Some families of bivariate distributions and their applications

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To My Family

CERTIFICATE

This is to certify that the thesis entitled **"Some families of bivariate distributions and their applications"** is a bonafide record of work done by Ms.Preethi John under our guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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Preliminaries

1.1 Introduction

In recent years stochastic modelling has become a convenient technique in many scientific studies to understand the basic characteristics of the random phenomenon under consideration. One of the basic problems in such situations is to identify the underlying stochastic model that is supposed to generate the observations. Generally it is not easy to isolate all the physical causes that contribute individually or collectively to the generation of data and to mathematically account for each. The task of determining the correct stochastic model representing the given data becomes very difficult. A standard practice in such contexts is to ascertain the physical properties of the process generating the observations, express them by means of mathematical equations or inequalities and then solve them to obtain the model. There are, however, situations when the system is so complex that the response derived from it may not be amenable to simple mathematical manipulations. One method that can be used in such occasions is to use a general family of probability distributions, one member of which could be a possible model that fits the data. The main reason to prefer this procedure is the desire to find the best possible approximation in a complex situation that generated the data rather than any reasonable evidence to the effect that the model explains the data generating mechanism. When the families of distributions are chosen for modelling, it is desirable that (i) the family contains enough members with different structures so that there is a member that can correspond to a given data situation, (ii) the members of the family should have a sufficient number of parameters to impart flexibility, (iii) there should be some simple criterion that distinguishes the various members of the family so that the choice of a member that fits the data becomes easy and (iv) efficient methods exist for the estimation of parameters. The above discussions clearly reveal that the family of distributions plays a pivotal role in statistical modelling. Statistical literature is abundant with various families of distributions that are employed in statistical data analysis.

In many scientific investigations, it is the rule rather than the exception to have multiple response variables. Multivariate data commonly arise in many scientific investigations and accordingly multivariate distributions are employed for modelling and analysis of data. Much of the early work in the literature on the analysis of bivariate (multivariate) data was focused on bivariate and multivariate normal distributions as there had been a tendency to regard all distributions as normal. However, the normal distributions are inappropriate in cases where the data exhibit multi-modality and skewness and hence significant developments have been made with regard to non-normal distributions. Bivariate(Multivariate) distributions with non-normal marginals arise in many fields. In lifetime data analysis, the variables of interest are non-negative that often have skewed marginal distributions like exponential, Pareto and Weibull distributions. In reliability, multivariate lifetime data arise when each study subject may experience several events. For example, the sequence of tumour recurrences, the occurrence of blindness in both eyes and the onset of a genetic disease among family members etc. The non-normal distributions are appropriate in such occasions. For various non-normal bivariate(multivariate) distributions, one may refer to Kotz et al. (2002).

The importance of exponential distribution in statistical theory and applications is established in literature. Motivated by the applicability of the distribution in the univariate case, there have been several attempts in statistical literature to construct exponential distribution in higher dimensions that have properties which generalize those in the one-dimensional situation. The work in this direction was initiated by Gumbel (1960) by presenting three different functional forms of bivariate exponential laws. Since then there have been spontaneous research on alternative forms of exponential distributions in higher dimensions with variety of applications. See Freund (1961), Marshall and Olkin (1967), Downton (1970), Nagao and Kadoya (1971), Hawkes (1972), Block and Basu (1974), Paulson (1973), Friday and Patil (1977), Tosch and Holmes (1980), Raftery (1985), Cowan (1987),Sarkar (1987),Ryu (1993), Hayakawa (1994), Iyer et al. (2002), Regoli (2009) and Balakrishna and Shiji (2014).

The Pareto distribution was first proposed in literature as a model for income analysis. Arnold (1985) has studied various properties of univariate Pareto distribution and its extensions using transformations of the random variable. As in the case of univariate Pareto distributions, mathematical simplicity and tractability have provided a lot of interest in the theory and applications of multivariate Pareto distributions. The bivariate Pareto distribution of first kind and the second kind was introduced by Mardia (1962). Later Lindley and Singpurwalla (1986) have introduced a bivariate Pareto II distribution which has simple joint survival function with Pareto II marginals. This distribution was further studied and generalized by Nayak (1987), Barlow and Mendel (1992), Sankaran and Nair (1993), Langseth (2002), Balakrishnan and Lai (2009) and Sankaran & Kundu (2014). For various other bivariate Pareto distributions and its generalizations, one may refer to Arnold (1990), Arnold (1992) and Kotz et al. (2002).

In literature, Weibull distribution has been employed for modelling lifetime data of various types of manufactured items. The distribution was first used for the analysis of data on breaking strength of materials. The bivariate distributions with Weibull marginals can be obtained from the bivariate exponential distributions by suitable transformations (Marshall and Olkin (1967) and Lee (1979)). One can visualize the ease and the usefulness of bivariate Weibull models for the analysis of lifetime data, such as the times to first and second failures of a device, the breakdown times of dual generators in a power plant, and the survival times of the organs in a two-organ system in the human body. An extensive literature is now available on the properties and applications of the bivariate and multivariate Weibull distributions; See, Lee and Thompson (1974), Clayton (1978), Lee (1979), Marshall and Olkin (1988), Crowder (1989), Castillo and Galambos (1990), Lu and Bhattacharyya (1990), Patra and Dey (1999), Kotz et al. (2002), Murthy et al. (2004) and Rinne (2008).

Bivariate (Multivariate) data commonly arise in many scientific investigations and accordingly we have discussed some commonly used bivariate(multivariate) distributions like exponential, Pareto and Weibull that can be employed for modelling and analysis of bivariate(multivariate)data. Measures of association often appear to be of great importance to study the dependence among the variables. The theory of copulas provides a flexible tool for identifying the nature and extent of dependence in multivariate models. Thus a discussion of these aspects are needed which will be taken up in the next section.

1.2 Copulas

Sklar (1959) introduced the notion of copula while answering a question raised by M. Frechet about the relationship between a multi-dimensional probability function and its lower dimensional marginals. Sklar was the person who first used the word "copula"in a mathematical or statistical sense in the theorem which bears his name, although similar ideas and results can be traced back to Hoeffding (1940). Copulas were initially used in the development of the theory of probabilistic metric spaces. Later, they were employed to define nonparametric measures of dependence between random variables, and since then, they began to play an important role in probability and mathematical statistics.

A copula is a function which "couples"a multivariate distribution function to its one-dimensional marginal distribution functions. Over the past forty years, copulas have played an important role in several areas of statistics. Copulas can be considered as a way of studying scale-free measures of dependence; and can be used as a starting point for constructing families of bivariate distributions (Fisher (1997)). Copulas are considered to be highly appealing in the non-Gaussian set up as they can capture dependence more broadly than the standard multivariate normal framework.

Let $\mathbf{R}$ denote the ordinary real line $(-\infty, \infty)$, $\bar{\mathbf{R}}$ denote the extended real line $[-\infty, \infty]$, and $\bar{\mathbf{R}}^n$ denote the extended n-space $\bar{\mathbf{R}} \times \bar{\mathbf{R}} \times ... \times \bar{\mathbf{R}}$. The unit n-cube $\mathbf{I}^n$

is the product $\mathbf{I}\times\mathbf{I}\times ... \times\mathbf{I}$ where $\mathbf{I} = [0,1]$. An n-place real function F is a function whose domain, $DomF$, is a subset of $\bar{\mathbf{R}}^n$ and whose range, $RanF$, is a subset of $\mathbf{R}$.

Definition 1.1. Let $S_1, ..., S_n$ be nonempty subsets of $\bar{\mathbf{R}}$. Let F be a real function of n variables such that $DomF = S_1 \times S_2 \times ... \times S_n$ and for $\mathbf{a} \leq \mathbf{b}(a_k \leq b_k$ for all k) let $B = [\mathbf{a}, \mathbf{b}]$ (=$[a_1, b_1] \times ... \times [a_n, b_n]$) be an n-box whose vertices are the points $\mathbf{c} = (c_1, c_2, ..., c_n)$ where each c_k is equal to either a_k or b_k and are in $DomF$. Then the F-volume of B is given by

$$V_F(B) = \Sigma sgn(\mathbf{c})F(\mathbf{c}),$$

where the sum is taken over all vertices $\mathbf{c}$ of B, and $sgn(\mathbf{c})$ is given by

$$sgn(\mathbf{c}) \quad = \quad \begin{cases} 1, & if\ c_k = a_k\ for\ an\ \ even\ number\ of\ k's, \\ -1, & if\ c_k = a_k\ for\ an\ odd\ number\ of\ k's. \end{cases}$$

The formal definition of copula is as follows:

Definition 1.2. An n-dimensional copula is a function $C : [0,1]^n \to [0,1]$, with the following properties:

1. C is grounded, it means that for every $\mathbf{u} = (u_1, u_2, ..., u_n) \in [0,1]^n, C(\mathbf{u}) = 0$ if at least one coordinate u_i is zero, i = 1, 2, . . . , n,

2. C is n-increasing, it means that for every $\mathbf{a} \in [0,1]^n$ and $\mathbf{b} \in [0,1]^n$ such that $\mathbf{a} \leq \mathbf{b}$, the C-volume $V_C([\mathbf{a}, \mathbf{b}])$ of the box $[\mathbf{a}, \mathbf{b}]$ is non-negative,

3. $C(1, ..., 1, u_i, 1, ..., 1) = u_i$, for all $u_i \in [0,1]$, $i = 1, 2, ..., n$.

The Frechet-Hoeffding bounds for the copula are namely $M^n(\mathbf{u})$ and $W^n(\mathbf{u})$(superscript denotes dimension of the copula rather than exponentiation), where

$$M^n(\mathbf{u}) = min(u_1, ..., u_n), \text{ Frechet-Hoeffding upper bound copula,}$$

$$W^n(\mathbf{u}) = max(u_1 + ... + u_n - n + 1, 0), \text{ Frechet-Hoeffding lower bound copula.}$$

Another important copula is

$$\Pi^n(\mathbf{u}) = u_1 ... u_n, \quad \text{product copula.}$$

The functions $M^n(\mathbf{u})$ and $\Pi^n(\mathbf{u})$ are n-copulas for all $n \geq 2$ whereas the function $W^n(\mathbf{u})$ is not a copula for any $n \geq 3$.

Theorem 1.3. *If C is any n-copula, then for every $\mathbf{u}$ in $[0, 1]^n$,*

$$W^n(\mathbf{u}) \leq C(\mathbf{u}) \leq M^n(\mathbf{u}).$$

This theorem is called the Frechet-Hoeffding bounds inequality (Frechet (1957)). For more details and geometrical interpretations one could refer to Mikusinski et al. (1992).

For $n = 2$, the above definition reduces to the following;

Definition 1.4. A two-dimensional (bivariate) copula is a function $C : [0, 1]^2 \rightarrow [0, 1]$, with the following properties;

1. C is grounded: For all $u, v \in [0, 1]$, $C(u, 0) = 0$ and $C(0, v) = 0$.

2. C is 2-increasing: for all $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

3. For all $u, v \in [0,1]$, $C(u,1) = u$ and $C(1,v) = v$.

The importance of copulas in statistics is described in Sklar's theorem and is perhaps the most important result which is used in all applications of copulas.

Theorem 1.5 (Sklar's theorem)**.** *Let $F(.)$ be an n-dimensional distribution function with marginals $F_1, ..., F_n$. Then there exists an n-copula C such that for all $\mathbf{x} = (x_1, x_2, ..., x_n)$ in $\bar{\mathbf{R}}^n$,*

$$F(x_1, ..., x_n) = C(F_1(x_1), ..., F_n(x_n)). \tag{1.1}$$

If $F_1, ..., F_n$ are continuous, then C is unique. Otherwise, the copula C is uniquely determined on $RanF_1 \times ... \times RanF_n$. Conversely, if C is an n-copula and $F_1, ..., F_n$ are distribution functions, then the function F defined above is an n-dimensional distribution function with marginals $F_1, ..., F_n$.

For the proof, see Sklar (1996). One can see from Sklar's theorem that for continuous multivariate distribution functions, the univariate marginals and the multivariate dependence structure can be separated and the dependence structure can be represented by a copula.

Corollary 1.1. *Let $F(.)$ be an n-dimensional distribution function with continuous marginals $F_1, ..., F_n$ and copula C and let $F_1^{(-1)}, F_2^{(-1)}, ..., F_n^{(-1)}$ be the quasi-inverses of $F_1, ..., F_n$ respectively. Then for any $\mathbf{u}$ in $[0,1]^n$,*

$$C(u_1, ..., u_n) = F(F_1^{(-1)}(u_1), ..., F_n^{(-1)}(u_n)).$$

Remark 1.1. The quasi-inverse of the univariate distribution function F_i is any function $F_i^{(-1)}$ with domain $\mathbf{I}$ such that

(i) if t is in $RanF_i$, then $F_i^{(-1)}(t)$ is any number x in $\bar{\mathbf{R}}$ such that $F_i(x) = t$ that is, for all t in $RanF_i$,

$$F_i(F_i^{(-1)}(t)) = t;$$

and

(ii) if t is not in $RanF_i$, then

$$F_i^{(-1)}(t) = inf\{x|F_i(x) \geq t\} = sup\{x|F_i(x) \leq t\}.$$

If F_i is strictly increasing, then there exists a single quasi-inverse which is of course the ordinary inverse notated by F_i^{-1}.

Sklar's theorem can be stated in terms of random variables and their distribution functions as follows:

Theorem 1.6. *Let $X_1, ..., X_n$ be random variables with distribution functions $F_1, F_2, ..., F_n$ and joint distribution function $F(.)$. Then there exists an n-copula C such that (1.1) holds. If $F_1, F_2, ...F_n$ are all continuous, C is unique. Otherwise, C is uniquely determined on $RanF_1 \times RanF_2 \times ... \times RanF_n$.*

In the two-dimensional case, we have the following theorem.

Theorem 1.7. *Let F be a joint distribution function with the marginals F_1 and F_2. Then there exists a copula C such that for all x and y in $\bar{\mathbf{R}}$,*

$$F(x, y) = C(F_1(x), F_2(y)). \tag{1.2}$$

If F_1 and F_2 are continuous, then the copula C is unique; otherwise it is uniquely determined on $RanF_1 \times RanF_2$. Conversely, if C is a copula, and F_1 F_2 are distribution functions, then the function F defined by equation (1.2) is a distribution function with marginals F_1 and F_2.

Example 1.1. For the Gumbel's bivariate exponential distribution (Gumbel (1960)), the joint distribution function is given by

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y+\theta xy)}, & x \geq 0, y \geq 0, 0 \leq \theta \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Then the marginal distribution functions are exponentials, with quasi-inverses $F_1^{(-1)}(u) = -\ln(1-u)$ and $F_2^{(-1)}(v) =$ -ln$(1-v)$ for u,v in $\mathbf{I}$. Hence the corresponding copula is

$$C(u,v) = u + v - 1 + (1-u)(1-v)e^{-\theta \ln(1-u)\ln(1-v)}. \tag{1.4}$$

Example 1.2. Consider the bivariate distribution function (Ali et al. (1978))

$$F(x,y) = (1 + e^{-x} + e^{-y} + (1-\theta)e^{-x-y})^{-1}; \ \ \theta \in [-1,1]. \tag{1.5}$$

By using the probability integral transform and algebraic methods we have

$$C(u,v) = uv + \theta uv(1-u)(1-v), \theta \in [-1,1], \tag{1.6}$$

which is referred to as the Ali-Mikhail-Haq copula.

1.3 Survival copulas

If we replace u by $1-u$ and v by $1-v$ in the bivariate copula $C(u,v)$, the resulting function is a copula denoted by $\hat{C}(u,v)$, called the survival copula or complementary copula, satisfying

$$\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v) \tag{1.7}$$

and the joint survival function of the random vector (X,Y) has the representation

$$\bar{F}(x,y) = \hat{C}(\bar{F}_1(x), \bar{F}_2(y)).$$

$\hat{C}(u,v)$ is a copula that couples the joint survival function $\bar{F}$ to the univariate marginal survival functions $\bar{F}_1$ and $\bar{F}_2$.
The copula for Gumbel's bivariate exponential distribution given in (1.4) has the survival copula, $\hat{C}(u,v) = uve^{-\theta \ln u \ln v}$. Various examples are given in Chapter 2 and in Chapter 7.

