag{4.12}$$

is a bivariate joint density with specified marginals $f_1(x_1)$ and $f_2(x_2)$, provided ω is a real number which satisfies the condition $1 + \omega\phi_1(x_1)\phi_2(x_2) \geq 0$ for all x_1 and x_2. This is called the Sarmanov family of bivariate distributions. For various applications of this family, we refer to Willett and Thomas (1985, 1987) and Lee (1996). When $\phi_1(x_1) = 1 - 2F_1(x_1)$ and $\phi_2(x_2) = 1 - 2F_2(x_2)$ the Sarmanov family becomes the Farlie-Gumbel-Morgenstern (FGM) family (see Bairamov et al. (2001)).

Theorem 4.2.7. *For a nonnegative random vector* (X_1, X_2)

$$\eta_1(x_1, x_2) = \eta_1(x_1) - \frac{\omega \phi_1'(x_1) \phi_2(x_2)}{1 + \omega \phi_1(x_1) \phi_2(x_2)} \tag{4.13}$$

and

$$\eta_2(x_1, x_2) = \eta_2(x_2) - \frac{\omega \phi_1(x_1) \phi_2'(x_2)}{1 + \omega \phi_1(x_1) \phi_2(x_2)} \tag{4.14}$$

if and only if $f(x_1, x_2)$ *is* (4.12).

Proof. The proof is similar to Theorem 4.2.1. □

The following examples give characterizations to vector-valued bivariate RCST for various members of Sarmanov family.

Example 4.2.8. Bivariate distributions with beta marginals. In this case, we have $\phi_1(x_1) = x_1 - \frac{a_1}{a_1+b_1}$ and $\phi_2(x_2) = x_2 - \frac{a_2}{a_2+b_2}$. Then (4.13) and (4.14) becomes

$$\eta_1(x_1, x_2) = \frac{x_1(a_1 + b_1 - 2) - a_1 + 1}{x_1(1 - x_1)} - \frac{\omega\left(x_2 - \frac{a_2}{a_2+b_2}\right)}{1 + \omega\left(x_1 - \frac{a_1}{a_1+b_1}\right)\left(x_2 - \frac{a_2}{a_2+b_2}\right)}$$

and

$$\eta_2(x_1, x_2) = \frac{x_2(a_2 + b_2 - 2) - a_2 + 1}{x_2(1 - x_2)} - \frac{\omega\left(x_1 - \frac{a_1}{a_1+b_1}\right)}{1 + \omega\left(x_1 - \frac{a_1}{a_1+b_1}\right)\left(x_2 - \frac{a_2}{a_2+b_2}\right)}.$$

Example 4.2.9. Bivariate distributions with gamma marginals. In this case, we have $\phi_1(x_1) = e^{-x_1} - \left(1 + \frac{1}{\lambda_1}\right)^{-\alpha_1}$ and $\phi_2(x_2) = e^{-x_2} - \left(1 + \frac{1}{\lambda_2}\right)^{-\alpha_2}$. Then

$$\eta_1(x_1, x_2) = \lambda_1 - \frac{\alpha_1 - 1}{x_1} + \frac{\omega e^{-x_1}\left(e^{-x_2} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}{1 + \omega\left(e^{-x_1} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)\left(e^{-x_2} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}$$

and

$$\eta_2(x_1, x_2) = \lambda_2 - \frac{\alpha_2 - 1}{x_2} + \frac{\omega e^{-x_2}\left(e^{-x_1} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)}{1 + \omega\left(e^{-x_1} - \left(\frac{\lambda_1}{\lambda_1+1}\right)^{\alpha_1}\right)\left(e^{-x_2} - \left(\frac{\lambda_2}{\lambda_2+1}\right)^{\alpha_2}\right)}.$$

Example 4.2.10. FGM family. In this case, we have $\phi_1(x_1) = 1 - 2F_1(x_1)$ and $\phi_2(x_2) = 1 - 2F_2(x_2)$. Then

$$\eta_1(x_1, x_2) = \eta_1(x_1) + \frac{2\omega f_1(x_1)\left(1 - 2F_2(x_2)\right)}{1 + \omega\left(1 - 2F_1(x_1)\right)\left(1 - 2F_2(x_2)\right)}$$

and

$$\eta_2(x_1, x_2) = \eta_2(x_2) + \frac{2\omega f_2(x_2)\,(1 - 2F_1(x_1))}{1 + \omega\,(1 - 2F_1(x_1))\,(1 - 2F_2(x_2))}.$$

4.2.3 Ali-Mikhail-Haq family of bivariate distributions

The family of bivariate distributions proposed by Ali et al. (1978) is given by

$$F(x_1, x_2) = \frac{F_1(x_1)F_2(x_2)}{1 - \alpha\bar{F}_1(x_1)\bar{F}_2(x_2)}, \quad -1 \leq \alpha \leq 1,$$

where $F_1(\cdot)$ and $F_2(\cdot)$ are the marginal distribution functions of X_1 and X_2, $\bar{F}_1(\cdot) = 1 - F_1(\cdot)$ and $\bar{F}_2(\cdot) = 1 - F_2(\cdot)$. The above family of bivariate distributions is indexed by a single parameter and contains Gumbel Type I distributions as well as the case of independent rv's. The parameter α is essentially a parameter of association between X_1 and X_2. A special case of the above model is Gumbels bivariate logistic distribution given by $F(x_1, x_2) = \frac{1}{1+e^{-x_1}+e^{-x_2}}$.

A simple way of describing the model would be through the joint distribution $F(u_1, u_2)$ for the rv's (U_1, U_2), where $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$, and we obtain the copula

$$F(u_1, u_2) = \frac{u_1\; u_2}{1 - \alpha\;\bar{u}_1\;\bar{u}_2}, \tag{4.15}$$

where $\bar{u}_1 = 1 - u_1$ and $\bar{u}_2 = 1 - u_2$. It can be verified that, for the model (4.15), the joint density is given by

$$f(u_1, u_2) = \frac{(1-\alpha)(1 - \alpha\;\bar{u}_1\;\bar{u}_2) + 2\alpha\; u_1\; u_2}{(1 - \alpha\;\bar{u}_1\;\bar{u}_2)^3}, \tag{4.16}$$

for $0 < u_1 < 1$ and $0 < u_2 < 1$.

Theorem 4.2.11. *For a nonnegative random vector* (X_1, X_2)*, the relationships*

$$\eta_1(u_1, u_2) = \frac{3\alpha\ \bar{u}_2}{1 - \alpha\ \bar{u}_1\ \bar{u}_2} - \frac{(1-\alpha)\alpha\ \bar{u}_2 + 2\alpha\ u_2}{(1-\alpha)(1 - \alpha\ \bar{u}_1\ \bar{u}_2) + 2\alpha\ u_1\ u_2}$$

and

$$\eta_2(u_1, u_2) = \frac{3\alpha\ \bar{u}_1}{1 - \alpha\ \bar{u}_1\ \bar{u}_2} - \frac{(1-\alpha)\alpha\ \bar{u}_1 + 2\alpha\ u_1}{(1-\alpha)(1 - \alpha\ \bar{u}_1\ \bar{u}_2) + 2\alpha\ u_1\ u_2}$$

are satisfied if and only if $f(u_1, u_2)$ *is* (4.16).

Proof. The proof is similar to Theorem 4.2.1. □

4.2.4 Local dependence function and RCST

Let (X_1, X_2) be a bivariate random vector in the support of $(a_1, b_1) \times (a_2, b_2), b_i > a_i, i = 1, 2$, where (a_i, b_i) is an interval on the real line with an absolutely continuous distribution function $F(x_1, x_2)$, and pdf $f(x_1, x_2)$. Assume that mixed partial derivative of $f(x_1, x_2)$ exists. The local dependence function (see Holland and Wang (1987)) of (X_1, X_2) is given by,

$$\gamma_f(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \log f(x_1, x_2).$$

The relation between local dependence function and RCST is

$$\gamma_f(x_1, x_2) = -\frac{\partial}{\partial x_2}\eta_1(x_1, x_2), where\ \eta_1(x_1, x_2) = -\frac{\partial}{\partial x_1} \log f(x_1, x_2)$$

or

$$\gamma_f(x_1, x_2) = -\frac{\partial}{\partial x_1}\eta_2(x_1, x_2), where\ \eta_2(x_1, x_2) = -\frac{\partial}{\partial x_2} \log f(x_1, x_2).$$

Theorem 4.2.12. *For a nonnegative random vector* (X_1, X_2) *with continuous RCST functions, the following conditions are equivalent:*

(i) (X_1, X_2) *follows a bivariate distribution with joint pdf*

$$f(x_1, x_2) = a(x_1;\theta)b(x_2;\theta)e^{\theta x_1 x_2},$$

for some appropriate functions $a(x_1;\theta)$ *and* $b(x_2;\theta)$*;*

(ii) $\eta_1(x_1, x_2) = -\theta x_2 - \frac{a'(x_1;\theta)}{a(x_1;\theta)}$ *and* $\eta_2(x_1, x_2) = -\theta x_1 - \frac{b'(x_2;\theta)}{b(x_2;\theta)}$*; and*

(iii) $\gamma_f(x_1, x_2)$ *is a constant.*

Proof. The sequence of relationships from (i) to (ii) and (iii) is direct. The proof of (i) from (iii) can be obtained from Jones (1998). □

4.3 Conditionally specified RCST for X_1 given $X_2 = x_2$ and for X_2 given $X_1 = x_1$

In this specification we consider conditioning on events of the forms $\{X_1 = x_1\}$ and $\{X_2 = x_2\}$. Then, let (X_1, X_2) be a bivariate random variable with support $S = (0,\infty)\times(0,\infty)$. Suppose $f_{(X_1|X_2=x_2)}(x_1|x_2)$ and $f_{(X_2|X_1=x_1)}(x_2|x_1)$ be the conditional pdf of $(X_1|X_2 = x_2)$ and $(X_2|X_1 = x_1)$ respectively, then a direct extension of RCST (4.1) and (4.2) to the these conditional rv's are given by

$$\eta_{(X_1|X_2=x_2)}(x_1|x_2) = -\frac{\partial}{\partial x_1}\log f_{(X_1|X_2=x_2)}(x_1|x_2) \tag{4.17}$$

and

$$\eta_{(X_2|X_1=x_1)}(x_2|x_1) = -\frac{\partial}{\partial x_2}\log f_{(X_2|X_1=x_1)}(x_2|x_1). \tag{4.18}$$

Integrating both sides of (4.17) with respect to x_1 over the integral 0 to x_1, we get

$$f_{(X_1|X_2=x_2)}(x_1|x_2) = C_1(x_2)\ \exp\left[-\int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du\right], \tag{4.19}$$

where $C_1(x_2)$ is constant of integration determined by$\int_{X_1} f_{(X_1|X_2=x_2)}(x_1|x_2)dx_1 = 1$. This implies that the conditionally specified RCST $\eta_{(X_1|X_2=x_2)}(x_1|x_2)$ uniquely determines the conditional pdf $f_{(X_1|X_2=x_2)}(x_1|x_2)$. Similarly by using (4.18), the conditional pdf $f_{(X_2|X_1=x_1)}(x_2|x_1)$ is determined by

$$f_{(X_2|X_1=x_1)}(x_2|x_1) = C_2(x_1)\ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right], \tag{4.20}$$

where $C_2(x_1)$ is constant of integration determined by $\int_{X_2} f_{(X_2|X_1=x_1)}(x_2|x_1)dx_2 = 1$.

Remark 4.3.1. Based on the definitions of conditional distributions, it can be easily seen that $\eta_{(X_1|X_2=x_2)}(x_1|x_2) = \eta_{X_1}(x_1, x_2)$ and $\eta_{(X_2|X_1=x_1)}(x_2|x_1) = \eta_{X_2}(x_1, x_2)$, and therefore the characterization result in Theorem 4.2.1 can be obtained from Theorem 2.1 in Navarro and Sarabia (2013).

Theorem 4.3.2. *A necessary condition for the existence for a random vector* (X_1, X_2) *with support* $S_{X_1} \times S_{X_2}$ *satisfying* (4.19) *and* (4.20) *is that*

$$\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv - \int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du = s^*(x_1) + t^*(x_2), \tag{4.21}$$

holds for all (x_1, x_2) *in* $S_{X_1} \times S_{X_2}$. *Moreover, in this case, the pdf of* (X_1, X_2) *can be obtained as*

$$f_{(X_1,X_2)}(x_1, x_2)\ = K\ \exp\left[s^*(x_1) - \int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right] \tag{4.22}$$

or

$$f_{(X_1,X_2)}(x_1,x_2) \ = K^* \ \exp\left[t^*(x_2) - \int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du\right] \tag{4.23}$$

where K and K^* are normalizing constants.

Proof. If (X_1, X_2) exists and satisfies (4.19) and (4.20), then the densities $f_{(X_1|X_2=x_2)}(x_1|x_2)$ and $f_{(X_2|X_1=x_1)}(x_2|x_1)$ satisfy the compatibility condition (1.20) in Theorem 1.2 of Arnold et al. (1999), that is,

$$\frac{f_{(X_1|X_2=x_2)}(x_1|x_2)}{f_{(X_2|X_1=x_1)}(x_2|x_1)} = s(x_1)t(x_2),$$

we have

$$\frac{C_1(x_2)\ \exp\left[-\int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du\right]}{C_2(x_1)\ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right]} = s(x_1)t(x_2).$$

Equivalently,

$$\exp\left[-\left(\int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du - \int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right)\right] = s(x_1)C_2(x_1)\frac{t(x_2)}{C_1(x_2)}.$$

Taking logarithm on both sides, we get

$$\begin{aligned} -\left(\int_0^{x_1} \eta_{(X_1|X_2=x_2)}(u|x_2)\ du - \int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right) &= \log\left(s(x_1)C_2(x_1)\frac{t(x_2)}{C_1(x_2)}\right) \\ &= s^*(x_1) + t^*(x_2), \end{aligned}$$

which gives (4.21), where

$$s^*(x_1) = \log\left(s(x_1)C_2(x_1)\right) \tag{4.24}$$

and

$$t^*(x_2) = \log\left(\frac{t(x_2)}{C_1(x_2)}\right).$$

From (4.20) and (4.24), if it exists, the joint pdf of (X_1, X_2) becomes

$$\begin{aligned} f_{(X_1,X_2)}(x_1,x_2) &= f_{X_1}(x_1) f_{(X_2|X_1=x_1)}(x_2|x_1) \\ &= f_{X_1}(x_1) C_2(x_1) \ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right] \\ &= f_{X_1}(x_1) \frac{\exp(s^*(x_1))}{s(x_1)} \ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right]. \end{aligned}$$

Then, using that $s(x_1)$ is proportional to $f_{X_1}(x_1)$ (see Arnold et al. (1999)), we have

$$f_{(X_1,X_2)}(x_1,x_2) = K \exp(s^*(x_1)) \ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right]$$

or

$$f_{(X_1,X_2)}(x_1,x_2) = K \ \exp\left[s^*(x_1) - \int_0^{x_2} \eta_{(X_2|X_1=x_1)}(v|x_1)\ dv\right],$$

thus obtains the form (4.22). In a similar fashion, we can obtain (4.23). □

Example 4.3.3. Suppose that $\eta_{(X_1|X_2=x_2)}(x_1|x_2)$ and $\eta_{(X_2|X_1=x_1)}(x_2|x_1)$ satisfies the relationships $\eta_{(X_1|X_2=x_2)}(x_1|x_2) = \alpha_1(x_2)\lambda_1(x_1) - \frac{\lambda_1'(x_1)}{\lambda_1(x_1)}$ and $\eta_{(X_2|X_1=x_1)}(x_2|x_1) = \alpha_2(x_1)\lambda_2(x_2) - \frac{\lambda_2'(x_2)}{\lambda_2(x_2)}$, where $\alpha_1(x_2) = a_1[1 + \phi a_2 \Lambda_2(x_2)]$ and $\alpha_2(x_1) = a_2[1 + \phi a_1 \Lambda_1(x_1)]$. Then we can easily show that it satisfies relationship (4.21) with $s^*(x_1) = \log \lambda_1(x_1) - a_1 \Lambda_1(x_1)$ and $t^*(x_2) = a_2 \Lambda_2(x_2) - \log \lambda_2(x_2)$. Now using (4.22) or (4.23), we have the model (4.6).

Obviously, from Remark 4.3.1, Theorem 4.3.2 can be used to obtain a compatibility condition for the bivariate RCST.

4.4 Conditionally specified RCST for $(X_1|X_2 > x_2)$ and for $(X_2|X_1 > x_1)$

In the case of bivariate survival models, instead of conditioning on a component failing at a specified time, it is sometimes more natural to condition on the component's having survived beyond a specified time (see Navarro and Sarabia (2013)). Then the conditional RCST for $(X_1|X_2 > x_2)$ and $(X_2|X_1 > x_1)$ are defined as

$$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = -\frac{\partial}{\partial x_1} \log f_{(X_1|X_2>x_2)}(x_1|x_2) \tag{4.25}$$

and

$$\eta_{(X_2|X_1>x_1)}(x_2|x_1) = -\frac{\partial}{\partial x_2} \log f_{(X_2|X_1>x_1)}(x_2|x_1), \tag{4.26}$$

where $P(X_1 > x_1|X_2 > x_2) = \int_{x_1}^{\infty} f_{(X_1|X_2>x_2)}(u|x_2)du$ and $P(X_2 > x_2|X_1 > x_1) = \int_{x_2}^{\infty} f_{(X_2|X_1>x_1)}(v|x_1)dv$ are the conditional sf's of $(X_1|X_2 > x_2)$ and $(X_2|X_1 > x_1)$ respectively and assume that $\frac{P(X_1>x_1|X_2>x_2)}{P(X_2>x_2|X_1>x_1)} = U(x_1)V(x_2)$ with $U(x_1)$ and $1/V(x_2)$ are two sf's (see Navarro and Sarabia (2010)). The conditional RCST functions given in (4.25) and (4.26) where used in Navarro (2008) to study ordering properties between series systems. Integrating both sides of (4.25) with respect to x_1 over the limit 0 to x_1, we get

$$f_{(X_1|X_2>x_2)}(x_1|x_2) = D_1(x_2)\ \exp\left[-\int_0^{x_1} \eta_{(X_1|X_2>x_2)}(u|x_2)\ du\right], \tag{4.27}$$

where $D_1(x_2)$ is constant of integration determined by $\int_{X_1} f_{(X_1|X_2>x_2)}(x_1|x_2)dx_1 = 1$. Similarly from (4.26), we have

$$f_{(X_2|X_1>x_1)}(x_2|x_1) = D_2(x_1)\ \exp\left[-\int_0^{x_2} \eta_{(X_2|X_1>x_1)}(v|x_1)\ dv\right], \tag{4.28}$$

where $D_2(x_1)$ is constant of integration determined by $\int_{X_2} f_{(X_2|X_1>x_1)}(x_2|x_1)dx_2 = 1$. Therefore, like the conditional RCST for $(X_1|X_2 = x_2)$ and $(X_2|X_1 = x_1)$, the conditional RCST for $(X_1|X_2 > x_2)$ and $(X_2|X_1 > x_1)$ uniquely determines the conditional pdf's $f_{(X_1|X_2>x_2)}(x_1|x_2)$ and $f_{(X_2|X_1>x_1)}(x_2|x_1)$ through the relationships (4.27) and (4.28).

Theorem 4.4.1. *The RCST functions $\eta_{(X_1|X_2>x_2)}(x_1|x_2)$ and $\eta_{(X_2|X_1>x_1)}(x_2|x_1)$ are the conditional RCST functions of a nonnegative random vector (X_1, X_2) with support $S_{X_1} \times S_{X_2}$ if and only if*

$$\frac{\int_{x_1}^{\infty} \exp\left[-\int_0^u \eta_{(X_1|X_2>x_2)}(z|x_2)dz\right] du}{\int_{x_2}^{\infty} \exp\left[-\int_0^v \eta_{(X_2|X_1>x_1)}(z|x_1)dz\right] dv} = \frac{U(x_1)}{V(x_2)}$$

holds for $S_{X_1} \times S_{X_2}$. Moreover, in this case, the sf of (X_1, X_2) can be obtained as

$$\bar{F}_{(X_1,X_2)}(x_1, x_2) = cV(x_2) \int_{x_1}^{\infty} \exp\left[-\int_0^u \eta_{(X_1|X_2>x_2)}(z|x_2)dz\right] du$$

or as

$$\bar{F}_{(X_1,X_2)}(x_1, x_2) = c^* U(x_1) \int_{x_2}^{\infty} \exp\left[-\int_0^v \eta_{(X_2|X_1>x_1)}(z|x_1)dz\right] dv,$$

where c and c^* are constants of integration.

Proof. The proof is a consequence of Theorem 11.1 in Arnold et al. (1999) and (4.27) and (4.28). □

Example 4.4.2. The model in (4.7) is characterized by

$$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \alpha_1(x_2)\lambda_1(x_1) - \frac{\lambda_1'(x_1)}{\lambda_1(x_1)}$$

and

$$\eta_{(X_2|X_1>x_1)}(x_2|x_1) \; = \alpha_2(x_1)\lambda_2(x_2) - \frac{\lambda_2'(x_2)}{\lambda_2(x_2)}.$$

Example 4.4.3. The functions $\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \theta(x_2)$ and $\eta_{(X_2|X_1>x_1)}(x_2|x_1) = \tau(x_1)$ are the conditional RCST functions of a random vector (X_1, X_2) with support $(0, \infty) \times (0, \infty)$ if and only if $\theta(x_2) = \alpha + \gamma x_2$ and $\tau(x_1) = \beta + \gamma x_1$ where $\alpha, \beta > 0$ and $\gamma \geq 0$. In this case they characterize the Gumbel's type I bivariate exponential distribution with sf $\bar{F}_{(X_1,X_2)}(x_1, x_2) = \exp(-\alpha x_1 - \beta x_2 - \gamma x_1 x_2)$ for $x_1, x_2 \geq 0$.

More examples can be obtained from that included in Arnold et al. (1999).

The FGM family specified by the joint sf of a two-dimensional random vector (X_1, X_2),

$$\bar{F}_{(X_1,X_2)}(x_1, x_2) = \bar{F}_1(x_1)\bar{F}_2(x_2)\left[1 + \omega\left(1 - \bar{F}_1(x_1)\right)\left(1 - \bar{F}_2(x_2)\right)\right], \; -1 \leq \omega \leq 1. \tag{4.29}$$

with specified marginal distributions through $\bar{F}_1(x_1)$ and $\bar{F}_2(x_2)$.

Theorem 4.4.4. *The relationships*

$$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \eta_1(x_1) - \frac{2\omega f_1(x_1)F_2(x_2)}{1 + \omega\left(2F_1(x_1) - 1\right)F_2(x_2)} \tag{4.30}$$

and

$$\eta_{(X_2|X_1>x_1)}(x_2|x_1) = \eta_2(x_2) - \frac{2\omega f_2(x_2)F_1(x_1)}{1 + \omega\left(2F_2(x_2) - 1\right)F_1(x_1)}, \tag{4.31}$$

holds if and only if (X_1, X_2) *follows the FGM family with joint pdf* (4.29).

Proof. Assume that (4.30) holds, then using (4.27) we have the conditional pdf

$$f_{(X_1|X_2>x_2)}(x_1|x_2) = \frac{D_1(x_2)f_1(x_1)\left[1 + \omega\left(2F_1(x_1) - 1\right)F_2(x_2)\right]}{\left[1 - \omega F_2(x_2)\right]}.$$

Now applying the boundary condition $\int_0^\infty f_{(X_1|X_2>x_2)}(x_1|x_2)dx_1 = 1$, we obtain

$$\frac{D_1(x_2)}{[1-\omega F_2(x_2)]}\left[(1-\omega F_2(x_2))\int_0^\infty f_1(x_1)dx_1 + 2\omega F_2(x_2)\int_0^\infty F_1(x_1)f_1(x_1)dx_1\right] = 1,$$

thus obtains $D_1(x_2) = 1 - \omega F_2(x_2)$, and therefore

$$f_{(X_1|X_2>x_2)}(x_1|x_2) = f_1(x_1)\left[1 + \omega(2F_1(x_1) - 1)F_2(x_2)\right], \tag{4.32}$$

the conditional pdf of $(X_1|X_2 > x_2)$ for FGM model given in (4.29). Integrating (4.32) between the limits x_1 to ∞, we get

$$\bar{F}_{(X_1|X_2>x_2)}(x_1|x_2) = (1-\omega F_2(x_2))\int_{x_1}^\infty f_1(u)du + 2\omega F_2(x_2)\int_{x_1}^\infty F_1(u)f_1(u)du,$$

where $\bar{F}_{(X_1|X_2>x_2)}(x_1|x_2)$ denote the sf of $(X_1|X_2 > x_2)$. On simplification, we further obtain

$$\bar{F}_{(X_1|X_2>x_2)}(x_1|x_2) = \bar{F}_1(x_1)[1 + \omega F_1(x_1)F_2(x_2)],$$

is the conditional sf of FGM with model (4.29). In a similar manner, using (4.31) and (4.28), we can obtain the conditional sf of $(X_2|X_1 > x_1)$ for FGM in (4.29). The other part is straightforward. □

In Theorem 4.4.4, if we consider identical marginals, *i.e.*, when $f_1(x_1) = f_2(x_2) = f(x_1)$, we have $\eta_1(x_1) = \eta_2(x_2) = \eta(x_1)$. In this case, (4.30) and (4.31) are reduced to a single relationship in either x_1 or x_2, which are illustrated in the following examples.