1.4 Archimedean copulas

In empirical modelling we make use of a particular group of copulas, called Archimedean copulas. The key characteristic of the Archimedean copulas is that all the information about n-dimensional dependence structure is contained in a univariate generator ϕ and hence the Archimedean representation reduces the study of a multivariate copula to a single univariate function. Archimedean copulas are highly appealing and they gained popularity due to the reason that they can produce wide ranges of dependence properties for different choices of the generator function.

Definition 1.8. A copula C is said to be Archimedean if there exists a representation of the form

$$C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v)) \tag{1.8}$$

where ϕ is a continuous, strictly decreasing function from $\mathbf{I}$ to $[0, \infty)$ such that $\phi(1) = 0$ and $\phi^{[-1]}$ is the pseudo-inverse of ϕ.

The pseudo-inverse of ϕ is defined as follows:

$$\phi^{[-1]}(t) \quad = \quad \begin{cases} \phi^{-1}(t), & 0 \leq t \leq \phi(0), \\ 0, & \phi(0) \leq t \leq \infty. \end{cases} \tag{1.9}$$

If $\phi(0) = \infty$, then $\phi^{[-1]}(t) = \phi^{-1}(t)$. In this case we say that ϕ is a *strict* generator and $C(u,v)$ is said to be a *strict* Archimedean copula.
For every Archimedean copula with generator ϕ, there exists

$$\bar{F^*}(t) = \phi^{-1}(t) \qquad \forall \ \ t \geq 0, \tag{1.10}$$

a univariate survival function taking values in $[0, \infty)$ with mode at 0.

Lemma 1.9. *Copula C is two-increasing if and only if whenever $u_1 \leq u_2$,*

$$C(u_2, v) - C(u_1, v) \leq u_2 - u_1 \tag{1.11}$$

for every v in $[0, 1]$.

Theorem 1.10. *Let ϕ be a continuous strictly decreasing function from* $\mathbf{I}$ *to* $[0,\infty]$ *such that* $\phi(1) = 0$*, and let* $\phi^{[-1]}$ *denote the "pseudo-inverse" of* ϕ *defined by* (1.9)*. Then* $C(u,v) = \phi^{[-1]}(\phi(u) + \phi(v))$ *is a copula if and only if* ϕ *is convex (proof see Nelsen (2006)).*

Example 1.3. Let $\phi(t) = \ln t$ for t in $[0,1]$.Then $\phi^{-1}(t) = e^{-t}$, $C(u,v) = uv = \Pi(u,v)$, say, a *strict* Archimedean copula.

Example 1.4. Let $\phi(t) = 1-t$ for t in $[0,1]$. Then $\phi^{[-1]}(t) = 1-t$ for t in $[0,1]$ and 0 for $t > 1$; i.e., $\phi^{[-1]}(t) = max(1-t,0)$ and $C(u,v) = max(u+v-1,0) = W(u,v)$, say. Hence W is also Archimedean.

Theorem 1.11. *Let C be an Archimedean copula with generator ϕ. Then*

1. *C is symmetric; i.e.,* $C(u,v) = C(v,u)$ *for all u,v in* $[0,1]$*;*
2. *C is associative, i.e.,* $C(C(u,v),w) = C(u,C(v,w))$ *for all u,v,w in* $[0,1]$*;*
3. *If* $c > 0$ *is any constant, then $c\phi$ is also a generator of C.*

Remark 1.2. Let U and V be uniform (0,1) random variables whose joint distribution function is the Archimedean copula C generated by ϕ in Ω, where Ω denotes the set of continuous strictly decreasing convex functions ϕ from $\mathbf{I}$ to $[0,\infty]$ with $\phi(1) = 0$. Then the function $K_C(w) = w - \dfrac{\phi(w)}{\phi'(w)}; 0 < w < 1$ is the distribution function of the random variable $W^* = C(U,V)$.

Theorem 1.12. *Let C be an Archimedean copula with generator ϕ in Ω. Then for almost all u,v in* $\mathbf{I}$*,*

$$\phi'(u)\frac{\partial C(u,v)}{\partial v} = \phi'(v)\frac{\partial C(u,v)}{\partial u} \tag{1.12}$$

where $\phi'(.)$ *is the derivative of* ϕ*.*

For the proof, see Nelsen (2006).

1.5 Dependence concepts

In bivariate(multivariate) set up, dependence concepts are employed to understand the nature of association among variables. The measures of association can be thought of as one-dimensional projections of the dependence structure onto the real line. Scarsini (1984) defined dependence as a matter of association between X and Y along any measurable function. That is, the more X and Y tend to cluster around the graph of a function the more they are dependent. From this definition, it is clear that there exists some freedom in how to define the extent to which X and Y cluster around the graph of a function.

1.5.1 A concordance function

Two observations (x_1, y_1) and (x_2, y_2) of a pair (X, Y) of continuous random variables are concordant if $x_1 > x_2$ and $y_1 > y_2$ or if $x_1 < x_2$ and $y_1 < y_2$, i.e., if $(x_1 - x_2)(y_1 - y_2) > 0$; and discordant if $x_1 > x_2$ and $y_1 < y_2$ or if $x_1 < x_2$ and $y_1 > y_2$, i.e., if $(x_1 - x_2)(y_1 - y_2) < 0$. Geometrically, two distinct points (x_1, y_1) and (x_2, y_2) in the plane are concordant if the line segment connecting them has positive slope, and discordant if the line segment has negative slope.

Let (X_1, Y_1) and (X_2, Y_2) be pairs of random vectors with (possibly) different joint distribution functions F_1 and F_2, but common marginals F_1^* (of X_1 and X_2) and F_2^* (of Y_1 and Y_2). Let C_1 and C_2 denote the copulas of (X_1, Y_1) and (X_2, Y_2), respectively. Then $F_1(x, y) = C_1(F_1^*(x), F_2^*(y))$ and $F_2(x, y) =$

$C_2(F_1^*(x), F_2^*(y))$. Let Q denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2),

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]. \tag{1.13}$$

We now have the following theorem, which demonstrates that Q depends only on the copulas C_1 and C_2 (Nelsen (2006)).

Theorem 1.13. *Under the conditions above,*

$$Q = Q(C_1, C_2) = 4 \iint_{\mathbf{I}^2} C_2(u, v) dC_1(u, v) - 1. \tag{1.14}$$

Some properties of Q are as follows:

(i) Q is symmetric in its arguments: $Q(C_1, C_2) = Q(C_2, C_1)$;

(ii) Q is non-decreasing in each argument: $C_1(u, v) \leq C_1^*(u, v)$ and $C_2(u, v) \leq C_2^*(u, v)$ for all (u, v) in $\mathbf{I}^2$ implies $Q(C_1, C_2) \leq Q(C_1^*, C_2^*)$;

(iii) Copulas can be replaced by survival copulas in Q, i.e., $Q(C_1, C_2) = Q(\hat{C}_1, \hat{C}_2)$;

(iv) $Q(M, M) = 1$, $Q(W, W) = -1$, $Q(\Pi, \Pi) = 0$, $Q(M, \Pi) = 1/3$, $Q(W, \Pi) = -1/3$, and $Q(M, W) = 0$;

(v) For any copula C, $Q(C, C) \in [-1, 1]$, $Q(C, \Pi) \in [-1/3, 1/3]$, $Q(C, M) \in [0, 1]$, and $Q(C, W) \in [-1, 0]$.

The inequality in (ii) above suggests an ordering $\prec$ of the set C of copulas:

Definition 1.14. For any pair of copulas C and C^*, we say that C is less concordant than C^*(and write $C \prec C^*$) whenever $C(u,v) \leq C^*(u,v)$ for all (u,v) in $\mathbf{I}^2$.

Remark 1.3. Let C_1 and C_2 be Archimedean copulas generated, respectively, by ϕ_1 and ϕ_2. Then $C_1 \prec C_2$, if $\dfrac{\phi_1}{\phi_2}$ is non-decreasing on (0,1), or if ϕ_1 and ϕ_2 are continuously differentiable on (0,1), and if $\dfrac{\phi_1^{'}}{\phi_2^{'}}$ is non-decreasing on (0,1).

The two important measures of dependence (concordance) Kendall's tau and Spearman's rho provide the best alternatives to the linear correlation coefficient as a measure of dependence for non-elliptical distributions.

1.5.2 Kendall's tau

Kendall's tau can capture non-linear dependences that were not possible to measure with linear correlation. If X and Y are continuous random variables with copula $C(u,v)$, then the population version of Kendall's tau has a succinct expression in terms of Q given by

$$\tau_{X,Y} = \tau_C = Q(C,C) = 4\iint_{\mathbf{I}^2} C(u,v)dC(u,v) - 1. \tag{1.15}$$

The integral can be interpreted as the expected value of the function $C(U,V)$ of uniform (0,1) random variables U and V whose joint distribution function is C, that is

$$\tau_C = 4E(C(U,V)) - 1.$$

Example 1.5. Let copula $C = C_\theta$ be a member of the Farlie-Gumbel-Morgenstern (FGM) family

$$C_\theta(u, v) = uv + \theta uv(1 - u)(1 - v), \quad \theta \in [-1, 1].$$

Then $\tau_C = 2\theta/9$. Since $\tau_C \in [-2/9, 2/9]$, FGM copulas can only model relatively weak dependence.

Remark 1.4. Let X and Y be random variables with an Archimedean copula C generated by ϕ in Ω. The population version τ_C of Kendall's tau for X and Y is given by

$$\tau_C = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt. \tag{1.16}$$

1.5.3 Spearman's rho

Let (X_1, Y_1), (X_2, Y_2), and (X_3, Y_3) be three independent random vectors with a common joint distribution function F (whose marginals are F_1 and F_2) and copula C. Then the population version of Spearman's rho is defined to be proportional to the difference between probabilities of concordance and discordance of the vectors (X_1, Y_1) and (X_2, Y_3), a pair of vectors with the same marginals, but one vector has distribution function F, while the components of the other are independent,

$$\rho_{X,Y} = 3(P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]). \tag{1.17}$$

Theorem 1.15. *Let X and Y be continuous random variables whose copula is C(u,v). Then the population version of Spearman's rho for X and Y is given by*

$$\begin{aligned} \rho_{X,Y} = \rho_C &= 3Q(C, \Pi), \\ &= 12 \iint_{\mathbf{I}^2} uv dC(u,v) - 3, \\ &= 12 \iint_{\mathbf{I}^2} C(u,v) dudv - 3, \\ &= 12 \iint_{\mathbf{I}^2} [C(u,v) - uv] dudv. \end{aligned}$$

For a pair of continuous random variables X and Y, Spearman's rho is identical to Pearson's product-moment correlation coefficient since

$$\begin{aligned} \rho_{X,Y} = \rho_C &= 12 \iint_{\mathbf{I}^2} uv dC(u,v) - 3, \\ &= 12E(UV) - 3, \\ &= \frac{E(UV) - {}^1\!/_4}{{}^1\!/_{12}}, \\ &= \frac{Cov(U,V)}{\sqrt{Var(U)}\sqrt{Var(V)}}. \end{aligned}$$

Theorem 1.16. *Let X and Y be continuous random variables, and let τ_C and ρ_C denote Kendall's tau and Spearman's rho respectively. Then*

$$-1 \leq 3\tau_C - 2\rho_C \leq 1, \frac{1+\rho_C}{2} \geq \left(\frac{1+\tau_C}{2}\right)^2, \frac{1-\rho_C}{2} \geq \left(\frac{1-\tau_C}{2}\right)^2. \tag{1.18}$$

Theorem 1.17. *Let X and Y be continuous random variables with copula C, and let k denote Kendall's tau or Spearman's rho. Then*

1. $k(X,Y) = 1 \Leftrightarrow C(u,v) = M(u,v) = min(u,v)$ *and*

2. $k(X,Y) = -1 \Leftrightarrow C(u,v) = W(u,v) = max(u+v-1,0)$.

The proof is given in Embrechts et al. (2002). For continuous random variables all values in the interval $[-1,1]$ can be obtained for Kendall's tau or Spearman's rho by a suitable choice of the copula.

1.5.4 Tail dependence

The concept of tail dependence measures the dependence in the upper-right-quadrant tail or lower-left-quadrant tail of a bivariate distribution. Tail dependence between two continuous random variables X and Y is a copula property and this concept is relevant for the study of dependence between extreme values. The amount of tail dependence is invariant under strictly increasing transformations of X and Y.

Definition 1.18. Let X and Y be continuous random variables with distribution functions F_1 and F_2 respectively. The coefficient of upper tail dependence is defined as

$$\lambda_U = \lim_{u \to 1^-} P[V > u | U > u], \tag{1.19}$$

provided this limit exists. Then $\lambda_U \in [0,1]$.
The coefficient of lower tail dependence is defined as

$$\lambda_L = \lim_{u \to 0^+} P[V \leq u | U \leq u], \tag{1.20}$$

provided this limit exists. Then $\lambda_L \in [0,1]$.

The coefficients λ_U and λ_L can be interpreted as follows:

1. If $\lambda_U = 0$, then X and Y are independent in the upper tail.

2. If $\lambda_U \in (0, 1]$,then X and Y are dependent in the upper tail.

3. If $\lambda_L = 0$, then X and Y are independent in the lower tail.

4. If $\lambda_L \in (0, 1]$,then X and Y are dependent in the lower tail.

Proposition 1.19. *Let C be a copula associated with (X, Y). If $\lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}$ and $\lim_{u \to 0^+} \frac{C(u, u)}{u}$ exist, then λ_U and λ_L are given by*

$$\lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}$$

and

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u}.$$

Remark 1.5. Let C be an Archimedean copula with generator $\phi \in \Omega$. Then

$$\lambda_U = 2 - 2 \lim_{u \to 1^-} \frac{\phi'(u)}{\phi'(\phi^{-1}(2\phi(u))}$$

and

$$\lambda_L = 2 \lim_{u \to 0^+} \frac{\phi'(u)}{\phi'(\phi^{-1}(2\phi(u))}.$$

1.5.5 Tail monotonicity

Definition 1.20. (Lehmann (1966)). The random variables X and Y are positively quadrant dependent(PQD) if for all (x, y) in $\mathbf{R}^2$,

$$P(X \leq x, Y \leq y) \geq P(X \leq x)P(Y \leq y) \tag{1.21}$$

or equivalently

$$P(X > x, Y > y) \geq P(X > x)P(Y > y). \tag{1.22}$$

The interpretation is that X and Y are positively quadrant dependent [$PQD(X, Y)$] if the probability that X and Y are simultaneously "small" is at least as great as the case when X and Y independent.

Example 1.6. (Barlow and Proschan (1981)). In many studies of reliability, components are assumed to have independent lifetimes however, it may be more realistic to assume some sort of dependence among components. For example, a system may have components that are subject to the same set of stresses or shocks, or in which the failure of one component results in an increased load on the surviving components. In such a two-component system with lifetimes X and Y, we may wish to use a model in which (regardless of the forms of the marginal distributions of X and Y) small values of X tend to occur with small values of Y, i.e., a model for which X and Y are PQD.

If X and Y have joint distribution function F and copula C, then (1.21) is equivalent to

$$F(x, y) \geq F_1(x)F_2(y) \quad \text{for all} \quad (x, y) \quad \text{in} \quad \mathbf{R}^2, \tag{1.23}$$

and to

$$C(u, v) \geq uv \quad \text{for all} \quad (u, v) \quad \text{in} \quad \mathbf{I}^2. \tag{1.24}$$

When the continuous random variables X and Y are PQD, the joint distribution function F or their copula C is PQD.

Negative quadrant dependence (NQD) is defined similarly, and is equivalent to $C(u, v) \leq uv$. Thus the quantity $[C(u, v) - uv]$ measures "local"positive (or negative) quadrant dependence at each point $(u, v) \in \mathbf{I}^2$, and thus $\iint_{\mathbf{I}^2}[C(u, v) - uv]dudv$ is a measure of "average"quadrant dependence.