Example 4.4.5. Uniform $[0, 1]$ marginals. In this case $f(x_1) = 1$, $F(x_1) = x_1$ and $\eta(x_1) = 0$, then $\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \eta_{(X_2|X_1>x_1)}(x_2|x_1) = -\frac{2\omega x_1}{1+\omega x_1(2x_1-1)}$.

Example 4.4.6. Exponential marginals, with $f(x_1) = \lambda e^{-\lambda x_1}$ we have $\eta(x_1) = \lambda$, then

$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \eta_{(X_2|X_1>x_1)}(x_2|x_1) = \lambda - \frac{2\omega\lambda e^{-\lambda x_1}(1-e^{-\lambda x_1})}{1+\omega(1-2e^{-\lambda x_1})(1-e^{-\lambda x_1})}$.

Example 4.4.7. Pareto marginals, with $f(x_1) = (1+x_1)^{-2}$ and $\eta(x_1) = 2(1+x_1)^{-1}$, then

$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \eta_{(X_2|X_1>x_1)}(x_2|x_1) = \frac{2}{1+x_1} - \frac{2\omega x_1(1+x_1)^{-3}}{1+\omega(2{x_1}^2(1+x_1)^{-2}-x_1(1+x_1)^{-1})}$.

Example 4.4.8. Weibull marginals, with $f(x_1) = c{x_1}^{c-1}e^{-{x_1}^c}$ and $\eta(x_1) = {x_1}^{-1}(1 - c(1 - {x_1}^c))$, then

$\eta_{(X_1|X_2>x_2)}(x_1|x_2) = \eta_{(X_2|X_1>x_1)}(x_2|x_1) = \frac{1-c(1-{x_1}^c)}{x_1} - \frac{2\omega c{x_1}^{c-1}e^{-{x_1}^c}(1-e^{-{x_1}^c})}{1+\omega(1-2e^{-{x_1}^c})(1-e^{-{x_1}^c})}$.

Chapter 5

Reciprocal coordinate subtangent in discrete time[4]

5.1 Introduction

Discrete distributions are important when the measurements are taken on a discrete scale. For example, number of copies made by a photocopier before it fails; number of road accidents in a city in a given month or devices that are observed or used for fixed duration of operation, discrete distributions provides better modelling, analysis and interpretation. Analogous to $\eta(\cdot)$ in the continuous case, a discrete version is due to Gupta et al. (1997). Let X be a nonnegative integer valued rv having a probability mass function (pmf) $p(x) = P(X = x)$. Gupta et al. (1997) introduced a new discrete measure that can be useful for measuring the shape of a failure rate

[4]Contents of this chapter have been published as entitled "A discrete analogue of reciprocal coordinate subtangent and its role in characterization problems", *Calcutta Statistical Association Bulletin*, 66:123–135 (see Sunoj and Sreejith (2014)).

function, given by

$$\phi(x) = \frac{p(x) - p(x+1)}{p(x)} = 1 - \frac{p(x+1)}{p(x)}. \tag{5.1}$$

Clearly equation (5.1) is a discrete analogue of $\eta(\cdot)$ and can be used an equivalent measure for determining the monotone behaviours of discrete failure rate functions (see Gupta et al. (1997)). $\phi(x)$ for different discrete distributions are given in Table 5.1. It is interesting to note that $E(\phi(X)) = p(0)$ for any nonnegative integer valued pmf, however, $E\left(\frac{1}{\phi(X)}\right) = \frac{1}{p(0)}$ for the geometric distribution. Like $\eta(\cdot)$, $\phi(\cdot)$ can also be represented in terms of the discrete failure rate $v(x) = 1 - \frac{\bar{F}(x+1)}{\bar{F}(x)}$, where $\bar{F}(x) = P(X \geq x)$ by $\phi(x) = 1 - v(x+1)\left(\frac{1}{v(x)} - 1\right)$. For further review on $\phi(\cdot)$ and its applications, we refer to Gupta et al. (1997), Sindu (2002), Kemp (2004) and Lai and Xie (2006).

The Chapter is organized as follows. In Section 5.2, we consider (5.1) and study its usefulness for the unique determination of the pmf's and characterizing some distributions. Characterizations are also proved for geometric, discrete Burr and modified power series family of distributions and obtained a distribution having linear $\phi(\cdot)$. In Section 5.3, we extend (5.1) to the weighted models and obtained characterization results to logarithmic-series, a distribution for which $\phi(\cdot)$ is linear, residual lifetime and partial sum distributions. Finally in Section 5.4, a new definition for discrete analogue of RCST is introduced and discrete proportional hazards model is characterized.

5.2 Characterizations using discrete RCST

In this section, we characterize some discrete distributions using $\phi(\cdot)$.

Table 5.1: $1-\phi(x)$ for different families of distributions

Distribution	**pmf**	$1-\phi(x)$
Geometric	$pq^x, x=0,1,...;0<p<1, q=1-p$	q
Binomial	$\binom{n}{x}p^x q^{n-x}, x=0,1,...,n; 0<p<1,$ $q=1-p$	$\frac{(n-x)p}{(x+1)q}$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}, x=0,1,...;\lambda>0$	$\frac{\lambda}{(x+1)}$
Extended Katz family	$\frac{p(x+1)}{p(x)}=\frac{\alpha+\beta x}{\gamma+x},\ \alpha>0, \beta<1, \gamma>0$	$\frac{\alpha+\beta x}{\gamma+x}$
Kemp family	$\frac{p(x+1)}{p(x)}=\frac{(a_1+x)(a_2+x)...(a_p+x)}{(b_1+x)(b_2+x)...(b_q+x)}.\frac{\theta}{x+1}$	$\frac{(a_1+x)(a_2+x)...(a_p+x)}{(b_1+x)(b_2+x)...(b_q+x)}.\frac{\theta}{x+1}$
Inverse linear failure rate	$\frac{p(x+1)}{p(x)}=\frac{a-1+bx}{a+b+bx},\ a>0, 0<b<1$	$\frac{a-1+bx}{a+b+bx}$
Logarithmic-series	$\frac{\theta^x}{-x\ln(1-\theta)},\ x=1,2,...;\ 0<\theta<1$	$\frac{\theta x}{1+x}$
Waring	$\frac{(a-b)(b+x-1)!a!}{a(b-1)!(a+x)!},\ x=0,1,...;\ a>b>0$	$\frac{b+x}{a+x+1}$
Cluster size	$\frac{p(x+1)}{p(x)}=\frac{x-\alpha}{(x+1)(k+1)},\ x=1,2,...;$ $0<\alpha<1, k\geq 0$	$\frac{x-\alpha}{(x+1)(k+1)}$

Theorem 5.2.1. *The discrete RCST $\phi(x)=C$, where $0<C<1$, a constant if and only if X follows a geometric distribution.*

We prove the following characterization theorems using unique determination of pmf due to Chechile (2003). That is, $\phi(\cdot)$ uniquely determines the pmf $p(\cdot)$

$$p(x)=\begin{cases}\alpha, & x=0,\\ \alpha\prod_{i=0}^{x-1}\left(1-\phi(i)\right), & x=1,2,...\\ 0 & otherwise\end{cases} \tag{5.2}$$

where α is a constant determined by $\sum_{x=0}^{\infty}p(x)=1$. In the continuous case, RCST is linear in x if the underlying distribution is truncated normal. Similarly, in the next theorem, we obtain a pmf for which $\phi(x)$ is linear.

Theorem 5.2.2. *Let X be a nonnegative rv with pmf $p(\cdot)$. Then $\phi(x) = a\,x + b$ if and only if the pmf of X is of the form*

$$p(x) = \frac{1}{{}_2F_0\left[1, \frac{b-1}{a}; -; -a\right]} a^x \left(\frac{1-b}{a}\right)^{(x)}, \quad x = 0, 1, ..., a > 0, b > 0, \tag{5.3}$$

where ${}_pF_q\left[a_1, ..., a_p; b_1, ..., b_q; x\right]$ is the generalized hypergeometric series, given by ${}_pF_q\left[a_1, ..., a_p; b_1, ..., b_q; x\right] = 1 + \frac{a_1 \cdots a_p}{b_1 \cdots b_q}\frac{x}{1!} + \frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)}\frac{x^2}{2!} + ...$ and $(n)^{(x)} = \frac{n!}{(n-x)!}$ is the descending factorial.

Proof. The if part is straightforward. To prove the only if part, suppose that $\phi(x) = a\,x + b$ holds, then from (5.2), we get

$$p(x) = \alpha \prod_{i=0}^{x-1} (1 - a\,i - b) \tag{5.4}$$

where α is determined by $\sum_{x=0}^{\infty} p(x) = 1$, which implies

$$\alpha\left(1 + \left(0 - \frac{1-b}{a}\right)(-a) + \left(0 - \frac{1-b}{a}\right)\left(1 - \frac{1-b}{a}\right)(-a)^2 + ...\right) = 1,$$

That is, $\alpha\ {}_2F_0\left[1, \frac{b-1}{a}; -; -a\right] = 1$, or $\alpha = \frac{1}{{}_2F_0\left[1, \frac{b-1}{a}; -; -a\right]}$ and hence equation (5.4) becomes

$$\begin{aligned} p(x) &= \frac{1}{{}_2F_0\left[1, \frac{b-1}{a}; -; -a\right]} \prod_{i=0}^{x-1} (1 - a\,i - b)\,, \\ &= \frac{1}{{}_2F_0\left[1, \frac{b-1}{a}; -; -a\right]} \left(a^x \frac{1-b}{a}\left(\frac{1-b}{a} - 1\right)\left(\frac{1-b}{a} - 2\right)\cdots\left(\frac{1-b}{a} - x + 1\right)\right), \end{aligned}$$

which gives (5.3). $\square$

In the next theorem, we characterize the discrete Burr family proposed by Nair and Asha (2004), with pmf

$$p(x) = \begin{cases} \frac{1}{c+1}, & x = 0 \\ \frac{cG(x-1)g(x)}{(1+cG(x))(1+cG(x-1))}, & x = 1, 2, ... \end{cases} \tag{5.5}$$

where $G(x) = \prod_{u=1}^{x} (1 - g(u))$ with $G(0) = 1$ and $G(\infty) = 0$ and $g(x) = (1 - v(x))$ $(1 - \bar{v}(x))$ with $\bar{v}(x) = \frac{p(x)}{F(x)}, F(x) = P(X \leq x)$ and c is determined such that $\sum_{0}^{\infty} p(x) = 1$. Some important members of (5.5) include uniform, geometric, discrete Weibull, power series and Waring, obtained by taking different functional forms to $g(x)$.

Theorem 5.2.3. *For a nonnegative rv X, $1 - \phi(x) = (1 - g(x)) \frac{g(x+1)}{g(x)} \frac{(1+cG(x-1))}{(1+cG(x+1))}$ if and only if the distribution of X belongs to discrete Burr family* (5.5) *with* $p(0) = \frac{1}{c+1}$.

Proof. Assume that, $1 - \phi(x) = (1 - g(x)) \frac{g(x+1)}{g(x)} \frac{(1+cG(x-1))}{(1+cG(x+1))}$ holds, then from (5.2) we have

$$\begin{aligned} p(x) &= \alpha \prod_{i=0}^{x-1} (1 - g(i)) \frac{g(i+1)}{g(i)} \frac{(1 + cG(i-1))}{(1 + cG(i+1))}, \\ &= \alpha \frac{g(x)}{g(0)} \frac{(1 + cG(-1))}{(1 + cG(x-1))} \frac{(1 + cG(0))}{(1 + cG(x))} (1 - g(0)) \left(\prod_{i=1}^{x-1} (1 - g(i)) \right), \\ &= \alpha A \frac{g(x)G(x-1)}{(1 + cG(x-1))(1 + cG(x))}, \end{aligned}$$

where $G(x) = \prod_{u=1}^{x} (1 - g(u))$and $A = \frac{(1-g(0))}{g(0)} (1 + cG(-1)) (1 + cG(0))$. Since $p(0) = \frac{1}{c+1}$, $\alpha = \frac{1}{c+1}$ and the constant A is determined by,

$$\frac{1}{c+1} + \frac{1}{c+1} A \sum_{x=1}^{\infty} \frac{G(x-1)g(x)}{(1 + cG(x-1))(1 + cG(x))} = 1,$$

which gives $A = c(c+1)$ and the form (5.5). The proof of the other part is direct. □

Theorem 5.2.4. *For a nonnegative rv X, $1 - \phi(x) = t(x)u(\theta)$, where $t(x) = \frac{a(x+1)}{a(x)} (> 0)$ is a function of x and $u(\theta) > 0$ if and only if the distribution of X belongs to modified power series family $p(x) = \frac{a(x)(u(\theta))^x}{k(\theta)}, x \in T$ where T is a subset of the set of nonnegative integers, $a(x) > 0$, and $u(\theta)$ and $k(\theta)$ are positive, finite and differentiable.*

Proof. The first part is straight forward and the converse part is direct from equation (5.2). □

We say that X is less than or equal in the likelihood ratio order $(X \underset{LR}{\leq} Y)$ than another random variable Y if $\frac{p_X(x)}{p_Y(x)}$ decreases in x, where $p_X(x)$ and $p_Y(x)$ denotes the pmf's of X and Y respectively. Then, it is easy to obtain the following theorem.

Theorem 5.2.5. *$X \underset{LR}{\leq} (\geq) Y$ if and only if $\phi_X(x) \geq (\leq) \phi_Y(x)$, where $\phi_X(\cdot)$ and $\phi_Y(\cdot)$ are $\phi(.)$ functions for the rv's X and Y.*

5.3 Characterizations of discrete RCST for weighted models

By virtue of the definition of discrete analogue of RCST in (5.1), the discrete analogue of RCST function for the weighted rv X^w is given by $\phi_w(x) = 1 - \frac{p_w(x+1)}{p_w(x)}$. Equivalently,

$$1 - \phi_w(x) = (1 - \phi(x)) \frac{w(x+1)}{w(x)}. \tag{5.6}$$

Following equation (5.2), it can be easily shown that $\phi_w(\cdot)$ also uniquely determines $p_w(\cdot)$. Further, $\phi_w(\cdot)$ uniquely determines the original pmf $p(\cdot)$ using the relationship

$$p(x) = \begin{cases} \gamma, & x = 0, \\ \frac{\gamma}{w(x)} \prod_{i=0}^{x-1} \left(1 - \phi_w(i)\right), & x = 1, 2, ..., \\ 0 & otherwise \end{cases} \tag{5.7}$$

where γ is a constant determined by $\sum_{x=0}^{\infty} p(x) = 1$.

Remark 5.3.1. If $w(\cdot)$ is monotonically increasing (decreasing), then $X \underset{LR}{\leq} (\geq) X_w$ if and only if $\phi_w(x) \leq (\geq)\, \phi(x)$.

It may be noted that for most of the weighted models, the probability distributions are characterized by specifying its weight functions (see Gupta and Kirmani (1990)). Likewise, equation (5.7) determines pmf uniquely using weight function $w(\cdot)$ and $\phi_w(\cdot)$. The following two theorems establish this idea.

Theorem 5.3.2. *Let X be a nonnegative rv with pmf $p(\cdot)$. Then for $w(x) = x$, $\phi_w(x) = \theta, 0 < \theta < 1$ if and only if X follows a logarithmic-series distribution with pmf*

$$p(x) = \frac{(1-\theta)^x}{-x \ln \theta} \quad , \quad 0 < \theta < 1, \quad x = 1, 2, ...$$

Proof. The first part easily follows from (5.6). Conversely, suppose that $\phi_w(x) = \theta$ holds, then using (5.7) we obtain $p(x) = \frac{\gamma}{x} \prod_{i=1}^{x-1} (1 - \theta) = \frac{K}{x}(1-\theta)^x$, where $K = \frac{\gamma}{1-\theta}$ and is determined by $\sum_{x=1}^{\infty} p(x) = 1$, implies that $\gamma + K \ \sum_{x=2}^{\infty} \frac{1}{x}(1-\theta)^x = 1$, *i.e.*, $K \ \sum_{x=1}^{\infty} \frac{1}{x}(1-\theta)^x = 1$ and therefore $K = \frac{1}{-\ln \theta}$, proves the theorem. □

Theorem 5.3.3. *Let X be a nonnegative rv with pmf $p(\cdot)$. Then for $w(x) = x$, $\phi_w(x) = a\,x + b$ if and only if the pmf of X is of the form*

$$p(x) = \frac{1}{{}_3F_1\left[1, 1, \frac{a+b-1}{a}; 2; -a\right]} \frac{a^{x-1}\left(\frac{1-a-b}{a}\right)^{(x-1)}}{x}, \quad x = 1, 2, ..., a > 0, b > 0.$$

Proof. The first part is straightforward from (5.6). The converse part is similar to the proof of Theorem 5.2.2. □

Theorem 5.3.4. *For a nonnegative rv X, $\phi_w(x) = \phi(x+t+1)$ if and only if $X^w = x < X \le x+t\,|X > x$ follows a residual life distribution.*

Proof. Suppose X^w follows a residual life distribution, *i.e.*, $w(x) = \frac{p(x+t+1)}{p(x)}$, then $\frac{w(x+1)}{w(x)} = \frac{1-\phi(x+t+1)}{1-\phi(x)}$. From (5.6), we have $\phi_w(x) = \phi(x+t+1)$. Conversely, suppose that $\phi_w(x) = \phi(x+t+1)$ holds, we have

$$p_w(x) = \beta \prod_{i=0}^{x-1} (1 - \phi(i+t+1)) = A(t)p(x+t+1), \tag{5.8}$$

where $A(t) = \frac{\beta}{p(t+1)}$, using the identity $\sum_{x=0}^{\infty} p_w(x) = 1$, we get $A(t) = \frac{1}{\bar{F}(t+1)}$ and therefore (5.8) becomes $p_w(x) = \frac{p(x+t+1)}{\bar{F}(t+1)}$, which is the residual life distribution. □

Theorem 5.3.5. *For a nonnegative rv X, $\phi_w(x) = v(x+1)$ if and only if X^w follows a partial sum (renewal) distribution.*

Proof. Suppose X^w follows a partial sum (renewal) distribution. *i.e.*, $w(x) = \frac{P(X>x)}{P(x)}$, then $\frac{w(x+1)}{w(x)} = \frac{1}{1-\phi(x)} \frac{P(X>x+1)}{P(X>x)}$. From (5.6), we get $\phi_w(x) = v(x+1)$. Conversely, suppose that $\phi_w(x) = v(x+1)$, we get

$$\begin{aligned} p_w(x) &= \beta \prod_{i=0}^{x-1} (1 - v(i+1)), \quad x = 1, 2, ..., \\ &= \beta\, P(X > x). \end{aligned} \tag{5.9}$$

Now using the identity $\sum_{x=0}^{\infty} p_w(x) = 1$, we have $\beta = \frac{1}{E(X)}$, where $E(X) = \sum_{x=0}^{\infty} P(X > x)$ and therefore (5.9) becomes $p_w(x) = \frac{P(X>x)}{E(X)}$. *i.e.*, X^w follows a partial sum (renewal) distribution. □

5.4 An alternative definition to discrete RCST

Although the definition of discrete failure rate function $v(x) = 1 - \frac{\bar{F}(x+1)}{\bar{F}(x)}$ has been widely used in literature, Xie et al. (2002) recently identified a few drawbacks of this definition and proposed an alternate definition to it, given by $v^*(x) = \ln \frac{\bar{F}(x)}{\bar{F}(x+1)}$. Xie et al. (2002) showed that $v^*(\cdot)$ is additive for series system and it has the same monotonicity properties as that of $v(\cdot)$. Motivated by this, we propose an alternate form for $\phi(\cdot)$ in the discrete case by

$$\phi^*(x) = \ln \frac{p(x)}{p(x+1)}.$$

$\phi^*(\cdot)$ can be easily represented in terms of $\phi(x)$ by $\phi^*(x) = -\ln(1 - \phi(x))$. Equivalently,

$$\phi(x) = 1 - e^{-\phi^*(x)}. \tag{5.10}$$

Also, we have,

$$\varDelta\phi^*(x) = \phi^*(x+1) - \phi^*(x) = \ln\left(\frac{p^2(x+1)}{p(x)p(x+2)}\right).$$

Recalling that the distribution is log-convex if $p(x)p(x+2) > p^2(x+1)$ and log-concave if $p(x)p(x+2) < p^2(x+1)$ for all x, we can say that log-concavity is equivalent to $\varDelta\phi^*(x) > 0$ and log-convexity is equivalent to $\varDelta\phi^*(x) < 0$. Thus, if $\varDelta\phi^*(x) > 0$, then $v(x)$ is nondecreasing (IFR); if $\varDelta\phi^*(x) < 0$, then $v(x)$ is nonincreasing (DFR) and if $\varDelta\phi^*(x) = 0$, then $\frac{p(x+1)}{p(x)} = \frac{p(x+2)}{p(x+1)}$ for all x. Therefore the monotonicity of $\varDelta\phi^*(x)$ is same as $\varDelta\phi(x)$ due to Gupta et al. (1997) in (5.1) and hence $\phi^*(x)$ is an alternate form for $\phi(x)$ and is equally useful in determining the monotone behaviours of different discrete distributions. Based on these, we have the following properties.

Property 5.4.1. The two measures $\phi(\cdot)$ and $\phi^*(\cdot)$ have the same monotonicity property. That is, $\phi(\cdot)$ is increasing (decreasing) if and only if $\phi^*(\cdot)$ is increasing (decreasing).

When $\phi^*(x)$ is small, we have

$$\phi(x) = 1 - e^{-\phi^*(x)} = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{(\phi^*(x))^j}{j!} = \phi^*(x) + O\left(\phi^*(x)\right),$$

where $O\left(\phi^*(x)\right)$ tends to zero when $\phi^*(x)$ tends to zero. Hence $\phi^*(\cdot)$ and $\phi(\cdot)$ are almost equal for small $\phi^*(\cdot)$.

Property 5.4.2. The ratio of two probabilities $\frac{p(x+1)}{p(x)} \geq 0$ for all x implies that $\phi(x) \leq 1$ *i.e.,* $\phi(\cdot)$ is bounded, whereas $\phi^*(\cdot)$ is not a bounded measure.

Based on Table 5.1, we have the following examples

Example 5.4.3 (Poisson). $\frac{p(x)}{p(x+1)} = \frac{(x+1)}{\lambda}$, then $\phi^*(x) = \ln \frac{(x+1)}{\lambda}$ and therefore $\Delta\phi^*(x) > 0$ and is IFR.

Example 5.4.4 (Logarithmic-series). $\phi^*(x) = \ln \frac{x+1}{\theta x}$, then $\Delta\phi^*(x) = \ln \frac{(x+2)x}{(x+1)^2} < 0$, and is DFR.

Example 5.4.5 (Waring). $\phi^*(x) = \ln \frac{a+x+1}{b+x}$, we have $\Delta\phi^*(x) = \ln \left[\frac{(a+x+2)(b+x)}{(a+x+1)(b+x+1)}\right] < 0$ and therefore DFR.

We can also express $\phi^*(\cdot)$ in terms of $v^*(\cdot)$ by, $\phi^*(x) = \ln\left(\frac{e^{v^*(x)}-1}{1-e^{-v^*(x+1)}}\right)$. In this fashion, $\phi(\cdot)$ for weighted distributions is $\phi^*_w(x) = \ln \frac{p_w(x)}{p_w(x+1)}$, and $\phi^*_w(\cdot)$ in terms of $\phi(\cdot)$ by

$$\phi^*_w(x) = -\ln\left((1-\phi(x))\,\frac{w(x+1)}{w(x)}\right)$$

whereas $\phi^*_w(\cdot)$ in terms of $\phi^*(\cdot)$ by

$$\phi^*_w(x) = \phi^*(x) + \ln \frac{w(x)}{w(x+1)}. \tag{5.11}$$

Further, the relationship between $\phi_w(\cdot)$ and $\phi^*_w(\cdot)$ is $\phi^*_w(x) = -\ln(1 - \phi_w(x))$, or

$$\phi_w(x) = 1 - e^{-\phi^*_w(x)}.$$

Theorem 5.4.6. *For a nonnegative rv X, $\phi^*(\cdot)$ uniquely determines the pmf $p(\cdot)$*

$$p(x) = \begin{cases} \alpha, & x = 0, \\ \alpha\, e^{-\sum_{i=0}^{x-1} \phi^*(i)}, & x = 1, 2, ... \\ 0 & otherwise \end{cases}$$

where α is a constant determined by $\sum_{x=0}^{\infty} p(x) = 1$.