Theorem 1.21. *If X and Y are PQD, then*

$$3\tau_{X,Y} \geq \rho_{X,Y} \geq 0.$$

Theorem 1.22. *The copula $C(u, v)$ is positive K-dependent (PKD) if and only if*

$$K_C(t) \leq t(1 - \log t)$$

where

$$K_C(t) = t - \frac{\phi(t)}{\phi'(t)};\ \ 0 < t < 1.$$

1.6 Motivation and present study

The multivariate distributions other than the normal distribution arise when the marginal distributions are not normal or when properties of the joint distribution differ from those of multivariate normal distribution. For example, when contours of constant density are not ellipses, or conditional expectations are not linear, variances and covariances of conditional distributions are affected by the values of the conditioning variables, the multivariate normal distribution is not appropriate. The incompatibility of normal distribution to explain theoretically and empirically many data situations led to the development of other distributions. Interesting applications of various bivariate(multivariate) distributions have been discussed in the statistical and applied literatures. For a comprehensive review, one could refer to Kotz et al. (2002).

The bivariate (multivariate) distributions like exponential, Pareto, and Weibull distributions discussed in literature are individual in nature, each based on specified properties so that they lack a uniform framework. The models have low flexibility in the sense that they cannot conform to different real data situation warranting inspection of each model separately. This motivated researchers to develop family of bivariate(multivariate) distributions with non-normal marginals. The family of distributions has sufficient richness in shape and other characteristics such as dependence to deal with various modelling problems. In many statistical models, the assumption of independence between two or more variables is often due to convenience rather than to the problem at hand. The study of dependence between variables can be done through copulas. Motivated by this, in the present work we introduce various families of bivariate distributions and study their properties. The proposed families are useful in different data modelling situations due to their flexibility and richness.

The thesis is organized into eight chapters. After this introductory chapter where the relevance and scope of the study are discussed, in Chapter 2, we introduce a family of bivariate Pareto distributions using a generalized version of dullness property. Some important bivariate Pareto distributions are derived as special cases. Distributional properties of the family are studied. The dependency structure of the family is investigated. The proposed family contains distributions having both positive as well as negative associations among variables. Finally, the family of distributions is applied to two real life data situations.

In Chapter 3, we study the characteristic properties of the family of bivariate Pareto distributions introduced in Chapter 2. Two measures of income inequality namely income gap ratio and mean left proportional residual income are defined in the bivariate case. We also introduce generalized bivariate failure rate useful in reliability analysis. Characterizations for various members of the family of bivariate Pareto distributions using the above concepts are also derived.

Traditionally, the modelling and analysis of lifetime data is carried out using the survival function and concepts derived from it. The basic concepts such as hazard rate and mean residual life function are widely employed in such situations since they determine distribution uniquely. These concepts are extended to higher dimensions for the analysis of bivariate lifetime data. In Chapter 4, we propose a variant approach by defining reliability measures directly from the copula rather than using the distribution-based measures in modelling survival data. We discuss the advantages of the proposed functions over the reliability measures already available in literature. Characterizations of some well known copulas using the proposed measures are also discussed. The results of the study are applied to case of the copulas of a bivariate exponential family of distributions.

In Chapter 5, we discuss one-parameter families of Archimedean copulas suitable for modelling negative dependent data. The distributional properties as well as the dependence measures such as tail dependence, Kendall's tau, Spearman's rho and measure based on Blomqvist's β are discussed. The local dependence measures such as ψ-measure and the Clayton-Oakes association measure (θ- measure) for the copulas are also discussed. The copula models are applied to a real data set.

In Chapter 6, we discuss a positive dependent Archimedean copula useful for modelling bivariate data sets. Various properties of the copula model such as the dependence structure, tail monotonicity, Kendall's measure and measure based on Blomqvist's β are discussed. The proposed model is fitted to a real data. A comparison with other positive dependent Archimedean copula is done using Akaike's Information Criterion (AIC).

As already mentioned, Weibull distribution is considered as a versatile family of life distributions. In Chapter 7, we discuss a class of bivariate Weibull distributions. This class include some of the existing models as members. Our choice of the marginal distributions as Weibull can lead to a copula for the proposed family. The general form of the copula is Archimedean which is popularly used in empirical modelling. The dependency structure of the family is investigated. Finally, the family of distributions is applied to two real life data sets. The comparison among the models using Akaike's Information Criterion (AIC) is done.
Chapter 8 summarizes the thesis with major conclusions of the study along with discussions on future research problems on this topic.

Chapter 2

A family of bivariate Pareto distributions

2.1 Introduction

Pareto distributions have been extensively employed for modelling and analysis of statistical data under different contexts. Originally, the distribution was first proposed as a model to explain the allocation of income among individuals. Later, various forms of the Pareto distribution have been formulated for modelling and analysis of data from engineering, environment, geology, hydrology etc. These diverse applications of the Pareto distributions lead researchers to develop different kinds of bivariate(multivariate)Pareto distributions. For various properties and applications of Pareto distributions, one could refer to Arnold (1985) and Johnson et al. (1994).

The models discussed in literature are individual in nature and are appropriate for a particular data set that meet the specified requirements. However, when there is little information about the data generating process, it is desirable to start with a family of distributions and then choose a member of the family that fits the given data. Motivated by this fact, we introduce a class of bivariate Pareto distributions arising from a generalization of the univariate dullness property which

[1]Some of the works in this chapter are published in Statistica (see Sankaran et al. (2014))

characterizes the Pareto law (Talwalker (1980)). It is shown that the marginal distributions of the proposed bivariate distribution are univariate Pareto I models. The proposed bivariate family includes some well known distributions as well as several new models. It also imparts enough flexibility in terms of desirable properties that are generally used in modelling problems.

The rest of the article is organized as follows. In Section 2.2, we introduce a family of bivariate Pareto distributions. Various members belonging to the family and their corresponding copulas are identified in Section 2.3. The distributional properties of the family are discussed in Section 2.4. In Section 2.5, we study dependence structure of the family of distributions. Section 2.6 discusses the inference procedure of the parameters of the model. We then apply the proposed class of models to two real data sets. Finally, Section 2.7 summarizes the major conclusions of the study.

2.2 A class of distributions

Let (X, Y) be a non-negative random vector having absolutely continuous survival function $\bar{F}(x, y) = P(X > x, Y > y)$. In order to construct the proposed family of bivariate Pareto distributions, we assume that Z is a non-negative random variable with continuous and strictly decreasing survival function $\bar{G}(z)$ and cumulative hazard function $H(z)$ defined by $H(z) = -\log \bar{G}(z)$. We require the following theorem to construct the proposed bivariate Pareto family.

Theorem 2.1. *The random variable Z satisfies the property*

$$P(Z > \log g(x, y) | Z > a \log x) = P(Z > b \log y) \tag{2.1}$$

for all $a, b > 0, x, y > 1$ and some $g(x,y) > x^a$ if and only if

$$H(\log\, g(x,y)) = H(a\log x) + H(b\log y). \tag{2.2}$$

Proof. Since $H^{-1}(t) = \bar{G}^{-1}(e^{-t})$ for all $t > 0$

$$\begin{aligned} H^{-1}(H(a\log x) + H(b\log y)) &= \bar{G}^{-1}(\exp[-H(a\log x) - H(b\log y)]) \\ &= \bar{G}^{-1}(\bar{G}(a\log x).\bar{G}(b\log y)) \end{aligned} \tag{2.3}$$

or

$$\bar{G}\,H^{-1}(H(a\log x) + H(b\log y)) = \bar{G}(a\log x).\bar{G}(b\log y). \tag{2.4}$$

To prove the theorem, we first assume (2.1). This is equivalent to

$$\bar{G}(\log g(x,y)) = \bar{G}(a\log x).\bar{G}(b\log y). \tag{2.5}$$

Then from (2.4), we have

$$\bar{G}(\log g(x,y)) = \bar{G}\,[H^{-1}(H(a\log x) + H(b\log y))] \tag{2.6}$$

which leads to (2.2).

To prove converse part, we assume (2.2). Now

$$\begin{aligned} P(Z > \log g(x,y)|\, Z > a\log x) &= \frac{\bar{G}(\log g(x,y))}{\bar{G}(a\log x)} \\ &= \frac{\exp[-H(\log g(x,y))]}{\exp[-H(a\log x)]} = \exp[-H(b\log y)] \\ &= \bar{G}(b\log y) = P[Z > b\log y]. \end{aligned}$$

This completes the proof. $\square$

We notice that $g(x,y)$ is a function of (x,y) in $\mathbf{R}^{2+} = \{(x,y)|x,y > 0\}$ satisfying the property (2.2). Further we have,

(a) $g(1,y) = y^b, g(x,1) = x^a$,

(b) $g(\infty,y) = \infty, g(x,\infty) = \infty$,

(c) since $H(.)$ is increasing and continuous, $g(x,y)$ is also increasing and continuous in x and y and

(d) it is assumed that $g(x,y)$ satisfies the inequality $\dfrac{2}{g(x,y)}\dfrac{\partial g}{\partial x}\dfrac{\partial g}{\partial y} - \dfrac{\partial^2 g}{\partial x \partial y} \geq 0.$

From properties (a) through (d) it follows that

$$\bar{F}(x,y) = [g(x,y)]^{-1},\ \ x,y\ > 1 \tag{2.7}$$

which is the survival function of a random vector (X,Y) with Pareto I marginals

$$\bar{F}_1(x) = x^{-a},\ x > 1\ \text{ and }\ \bar{F}_2(y) = y^{-b},\ y > 1.$$

This completes the procedure for constructing the family of bivariate Pareto distributions based on $g(x,y)$ arising from a property characterizing a class of univariate distributions. We designate $\bar{G}(z)$ as the baseline distribution that corresponds to $\bar{F}(x,y)$, since the members of the family are generated through the functional equation (2.2) based on $H(z)$, the cumulative hazard rate of Z.

2.3 Members of the family and their copulas

We derive some members of the family along with their copulas.

1. Let Z be exponential with $\bar{G}_1(z) = \exp(-\lambda\, z), z > 0$ so that $H(z) = \lambda z$. Then $g(x, y) = x^a y^b$. The bivariate survival function is given by

$$\bar{F}^{(1)}(x, y) = x^{-a}y^{-b};\ x, y > 1; a, b > 0. \tag{2.8}$$

The copula of the model (2.8) is the product copula,

$$\hat{C}_1(u, v) = uv,\ \ 0 \leq u, v \leq 1.$$

2. When Z has Gompertz distribution $\bar{G}_2(z) = \exp[-\theta(e^{\alpha z} - 1)]; z \geq 0; \alpha, \theta > 0$ $H(z) = \theta(e^{\alpha z} - 1)$ and the resulting bivariate survival function is

$$\bar{F}^{(2)}(x, y) = (x^{a\alpha} + y^{b\alpha} - 1)^{\frac{-1}{\alpha}}; x, y > 1, \alpha, a > 0. \tag{2.9}$$

Setting $\alpha = \frac{1}{a} = \frac{1}{b}$, we obtain

$$\bar{F}^{(3)}(x, y) = (x + y - 1)^{-a}; x, y > 1, \tag{2.10}$$

the well known Mardia (1962) type I bivariate Pareto model.

The copula is

$$\hat{C}_3(u, v) = (u^{-\frac{1}{a}} + v^{-\frac{1}{b}} - 1)^{-a},\ \ 0 \leq u, v \leq 1.$$

3. Take Z to be a Pareto II variable with $\bar{G}_4(z) = (1+\beta z)^{-\alpha}$ to get $H(z) = \alpha \log(1+\beta z)$. Then we have the bivariate law

$$\bar{F}^{(4)}(x,y) = x^{-a-c\log y} y^{-b}, x, y > 1, a, b > 0; 0 \leq c \leq 1. \tag{2.11}$$

   The corresponding copula is,

$$\hat{C}_4(u,v) = u^{1-\frac{c}{ab}\log v} v, \; 0 \leq u, v \leq 1.$$

4. If Z has half-logistic distribution specified by the survival function

$$\bar{G}_5(z) = 2(1+e^{\frac{z}{\sigma}})^{-1}, \; z > 0, \sigma > 0.$$

   The bivariate model is

$$\bar{F}^{(5)}(x,y) = [\frac{1}{2}(x^{\alpha} + y^{\beta} + x^{\alpha} y^{\beta} - 1)]^{-\sigma}; \alpha = \frac{a}{\sigma} > 0, \sigma > 0, \beta = \frac{b}{\sigma} > 0 \tag{2.12}$$

   and copula of the model is

$$\hat{C}_5(u,v) = \max\left[\frac{1}{2}\left(u^{\frac{-1}{\sigma}} + v^{\frac{-1}{\sigma}} + (uv)^{\frac{-1}{\sigma}} - 1\right)^{-\sigma}, 0\right], \sigma > 0.$$

5. The Burr XII distribution (Pareto IV), $\bar{G}_6(z) = (1+z^c)^{-k}, \; z > 0; \; c, \; k > 0$ with $H(z) = k\log(1+z^c)$ leads to the bivariate model as

$$\bar{F}^{(6)}(x,y) = \exp[-(a\log x)^c - (b\log y)^c - (ab\log x \log y)^c]^{\frac{1}{c}}, \tag{2.13}$$

with the copula

$$\hat{C}_6(u,v) = \exp[-\{(-(\log u)^c + (-\log v)^c + (-\log u)^c(-\log v)^c\}^{\frac{1}{c}}], \text{valid for } c > 1.$$

6. Suppose Z follows the distribution $\bar{G}_7(z) = (2e^z - 1)^{-\sigma}, z > 0; \sigma > 0$, then $H(z) = \sigma \log(2e^z - 1)$ and the bivariate model is

$$\bar{F}^{(7)}(x,y) = (1 + 2x^a y^b - x^a - y^b)^{-1}, \tag{2.14}$$

and the copula is

$$\hat{C}_7(u,v) = \frac{uv}{1 + (1-u)(1-v)}.$$

7. When Z is distributed as Weibull $\bar{G}_8(z) = e^{-(\lambda z)^\alpha}$ $\alpha, \lambda > 0, z > 0$

gives $H(z) = (\lambda z)^\alpha$ and

$$\bar{F}^{(8)}(x,y) = \exp[\frac{-1}{\lambda}\{(\lambda a \log x)^\alpha + (\lambda b \log y)^\alpha\}^{\frac{1}{\alpha}}]. \tag{2.15}$$

The survival copula is given by

$$\hat{C}_8(u,v) = \exp\left[-\{(-\log u)^\alpha + (-\log v)^\alpha\}^{\frac{1}{\alpha}}\right], \text{valid for } \alpha \geq 1.$$

8. If Z has generalized exponential distribution $\bar{G}_9(z) = \frac{p}{e^{\lambda z} - q}$, $z > 0$; $\lambda > 0$, $0 < p < 1$, $q = 1 - p$, we have $H(z) = \log \dfrac{e^{\lambda z} - q}{p}$ and

$$\bar{F}^{(9)}(x,y) = (q + p^{-1}(x^{a\lambda} - q)(y^{b\lambda} - q))^{\frac{-1}{\lambda}}. \tag{2.16}$$

The corresponding copula is,

$$\hat{C}_9\,(u,v) = (q + p^{-1}(u^{-\lambda} - q)(v^{-\lambda} - q))^{\frac{-1}{\lambda}}.$$

9. Taking $\bar{G}_{10}(z) = (1 + \frac{e^{\lambda z}-1}{\alpha})^{-1}, \; \alpha, \lambda > 0$, the cumulative hazard function $H(z) = \log(1 + \alpha^{-1}(e^{\lambda z} - 1))$ provides the bivariate Pareto

$$\bar{F}^{(10)}(x,y) = (1 + \alpha^{-1}(\alpha + x^{a\lambda} - 1)(\alpha + y^{b\lambda} - 1) - \alpha)^{\frac{-1}{\lambda}}. \tag{2.17}$$

The copula is

$$\hat{C}_{10}\,(u,v) = (1 + \alpha^{-1}(\alpha + u^{-\lambda} - 1)(\alpha + v^{-\lambda} - 1) - \alpha)^{\frac{-1}{\lambda}}.$$

Remark 2.1. The method of construction provides a class of bivariate Pareto distributions. Any $\bar{G}(z)$ which is strictly increasing and a $g(x,y)$ satisfying conditions (a) to (d) give rise to a bivariate Pareto model. The bivariate models 1 to 9 comprise some simple forms that do not exhaust the members of the family.

Remark 2.2. When $a = b$ in $g(x,y)$, we have an exchangeable family of Pareto distributions. Such a restriction becomes quite handy in inference problems using Bayesian approach. In that case, $\bar{F}^{(1)}(x,y)$ is the only Schur-constant model belonging to the family.