Proof. The proof is obtained by substituting (5.10) in equation (5.2). □

Theorem 5.4.7. *For a nonnegative rv X, $\phi^*_w(\cdot)$ uniquely determines the pmf $p_w(\cdot)$*

$$p_w(x) = \begin{cases} \beta, & x = 0, \\ \beta\, e^{-\sum_{i=0}^{x-1} \phi^*_w(i)}, & x = 1, 2, ..., \\ 0 & otherwise \end{cases} \tag{5.12}$$

where β is a constant determined by $\sum_{x=0}^{\infty} p_w(x) = 1$.

Corollary 5.4.8. *For a nonnegative rv X, $\phi^*_w(\cdot)$uniquely determines the pmf $p(\cdot)$*

$$p(x) = \begin{cases} \gamma, & x = 0, \\ \frac{\gamma}{w(x)}\, e^{-\sum_{i=0}^{x-1} \phi^*_w(i)}, & x = 1, 2, ..., \\ 0 & otherwise \end{cases}$$

where γ *is a constant determined by* $\sum_{x=0}^{\infty} p(x) = 1$.

For some probability models, we will not get a closed form characterizing relationships using $\phi(\cdot)$ while $\phi_w^*(\cdot)$ obtains. To illustrate this, in the next theorem we characterize the discrete proportional hazards model proposed by Dewan and Sudheesh (2009) using the functional form of $\phi_w^*(\cdot)$.

Theorem 5.4.9. *For a nonnegative rv* X, $\phi_w^*(x) = \ln\left(\frac{e^{\theta v^*(x)}-1}{1-e^{-\theta v^*(x+1)}}\right)$, $\theta > 0$ *if and only if ,* X^w *follows discrete proportional hazards model.*

Proof. Suppose X^w follows proportional hazards model, *i.e.*, $w(x) = \frac{\left(\bar{F}(x)\right)^{\theta}-\left(\bar{F}(x+1)\right)^{\theta}}{P(x)}$, then $\ln \frac{w(x)}{w(x+1)} = \ln \frac{\left(\bar{F}(x)\right)^{\theta}-\left(\bar{F}(x+1)\right)^{\theta}}{\left(\bar{F}(x+1)\right)^{\theta}-\left(\bar{F}(x+2)\right)^{\theta}} - \phi^*(x)$. From (5.11), we get

$$\phi_w^*(x) = \ln \frac{\left(\bar{F}(x)\right)^{\theta} - \left(\bar{F}(x+1)\right)^{\theta}}{\left(\bar{F}(x+1)\right)^{\theta} - \left(\bar{F}(x+2)\right)^{\theta}} = \ln \left(\frac{\left(\frac{\bar{F}(x)}{\bar{F}(x+1)}\right)^{\theta} - 1}{1 - \left(\frac{\bar{F}(x+2)}{\bar{F}(x+1)}\right)^{\theta}} \right),$$

i.e., $\phi_w^*(x) = \ln\left(\frac{e^{\theta v^*(x)}-1}{1-e^{-\theta v^*(x+1)}}\right)$, $\theta > 0$.

Conversely, suppose that $\phi_w^*(x) = \ln\left(\frac{e^{\theta v^*(x)}-1}{1-e^{-\theta v^*(x+1)}}\right)$ holds, then from (5.12) we have

$$\begin{aligned} p_w(x) &= \beta \ \exp\left(-\sum_{i=0}^{x-1} \ln\left(\frac{e^{\theta v^*(i)} - 1}{1 - e^{-\theta v^*(i+1)}} \right) \right) = \beta \ \exp\left(-\sum_{i=0}^{x-1} \ln \left(\frac{\left(\frac{\bar{F}(i)}{\bar{F}(i+1)}\right)^{\theta} - 1}{1 - \left(\frac{\bar{F}(i+2)}{\bar{F}(i+1)}\right)^{\theta}} \right) \right), \\ &= \beta \ \exp\left(\sum_{i=0}^{x-1} \ln \left(\left(\bar{F}(i+1)\right)^{\theta} - \left(\bar{F}(i+2)\right)^{\theta} \right) - \sum_{i=0}^{x-1} \ln \left(\left(\bar{F}(i)\right)^{\theta} - \left(\bar{F}(i+1)\right)^{\theta} \right) \right), \\ &= A\left(\left(\bar{F}(x)\right)^{\theta} - \left(\bar{F}(x+1)\right)^{\theta} \right), \end{aligned} \tag{5.13}$$

where $A = \frac{\beta}{\left(\left(\bar{F}(0)\right)^{\theta}-\left(\bar{F}(1)\right)^{\theta}\right)}$ and is determined by $\sum_{x=0}^{\infty} p_w(x) = 1$, which implies that $A = 1$, therefore (5.13) becomes $p_w(x) = \left(\bar{F}(x)\right)^{\theta} - \left(\bar{F}(x+1)\right)^{\theta}$, the discrete proportional hazards model. □

Chapter 6

Nonparametric estimation of RCST for censored dependent observations[5]

6.1 Introduction

Nonparametric estimation is a very effective and useful technique for obtaining properties having to do with general aspects of a curve (density, regression, etc.). Nonparametric estimation methods typically involve some kind of approximation or smoothing method. One of the main smoothing methods used in nonparametric estimation of density is that of kernel estimation (see Silverman (1986)). Kernel estimation involves a smoothing parameter or bandwidth which controls the orientation and amount of smoothing induced. It is quite common that there is no explicit data-dependent rule for selecting the bandwidth. This is due to the difficulty in finding rigorous rules for bandwidth selection. Usually in these cases the bandwidth

[5]Contents of this chapter have been communicated to an International Journal.

is selected based on a related statistical problem. This is a practically feasible yet worrisome compromise.

Accordingly, we propose a nonparametric kernel type estimation for $\eta(\cdot)$ under right censored dependent data. We consider the situation where the data under study are dependent. In this situation, the underlying lifetimes are assumed to be α-mixing (see Rosenblatt (1956)) and its definition is given below.

Definition 6.1.1. Let $\{X_i; i \geq 1\}$ denote a sequence of random variables. Given a positive integer n, set

$$\alpha(n) = \sup_{k \geq 1} \left\{ |P(A \cap B) - P(A)P(B)| \, ; A \in \Im_1^k, B \in \Im_{k+n}^\infty \right\},$$

where $\Im_i^k$ denote the σ-field of events generated by $\{X_j; i \leq j \leq k\}$. The sequence is said to be α-mixing (strong mixing), if the mixing coefficient $\alpha(n) \to 0$ as $n \to \infty$.

Many stochastic processes satisfy the α-mixing condition, see, for example, Doukhan (1994) and Carrasco et al. (2007). Fakoor (2010) examined the strong uniform consistency of kernel density estimators for censored dependent data. Cai (1998b) proposed hazard rate estimation for censored dependent data and Cai (1998a) established the asymptotic properties of Kaplan-Meier estimator for censored dependent data. Rajesh et al. (2015) and Rajesh et al. (2016) respectively proposed nonparametric estimators for the residual entropy function and for the inaccuracy measure based on the right censored dependent data.

The Chapter is organized as follows. In Section 6.2, we present a nonparametric estimator for $\eta(\cdot)$ under right censored sample and obtained the bias and mean squared error (MSE). In Section 6.3, the asymptotic properties of the estimator are studied under suitable regularity conditions. In Section 6.4, a simulation study is carried out to illustrate the performance of the estimator. The usefulness of the estimator for real data set is also investigated.

6.2 Nonparametric estimation of RCST

In this section, we propose a nonparametric estimator for $\eta(\cdot)$ for censored data sets. In reliability and life testing, due to time constraints or cost considerations the experimenter is forced to terminate the experiment after a specific period of time or after the failure of a specified number of units. Such nonavailability of the complete information results the underlying data censored. There are different censoring mechanisms adopted by the experimenters, however, the more commonly encountered one is the random right censoring. In random right censoring, the individuals start at random times such that both the lifetimes and the censoring times are random and it occurs when a subject leaves the study before an event occurs, or the study ends before the event has occurred.

Let $\{X_i; 1 \leq i \leq n\}$ be a sequence of nonnegative random variables representing the life times for n components/devices. The random variables are not assumed to be mutually independent. However, X_i have a common unknown continuous marginal distribution function $F(\cdot)$ with a pdf $f(\cdot) = F'(\cdot)$. Let the random variable X_i be censored on the right and Y_i denotes the censoring time associated with X_i. In this random censorship model, the censoring times $Y_1, Y_2, ..., Y_n$ are assumed to be independently and identically distributed random variables with common distribution function $G(\cdot)$ and are independent of $X_1, X_2, ..., X_n$. Let $Z_i = \min(X_i, Y_i)$ and $\delta_i = I(X_i \leq Y_i)$, where $I(\cdot)$ denotes the indicator function contain the censoring information. The actually observed Z_i's have a distribution function $L(\cdot)$ satisfying

$$1 - L(t) = (1 - F(t))(1 - G(t)), \quad t \in R_+ = [0, \infty).$$

Let $L^*(t) = P(Z_1 \leq t; \delta_1 = 1)$ be the corresponding sub-distribution function for the uncensored observations and $l^*(t) = f(t)(1 - G(t))$ be the corresponding sub-density. A reasonable estimator of $f(\cdot)$ should behave like $\frac{l_n^*(t)}{(1-G(t))}$ where $l_n^*(t) = \frac{1}{h_n} \int\limits_{R^+} K\left(\frac{t-x}{h_n}\right) dL_n^*(x)$ is the kernel estimator pertaining to $L_n^*(t) = \frac{1}{n} \sum\limits_{i=1}^{n} I(Z_i \leq t; \delta_i = 1)$.

The nonparametric estimator for (1.2) under censoring is defined as

$$\eta_n(x) = -\frac{d}{dx} \log f_n(x),$$

where

$$f_n(x) = \frac{1}{h_n} \int\limits_{R^+} \frac{K\left(\frac{x-u}{h_n}\right)}{1 - G(u)} dL_n^*(u), \tag{6.1}$$

is the nonparametric estimator for $f(x)$ under right censoring (see Cai (1998b)), where $K(\cdot)$ is a kernel function and h_n is the smoothing parameter or bandwidth.

The Taylor expansion of $\log f_n(x)$ with integral remainder form (see Wade (2004)) is

$$\log f_n(x) = \log f(x) + \frac{f_n(x) - f(x)}{f(x)} + T_n, \tag{6.2}$$

where

$$T_n = \int\limits_0^1 \frac{(\tau - 1)}{\{f(x) + \tau\,[f_n(x) - f(x)]\}^2} \,[f_n(x) - f(x)]^2 \, d\tau.$$

Taking negative differentiation of (6.2), we get

$$-\frac{d}{dx}\log f_n(x) = -\frac{d}{dx}\left(\log f(x) + \frac{f_n(x)-f(x)}{f(x)} + T_n\right)$$

$$-\frac{d}{dx}\log f_n(x) = -\frac{d}{dx}\log f(x) - \frac{d}{dx}\left(\frac{f_n(x)-f(x)}{f(x)}\right) - \frac{d}{dx}T_n$$

$$\eta_n(x) = \eta(x) - \left(\frac{f_n'(x)-f'(x)}{f(x)} - \frac{f_n(x)-f(x)}{[f(x)]^2}f'(x)\right) - \frac{d}{dx}T_n$$

$$\eta_n(x) - \eta(x) = -\frac{f_n'(x)-f'(x)}{f(x)} + \frac{f_n(x)-f(x)}{[f(x)]^2}f'(x) + R_n \tag{6.3}$$

where $R_n = \frac{d}{dx}(-T_n)$

i.e.,

$$\begin{aligned}
R_n =& \int_0^1 \left(\frac{d}{dx}\frac{(1-\tau)[f_n(x)-f(x)]^2}{\{f(x)+\tau[f_n(x)-f(x)]\}^2}\right)d\tau \\
=& \int_0^1 \frac{2(1-\tau)[f_n(x)-f(x)][f_n'(x)-f'(x)]}{\{f(x)+\tau[f_n(x)-f(x)]\}^2} \\
&- \frac{2(1-\tau)[f_n(x)-f(x)]^2\{f'(x)+\tau[f_n'(x)-f'(x)]\}}{\{f(x)+\tau[f_n(x)-f(x)]\}^3}d\tau \\
=& \int_0^1 \frac{2(1-\tau)[f_n(x)-f(x)][f_n'(x)-f'(x)]}{\{f(x)+\tau[f_n(x)-f(x)]\}^2}d\tau \\
&- \int_0^1 \frac{2(1-\tau)[f_n(x)-f(x)]^2\{f'(x)+\tau[f_n'(x)-f'(x)]\}}{\{f(x)+\tau[f_n(x)-f(x)]\}^3}d\tau \\
=& R_{n_1} - R_{n_2},
\end{aligned}$$

where

$$R_{n_1} = \int_0^1 \frac{2(1-\tau)[f_n(x)-f(x)][f_n'(x)-f'(x)]}{\{f(x)+\tau[f_n(x)-f(x)]\}^2}d\tau \tag{6.4}$$

and

$$R_{n_2} = \int_0^1 \frac{2(1-\tau)\left[f_n(x)-f(x)\right]^2\left\{f'(x)+\tau\left[f_n'(x)-f'(x)\right]\right\}}{\left\{f(x)+\tau\left[f_n(x)-f(x)\right]\right\}^3} d\tau. \tag{6.5}$$

The following theorem is due to Chen et al. (2009).