Remark 2.3. A random variable Z_1 (or its probability distribution) satisfies dullness property (Talwalker (1980)) if for all $x, y \geq 1$

$$P(Z_1 > xy | Z_1 > x) = P(Z_1 > y). \tag{2.18}$$

It may be easy to observe that the property (2.1) reduces to the dullness property (2.18) when $Z = \log Z_1$, $g(x, y) = xy$ and $a = b = 1$.

Remark 2.4. Although the family (2.7) comprises of a large number of members, every bivariate Pareto distribution does not belong to it. For example, the survival function

$$\bar{F}(x, y) = x^{\frac{-a}{2}} y^{\frac{-a}{2}} \exp[-\frac{1}{2}((a \log x)^2 + (a \log y)^2)^{\frac{1}{2}}] \quad x, y > 1, a > 0 \tag{2.19}$$

represents a bivariate Pareto model with Pareto I marginals. If it belongs to the family one must have

$$g(x, y) = x^{\frac{a}{2}} y^{\frac{a}{2}} \exp[\frac{1}{2}((a \log x)^2 + (a \log y)^2)^{\frac{1}{2}}] \tag{2.20}$$

that satisfies (2.2) for some cumulative hazard function $H(.)$ of a non-negative random variable Z, for all x, y. If (2.20) is true for all x, y, it should also hold for

$$H(\log g(x, x)) = 2H(a \log x)$$

or

$$H \log(x^{(\frac{\sqrt{2}+1}{\sqrt{2}} a)}) = 2H(a \log x)$$

or

$$\frac{1}{2} H(\frac{\sqrt{2}+1}{\sqrt{2}} t) = H(t)\,;\, t = a \log x \tag{2.21}$$

for all $t > 0$. It is known from Kagan et al. (1973) that the functional equation

$$A(x) = k^* A(\theta x),\ \theta > 0\ ;\ A(0) = 0 \tag{2.22}$$

has a solution only if $0 < \theta < 1 < k^*$. By analogy, (2.21) is a particular case of (2.22) with $\theta = \frac{\sqrt{2}+1}{\sqrt{2}} > 1$ and hence there is no admissible $H(x)$ that satisfy (2.21). Thus (2.19) does not belong to the proposed family (2.7).

2.4 Properties

The joint density functions of the various models are presented in Table 2.1.

2.4.1 Conditional distributions

There are two kinds of conditional distributions of interest. One is the usual $f_1^*(x|y) = \frac{f(x,y)}{a_2(y)}$ and $f_2^*(y|x) = \frac{f(x,y)}{a_1(x)}$ where $f(x, y)$ is the joint density function and $a_1(x)$ and $a_2(y)$ are respectively the marginal density functions of X and Y. These conditional density functions are given respectively in Table 2.2. The second type of conditional distributions required in the sequel are conditional distributions of X (Y) given $Y > y$ $(X > x)$. The corresponding conditional survival functions are $P(X > x|Y > y)$ and $P(Y > y|X > x)$. These are exhibited in Table 2.3.

2.4.2 Regression functions

The bivariate Pareto family (2.7) is rich enough in the sense that it contains a large number of members that could be candidates for different data situations. The members of the family are highly flexible in various distributional characteristics to represent a wide variety of models.

Table 2.1: Joint density functions for various types of Pareto models

Model type	Joint density function
1	$a\,b\,x^{-a-1}\,y^{-b-1};\ x, y > 1; a, b > 0.$
2	$a\,b\,(1+\alpha)x^{\alpha a-1}\,y^{\alpha b-1}(x^{\alpha a}+y^{\alpha b}-1)^{\frac{-1}{\alpha}-2}; x, y > 1, \alpha, a > 0.$
3	$x^{-a-c\log y-1}y^{-b-1}[(a+c\log y)(b+c\log x)-c]; x, y > 1, a, b > 0; 0 \leq c \leq a\,b.$
4	$\frac{\alpha\,\beta\,\sigma}{4}\ [\frac{1}{2}(1+x^{\alpha})(1+y^{b})-1]^{-\sigma-2}(1+\sigma(1+x^{\alpha})(1+y^{b})x^{\alpha-1}y^{\beta-1};$ $x, y > 1, \alpha > 0, \sigma > 0, \beta > 0.$
6	$2\,ab\dfrac{(2x^{a}y^{b}-x^{a}-y^{b})}{(1+2x^{a}y^{b}-x^{a}-y^{b})^{3}}x^{a-1}y^{b-1}; x, y > 1, a, b > 0.$
7	$a\,b\,\exp[-\frac{1}{\lambda}\{(\lambda\,a\,\log x)^{\alpha}+(\lambda\,b\,\log y)^{\alpha}\}^{\frac{1}{\alpha}}][(\lambda\,a\,\log x)^{\alpha}+(\lambda\,b\,\log y)^{\alpha}]^{\frac{1}{\alpha}-2}$ $\dfrac{(\lambda\,a\,\log x)^{\alpha-1}(\lambda\,b\,\log y)^{\alpha-1}}{x\,y}; x, y > 1, \alpha, \lambda > 0; a, b > 0.$
8	$\frac{\lambda\,ab}{p}[q+\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{p}]^{-\frac{1}{\lambda}-2}[\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{\lambda p}-q]x^{\lambda\,a-1}y^{\lambda\,b-1};$ $x, y > 1, \lambda > 0,\ 0 < p < 1; a, b > 0.$

Table 2.2: Conditional densities $f_1^*(x|y)$ and $f_2^*(y|x)$

Joint density function	$f_1^*(x\|y)$	$f_2^*(y\|x)$
$f_1(x,y)$	$a\,x^{-a-1}$	$b\,y^{-b-1}$
$f_2(x,y)$	$(1+\alpha)a(x^{a\alpha}+y^{b\alpha}-1)^{-\frac{1}{\alpha}-2}x^{a\alpha-1}y^{(1+\alpha)b}$	$(1+\alpha)b(x^{a\alpha}+y^{b\alpha}-1)^{-\frac{1}{\alpha}-2}y^{b\alpha-1}x^{(1+\alpha)a}$
$f_3(x,y)$	$(a+1)\,(x+y-1)^{-a-2}y^{a+1}$	$(a+1)\,(x+y-1)^{-a-2}x^{a+1}$
$f_4(x,y)$	$a^{-1}\,[(a+c\log y)(b+c\log x)-c]\,x^{-c\log y}y^{-b-1}$	$b^{-1}[(a+c\log y)(b+c\log x)-c]x^{-a-c\log y-1}$
$f_5(x,y)$	$\frac{1}{4}\beta[\frac{1}{2}(1+x^{\alpha})(1+y^{\beta})-1]^{-\sigma-2}$ $(1+\sigma(1+x^{\alpha})(1+y^{\beta}))x^{\alpha-1}y^{\beta(\sigma+1)}$	$\frac{1}{4}\beta[\frac{1}{2}(1+x^{\alpha})(1+y^{\beta})-1]^{-\sigma-2}(1+\sigma(1+x^{\alpha})(1+y^{\beta}))$ $x^{(\sigma+1)\alpha}y^{\beta-1)}$
$f_7(x,y)$	$2\,b\frac{(2x^ay^b-x^a-y^b)\,x^{a-1}y^{2b}}{1+2x^ay^b-x^a-y^b}$	$2\,a\frac{(2x^ay^b-x^a-y^b)\,x^{2a}y^{b-1}}{1+2x^ay^b-x^a-y^b}$
$f_8(x,y)$	$\exp[-\frac{1}{\lambda}\{(\lambda\,a\log x)^{\alpha}+(\lambda\,b\log y)^{\alpha}\}^{\frac{1}{\alpha}}[(\lambda\,a\log x)^{\alpha}$ $+(\lambda\,b\log y)^{\alpha}]^{\frac{1}{\alpha}-2}\{(\lambda\,a\log x)^{\alpha}+(\lambda\,b\log y)^{\alpha}\}^{\frac{2}{\alpha}}$ $-(1-\alpha)\lambda](\lambda\,a\log x)^{\alpha-1}(\lambda\,b\log y)^{\alpha-1}x^{-1}y^{b}$	$\exp[-\frac{1}{\lambda}\{(\lambda\,a\log x)^{\alpha}+(\lambda\,b\log y)^{\alpha}\}^{\frac{1}{\alpha}}$ $[(\lambda\,a\log x)^{\alpha}+(\lambda\,b\log y)^{\alpha}]^{\frac{1}{\alpha}-2}\{[(\lambda\,a\log x)^{\alpha}+(\lambda\,b\log y)^{\alpha}]^{\frac{2}{\alpha}}$ $-(1-\alpha)\lambda\}x^{a}y^{-1}$
$f_9(x,y)$	$[q+\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{p}]^{-\frac{1}{\lambda}-2}[\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{\lambda p}-q]$ $p^{-1}\lambda\,a\,x^{\lambda a-1}y^{(1+\lambda)b}$	$[q+\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{p}]^{-\frac{1}{\lambda}-2}[\frac{(x^{\lambda\,a}-q)(y^{\lambda\,b}-q)}{\lambda p}-q]p^{-1}\lambda\,b\,x^{(1+\lambda)a}y^{\lambda b-1}$

TABLE 2.3: Conditional survival functions

Distribution	$P(X > x \mid Y > y)$	$P(Y > y \mid X > x)$
$\bar{F}^{(1)}(x, y)$	x^{-a}	y^{-b}
$\bar{F}^{(2)}(x, y)$	$(x^{a\alpha} + y^{b\alpha} - 1)^{-\frac{1}{\alpha}} y^b$	$(x^{a\alpha} + y^{b\alpha} - 1)^{-\frac{1}{\alpha}} x^a$
$\bar{F}^{(3)}(x, y)$	$(\frac{x+y-1}{y})^{-a}$	$(\frac{x+y-1}{x})^{-a}$
$\bar{F}^{(4)}(x, y)$	$x^{-a-c\log y}$	$x^{-c\log y} y^{-b}$
$\bar{F}^{(5)}(x, y)$	$[\frac{1}{2}(\frac{x^\alpha + y^\beta + x^\alpha y^\beta - 1}{y^\beta})]^{-\sigma}$	$[\frac{1}{2}(\frac{x^\alpha + y^\beta + x^\alpha y^\beta - 1}{x^\alpha})]^{-\sigma}$
$\bar{F}^{(7)}(x, y)$	$\frac{y^b}{1 + 2x^a y^b - x^a - y^b}$	$\frac{x^a}{1 + 2x^a y^b - x^a - y^b}$
$\bar{F}^{(8)}(x, y)$	$\exp[-\frac{1}{\lambda}\{(\lambda\, a \log x)^\alpha + (\lambda\, b \log x)^\alpha\}^{\frac{1}{\alpha}}] y^b$	$\exp[-\frac{1}{\lambda}\{(\lambda\, a \log x)^\alpha + (\lambda\, b \log x)^\alpha\}^{\frac{1}{\alpha}}] x^a$
$\bar{F}^{(9)}(x, y)$	$[q + \frac{(x^{\lambda\, a} - q)(y^{\lambda\, b} - q)}{p}]^{-\frac{1}{\lambda}} y^b$	$[q + \frac{(x^{\lambda\, a} - q)(y^{\lambda\, b} - q)}{p}]^{-\frac{1}{\lambda}} x^a$

We represent the regression functions $A(x) = E(Y|X = x)$ and $B(y) = E(X|Y = y)$ with suffixes corresponding to the member distributions. Accordingly for $\bar{F}^{(1)}(x, y)$, the regression functions are constants, being the respective means. In the case of $\bar{F}^{(3)}(x, y)$, we obtain

$$A_3(x) = (1 + \frac{x}{a})$$

and

$$B_3(y) = (1 + \frac{y}{b}),$$

both linearly increasing functions. They intersect on the means $(E(X), E(Y))$ of the distributions. However for $\bar{F}^{(2)}(x, y)$, the regression functions are

$$A_2(x) = \frac{1+\alpha}{\alpha} \frac{x^{a(1+\alpha)}}{(x^a - 1)^{1+\frac{b-1}{b\alpha}}}$$

and

$$B_2(y) = \frac{1+\alpha}{\alpha} \frac{y^{b(1+\alpha)}}{(y^b - 1)^{1+\frac{a-1}{a\alpha}}}$$

which are non-linear in character. These functions do not intersect at the means. All the remaining distributions also have non-linear regressions, but with different functional forms. For instance, $\bar{F}^{(4)}(x, y)$ has

$$B_4(y) = \frac{b(a + c\log y)(a + c\log y - 1) + c}{b(a + c\log y - 1)^2}$$

which is a logarithmic function, where as for $\bar{F}^{(7)}(x, y)$

$$B_7(y) = \frac{2(1+t)^2}{t^{\frac{a-1}{a}}(2t+1)^{\frac{1}{a}+1}}[B_u(\frac{1}{a}+1, \frac{a-1}{a}) - \frac{1}{t}B_u(\frac{1}{a}+1, \frac{2a-1}{a})]$$

where $u = \frac{3-2y^b}{y^b-2}$, $t = 1 - y^b$ and $B_u(p,q) = \int_u^1 z^{p-1}(1-z)^{q-1}dz$ is the incomplete beta function. The expressions for $A_4(x)$ and $A_7(x)$ are obtained by changing a to b, b to a and y to x.

2.5 Dependence structure

Since the bivariate distributions in the proposed family have identical marginal distributions, a crucial aspect that differentiate them in a practical situation is the differences in the dependence or association between the constituent random variables. Thus a study of various dependence concepts and measures become crucial when discussing family properties, as they tell us the extent to which the variables are associated and also the nature of their relationships. There are three distinct approaches in the study of association. The first one is through global measures like the Pearson's correlation coefficient, the Kendall's tau, Spearman's rho, Gini's measure and Blomqvist's β. A second approach is to study the dependence properties. The six basic properties of positive dependence are (1) total positivity of order 2 (2) stochastic increasing (3) right tail increasing (4) positive association (5) positive quadrant dependence and (6) positive correlation or $Cov(X,Y) \geq 0$. Negative dependence properties are defined as the duals of these. The implications among these are expressed as follows:

$TP_2 \Rightarrow SI \Rightarrow RTI \Rightarrow$ positive association $\Rightarrow PQD \Rightarrow Cov(X,Y) \geq 0$.

Finally, we have local measures of dependence, which measures the dependence structure at specific values of x and y. These become important in survival studies where the duration spent in a specific state of a disease is crucial and also in economics where income of individuals below the poverty line or above the affluence level is of importance. We find global measures of association for various members of the family.

2.5.1 Correlation coefficient

When (X, Y) follow the survival function $\bar{F}^{(1)}(x,y)$, the variables have zero correlation coefficient. The Mardia form $\bar{F}^{(3)}(x,y)$, has coefficient of correlation $R_3 = \frac{1}{a}$ (Kotz et al. (2002)). Since the variances of X and Y exist only when $a > 2$, we see that the model exhibits a low correlation lying in $(0, \frac{1}{2})$.
As regards $\bar{F}^{(4)}(x,y)$, the correlation coefficient (R_4) has the form

$$R_4 = [\frac{(a-2)(b-2)}{a\,b}]^{\frac{1}{2}}[\frac{(a-1)(b-1)}{c}\, e^{\frac{(a-1)(b-1)}{c}} E_1(\frac{(a-1)(b-1)}{c}) - 1]$$

where $E_1(z) = \int\limits_1^\infty \frac{e^{-z\,t}}{t}\,dt\,,\;\; Rez > 0$ is the exponential integral discussed in Abramowitz and Stegun (1966). When $c = 0$, the distribution $\bar{F}^{(4)}(x,y)$ is the product of the marginal distributions of X and Y which means that X and Y are independent and hence $R_4 = 0$. For any fixed values of $a, b > 2$, R_4 is a decreasing function of c. Thus as c runs through $[0, ab]$, the correlation coefficient becomes increasingly negative. When $c = ab$

$$R_4 = (\frac{(a-2)(b-2)}{a\,b})^{\frac{1}{2}}[p\,e^{p}E_1(p) - 1]$$

where $p = \frac{(a-1)(b-1)}{ab}$. As a, b tend to infinity, $\lim_{a,b\to\infty} R_4 = [eE_1(1) - 1]$ which is always negative. Thus (X, Y) is always negatively correlated.

All other cases involve integrals of incomplete beta function to enable an algebraic analysis of R difficult. However, the nature of the correlation will be deduced below using other dependence concepts.

2.5.2 Dependence concepts

While studying the dependence concepts in relation to the members of the bivariate Pareto family, we begin with the strongest concepts in view of the implications to others already considered. We say that a bivariate probability density function $f(x, y)$ is totally positive of order 2-TP_2 (reverse regular of order 2-RR_2) if and only if for all $x_1 < x_2, y_1 < y_2$

$$f(x_1, y_1)f(x_2, y_2) \geq (\leq) f(x_1, y_2)f(x_2, y_1) \tag{2.23}$$

(Barlow and Proschan (1975)).

In the case of the Mardia form $f_3(x, y)$ from Table 2.1, we consider the difference

$$f_3(x_1, y_1)f_3(x_2, y_2) - f_3(x_1, y_2)f_3(x_2, y_1)$$

$$= \frac{a(a+1)}{(x_1 + y_1 - 1)^{a+2}} \frac{a(a+1)}{(x_2 + y_2 - 1)^{a+2}} - \frac{a(a+1)}{(x_1 + y_2 - 1)^{a+2}} \frac{a(a+1)}{(x_2 + y_1 - 1)^{a+2}}.$$

The sign of the above expression depends on $(x_1 - x_2)(y_1 - y_2)$ which is non-negative for all $x_1 < x_2$ and $y_1 < y_2$. Hence $f_3(x, y)$ is TP_2. For the more general model $f_2(x, y)$, the difference leads to the determination of the sign from $(x_1^{a\alpha} - x_2^{a\alpha})(y_1^{b\alpha} - y_2^{b\alpha})$ which is positive. Thus $f_2(x, y)$ is TP_2. Since $TP_2 \Rightarrow Cov(X, Y) \geq 0$, we deduce that in the case of $f_2(x, y)$, X and Y are positively correlated.

The density function $f_4(x,y)$ is neither TP_2 nor RR_2. However $\bar{F}^{(4)}(x,y)$ is RR_2 as evidenced from

$$\begin{aligned}\bar{F}^4(x_1,y_1)\bar{F}^4(x_2,y_2) - \bar{F}^4(x_1,y_2)\bar{F}^4(x_2,y_1) \\ = x_1^{-a}x_2^{-a}y_1^{-b}y_2^{-b}(x_1^{-c\log y_1}x_2^{-c\log y_2} - x_1^{-c\log y_2}x_2^{-c\log y_1}) \leq 0\end{aligned} \tag{2.24}$$

for all $x_1 < x_2$ and $y_1 < y_2$. Recall that X and Y are positive(negative) quadrant dependent- *PQD (NQD)* if and only if

$$\bar{F}(x,y) \geq (\leq)\bar{F}_1(x)\bar{F}_2(y)$$

and that $\bar{F}(x,y)$ is RR_2 implies *NQD*. Thus $\bar{F}^{(4)}(x,y)$ possesses negative dependence.

In the case of $\bar{F}^{(5)}(x,y)$, it is TP_2 since the sign of the expressions on the left of (2.24) with respect to $\bar{F}_5(x,y)$ depends on $(2x_2^{\alpha}+x_1^{\alpha})(y_2^{\beta}-y_1^{\beta})$ which is positive for $y_1 < y_2$. We conclude that $\bar{F}^{(5)}(x,y)$ has positive dependence through *PQD* and further that this implies positive correlation.

While considering the nature of dependence in $\bar{F}^{(7)}(x,y)$ we note that

$$\begin{aligned}\bar{F}^{(7)}(x_1,y_1)\bar{F}^{(7)}(x_2,y_2) - \bar{F}^{(7)}(x_1,y_2)\bar{F}^{(7)}(x_2,y_1) \\ = (x_1^a - x_2^a)(y_2^b - y_1^b)\bar{F}^{(7)}(x_1,y_1)\bar{F}^{(7)}(x_2,y_2)\bar{F}^{(7)}(x_1,y_2)\bar{F}^{(7)}(x_2,y_1)\end{aligned}$$

which is positive for $x_1 < x_2, y_1 < y_2; x_1 > x_2, y_1 > y_2$ and negative for $x_1 < x_2, y_1 > y_2; x_1 > x_2, y_1 < y_2$. Accordingly we see that $\bar{F}^{(7)}(x,y)$ is RR_2 with the consequent implication that the distribution is *NQD* and the associated random variables are negatively correlated in the region $x_1 < x_2, y_1 > y_2; x_1 > x_2, y_1 < y_2$ and $\bar{F}^{(7)}(x,y)$ is TP_2, the distribution is PQD and the associated random variables are positively correlated in the region $x_1 < x_2, y_1 < y_2; x_1 > x_2, y_1 > y_2$. Similar calculations show that $\bar{F}^{(9)}(x,y)$ is PQD and hence the corresponding variables are positively correlated.

A more interesting result emerges for the bivariate distribution $\bar{F}^{(8)}(x,y)$ when Z has Weibull distribution. The TP_2 nature of the survival function depends on α. For example $\alpha = \frac{1}{2}$, $\bar{F}^{(8)}(x,y)$ is RR_2 and for $\alpha = 2$, $\bar{F}^{(8)}(x,y)$ is TP_2. Accordingly the distribution can be NQD or PQD depending on values of α. Thus the random variables X and Y have negative correlation as well as positive correlation depending on values of α.

2.5.3 Dependence functions

Among the various dependence functions available in literature we choose the Clayton function (Clayton (1978)), which seems to be more popular. It is defined as

$$\theta(x,y) = \frac{\bar{F}(x,y)\dfrac{\partial^2 \bar{F}(x,y)}{\partial x \partial y}}{\dfrac{\partial \bar{F}(x,y)}{\partial x}\dfrac{\partial \bar{F}(x,y)}{\partial y}}. \tag{2.25}$$

The interpretation of $\theta(x,y)$ is that when X and Y are positively (negatively) associated $\theta(x,y) > (<) 1$ and $\theta(x,y) = 1$ implies independence of X and Y. For a detailed study of the measure, we refer to Oakes (1989), Anderson et al. (1992), Gupta (2003) and Nair and Sankaran (2014 a).

Table 2.4: Clayton measure for bivariate Pareto models

Distribution	$\theta(x, y)$	dependence
$\bar{F}^{(1)}(x, y)$	1	independent
$\bar{F}^{(2)}(x, y)$	$1 + \alpha$	positive
$\bar{F}^{(3)}(x, y)$	$1 + \frac{1}{\theta}$	positive
$\bar{F}^{(4)}(x, y)$	$1 - \frac{c}{(a+c\log y)(b+c\log x)}$	negative
$\bar{F}^{(5)}(x, y)$	$1 + \frac{1}{\sigma(1+x)^{\alpha}(1+y)^{\beta}}$	positive
$\bar{F}^{(7)}(x, y)$	$\frac{4x^a y^b - 2x^a - 2y^b}{1+4x^a y^b - 2x^a - 2y^b}$	negative

The values of $\theta(x, y)$ and the nature of dependence for various models are presented in Table 2.4. Other dependence functions mentioned in Nair and Sankaran (2010) can be obtained in closed forms for certain bivariate Pareto models. As the nature of dependence is similar to the one based on $\theta(x, y)$, we do not present details on the dependence using other functions.

2.6 Inference and data analysis

The estimators of parameters of the models belonging to the family (2.7) can be generally derived using the method of maximum likelihood. When the number of parameters is not large, one can easily get estimates by solving likelihood equation.

If the model involves more than three parameters, as in the case of (2.17), we need to solve a four dimensional optimization problem which may not give unique solutions. Alternatively one can use a computationally efficient two-stage estimation procedure as suggested by Xu (1996), see also Joe (1997), Joe (2005), in this respect. In the two-stage estimation procedure, the first stage involves the maximum likelihood estimation from univariate marginals and the second stage involves the maximum likelihood estimation of the dependent parameters keeping the univariate parameters held fixed obtained from the first stage. It is proved that the estimators so obtained satisfy large sample properties of the maximum likelihood estimators (MLE).

We now apply the proposed family of distributions to two real life data sets. We first apply the model (2.16) to the American football league data obtained from the matches played on three consecutive week ends in 1986. The data were first published in 'Washington Post' and they are also available in Csörgă and Welsh (1989). In this bivariate data set, the variables X and Y are defined as follows; X represents the game time to the first points scored by kicking the ball between goal posts and Y represents the game time to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game. The data were first analyzed by Csörgă and Welsh (1989), by converting the seconds to decimal minutes i.e 2:03 has been converted to 2.05. We have adopted the same procedure. The data are presented in Table 2.5. We use exponential transformation to the data to make observations larger than one. The Kendall's tau and Spearman's rho of the data are 0.680 and 0.804 respectively and hence the variables X and Y are positively correlated.

The log-likelihood function of the model (2.16) is obtained as

$$l(\lambda, a, b, p) = \left(-\frac{1}{\lambda} - 2\right) \sum_{i=1}^{n} \log\left(\frac{ab\lambda}{p}\left(\frac{\left(x_i^{a\lambda} - q\right)\left(y_i^{b\lambda} - q\right)}{p} + q\right)\right) + \sum_{i=1}^{n} \log\left(\frac{\left(x_i^{a\lambda} - q\right)\left(y_i^{b\lambda} - q\right)}{\lambda p} - q\right) + (a\lambda - 1)\sum_{i=1}^{n} \log x_i + (b\lambda - 1)\sum_{i=1}^{n} \log y_i. \quad (2.26)$$

We do not have an analytically closed form expressions for the estimator. Thus one has to use the numerical method. The maximum likelihood estimates of the parameters of the model (2.16) are obtained as $\hat{a} = 0.1128$, $\hat{b} = 0.0750$, $\hat{p} = 0.991$ and $\hat{\lambda} = 67.913$.

To test the goodness of fit, we use the bivariate version of Kolmogrov-Smirnov (K.S.) test given in Justel et al. (1997). The K.S. statistic values are $D_1 = 0.1976$, $D_2 = 0.2085, D_3 = 0.0474, D_4 = 0.0237$ and $D_5 = 0.0183$ and thus D^*=0.2085 (Max$(D_1, D_2, D_3, D_4, D_5)$). The above value is less than the value 0.2103 at 25^{th} percentile so that the model (2.16) is appropriate for the given data.

The second data set is taken from the official website of ESPN Cricinfo (www.stats. espncricinfo.com). Here we consider a system consisting of two opening batsmen who have been playing together for India since 2001. First opener has played so far 96 Innings and Second opener has played 180 Innings so far. Both of them together opened the Innings for India on 87 occasions. We have randomly chosen 28 Innings score cards. The data are presented in Table 2.6. The Kendall's tau and Spearman's rho of the data are 0.42 and 0.55 respectively and hence the variables X and Y are positively correlated.

Table 2.5: American football league data

Sl.No.	X	Y	Sl.No.	X	Y
1	2.05	3.98	22	10.85	38.07
2	7.78	7.78	23	0.85	0.85
3	7.23	9.68	24	7.05	7.05
4	31.13	49.88	25	32.45	42.35
5	7.25	7.25	26	5.78	25.98
6	4.22	9.48	27	1.65	1.65
7	6.42	6.42	28	2.90	2.90
8	10.40	14.25	29	10.15	10.15
9	11.63	17.37	30	3.88	6.43
10	14.58	14.58	31	10.35	10.35
11	17.83	17.83	32	5.52	11.27
12	9.05	9.05	33	3.43	3.43
13	10.57	14.28	34	2.58	2.58
14	6.85	34.58	35	8.53	14.57
15	14.58	20.57	36	13.80	49.75
16	4.25	4.25	37	6.42	15.08
17	15.53	15.53	38	7.02	7.02
18	8.98	8.98	39	8.87	8.87
19	2.98	2.98	40	0.75	0.75
20	1.38	1.38	41	12.13	12.13
21	11.82	11.82	42	19.65	10.70

The model (2.9) is applied to the data and the log-likelihood function is

$$\begin{aligned} l(\alpha, a, b) = \frac{1}{\alpha} - 2\alpha \sum_{i=1}^{n} \log\left(x_i^{a\alpha} + y_i^{\alpha b} - 1\right) - \\ \sum_{i=1}^{n} \log\left(x_i^{a\alpha} + y_i)^{\alpha b} - 1\right) + \alpha(a\alpha - 1) \sum_{i=1}^{n} \log x_i + \\ \alpha(\alpha b - 1) \sum_{i=1}^{n} \log y_i + \alpha n \log a + \alpha n \log b + \alpha n \log(\alpha + 1) \end{aligned} \tag{2.27}$$

Here also we do not have an analytical closed form expressions for the estimator. Thus one has to use the numerical method. The method of maximum likelihood provides the estimates of the parameters as $\hat{a} = 0.0178, \hat{b} = 0.0199$ and $\hat{\alpha} = 2.53$. The goodness of fit of Justel et al. (1997) is applied and the test statistic value $D^* = 0.196$. This value is less than the value 0.2420 at 25^{th} percentile, we conclude the model (2.9) is a good fit for the given data set.

TABLE 2.6: Cricket data

Sl.No.	X	Y
1	13	16
2	23	73
3	70	60
4	82	91
5	43	11
6	23	74
7	115	96
8	24	65
9	42	77
10	25	22
11	52	81
12	65	47
13	14	2
14	17	57
15	25	13
16	23	86
17	17	6
18	9	7
19	62	41
20	48	55
21	183	219
22	39	23
23	77	74
24	63	64
25	79	64
26	28	35
27	21	40
28	83	52

2.7 Conclusion

In this chapter we have introduced a class of bivariate Pareto distributions and studied various distributional properties of the class. The class includes several well known as well as new bivariate Pareto distributions. It also contains distributions having both positive as well as negative correlations among variables. The dependence structure of the class of distributions were discussed. The proposed class of distributions was applied to two real life data situations.

Chapter 3

Characterizations of a family of bivariate Pareto distributions

3.1 Introduction

Characterizations of probability distributions play a vital role in modelling and analysis of statistical data. The tool that enables the exact determination of a probability model is the characterization theorem. The characterization theorem makes a conclusion that if X exhibits a property $\mathbf{P}^*$ then the distribution belongs to a family of distributions say, $\mathbf{F}^*$. One of the problems that is usually addressed while examining the characteristic properties of a bivariate distribution is to investigate how far the characterizations of the corresponding univariate version can be extended to the bivariate forms. A well known characterization of the Pareto I law is the dullness property. The bivariate version of the dullness property is employed to characterize the family of Pareto distributions, discussed in Chapter 2.

A second concept that has applications in economics is income gap ratio which is used for developing indices of affluence and poverty(Sen (1988)). The bivariate generalization of the concept is proposed and characterizations using this concept

[1]The results in this chapter are published in Statistica (see Sankaran et al. (2015))

are derived. Another function of interest that has applications in reliability is the bivariate generalized failure rate. Characterizations of the family of distributions using the generalized failure rate are also discussed. As the bivariate versions of these functions are not unique, different versions of these concepts lead to different bivariate Pareto distributions which are members of the family.

The rest of the chapter is organized as follows. In Section 3.2, we introduce bivariate versions of dullness property and present characterizations using these versions. The bivariate version of income gap ratio and related concepts in economics are discussed in Section 3.3. The forms of these functions for various bivariate Pareto distributions are given. Section 3.4 discusses bivariate generalized failure rate and characterizations using the generalized failure rate are also developed. Finally, Section 3.5 provides brief conclusions of the study.

3.2 Dullness property

We first discuss the univariate dullness property. Let Z_1 be a non-negative random variable representing the income in a population. As we have already discussed in Remark 2.3, the distribution of Z_1 is said to have dullness property if

$$P(Z_1 > xy|Z_1 > x) = P(Z_1 > y) \tag{3.1}$$

for all $x, y \geq 1$. This means that the conditional probability that true income Z_1 is at least y times the reported value x is the same as the unconditional probability that Z_1 has at least income y. It is proved that the property (3.1) holds if and only if the distribution of Z_1 is Pareto I (Talwalker (1980)). We introduce an equivalent form of dullness property, since (3.1) is not helpful in practice to verify whether a given data set follows Pareto distribution or not.