Theorem 6.2.1. *For positive integers i and j, assume that*

(i) $f^{(p)}(x), 1 \leq p \leq 2j$, *exists and* $f^{(2j)}(x)$ *is bounded.*

(ii) $K(s) \geq 0, -\infty < s < \infty, \int K(s)ds = 1, \int s^a K^i(s)ds = 0$ *for positive odd integers a and* $\int s^b K^i(s)ds < \infty$ *for positive even integer b.*

Then

$$\begin{aligned}
&E\left[\frac{1}{h_n} K\left(\frac{x-Z_1}{h_n}\right)\right]^i \\
&= \frac{1}{h_n^{i-1}}\left\{f(x)\int K^i(u)du + \left[\frac{f''(x)}{2}\int u^2 K^i(u)du\right] h_n^2\right. \\
&\quad + \left[\frac{f^{(4)}(x)}{4!}\int u^4 K^i(u)du\right] h_n^4 + \ldots + \left[\frac{f^{(2j-2)}(x)}{(2j-2)!}\int u^{2j-2} K^i(u)du\right] h_n^{2j-2} + O(h_n^{2j}).
\end{aligned}$$

Now we have the following theorem.

Theorem 6.2.2. *For positive integers i and j, assume that*

(i) $f^{(p)}(x), 1 \leq p \leq 2j$, *exists and* $f^{(2j)}(x)$ *is bounded.*

(ii) $K(s) \geq 0, -\infty < s < \infty, \int K(s)ds = 1, \int s^a K^i(s)ds = 0$ *for positive odd integers a and* $\int s^b K^i(s)ds < \infty$ *for positive even integer b.*

Then

$$E\left[\frac{1}{h_n}\frac{K\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right] = f(x) + \left(\frac{f''(x)}{2}\int u^2K(u)du\right)h_n^2 + \left(\frac{f^{(4)}(x)}{4!}\int u^4K(u)du\right)h_n^4 + \ldots + \left(\frac{f^{(2j-2)}(x)}{(2j-2)!}\int u^{2j-2}K(u)du\right)h_n^{2j-2} + O(h_n^{2j})$$

and

$$E\left[\frac{1}{h_n}\frac{K\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right]^2 = \frac{C(k)}{h_n}\frac{f(x)}{1-G(x)} + O(h_n),$$

where $C(k) = \int K^2(u)du$.

The proof follows from Theorem 6.2.1.

Theorem 6.2.3. *Assume that* $f'(x), f''(x)$ *and* $f^{(3)}(x)$ *exists and* $f^{(4)}(x)$ *is bounded. If* $K(\cdot)$ *satisfies condition (ii) in Theorem 6.2.2 and* $h_n \to 0, nh_n \to \infty$, *then*

$$E\left[f_n(x) - f(x)\right] = \left(\frac{f''(x)}{2}\int u^2K(u)du\right)h_n^2 + O(h_n^4) = O(h_n^2)$$

and

$$E\left[f_n(x) - f(x)\right]^2 = \frac{C_k}{nh_n}\frac{f(x)}{1-G(x)} + \left(\frac{f''(x)}{2}\int u^2K(u)du\right)^2 h_n^4 + O(h_n^5) = O\left(\frac{1}{nh_n}\right) + O(h_n^4).$$

Proof. The proof follows directly by using (6.1) and Theorem 6.2.2. □

An estimator for estimating the r^{th} order derivative of the density $f^{(r)}(x) = -\frac{d^r}{dx^r}f(x)$

is found by taking derivatives of the nonparametric estimator.

$$f_n^{(r)}(x) = -\frac{d^r}{dx^r} f_n(x) = \frac{1}{nh_n{}^{r+1}} \sum_{i=1}^{n} \frac{K^{(r)}\left(\frac{x-Z_i}{h_n}\right)}{1-G(Z_i)}, \tag{6.6}$$

where $K^{(r)}(x) = \frac{d^r}{dx^r} K(x)$, provided $K^{(r)}(x)$ exist and is nonzero.

Theorem 6.2.4. *For positive integers i and j, assume that $f^{(r+p)}(x), 1 \leq p \leq 2j$, exists and $f^{(r+2j)}(x)$ is bounded, and $K(\cdot)$ satisfies condition (ii) in Theorem 6.2.2, then*

$$E\left[\frac{1}{h_n^{r+1}} \frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right] = f^{(r)}(x) + \frac{f^{(r+2)}(x)}{2} h_n^2 \int u^2 K(u) du + ... + \frac{f^{(r+2j-2)}(x)}{(2j-2)!} h_n^{2j-2} \int u^{2j-2} K(u) du + O(h_n^{2j}),$$

and

$$E\left[\frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right]^2 = C(k^{(r)}) \frac{f(x)}{1-G(x)} + O(h_n),$$

where $C(k^{(r)}) = \int \left[K^{(r)}(u)\right]^2 du$.

Proof.

$$E\left[\frac{1}{h_n^{r+1}} \frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right] = \int \frac{1}{h_n^{r+1}} K^{(r)}\left(\frac{x-z_1}{h_n}\right) f(z_1) dz_1.$$

Using the fact that $\int \frac{1}{h_n} K^{(r)}\left(\frac{x-z_1}{h_n}\right) dz_1 = -K^{(r-1)}\left(\frac{x-z_1}{h_n}\right)$, the above expression becomes

$$E\left[\frac{1}{h_n^{r+1}} \frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right] = \int \frac{1}{h_n^{r}} K^{(r-1)}\left(\frac{x-z_1}{h_n}\right) f'(z_1) dz_1.$$

Repeating this a total of r times, we obtain

$$
\begin{aligned}
E\left[\frac{1}{h_n^{r+1}}\frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right] &= \int \frac{1}{h_n}K\left(\frac{x-z_1}{h_n}\right)f^{(r)}(z_1)dz_1 \\
&= \int K(u)f^{(r)}(x-uh_n)du \\
&= f^{(r)}(x) + f^{(r+1)}(x)h_n\int uK(u)du + \\
&\quad \frac{f^{(r+2)}(x)}{2!}h_n^2\int u^2K(u)du + ... + \\
&\quad \frac{f^{(r+2j-2)}(x)}{(2j-2)!}h_n^{2j-2}\int u^{2j-2}K(u)du + O(h_n^{2j}).
\end{aligned}
$$

$$
\begin{aligned}
E\left[\frac{K^{(r)}\left(\frac{x-Z_1}{h_n}\right)}{1-G(Z_1)}\right]^2 &= \int \frac{\left[K^{(r)}\left(\frac{x-z_1}{h_n}\right)\right]^2}{1-G(Z_1)}f(z_1)dz_1 \\
&= \int \left[K^{(r)}(u)\right]^2\frac{f(x-uh_n)}{1-G(x-uh_n)}hdu \\
&= \frac{f(x)}{1-G(x)}h_n\int \left[K^{(r)}(u)\right]^2du + O(h_n) \\
&= C(k^{(r)})\frac{f(x)}{1-G(x)}h_n + O(h_n),
\end{aligned}
$$

where $C(k^{(r)}) = \int \left[K^{(r)}(u)\right]^2du$. □

Theorem 6.2.5. *Assume that $f'(x), f''(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$ exists and $f^{(5)}(x)$ is bounded. If $K(\cdot)$ satisfies condition (ii) in Theorem 6.2.2 and $h_n \to 0, nh_n \to \infty$, then*

$$
E\left[f_n'(x) - f'(x)\right] = \frac{f^{(3)}(x)}{2}h_n^2\int u^2K(u)du + O(h_n^4) = O(h_n^4)
$$

and

$$E\left[f_n'(x) - f'(x)\right]^2 = \frac{C(k^{(1)})}{n{h_n}^3}\frac{f(x)}{1-G(x)} + \left(\frac{f^{(3)}(x)}{2}\int u^2K(u)du\right)^2 h_n^4 + O(h_n^5)$$
$$= O\left(\frac{1}{nh_n}\right) + O(h_n^4).$$

Proof. The proof straightforward from (6.6) and Theorem 6.2.4. □

In the next theorem we derive the bias and MSE of $\eta_n(x)$ in which the consistency of $\eta_n(x)$ is evident.

Theorem 6.2.6. *Assume that $f'(x), f''(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$ exists and $f^{(5)}(x)$ is bounded. If $K(\cdot)$ satisfies condition (ii) in Theorem 6.2.2 and $h_n \to 0, nh_n \to \infty$ and if $m_1 < K(x) < M_1$, $x \in \{y : K(y) \neq 0\}$, for some positive constants m_1 and M_1, and $m_2 < K'(x) < M_2$, $x \in \{y : K'(y) \neq 0\}$, for some positive constants m_2 and M_2, then the bias of $\eta_n(x)$ is given by*

$$Bias\left(\eta_n(x)\right) = \left[\frac{f'(x)f''(x)}{2\left[f(x)\right]^2} - \frac{f^{(3)}(x)}{2f(x)}\right]\left[\int u^2K(u)du\right]h_n^2 + O\left(\frac{1}{nh_n}\right) + O(h_n^4).$$

The variance of $\eta_n(x)$ is given by

$$Var\left(\eta_n(x)\right) = \left[\frac{C(k^{(1)})}{h_n^2 f(x)\left[1-G(x)\right]} + \frac{C_k\left[f'(x)\right]^2}{\left[f(x)\right]^3\left[1-G(x)\right]}\right]\frac{1}{nh_n} + O(h_n^4)$$

and the mean squared error of $\eta_n(x)$ is given by

$$MSE\left(\eta_n(x)\right) = \left[\frac{C(k^{(1)})}{h_n^2 f(x)\left[1-G(x)\right]} + \frac{C_k\left[f'(x)\right]^2}{\left[f(x)\right]^3\left[1-G(x)\right]}\right]\frac{1}{nh_n} +$$
$$\left[\frac{f'(x)f''(x)}{2\left[f(x)\right]^2} - \frac{f^{(3)}(x)}{2f(x)}\right]^2\left[\int u^2K(u)du\right]^2 h_n^4 + O\left(\frac{1}{nh_n}\right) + O(h_n^4). \tag{6.7}$$

Proof. Let $S = \left\{\omega : |f_n(x) - f(x)| \leq \frac{f(x)}{2} \ and \ |f_n'(x) - f'(x)| \leq \frac{f(x)}{2}\right\}$ and S^c is the compliment of S. For positive integer j, we will find the integral $E\left|R_n\right|^j$ over S and S^c. Clearly, for $\omega \in S$, we have

$$0 < f(x)\left(1 - \frac{\tau}{2}\right) \leq f(x) + \tau\left[f_n(x) - f(x)\right] \leq f(x)\left(1 + \frac{\tau}{2}\right),$$

and

$$0 < f(x)\left(1 - \frac{\tau}{2}\right) \leq f'(x) + \tau\left[f_n'(x) - f'(x)\right] \leq f(x)\left(1 + \frac{\tau}{2}\right)$$

for $0 \leq \tau \leq 1$.

Let I_S denote indicator function of S, then for positive integer j

$$\begin{aligned}
E\left|R_{n_1}\right|^j \mathrm{I}_S &= \int\limits_S \left|\int\limits_0^1 \frac{2(1-\tau)\left[\upsilon - f(x)\right]\left[\upsilon' - f'(x)\right]}{\left\{f(x) + \tau\left[\upsilon - f(x)\right]\right\}^2} d\tau\right|^j dG(\upsilon) \\
&\leq \int\limits_S 2^j\left|\upsilon - f(x)\right|^j\left|\upsilon' - f'(x)\right|^j \left|\int\limits_0^1 \frac{(1-\tau)}{\left\{\left(1 - \frac{\tau}{2}\right) f(x)\right\}^2} d\tau\right|^j dG(\upsilon) \\
&\leq \int\limits_S \frac{2^j\left|\upsilon - f(x)\right|^j\left|\upsilon' - f'(x)\right|^j}{\left[f(x)\right]^{2j}} dG(\upsilon) \\
&\leq \int\limits_S \frac{2^j\left|\upsilon - f(x)\right|^{2j}}{\left[f(x)\right]^{2j}} dG(\upsilon) \\
&\leq \int\limits_{\mathbb{R}} \frac{2^j\left|\upsilon - f(x)\right|^{2j}}{\left[f(x)\right]^{2j}} dG(\upsilon) \\
&= \frac{2^j}{\left[f(x)\right]^{2j}} E\left[f_n(x) - f(x)\right]^{2j}
\end{aligned}$$

since $\int\limits_0^1 \frac{1-\tau}{\left(1-\frac{\tau}{2}\right)^2} d\tau = \log(16) - 2 < 1$. Therefore

$$E\left|R_{n_1}\right|^j \mathrm{I}_S \leq \frac{2^j}{\left[f(x)\right]^{2j}} E\left[f_n(x) - f(x)\right]^{2j}. \tag{6.8}$$

Now

$$\begin{aligned}
E\left|R_{n_2}\right|^j \mathrm{I}_S =& \int\limits_S \left| \int\limits_0^1 \frac{2(1-\tau)\left[\upsilon - f(x)\right]^2 \left\{f'(x) + \tau\left[\upsilon' - f'(x)\right]\right\}}{\left\{f(x)+\tau\left[\upsilon - f(x)\right]\right\}^3} d\tau \right|^j dG(\upsilon) \\
\leq& \int\limits_S \left| \int\limits_0^1 \frac{2(1-\tau)\left[\upsilon - f(x)\right]^2 f(x)\left(1+\frac{\tau}{2}\right)}{\left\{\left(1-\frac{\tau}{2}\right) f'(x)\right\}^3} d\tau \right|^j dG(\upsilon) \\
\leq& \int\limits_S \left| \int\limits_0^1 \frac{2(1-\tau)\left[\upsilon - f(x)\right]^2 f'(x)\left(1+\frac{\tau}{2}\right)}{\left\{\left(1-\frac{\tau}{2}\right) f'(x)\right\}^3} d\tau \right|^j dG(\upsilon) \\
\leq& \int\limits_S \frac{4^j\left[\upsilon - f(x)\right]^{2j}}{\left[f'(x)\right]^{2j}} dG(\upsilon) \\
\leq& \int\limits_{\mathbb{R}} \frac{4^j\left[\upsilon - f(x)\right]^{2j}}{\left[f'(x)\right]^{2j}} dG(\upsilon) \\
=& \frac{4^j}{\left[f'(x)\right]^{2j}} E\left[f_n(x) - f(x)\right]^{2j}
\end{aligned}$$

since $\int_0^1 \frac{(1-\tau)\left(1+\frac{\tau}{2}\right)}{\left(1-\frac{\tau}{2}\right)^3} d\tau = 4 - 4\log(2) < 2$. Therefore

$$E\left|R_{n_2}\right|^j \mathrm{I}_S \leq \frac{4^j}{\left[f'(x)\right]^{2j}} E\left[f_n(x) - f(x)\right]^{2j}. \tag{6.9}$$

$$\left|R_n\right| = \left|R_{n_1} - R_{n_2}\right| \leq \left|R_{n_1}\right| + \left|R_{n_2}\right|$$

$$E\left|R_n\right|^j \mathrm{I}_S \leq E\left|R_{n_1}\right|^j \mathrm{I}_S + E\left|R_{n_2}\right|^j \mathrm{I}_S$$

Therefore from (6.8) and (6.9)

$$E\left|R_n\right|^j \mathrm{I}_S \leq \left\{\frac{2^j}{\left[f(x)\right]^{2j}} + \frac{4^j}{\left[f'(x)\right]^{2j}}\right\} E\left[f_n(x) - f(x)\right]^{2j}.$$

We proceed to find $E\left|R_n\right|^j \mathrm{I}_{S^c}$. In this case $K\left(\frac{x-X_i}{h_n}\right) \neq 0$ and $K'\left(\frac{x-X_i}{h_n}\right) \neq 0$ for some positive integer i. Since $m_1 < K(u) < M_1$, we have $f_n(x) \geq \frac{m_1}{nh_n}$, or equivalently, $\frac{1}{f_n(x)} \leq \frac{nh_n}{m_1}$ and $f_n(x) \leq \frac{M_1}{h_n}$. Similarly for $m_2 < K'(u) < M_2$, we have $f_n'(x) \geq \frac{m_2}{nh_n}$, that is, $\frac{1}{f_n'(x)} \leq \frac{nh_n}{m_2}$. Moreover $f_n'(x) \leq \frac{M_2}{h_n}$ and $nh_n \to \infty$ implies for sufficiently large n.

$$
\begin{aligned}
E\left|R_n\right|^j \mathrm{I}_{S^c} =& E\left[\left|\eta_n(x) - \eta(x) + \frac{f_n'(x) - f'(x)}{f(x)} - \frac{f_n(x) - f(x)}{[f(x)]^2} f'(x)\right|^j \mathrm{I}_{S^c}\right] \\
=& E\left[\left|-\frac{f_n'(x)}{f_n(x)} + \frac{f'(x)}{f(x)} + \frac{f_n'(x) - f'(x)}{f(x)} - \frac{f_n(x) - f(x)}{[f(x)]^2} f'(x)\right|^j \mathrm{I}_{S^c}\right] \\
=& E\left[\left|-\frac{f_n'(x)}{f_n(x)} + \frac{f'(x)}{f(x)} + \frac{f_n'(x)}{f(x)} - \frac{f_n(x) f'(x)}{[f(x)]^2}\right|^j \mathrm{I}_{S^c}\right] \\
\leq& E\left[\left|-\frac{m_2}{nM_1} + \frac{f'(x)}{f(x)} + \frac{M_2}{hf(x)} - \frac{m_1 f'(x)}{nhf^2(x)}\right|^j \mathrm{I}_{S^c}\right] \\
=& \left|-\frac{m_2 h_n}{M_1} + \frac{nhf'(x)}{f(x)} + \frac{nM_2}{f(x)} - \frac{m_1 f'(x)}{f^2(x)}\right|^j \left[\frac{1}{nh_n}\right]^j E\mathrm{I}_{S^c} \\
=& O(n^j h_n^j)\left[\frac{1}{n^j h_n^j}\right] E\mathrm{I}_{S^c} \\
=& P\left(|f_n(x) - f(x)| \geq \frac{f(x)}{2}\right) + P\left(|f_n'(x) - f'(x)| \geq \frac{f(x)}{2}\right) \\
\leq& P\left(|f_n(x) - Ef_n(x)| \geq \frac{f(x)}{4}\right) + P\left(|Ef_n(x) - f(x)| \geq \frac{f(x)}{4}\right) + \\
& P\left(|f_n'(x) - Ef_n'(x)| \geq \frac{f(x)}{4}\right) + P\left(|Ef_n'(x) - f'(x)| \geq \frac{f(x)}{4}\right)
\end{aligned}
$$

For sufficiently large n,

$$P\left(|Ef_n(x) - f(x)| \geq \frac{f(x)}{4}\right) \to 0$$

$$P\left(|Ef_n'(x) - f'(x)| \geq \frac{f(x)}{4}\right) \to 0$$

and

$$P\left(|f_n(x) - Ef_n(x)| \geq \frac{f(x)}{4}\right) \leq 2exp\left\{-C_1 nh_n\right\}$$

$$P\left(|f_n'(x) - Ef_n'(x)| \geq \frac{f(x)}{4}\right) \leq 2exp\left\{-C_2 n{h_n}^2\right\},$$

for constant $C_1 > 0$ and $C_2 > 0$, (see Rao (1983)).

Therefore

$$E\left|R_n\right|^j \mathrm{I}_{S^c} \leq 2exp\left\{-C_1 nh_n\right\} + 2exp\left\{-C_2 n{h_n}^2\right\}.$$

$$\begin{aligned}
E\left|R_n\right|^j =& E\left|R_n\right|^j \mathrm{I}_S + E\left|R_n\right|^j \mathrm{I}_{S^c} \\
\leq& \left\{\frac{2^j}{\left[f(x)\right]^{2j}} + \frac{4^j}{\left[f'(x)\right]^{2j}}\right\} E\left[f_n(x) - f(x)\right]^{2j} + \\
& 2exp\left\{-C_1 nh_n\right\} + 2exp\left\{-C_2 n{h_n}^2\right\}.
\end{aligned}$$

In particular for $j = 1, 2$

$$\begin{aligned}
E\left|R_n\right| \;&\leq\; \left\{\frac{2}{\left[f(x)\right]^{2}} + \frac{4}{\left[f'(x)\right]^{2}}\right\} E\left[f_n(x) - f(x)\right]^{2} + \\
&\quad\; 2exp\left\{-C_1 nh_n\right\} + 2exp\left\{-C_2 n{h_n}^2\right\} \\
&=\; O\left(\frac{1}{nh_n}\right) + O(h_n^4) \qquad (6.10) \\
E\left|R_n\right|^2 \;&\leq\; \left\{\frac{4}{\left[f(x)\right]^{4}} + \frac{16}{\left[f'(x)\right]^{4}}\right\} E\left[f_n(x) - f(x)\right]^{4} + \\
&\quad\; 2exp\left\{-C_1 nh_n\right\} + 2exp\left\{-C_2 n{h_n}^2\right\} \\
&=\; O\left(\frac{h_n^3}{n}\right) + O\left(\frac{1}{n^2 h_n^2}\right) + O(h_n^8). \qquad (6.11)
\end{aligned}$$

Since $E\left[f_n(x) - f(x)\right]^4 = O\left(\frac{h_n^3}{n}\right) + O\left(\frac{1}{n^2 h_n^2}\right) + O(h_n^8)$ (see Rajesh et al. (2016)) and since $nh_n \to \infty$, $2exp\left\{-C_1 nh_n\right\}$ and $2exp\left\{-C_2 n{h_n}^2\right\}$ will have orders smaller than those of $E\left[f_n(x) - f(x)\right]^2$ and $E\left[f_n(x) - f(x)\right]^4$, respectively. Consequently, $E\left|R_n\right|$ and $E\left|R_n\right|^2$ have the same orders as $E\left[f_n(x) - f(x)\right]^2$ and $E\left[f_n(x) - f(x)\right]^4$, respectively. The desired results follows from (6.3), (6.10), (6.11) and Theorem 6.2.3 and 6.2.5. □

6.3 Asymptotic properties

In this section, we look into consistency, asymptotic normality and the strong convergence of $\eta_n(x)$.

Definition 6.3.1. A sequence θ_n of estimators is integratedly consistent in quadratic mean if the mean integrated squared error (MISE) tends to zero for every $\theta \in \Theta$, a family of univariate θ, that is,

$$\lim_{n\to\infty} E \int [\theta_n(x) - \theta(x)]^2 dx = 0$$

(see Rao (1983)).

Theorem 6.3.2. *Under the assumptions in Theorem 6.2.6, MISE of $\eta_n(x)$ is tends to zero as $n \to \infty$.*

Proof.

$$\begin{aligned} MISE\left(\eta_n(x)\right) &= E \int \left[\eta_n(x) - \eta(x)\right]^2 dx \\ &= \int E\left[\eta_n(x) - \eta(x)\right]^2 dx \\ &= \int \left\{Var\left[\eta_n(x)\right] + Bias^2\left[\eta_n(x)\right]\right\} dx \\ &= \int \left\{MSE\left[\eta_n(x)\right]\right\} dx \to 0, \end{aligned}$$

as $n \to \infty$, which is immediate from (6.7). $\square$

The following theorem establishes the asymptotic normality of $\eta_n(x)$.

Theorem 6.3.3. *Let $\eta_n(x)$ be nonparametric estimator of $\eta(x)$, $K(x)$ be a kernel* *and h_n satisfying the conditions for bandwidth. Then*

$$\eta_n(x) - \eta(x) \xrightarrow{d} N(0, \sigma_d^2)$$

where

$$\sigma_d^2 = \frac{C(k^{(1)})}{n{h_n}^3 f(x)\left[1 - G(x)\right]} + \frac{C_k \left[f'(x)\right]^2}{nh_n \left[f(x)\right]^3 \left[1 - G(x)\right]}.$$

Proof.

$$\begin{aligned}
(nh_n)^{\frac{1}{2}} \left[\eta_n(x) - \eta(x)\right] &= (nh_n)^{\frac{1}{2}} \left\{-\frac{d}{dx} \log f_n(x) + \frac{d}{dx} \log f(x)\right\} \\
&= -(nh_n)^{\frac{1}{2}} \frac{d}{dx} \left\{\log f_n(x) - \log f(x)\right\}
\end{aligned}$$

By the asymptotic normality of $f_n(x)$ given in Cai (1998b), the proof is immediate.