Theorem 3.1. *The random variable Z_1 with support $[1,\infty)$ satisfies dullness property (3.1) if and only if*

$$m(x) = E(Z_1|Z_1 > x) = \mu x \tag{3.2}$$

for all $x > 1$, where $\mu = E(Z_1) < \infty$.

Proof. Assume that (3.1) holds. Then we can write

$$\bar{G}^*(xy) = \bar{G}^*(x)\bar{G}^*(y) \;\; for \;\; all \;\; x, y > 1, \tag{3.3}$$

where $\bar{G}^*(x) = P(Z_1 > x)$. Integrating (3.3) from 1 to ∞, we obtain

$$\int_1^\infty \bar{G}^*(xy)\,dy = \bar{G}^*(x)\int_1^\infty \bar{G}^*(y)\,dy$$

or

$$\frac{\int_x^\infty \bar{G}^*(t)\,dt}{\bar{G}^*(x)} = x\,(\mu - 1). \tag{3.4}$$

Since left side of (3.4) is $E(Z_1 - x|Z_1 > x)$, we get $m(x) - x = x(\mu - 1)$ which leads to (3.2).

Conversely, when (3.2) holds, we obtain (3.4) and hence

$$\frac{\bar{G}^*(x)}{\int_x^\infty \bar{G}^*(t)\,dt} = \frac{1}{x(\mu - 1)}. \tag{3.5}$$

Integrating (3.5), we obtain

$$-\log \int_x^{\infty} \bar{G}^*(t)\, dt = \frac{1}{\mu-1}\log x + \log c$$

$$\int_x^{\infty} \bar{G}^*(t)\, dt = cx^{-\frac{1}{\mu-1}} \tag{3.6}$$

where c is the integrating constant. Differentiating (3.6) with respect to x, we get

$$\bar{G}^*(x) = \frac{c}{\mu-1}x^{-\frac{\mu}{\mu-1}}. \tag{3.7}$$

Since $\bar{G}^*(1)$ =1, we obtain $c = \mu - 1$ and thus

$$\begin{aligned}\bar{G}^*(x) &= x^{-\frac{\mu}{\mu-1}} \\ &= x^{-(1+\frac{1}{\mu-1})}.\end{aligned}$$

Then

$$\bar{G}^*(xy) = (xy)^{-(1+\frac{1}{\mu-1})} = \bar{G}^*(x)\bar{G}^*(y),$$

which implies (3.1). □

Corollary 3.2.1. The distribution of Z_1 is Pareto I if and only if (3.2) is satisfied.

We propose bivariate versions of (3.2) and examine whether they characterize members of the bivariate Pareto family. A natural extension of $m(x)$ with respect to the bivariate random vector (X, Y) is the vector $(m_1^*(x, y), m_2^*(x, y))$ where

$$m_1^*(x, y) = E(X|X > x, Y > y) \tag{3.8}$$

and

$$m_2^*(x,y) = E(Y|X > x, Y > y). \tag{3.9}$$

Remark 3.1. A closely related function to $(m_1^*(x,y), m_2^*(x,y))$ used extensively in reliability analysis is the bivariate mean residual life vector $(m_1(x,y), m_2(x,y))$ defined as

$$(E(X - x|X > x, Y > y), E(Y - y|X > x, Y > y)) = (m_1^*(x,y) - x, m_2^*(x,y) - y). \tag{3.10}$$

We first observe that the joint survival function $\bar{F}(x,y)$ of (X,Y) can be determined from (3.8) and (3.9).

The joint survival function $\bar{F}(x,y)$ is obtained as

$$\bar{F}(x,y) = \exp\left[-\int_1^x \frac{\frac{\partial m_1^*(t,1)}{\partial t}}{m_1^*(t,1) - t}dt - \int_1^y \frac{\frac{\partial m_2^*(x,t)}{\partial t}}{m_2^*(x,t) - t}dt\right] \tag{3.11}$$

$$= \exp\left[-\int_1^y \frac{\frac{\partial m_2^*(1,t)}{\partial t}}{m_2^*(1,t) - t}dt - \int_1^x \frac{\frac{\partial m_1^*(t,y)}{\partial t}}{m_1^*(t,y) - t}dt\right]. \tag{3.12}$$

The proof follows from Nair and Nair (1989) by using the relationship between $(m_1^*(x,y), m_2^*(x,y))$ and bivariate mean residual life functions.

Definition 3.2. The distribution of the random vector (X,Y) is said to have bivariate dullness property- 1 $(BDP - 1)$ if for $x, y > 1$

$$P(X > xt|X > x, Y > y) = P(X > t|Y > y), \;\; t > 1$$

and

$$P(Y > ys|X > x, Y > y) = P(Y > s|X > x), \;\; s > 1. \tag{3.13}$$

Definition 3.3. The distribution of (X, Y) satisfies bivariate dullness property -2 $(BDP-2)$ if for $x, y > 1$

$$P(X > xt, Y > ys | X > x, Y > y) = P(X > t, Y > s). \tag{3.14}$$

The following theorems characterize bivariate Pareto distributions belonging to the family by the bivariate dullness properties (3.13) and (3.14).

Theorem 3.4. *Let (X, Y) be a bivariate random vector as described in Section 2.2, with $\mu_X = E(X) < \infty$ and $\mu_Y = E(Y) < \infty$. Denote $\mu_X(y) = E(X|Y > y)$ and $\mu_Y(x) = E(Y|X > x)$. Then the following statements are equivalent.*

(a) $\bar{F}^{(4)}(x, y) = x^{-(a - c\log y)} y^{-b}; \qquad x, y > 1, \quad a, b > 1, \quad 0 < c \leq a\, b.$

(b) (X, Y) *satisfies* $BDP - 1$.

(c) $m_1^*(x, y) = x\mu_X(y)$ *and* $m_2^*(x, y) = y\mu_Y(x)$.

Proof. To prove $(a) \Rightarrow (b)$, we have

$$P[X > xt | X > x, Y > y] = \frac{(xt)^{-(a - c\log y)} y^{-b}}{x^{-(a - c\log y)} y^{-b}} = t^{-(a - c\log y)} = P(X > t | Y > y).$$

The second probability identity in (3.13) can be proved similarly.

To establish $(b) \Rightarrow (c)$, we note that the first probability identity in (3.13) is same as

$$\frac{\bar{F}(xt, y)}{\bar{F}(x, y)} = \frac{\bar{F}(t, y)}{P(Y > y)} = \bar{F}_X(t | Y > y) \tag{3.15}$$

where $\bar{F}_X(t|Y > y)$ is the conditional survival function of X given $Y > y$. Integrating (3.15) with respect to t over $(1, \infty)$, we obtain

$$\begin{aligned}
\frac{1}{\bar{F}(x,y)} \int_1^\infty \bar{F}(x\,t, y)\, dt &= \int_1^\infty \bar{F}_X(t|Y > y) dt \\
\Rightarrow \frac{1}{\bar{F}(x,y)} \int_x^\infty \bar{F}(u, y)\, du &= x(-1 + E(X|Y > y)) \\
\Rightarrow E(X|X > x, Y > y) - x &= x(-1 + E(X|Y > y)) \\
\Rightarrow m_1^*(x,y) - x &= x(-1 + E(X|Y > y)) \\
\Rightarrow m_1^*(x,y) &= x\mu_X(y).
\end{aligned}$$

The expression for $m_2^*(x,y)$ can be established in a similar way.

To establish $(c) \Rightarrow (a)$, we have from (c) that

$$m_1^*(t,y) - t = t\mu_X(y) - t$$

and

$$m_2^*(x,t) - t = t\mu_Y(x) - t.$$

Substituting the above expressions in (3.11) and (3.12) we have,

$$\begin{aligned}
\bar{F}(x,y) &= \exp\left[-\int_1^x \frac{\mu_X}{t\mu_X - t} dt - \int_1^y \frac{\mu_Y(x)}{t\mu_Y(x) - t} dt \right] \\
&= \exp\left[-\int_1^y \frac{\mu_Y}{t\mu_Y - t} dt - \int_1^x \frac{\mu_X(y)}{t\mu_X(y) - t} dt \right].
\end{aligned}$$

Equating the resulting expressions, we get,

$$\int_1^x \frac{\mu_X}{t\mu_X - t}dt + \int_1^y \frac{\mu_Y(x)}{t\mu_Y(x) - t}dt = \int_1^y \frac{\mu_Y}{t\mu_Y - t}dt + \int_1^x \frac{\mu_X(y)}{t\mu_X(y) - t}dt.$$

After some simplifications, we obtain,

$$\frac{\mu_X}{\mu_X - 1}\log x + \frac{\mu_Y(x)}{\mu_Y(x) - 1}\log y = \frac{\mu_Y}{\mu_Y - 1}\log y + \frac{\mu_X(y)}{\mu_X(y) - 1}\log x$$

which leads to the functional equation

$$\left(\frac{\mu_X}{\mu_X - 1} - \frac{\mu_X(y)}{\mu_X(y) - 1}\right)\log x = \left(\frac{\mu_Y}{\mu_Y - 1} - \frac{\mu_Y(x)}{\mu_Y(x) - 1}\right)\log y. \tag{3.16}$$

To solve (3.16), we rewrite it as

$$\frac{\log x}{\frac{\mu_Y}{\mu_Y - 1} - \frac{\mu_Y(x)}{\mu_Y(x) - 1}} = \frac{\log y}{\frac{\mu_X}{\mu_X - 1} - \frac{\mu_X(y)}{\mu_X(y) - 1}}. \tag{3.17}$$

The right(left) side of (3.17) is a function of $y(x)$ alone and therefore the equality of the two sides hold good for all $x, y > 1$ if and only if each side must be a constant say $\frac{1}{c}$. Hence

$$\frac{\mu_Y(x)}{\mu_Y(x) - 1} = \frac{\mu_Y}{\mu_Y - 1} - c\log x$$

and

$$\frac{\mu_X(y)}{\mu_X(y) - 1} = \frac{\mu_X}{\mu_X - 1} - c\log y. \tag{3.18}$$

Substituting (3.18) in (3.11) or (3.12), we have

$$\begin{aligned}
-\log \bar{F}(x,y) &= \int_1^x \frac{\mu_X}{t\mu_X - t}dt + \int_1^y \frac{\mu_Y(x)}{t\mu_Y(x) - t}dt \\
&= \frac{\mu_X}{\mu_X - 1}\log x + \frac{\mu_Y(x)}{\mu_Y(x) - 1}\log y \\
&= \frac{\mu_X}{\mu_X - 1}\log x + \left(\frac{\mu_Y}{\mu_Y - 1} - c\log x\right)\log y.
\end{aligned}$$

Thus we get the survival function as

$$\bar{F}(x,y) = x^{-\frac{\mu_X}{\mu_X - 1}} y^{-\frac{\mu_Y}{\mu_Y - 1} + c\log x}.$$

Taking $\mu_X = \frac{a}{a-1}$ and $\mu_Y = \frac{b}{b-1}$, $a, b > 1$, we obtain

$$\bar{F}^{(4)}(x,y) = x^{-a} y^{-(b - c\log x)}.$$

Since $x^{-c\log y} = y^{-c\log x}$, we have (a) and the theorem is completely proved. The parameter values $a, b > 1$ is required for the existence of the means. $\square$

Theorem 3.5. *Setting $c = 0$ in Theorem 3.4, the following statements are equivalent;*

(a) $\bar{F}^{(1)}(x,y) = x^{-a}y^{-b}; \qquad x, y > 1, \qquad a, b > 1.$

(b) (X,Y) *satisfies* $BDP-2$.

(c) $(m_1^*(x,y), m_2^*(x,y)) = (x\mu_X, y\mu_Y)$.

Proof. To prove $(a) \Rightarrow (b)$, we have

$$P(X > xt, Y > ys | X > x, Y > y) = \frac{(xt)^{-a}(ys)^{-b}}{x^{-a}y^{-b}} = t^{-a}s^{-b} = P(X > t, Y > s).$$

To establish $(b) \Rightarrow (c)$, we note that the probability identity in (3.14) is same as

$$\frac{\bar{F}(xt, ys)}{\bar{F}(x, y)} = \bar{F}(t, s). \tag{3.19}$$

Integrating (3.19) with respect to t and s over $(1, \infty)$, we obtain

$$\begin{aligned}
\frac{1}{\bar{F}(x, y)} \int_1^\infty \int_1^\infty \bar{F}(xt, ys)\, dt\, ds &= \int_1^\infty \int_1^\infty \bar{F}(t, s)\, dt\, ds \\
\Rightarrow m_1^*(x, y) - x &= x(-1 + E(x)) \\
\Rightarrow m_1^*(x, y) &= x\mu_X.
\end{aligned}$$

The expression for $m_2^*(x, y)$ is obtained in an analogous manner.

Now we prove $(c) \Rightarrow (a)$. From (c), we have

$$m_1^*(t, y) - t = t\mu_X - t$$

and

$$m_2^*(x, t) - t = t\mu_Y - t.$$

Substituting the above expressions in (3.11), we have

$$\bar{F}(x, y) = \exp\left[-\int_1^x \frac{\mu_X}{t\mu_X - t} dt - \int_1^y \frac{\mu_Y}{t\mu_Y - t} dt \right]$$

which implies

$$\begin{aligned} -\log \bar{F}(x,y) &= \int_1^x \frac{\mu_X}{t\mu_X - t}dt + \int_1^y \frac{\mu_Y}{t\mu_Y - t}dt \\ &= \frac{\mu_X}{\mu_X - 1}\log x + \frac{\mu_Y}{\mu_Y - 1}\log y. \end{aligned}$$

Taking $\mu_X = \frac{a}{a-1}$ and $\mu_Y = \frac{b}{b-1}$, $a, b > 1$, we obtain

$$\bar{F}^{(1)}(x,y) = x^{-a}y^{-b},$$

which completes the proof. □

Remark 3.2. It may be noticed that $BDP-1$ is stronger than $BDP-2$.

Remark 3.3. The properties $BDP-1$ and $BDP-2$ can be interpreted in income analysis as follows. Let X and Y be the incomes from two different sources of a unit in a population. Assume that the incomes of X and Y are at least x and y respectively. The average under-reporting error is proportional to the amount by which the income exceeds the tax exemption level. The under-reporting error in $X(Y)$ is a linear function of the reported income if and only if the incomes (X, Y) follow bivariate Pareto law. In the case of $BDP-1$, the proportionality is independent of x and y, while in $BDP-2$, it is independent of x in the case of X and independent of y in the case of Y.

Theorem 3.6. *Let (X, Y) be a non-negative exchangeable random vector with absolutely continuous survival function and $\mu = E(X) = E(Y) < \infty$. Then*

$$(m_1^*(x,y), m_2^*(x,y)) = (\mu x + (\mu-1)p(y), \mu y + (\mu-1)p(x)) \tag{3.20}$$

for some non-negative function $p(.)$ *with* $p(1) = 0$ *if and only if the survival function of* (X, Y) *is*

$$\bar{F}^{(10)}(x, y) = \left(\frac{x + y - cxy - 1}{1 - c}\right)^{-\frac{\mu}{\mu - 1}} ; \ x, y > 1 \tag{3.21}$$

where $0 < c < 1$.

Proof. Assume that (X, Y) has the distribution (3.21). Then

$$\begin{aligned} m_1^*(x, y) &= x + \frac{1}{\bar{F}(x, y)} \int_x^\infty \bar{F}(t, y) dt \\ &= x + \frac{x + y - cxy - 1}{1 - cy} (\mu - 1) \\ &= \mu x + \frac{y - 1}{1 - cy} (\mu - 1) \end{aligned}$$

which is of the form (3.20) with $p(y) = \frac{y-1}{1-cy}$ and $p(1) = 0$. The proof for $m_2^*(x, y)$ is similar.