$\square$

In following theorem we prove the strong consistency of the estimator $\eta_n(x)$.

Theorem 6.3.4. *Let $\eta_n(x)$ be nonparametric estimator of $\eta(x)$, suppose that $f(\cdot)$ and $f'(\cdot)$ satisfy the Lipschitz conditions and the kernel $K(\cdot)$ satisfies the requirements and for $0 < \tau < \infty$, the marginal distribution function of $L(.)$ of Z satisfies $L(\tau) < 1$ (see Cai (1998a)) then $\sup_{0 \le x \le \tau} |\eta_n(x) - \eta(x)| \to 0$ a.s.*

Proof.

$$\begin{aligned}
\eta_n(x) - \eta(x) &= -\frac{f_n'(x)}{f_n(x)} + \frac{f'(x)}{f(x)} \\
&= \frac{1}{f_n(x)} \left(-f_n'(x) + \frac{f_n(x) f'(x)}{f(x)}\right) \\
&= \frac{1}{f_n(x)} \left(-\left(f_n'(x) - f'(x)\right) + \frac{f'(x)}{f(x)} \left(f_n(x) - f(x)\right)\right)
\end{aligned}$$

Since $\sup_{0\leq x\leq\tau} |f_n(x) - f(x)| \to 0$ a.s (Cai (1998)) and $\sup_{0\leq x\leq\tau} |f_n'(x) - f'(x)| \to 0$ a.s (del Ríoz (1997)), the proof completes. $\square$

6.4 Numerical examples

Example 6.4.1 (Simulated data)**.** In order to examine the performance of nonparametric estimator $\eta_n(x)$, we generated data as follows. Let $X_i = \sqrt{1-\rho^2}\,|T_i|$, where $\{T_i\}$ were generated from AR(1) with correlation coefficient $\rho = 0.2$ and censoring time $\{Y_i\}$ were generated independently from exponential distribution with parameter 2.5. We considered three sample sizes, $n = 50, 70$ and 100. Also, the kernel function is taken to be Gaussian kernel, $K(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$ and the bandwidth $h_n = n^{-\frac{1}{7}}$. Clearly, $\{X_i\}$ are stationary and pdf $f(x) = 2\phi(x)$, and $\eta_n(x) = x$, where $\phi(x)$ is the standard normal pdf. Figures 6.1.1, 6.1.2 and 6.1.3 are corresponding to sample sizes $n = 50, 70$ and 100. Each figure has two graphs: the true underlying function is denoted by a solid line, and the estimate is denoted by a dashed line. The bias and MSE of $\eta_n(x)$ is given in Table 6.1 and MSE is decreasing with increasing sample size.

Figure 6.1: Simulation results for $n = 50$ (Figure 6.1.1), $n = 70$ (Figure 6.1.2) and $n = 100$ (Figure 6.1.3)

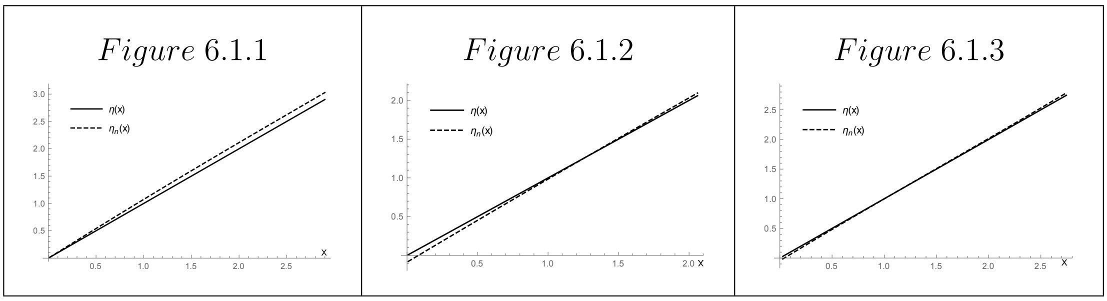


Table 6.1: Bias and MSE of $\eta_n(x)$

Sample size	Censored observations	Bias	MSE
50	35	0.0656	0.0058
70	42	-0.0395	0.0027
100	77	-0.0108	0.0004

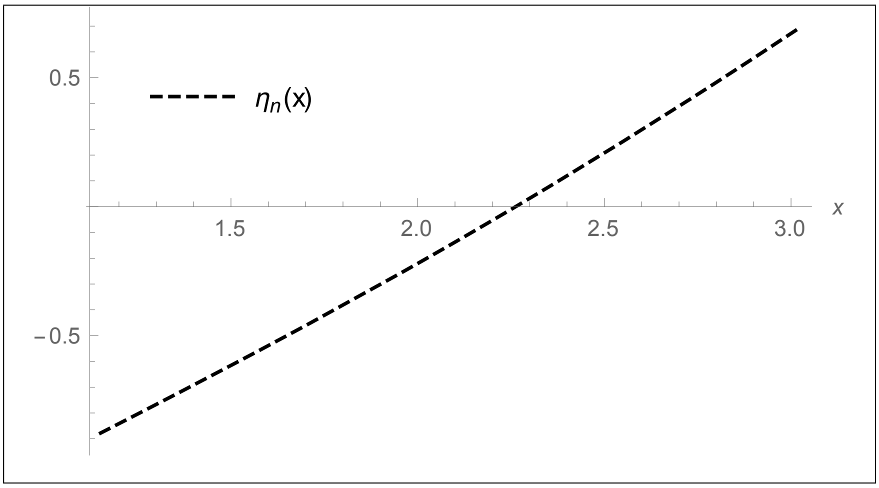
Example 6.4.2 (Real data)**.** To illustrate the usefulness of the proposed nonparametric estimate $\eta_n(x)$ for real data, we consider the failure times (measured in millions of operations) of 40 randomly selected mechanical switches given in Nair (1984) and Nair (1993) and is reproduced in Table 6.2.

Table 6.2: Failure times (in millions of operations) for a mechanical switch life test

1.151	1.170	1.248	1.331	1.381	1.499	1.508	1.534	1.577	1.584
1.667	1.695	1.710	1.955	1.965	2.012	2.051	2.076	2.109	2.116
2.119	2.135	2.197	2.199	2.227	2.250	2.254	2.261	2.349	2.369
2.547	2.548	2.738	2.794	2.883+	2.883+	2.910	3.015	3.017	3.793+

Three of the test positions became available much later than the others, so the three switches tested at these positions were still operating at the termination of the test. The corresponding censored observations are indicated by the code +. Here the censoring mechanisms are dependent, in fact, since the components are subject to the same stress and operating environment, it is likely that the failure times of the components are positively dependent. The Gaussian kernel is used as the kernel function for the estimation. Figure 6.2 shows the plot of $\eta_n(x)$ calculated using Gaussian kernel.

Figure 6.2: $\eta_n(x)$ for the real data set given in Table 6.2



From Figure 6.2, it is easy to see that for the data set considered $\eta_n(x)$ is increasing. Therefore the failure rate is increasing for the real data set considered.

Chapter 7

Conclusions and future research

In view of the importance and usefulness of RCST in examining the monotone properties of a function and for model identification, we further studied its properties in the context of various other situations. Chapter 1 gave an introduction to the notion of RCST and review of literature on various measures that has been used in the successive chapters. In Chapter 2, we further studied the usefulness of RCST in the context of weighted models. We have proved characterizations to some important distributions such as gamma and Rayleigh, under the inversed length-biased model. We have also obtained characterizations to equilibrium, residual lifetime (reversed residual lifetime) and proportional hazards model in the context of weighted distributions. We derived an identity for weighted distribution when RCST takes the form of a general class of distributions which contains many important moment relationships, and a generalization of the result due to Nair and Sankaran (2008). We further extended RCST of weighted distributions to bivariate and multivariate cases and obtained some characterization theorems arising out of it.

In Chapter 3, we proved that monotone failure properties of probability models are invariant under nonsingular transformation by using RCST. We studied the monotone properties of weighted random variable based on RCST and illustrated it through examples. By using a relationship between RCST and MRCST, a characterization result is proved for mixtures of exponential, Lomax and beta distributions. The different stochastic orderings of two random variables based on RCST and MRCST are also studied. We proved certain characterizations of probability models based on RCST of record values. Also we studied some properties of RCST in the context of circular distributions.

In Chapter 4, RCST is studied in context of bivariate and conditionally specified models. We have obtained characterization results for a general bivariate model proposed by Navarro and Sarabia (2013), Sarmanov family, Farlie-Gumbel-Morgenstern (FGM) family and Ali-Mikhail-Haq family proposed by Ali et al. (1978). We defined RCST for conditionally specified distributions and proved characterization results based on it. We also obtained a relationship between local dependence function of Holland and Wang (1987) and RCST.

Chapter 5 is focused on finding the properties of RCST in discrete time. We studied the usefulness of discrete analogue of RCST introduced by Gupta et al. (1997) in modelling different discrete distributions/families. Characterizations are also proved for geometric, discrete Burr (Nair and Asha (2004)) and modified power series family of distributions. We also obtained a characterization to a new discrete distribution for which RCST is linear. We have extended RCST to the weighted models in discrete case and proved characterization results to logarithmic series, a distribution for which RCST is linear, residual lifetime and partial sum distributions. A new definition for discrete analogue of RCST is also introduced that possess certain new properties than the discrete analogue of RCST proposed by Gupta et al. (1997). Discrete proportional hazards model (Dewan and Sudheesh (2009)) is characterized using this new definition.

Finally, in Chapter 6, we proposed a nonparametric estimator for RCST under right censored dependent case. We have examined the asymptotic property of the estimator. A simulation study is carried out to illustrate the performance of the estimator which in turn implies the nature of failure rate and also evaluated the estimator using a real data set.

More studies of RCST in the context of bivariate and multivariate setup will be a potential future work. Since the present study restricts only preliminary investigations of RCST based on record values and circular distributions, a detailed study in this direction will be worthwhile.

The extension of RCST in discrete time to the bivariate and its multivariate cases are yet to be explored. The nonparametric estimation of bivariate RCST is another interesting problem that can be considered in the future study.

The modelling and analysis of lifetime data can be done by emphasizing the df $F(\cdot)$ of a rv X that is assumed to generate the observations or through the quantile function

$$Q(p) = \inf\left\{x\left|F(x) \geq p\right.\right\}, 0 \leq p \leq 1$$

with $Q(0) = 0$ and $Q(1) = \infty$. Since$F(x)$ is continuous $F\left(Q(p)\right) = p$. Parzen (1979) defined $f\left(Q(p)\right)$ as the density quantile function and $q(p) = Q'(p)$ as the quantile density function. The quantile density function $q(p)$ satisfies the relationship

$$q(p)f\left(Q(p)\right) = 1.$$

Nair and Sankaran (2009) expressed various reliability measures in terms of quantile functions like the hazard quantile function,

$$H(p) = \frac{f(Q(p))}{\bar{F}(Q(p))} = \frac{1}{(1-p)q(p)},$$

and mean residual quantile function,

$$M(p) = (1-p)^{-1}\int_p^1 Q(u)\, du - Q(p).$$

Parzen (1979) introduced the score function $J(p)$ equivalent to the RCST function defined in (1.1) as

$$J(p) = -\frac{f'(Q(p))}{f(Q(p))} = \frac{q'(p)}{q^2(p)} = -\frac{d}{du}\left(\frac{1}{q(p)}\right).$$

Nair et al. (2012) found that $J(p)$ is uniquely determines the distribution of X through (provided $f(\infty) = 0$),

$$q(p) = \left(\int_p^1 J(u)du\right)^{-1}.$$

They also studied the relationships between $J(p)$ and $H(p)$ that characterize many life distributions. Motivated with this we propose a quantile-based study on the concept of $J(p)$ in the context of weighted models. This also involves the usefulness of $J(p)$ for length-biased and equilibrium distributions in the quantile setup.

List of published/ accepted papers

1. Sunoj, S. M. and Sreejith, T. B. (2012). Some results on reciprocal subtangent in the context of weighted models. *Communications in Statistics–Theory and Methods*, 41(8):1397–1410.

2. Sunoj, S. M. and Sreejith, T. B. (2014). A discrete analogue of reciprocal coordinate subtangent and its role in characterization problems. *Calcutta Statistical Association Bulletin*, 66:123–135.

3. Sunoj, S. M., Sreejith, T. B., and Navarro, J. (2014). Characterizations of some bivariate models using reciprocal coordinate subtangents. *Statistica*, 74(2):153–170.

4. Sunoj, S. M. and Sreejith, T. B. (2017). Some properties of reciprocal coordinate subtangents in the context stochastic modelling. *Journal of the Indian Statistical Association, (accepted).*

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