We have,

$$\begin{aligned} \frac{\partial m_1^*(t, 1)}{\partial t} &= \frac{\partial}{\partial t}(t\mu) = \mu, \\ \frac{\partial m_2^*(x, t)}{\partial t} &= \frac{\partial}{\partial t}(t\mu + p(x)(\mu - 1)) = \mu. \end{aligned}$$

$$m_1^*(t, 1) - t = t\mu - t$$

and

$$m_2^*(x, t) - t = t\mu - t + p(x)(\mu - 1) = (\mu - 1)(t + p(x)).$$

Conversely, if the relation (3.20) holds, from (3.12),

$$\begin{aligned}\bar{F}(x,y) &= \exp\left[-\int_1^x \frac{\mu}{(\mu-1)t}dt - \int_1^y \frac{\mu}{(\mu-1)t+(\mu-1)p(x)}dt\right]\\ &= \exp\left[-\frac{\mu}{\mu-1}(\log x + \log(y+p(x)) - \log(1+p(x)))\right]\\ &= \left(\frac{x(y+p(x))}{1+p(x)}\right)^{-\frac{\mu}{\mu-1}}. \qquad (3.22)\end{aligned}$$

From (3.12) and (3.20), we obtain

$$\bar{F}(x,y) = \left(\frac{y(x+p(y))}{1+p(y)}\right)^{-\frac{\mu}{\mu-1}}. \qquad (3.23)$$

Equating (3.22) and (3.23) and simplifying,

$$\frac{xp(x)}{1+p(x)-x} = \frac{yp(y)}{1+p(y)-y}. \qquad (3.24)$$

Since (3.24) holds for all $x, y > 1$, one should have

$$\frac{xp(x)}{1+p(x)-x} = \frac{1}{c},$$

a constant independent of x and y, which leads to $p(x) = \dfrac{x-1}{1-cx}$.
Substituting $p(x)$ in (3.22), we have

$$\begin{aligned}\bar{F}^{(10)}(x,y) &= \left(\frac{x(y+\frac{x-1}{1-cx})}{1+\frac{x-1}{1-cx}}\right)^{-\frac{\mu}{\mu-1}} \; ; \; x, y > 1\\ &= \left(\frac{x+y-cxy-1}{1-c}\right)^{-\frac{\mu}{\mu-1}}\end{aligned}$$

which completes the proof. □

Remark 3.4. The distribution specified by (3.21) is a bivariate distribution with Pareto I marginals. It contains some members of the family (2.7). When $c = 0$,

$$m_1^*(x, y) = \mu x + (\mu - 1)(y - 1)$$

and

$$m_2^*(x, y) = \mu y + (\mu - 1)(x - 1)$$

which characterizes the bivariate Pareto distribution

$$\bar{F}^{(3)}(x, y) = (x + y - 1)^{-a}; x, y > 1,$$

the well known Mardia (1962) type I bivariate Pareto model. Similarly when $c = -1$, we have

$$(m_1^*(x, y), m_2^*(x, y)) = \left(\mu x + (\mu - 1)\frac{y - 1}{y + 1}, \mu y + (\mu - 1)\frac{x - 1}{x + 1}\right),$$

that characterizes the bivariate Pareto distribution

$$\bar{F}^{(11)}(x, y) = [\frac{1}{2}(x + y + xy - 1)]^{-a}, x, y > 1, a > 1$$

which is a special case of $\bar{F}^{(5)}(x, y)$ in (2.12) when $\alpha = \beta = 1$ so that $\sigma = a$. Finally $c = \frac{1}{q},\ q > 0$ gives

$$m_1^*(x, y) = \mu x + \frac{(\mu - 1)q(y - 1)}{q - y}$$

and

$$m_2^*(x, y) = \mu y + \frac{(\mu - 1)q(x - 1)}{q - x},$$

provides

$$\bar{F}^{(12)}(x,y) = (q + p^{-1}(x-q)(y-q))^{-a}$$

a special case of $\bar{F}^{(9)}(x,y)$ obtained by taking $\lambda a = \lambda b = 1$.

Remark 3.5. It is easy to see that all the bivariate distributions discussed in Remark 3.4, including the models $\bar{F}^{(3)}(x,y)$, $\bar{F}^{(5)}(x,y)$ and $\bar{F}^{(9)}(x,y)$ do not satisfy the dullness properties $BDP-1$ and $BDP-2$. The extent to which they depart from $BDP-1$ is accounted for by the terms $\mu p(x)$ and $\mu p(y)$.

It follows that Theorems 3.4 - 3.6 provide useful characterizations of the Pareto distributions by the form of the bivariate mean residual life function that can be easily deduced from the relationship (3.10).

3.3 Bivariate income gap ratio

In the context of applications in economics, two functions that are closely related to $m(x)$ are the income gap ratio and the left proportional residual income. For a continuous non-negative random variable Z_1 which represents the income of a population, those with income exceeding x are deemed to be affluent or rich. We call $Z_1 = x$ to be the affluence line. Then $\bar{G}^*(x) = P(Z_1 > x)$ represents the proportion of rich in the population. The proportion of rich, their average income and the measures of income inequality are important indices discussed in connection with income analysis and also for comparison between the rich and poor. Of these, Sen (1988) defined the income gap ratio among the affluent as

$$i(x) = 1 - \frac{x}{E(Z_1|Z_1 > x)}. \tag{3.25}$$

The measure $i(x)$ is used in defining indices of affluence in Sen (1988). On the other hand, Belzunce et al. (1998) defined the mean left proportional residual income($MLPRI$) as

$$l(x) = E\left(\frac{Z_1}{x}|Z_1 > x\right) = 1 - \frac{1}{i(x)}. \tag{3.26}$$

We propose bivariate generalizations of these concepts. For a non-negative random vector (X, Y), the bivariate income gap ratio is defined by the vector

$$\begin{aligned}(i_1(x,y), i_2(x,y)) &= (1 - \frac{x}{E(X|X > x, Y > y)}, 1 - \frac{y}{E(Y|X > x, Y > y)}) \\ &= (1 - \frac{x}{m_1^*(x,y)}, 1 - \frac{y}{m_2^*(x,y)}).\end{aligned} \tag{3.27}$$

Equation (3.27) shows that there is one-to-one relationship between $(i_1(x,y), i_2(x,y))$ and $(m_1^*(x,y), m_2^*(x,y))$, so that each determines other and the corresponding distribution uniquely. The functional forms of $(i_1(x,y), i_2(x,y))$, characterizing some members of the family of Pareto distributions, are given in Table 3.1.

The bivariate generalization of $MLPRI$ is proposed as the vector

$$\begin{aligned}(l_1(x,y), l_2(x,y)) &= \left(E\left(\frac{X}{x}|X > x, Y > y\right), E\left(\frac{Y}{y}|X > x, Y > y\right)\right) \\ &= \left(\frac{m_1^*(x,y)}{x}, \frac{m_2^*(x,y)}{y}\right).\end{aligned} \tag{3.28}$$

TABLE 3.1: Bivariate income gap ratios

Distribution	$(i_1(x,y), i_2(x,y))$
$\bar{F}^{(1)}(x,y)$	$\left(\frac{\mu_X-1}{\mu_X}, \frac{\mu_Y-1}{\mu_Y}\right)$
$\bar{F}^{(2)}(x,y)$	$\left(\frac{\mu_X(y)-1}{\mu_X(y)}, \frac{\mu_Y(x)-1}{\mu_Y(x)}\right)$
$\bar{F}^{(3)}(x,y)$	$\left(\frac{(\mu-1)(x+y-2)}{\mu x+(\mu-1)(y-1)}, \frac{(\mu-1)(x+y-2)}{\mu x+(\mu-1)(y-1)}\right)$
$\bar{F}^{(11)}(x,y)$	$\left(\frac{(\mu-1)(x+y+xy-1)}{\mu x(y+1)+(\mu-1)(y-1)}, \frac{(\mu-1)(x+y+xy-1)}{\mu x(y+1)+(\mu-1)(y-1)}\right)$
$\bar{F}^{(12)}(x,y)$	$\left(\frac{(\mu-1)(x(q-y)+q(y-1))}{\mu x(q-y)+q(\mu-1)(y-1)}, \frac{(\mu-1)(y(q-x)+q(x-1))}{\mu x(q-y)+q(\mu-1)(y-1)}\right)$

The calculation of $(l_1(x,y), l_2(x,y))$ is easily facilitated from those of $(m_1^*(x,y), m_2^*(x,y))$. Thus the characterizations established in Section 3.2 using $(m_1^*(x,y), m_2^*(x,y))$ can be translated in terms of $(l_1(x,y), l_2(x,y))$.

3.4 Bivariate generalized failure rate

For a non-negative random variable Z_1, the generalized failure rate is given by

$$r(x) = -x\frac{dlog\bar{G}^*(x)}{dx}. \tag{3.29}$$

Lariviere and Porteus (2001) and Lariviere (2006) discussed properties of $r(x)$ and its applications in operations management. The well known income model derived by Singh and Maddala (1976) is based on a relationship between $r(x)$ and $\bar{G}^*(x)$ as

$$r(x) = \alpha x^{\beta}(\bar{G}^*(x))^{\gamma}, \alpha, \beta, \gamma > 0.$$

For a non-negative random vector (X, Y), the bivariate generalized failure rate is defined by the vector

$$(r_1(x,y), r_2(x,y)) = \left(-x\frac{\partial \log \bar{F}(x,y)}{\partial x}, -y\frac{\partial \log \bar{F}(x,y)}{\partial y}\right). \tag{3.30}$$

There exists an identity connecting $(r_1(x,y), r_2(x,y))$ and $(m_1^*(x,y), m_2^*(x,y))$. Differentiating

$$m_1^*(x,y) = x + \frac{1}{\bar{F}(x,y)}\int_x^{\infty} \bar{F}(t,y)\,dt$$

with respect to x and rearranging terms, we obtain

$$\bar{F}(x,y)\frac{\partial m_1^*(x,y)}{\partial x} = (m_1^*(x,y) + x)\,\frac{\partial \bar{F}(x,y)}{\partial x}.$$

This gives

$$r_1(x,y) = \frac{x\frac{\partial m_1^*(x,y)}{\partial x}}{m_1^*(x,y) - x}. \tag{3.31}$$

Similarly we obtain

$$r_2(x,y) = \frac{y\frac{\partial m_2^*(x,y)}{\partial y}}{m_2^*(x,y) - y}. \tag{3.32}$$

A redeeming feature of $(r_1(x,y), r_2(x,y))$ is that it allows simple analytically tractable expression for various distributions in the bivariate Pareto family, while the other functions can be expressed in terms of special functions only for many members. See

Table 3.2 for expressions of $(r_1(x,y), r_2(x,y))$. It may be noted that the characterizations developed in Section 3.2 can be transformed in terms of $(r_1(x,y), r_2(x,y))$.

3.5 Conclusion

In this chapter, we have developed characterizations of the family of bivariate Pareto distributions discussed in Chapter 2. The well known dullness property was extended to the bivariate set up and characterizations of bivariate Pareto distributions using this property were derived. The measures of income inequality such as income gap ratio and mean left proportional residual income were proposed and studied in the bivariate case. The generalized failure rate has been extended to the bivariate set up and characterizations using this concept were derived.

Table 3.2: Bivariate generalized failure rates

Distribution	$(r_1(x,y), r_2(x,y))$
$\bar{F}^{(1)}(x,y)$	$\left(a = \dfrac{\mu_X}{\mu_X - 1}, b = \dfrac{\mu_Y}{\mu_Y - 1}\right)$
$\bar{F}^{(2)}(x,y)$	$\left(\dfrac{ax^{a\alpha}}{x^{a\alpha} + y^{b\alpha} - 1}, \dfrac{by^{b\alpha}}{x^{a\alpha} + y^{b\alpha} - 1}\right)$
$\bar{F}^{(3)}(x,y)$	$\left(\dfrac{ax}{x + y - 1}, \dfrac{by}{x + y - 1}\right)$
$\bar{F}^{(4)}(x,y)$	$\left(\dfrac{\mu_X(y)}{\mu_X(y) - 1}, \dfrac{\mu_Y(x)}{\mu_Y(x) - 1}\right)$
$\bar{F}^{(5)}(x,y)$	$\left(\dfrac{\sigma\alpha x^{\alpha}(1 + y^{\beta})}{(x^{\alpha} + y^{\beta} + x^{\alpha}y^{\beta} - 1)}, \dfrac{\sigma\alpha y^{\beta}(1 + x^{\alpha})}{(x^{\alpha} + y^{\beta} + x^{\alpha}y^{\beta} - 1)}\right)$
$\bar{F}^{(6)}(x,y)$	$(ca^c(\log x)^{c-1}(1 - (b\log y)^c), cb^c(\log y)^{c-1}(1 - (a\log x)^c))$
$\bar{F}^{(7)}(x,y)$	$(1 + 2x^a y^b - x^a - y^b)^{-1}\left(ax^a(2y^b - 1), by^b(2x^a - 1)\right)$
$\bar{F}^{(8)}(x,y)$	$\lambda^{\alpha-1}\left\{\lambda^{\alpha}a^{\alpha}\,(\log x)^{\alpha} + \lambda^{\alpha}b^{\alpha}(\log y)^{\alpha}\right\}^{1-\frac{1}{\alpha}}\,(a^{\alpha}(\log x)^{\alpha-1}, b^{\alpha}(\log y)^{\alpha-1})$
$\bar{F}^{(9)}(x,y)$	$\left[pq + (x^{\lambda a} - q)(y^{\lambda b} - q)\right]^{-1} ap\left(x^{\lambda a}(y^{\lambda b} - q), y^{\lambda b}(x^{\lambda a} - q)\right)$
$\bar{F}^{(11)}(x,y)$	$a(x + y + xy - 1)^{-1}\,(x(1 + y), y(1 + x))$
$\bar{F}^{(12)}(x,y)$	$ap\left[pq + (x - q)(y - q)\right]^{-1}(x(y - q), y(x - q))$

Chapter 4

Copula-based reliability concepts

4.1 Introduction

The role of copulas in the analysis of lifetime data has been emphasised either implicitly or explicitly during the past thirty years. This can be seen from the works of various researchers like Georges et al. (2001), Romeo et al. (2006), Kaishev et al. (2007), Pellerey (2008), Navarro and Spizzichino (2010) and Louzada et al. (2012). The methodology adopted to analyse bivariate data in these works is to infer the copula directly from the observations or by appealing to reliability functions like the bivariate hazard rate or mean residual life based on the survival function to identify the appropriate copula. In the present work an alternative approach is proposed by considering bivariate copulas instead of bivariate distributions. We define the analogues of reliability functions that are expressed in terms of copulas and study their properties. The proposed copula functions possesses several advantages over the usual reliability functions defined in the literature. We can generate new copulas through appropriate choices of the copula-based reliability functions and the proposed copula functions satisfy certain properties that are not shared by their distribution-based counterparts.

[1]The results in this chapter have been communicated as entitled "Modelling bivariate lifetime data using copula"(see Nair et al. (2016))

The chapter is organized as follows. In Section 4.2, we study the copula-based hazard function. The mean residual function in terms of copula is introduced in Section 4.3. The proposed measures are employed to develop characterizations of various copulas. The application of the results in case of a bivariate exponential family is investigated in Section 4.4 and is illustrated using a real data set in Section 4.5. Finally the study is concluded in Section 4.6.

4.2 Hazard rate function of copula

Let (X, Y) be a non-negative random vector with survival function $\bar{F}(x, y)$. There are several ways of defining a bivariate hazard rate function. We now consider the vector-valued failure rate function (Johnson and Kotz (1975)) defined by the vector

$$(h_1(x, y), h_2(x, y)) = \nabla(-\log \bar{F}(x, y)) \tag{4.1}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the gradient operator. Thus

$$\begin{aligned} h_1(x, y) &= -\frac{\partial \log\, \bar{F}(x, y)}{\partial x}, \text{and} \\ h_2(x, y) &= -\frac{\partial \log\, \bar{F}(x, y)}{\partial y}. \end{aligned} \tag{4.2}$$

Now we introduce copula-based hazard rate function. From the representation given in (4.2) for the hazard rate, we can write

$$A_1(u, v) = h_1(F_1^{-1}(u), F_2^{-1}(v)) = \frac{-\partial \log \hat{C}(u, v)}{\partial u} \frac{du}{dF_1^{-1}(u)}. \tag{4.3}$$

Note that

$$A_1(u,1) = h_1(F_1^{-1}(u),0) = -\frac{1}{F_1(x)}\frac{\partial F_1(x)}{\partial x} = -\frac{1}{u}\frac{du}{dF_1^{-1}(u)}.$$

Thus from (4.3),

$$\frac{\partial \log\, \hat{C}(u,v)}{\partial u} = \frac{A_1(u,v)}{u\, A_1(u,1)}. \tag{4.4}$$

Similarly if

$$A_2(u,v) = h_2(F_1^{-1}(u), F_2^{-1}(v)),$$

$$\frac{\partial \log\, \hat{C}(u,v)}{\partial v} = \frac{A_2(u,v)}{v\, A_2(1,v)}. \tag{4.5}$$

Thus we can write

$$G_1(u,v) = u\frac{\partial \log\, \hat{C}(u,v)}{\partial u} = \frac{A_1(u,v)}{A_1(u,1)} \tag{4.6}$$

and

$$G_2(u,v) = v\frac{\partial \log\, \hat{C}(u,v)}{\partial v} = \frac{A_2(u,v)}{A_2(1,v)}. \tag{4.7}$$

From (4.6), we obtain

$$\int_u^1 \frac{G_1(p,v)}{p}\, dp = \int_u^1 \frac{\partial \log\, \hat{C}(p,v)}{\partial\, p}\, dp = \int_u^1 \frac{A_1(p,v)}{A_1(p,1)}\, dp.$$

Thus we get

$$\log \hat{C}(u,v) - \log \hat{C}(1,v) = -\int_u^1 \frac{A_1(p,v)}{A_1(p,1)}\, dp$$

which leads to

$$\frac{\hat{C}(u,v)}{v} = \exp[-\int_u^1 \frac{A_1(p,v)}{A_1(p,1)}\,dp] = \exp[-\int_u^1 \frac{G_1(p,v)}{p}\,dp].$$

Similarly from (4.7), we obtain

$$\frac{\hat{C}(u,v)}{u} = \exp[-\int_v^1 \frac{G_2(u,p)}{p}\,dp].$$

Definition 4.1. The vector $(G_1(u,v), G_2(u,v))$ is defined as the hazard rate of the survival copula $\hat{C}(u,v)$.

We give a physical interpretation of $G_1(u,v)$ and $G_2(u,v)$ in the following manner. Assume that there are N two-component devices with lifetimes (X,Y), of which m of the first components fail instantaneously after surviving the $100(u)\%$ point of the distribution of X given that the second component has survived $100(v)\%$ point of the distribution of Y. Then $A_1(u,v)$ is approximately $\frac{m}{N}$, when N is large. Similarly if n of the first components fail irrespective of the lifetime Y, $A_1(u,1)$ is approximately $\frac{n}{N}$. Note that n is the number of first components exposed to the risk of failure and m is the number exposed to the same risk given v. Thus $G_1(u,v)$ represents a rate of failure of the first component taking into consideration the lifetime of the second component. The interpretation of $G_2(u,v)$ is similar.

Proposition 4.2. *A survival copula $\hat{C}(u,v)$ is uniquely determined by one of components $G_1(u,v)$ or $G_2(u,v)$ by the formula*

$$\hat{C}(u,v) = v\,\exp[-\int_u^1 \frac{G_1(p,v)}{p}\,dp] \tag{4.8}$$

or

$$\hat{C}(u,v) = u \exp[-\int_v^1 \frac{G_2(u,p)}{p}\, dp]. \tag{4.9}$$

Notice that Proposition 4.2 enables a unique determination of the survival copula from the functional form of one of the components $G_1(u,v)$ or $G_2(u,v)$. At the same time in the distribution-based approach the survival function is determined from $h_1(x,y)$ and $h_2(x,y)$ using the formula

$$\bar{F}(x,y) = \exp\left[-\int_0^x h_1(t,0)\, dt - \int_0^y h_2(x,t)\, dt\right] \tag{4.10}$$

given in Galambos and Kotz (1978) which requires the knowledge of the forms of both $h_1(x,y)$ and $h_2(x,y)$.

For example, the expressions $h_1(x,y) = \lambda_1 + \theta y$ and $h_2(x,y) = \lambda_2 + \theta x$ need to be used in (4.10) to determine the Gumbel (1960) bivariate exponential distribution

$$\bar{F}(x,y) = \exp\left[-\lambda_1 x - \lambda_2 y - \theta xy\right], x > 0, y > 0, \lambda_1, \lambda_2 > 0, 0 \leq \theta \leq \lambda_1\lambda_2.$$

On the other hand the corresponding Gumbel-Barnett copula

$$\hat{C}(u,v) = uv \exp\left[-\beta \log u \log v\right], \;\; 0 \leq \beta \leq 1, \tag{4.11}$$

is characterized by $G_1(u,v) = (1 - \beta \log v)$.

The following proposition provides the criterion to verify whether a given function can be the hazard rate of a copula.

Proposition 4.3. *If a function* $G_1 : \mathbf{I}^2 \to [0,\infty)$ *satisfies* *(i)* $G_1(u,v) \geq 0$

(ii) $\lim_{u \to 0} \int_u^1 \frac{G_1(p,v)}{p} dp = \infty$ *and*

(iii) $A(u_2,v_2) - A(u_1,v_2) \geq A(u_2,v_1) - A(u_1,v_1), u_2 \geq u_1, v_2 \geq v_1$ *where*

$$A(u,v) = \exp[-\int_u^1 \frac{G_1(p,v)}{p} dp],$$

then

$$\hat{C}(u,v) = v \exp[-\int_u^1 \frac{G_1(p,v)}{p} dp]$$

is a survival copula with hazard rate component $G_1(u,v)$*.*

Proof. Since $G_1(u,v) = \frac{A_1(u,v)}{A_1(u,1)}$, we have $G_1(p,1) = 1$ and

$$\hat{C}(u,1) = \exp[-\int_u^1 \frac{dp}{p}] = u.$$

The properties $\hat{C}(1,v) = v$ and $\hat{C}(u,0) = 0$ are obvious. Further,

$$\hat{C}(0,v) = v \exp[-\int_0^1 \frac{G_1(p,v)}{p} dp] = 0, \text{ by (ii)}.$$

Condition (iii) implies that for $v_2 \geq v_1$,

$$v_2[\exp(-\int_{u_2}^1 \tfrac{G_1(p,v_2)}{p} dp) - \exp(-\int_{u_1}^1 \tfrac{G_1(p,v_2)}{p} dp)] \geq v_1[\exp(-\int_{u_2}^1 \tfrac{G_1(p,v_1)}{p} dp) - \exp(-\int_{u_1}^1 \tfrac{G_1(p,v_1)}{p} dp)]$$

which is the same as

$$\hat{C}(u_2, v_2) - \hat{C}(u_1, v_2) - \hat{C}(u_2, v_1) + \hat{C}(u_1, v_1) \geq 0$$

and thus $\hat{C}(u, v)$ is a copula. □

Example 4.1. Assume that $G_1(u, v) = \dfrac{1 + \theta\,(1 - 2u)\,(1 - v)}{1 + \theta\,(1 - u)(1 - v)}$, $-1 \leq \theta \leq 1$.

Writing

$$\frac{G_1(u, v)}{u} = \frac{1}{u} - \frac{\theta\ (1 - v)}{1 + \theta\,(1 - u)(1 - v)}$$

and substituting in (4.8)

$$\begin{aligned}
\hat{C}(u, v) &= v\ \exp[-\int_u^1 \left(\frac{1}{p} - \frac{\theta\ (1 - v)}{1 + \theta\,(1 - p)(1 - v)}\right) du] \\
&= v\ \exp[-(-\log u - \log(1 + \theta\,(1 - u)(1 - v)))] \\
&= v \exp[\log u(1 + \theta\,(1 - u)(1 - v))].
\end{aligned}$$

Thus

$$\hat{C}(u, v) = uv[1 + \theta(1 - u)(1 - v)], \tag{4.12}$$

the Farlie-Gumbel-Morgenstern (FGM) survival copula.

Remark 4.1. Proposition 4.3 enables us to generate new copulas based on functional forms of G_1.

In identifying the appropriate copula for a given set of observations, the usual practice is to employ inferential techniques such as estimation of the parameters of assumed marginal distributions and dependence parameters and then to validate the model. Though characteristic properties of models provide unique determination of model $\hat{C}(u,v)$, there have been only a few attempts in this direction in copula theory. Proposition 4.2 enables characterizations through the functional form of $(G_1(u,v), G_2(u,v))$. The results characterizing well known copulas are given below.

Proposition 4.4. *The copula hazard rate is of the form*

$$(G_1(u,v), G_2(u,v)) = (B_1(v), B_2(u)) \tag{4.13}$$

where $B_1(v)$ does not depend on u and $B_2(u)$ does not depend on v if and only if the survival copula is Gumbel-Barnett in (4.11).

Proof. The proof of the necessary part is obtained by assuming that the copula has the form (4.11). Then by direct calculation,

$$(G_1(u,v), G_2(u,v)) = (1 - \beta \log v, 1 - \beta \log u)$$

which is of the form (4.13). To prove the converse part, we assume that (4.13) holds. Then formulas (4.8) and (4.9) lead to the functional equation

$$v\ u^{B_1(v)} = u\ v^{B_2(u)}, \tag{4.14}$$

equivalent to

$$u^{\frac{1}{B_2(u)-1}} = v^{\frac{1}{B_1(v)-1}}.$$

The solution is

$$u^{\frac{1}{B_2(u)-1}} = k^* = v^{\frac{1}{B_1(v)-1}}$$

giving

$$B_1(v) = 1 - \beta \log v \quad \text{and} \quad B_2(u) = 1 - \beta \log u \quad \text{where} \quad \beta = (-\log k^*)^{-1}. \tag{4.15}$$

Substituting $B_1(v)$ and $B_2(u)$ in (4.14), we recover the copula (4.11). $\square$

Remark 4.2. The value $G_1(u,v) = 1(G_2(u,v) = 1)$, characterizes the product survival copula $\hat{C}(u,v) = uv$.

Remark 4.3. Proposition 4.4 extends the characterization of Gumbel (1960) type I bivariate exponential distribution given in Theorem 5.4.11 of Galambos and Kotz (1978) to the copula (4.9). As a result, apart from the above exponential case we can deduce that the bivariate Weibull law

$$\bar{F}(x,y) = \exp[-x^{\lambda_1} - y^{\lambda_2} - \beta x^{\lambda_1} y^{\lambda_2}], \;\; x > 0, y > 0, 0 \leq \beta \leq 1, \lambda_i > 0, i = 1, 2$$

which is characterized by

$$(h_1(x,y), h_2(x,y)) = \left((1 + \beta y^{\lambda_2})\lambda_1 x^{\lambda_1 - 1}, (1 + \beta x^{\lambda_1})\lambda_2 y^{\lambda_2 - 1}\right)$$

with marginals as Weibull, $\bar{F}_1(x) = \exp[-x^{\lambda_1}]$ and $\bar{F}_2(y) = \exp[-y^{\lambda_2}]$.

Proposition 4.5. *The components of the copula hazard rate of an Archimedean copula satisfies*

$$\frac{G_1(u,v)}{G_2(u,v)} = \left(\frac{v}{u}\right)^{\frac{1}{\theta}}, \;\; -1 \leq \theta < \infty, \;\; \theta \neq 0 \tag{4.16}$$

for all u,v in $\mathbf{I}$ *if and only if*

$$\hat{C}(u,v) = \left(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1\right)^{-\theta}, \tag{4.17}$$

the Clayton survival copula.

Proof. We first note that for the Clayton survival copula,

$$G_1(u,v) = \frac{u^{-\frac{1}{\theta}}}{u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1}$$

and

$$G_2(u,v) = \frac{v^{-\frac{1}{\theta}}}{u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1}$$

so that (4.16) holds. To prove the converse part, we observe that (4.17) is an Archimedean copula with generator $\phi(u) = \theta(u^{\frac{-1}{\theta}} - 1)$. From Theorem 1.12, for an Archimedean copula with generator $\phi(.)$, we have

$$\phi^{'}(u)\frac{\partial \hat{C}(u,v)}{\partial v} = \phi^{'}(v)\frac{\partial \hat{C}(u,v)}{\partial u} \tag{4.18}$$

for all u,v in $\mathbf{I}$, where $\phi^{'}(.)$ is the derivative of $\phi(.)$.

Using

$$\frac{G_1(u,v)}{G_2(u,v)} = \frac{u\dfrac{1}{\hat{C}(u,v)}\dfrac{\partial \hat{C}(u,v)}{\partial u}}{v\dfrac{1}{\hat{C}(u,v)}\dfrac{\partial \hat{C}(u,v)}{\partial v}} = \frac{u^{-\frac{1}{\theta}}}{v^{-\frac{1}{\theta}}},$$

(4.18) becomes

$$\phi^{'}(u)u^{\frac{1}{\theta}+1} = \phi^{'}(v)v^{\frac{1}{\theta}+1},$$

for all u, v.

The solution of the above functional equation is

$$\phi^{'}(u)u^{\frac{1}{\theta}+1} = \phi^{'}(v)v^{\frac{1}{\theta}+1} = k^{*} \tag{4.19}$$

where k^* is a constant that may depend on θ. Also k^* is less than zero since ϕ is decreasing. Solving (4.19) with the boundary condition $\phi(1) = 0$ and setting $k^* = -1$,

$$\phi(u) = \theta(u^{\frac{-1}{\theta}} - 1)$$

and accordingly $\hat{C}(u, v)$ is a Clayton survival copula. $\square$

Remark 4.4. Similar results can be derived for characterizing other Archimedean copulas. For example, Gumbel-Barnett survival copula is characterized by,

$$\frac{G_1(u, v)}{G_2(u, v)} = \frac{1 - \log v}{1 - \log u}$$

and Ali-Mikhail-Haq survival copula is characterized by,

$$\frac{G_1(u, v)}{G_2(u, v)} = \frac{1 - \theta + \theta v}{1 - \theta + \theta u}.$$

The behaviour of the hazard rate is a good indicator of the ageing patterns of the device as well as a tool for model selection.

Definition 4.6. The lifetime (X, Y) is said to have increasing hazard rate $(IHR(x, y))$ if the component $h_1(x, y)$ is an increasing function of x for a fixed $y > 0$ and $h_2(x, y)$ is an increasing function of y for a fixed $x > 0$. The decreasing hazard rate $(DHR(x, y))$ property is defined by reversing the monotonicity of $h_i(x, y), \quad i = 1, 2$.

In the same manner we define the monotonicity of the copula-hazard rate as follows.

Definition 4.7. The copula-hazard rate $(G_1(u, v), G_2(u, v))$ is said to be increasing $(IHR^*(u, v))$ if $G_1(u, v)$ is increasing in u for a fixed v and $G_2(u, v)$ is increasing in v for a fixed u. By reversing the monotonicity of $G_i(u,v)$, $i = 1,2$ we have decreasing hazard rate ($DHR^*(u,v)$).

We now seek the implication between $IHR(x,y)$ and $DHR^*(u,v)$.

Remark 4.5. We have $\frac{\partial}{\partial x}h_1(x,y) = \frac{\partial}{\partial F_1^{-1}(u)}A_1(u,v) = \frac{\partial}{\partial u}A_1(u,v)\frac{\partial u}{\partial F_1^{-1}(u)}$.
Since $\frac{\partial F_1^{-1}(u)}{\partial u} \leq 0$, $\frac{\partial h_1}{\partial x} \geq 0 \Leftrightarrow \frac{\partial}{\partial u}A_1(u,v) \leq 0$.
Thus $IHR(x,y)(DHR(x,y)) \Leftrightarrow A_1(u,v)$ is decreasing (increasing) in u for all v.

For many known survival copulas such as the Clayton copula, the Farlie-Gumbel-Morngestern copula, etc., $A_1(u,1) = 1$ so that $A_1(u,v) = G_1(u,v)$ and similarly $A_2(u,v) = G_2(u,v)$. In such cases

$$IHR(x,y)(DHR(x,y)) \Leftrightarrow DHR^*(u,v)(IHR^*(u,v)). \tag{4.20}$$

Remark 4.6. In general the implication (4.20) is not true. Since

$$\frac{\partial G_1(u,v)}{\partial u} = u\frac{\partial^2 \log \hat{C}(u,v)}{\partial u^2} + \frac{\partial \log \hat{C}(u,v)}{\partial u}$$

a sufficient condition for $IHR^*(u,v)$ is that $\log \hat{C}(u,v)$ is increasing and convex.

4.3 Mean residual quantile function

The bivariate mean residual life is the vector $(m_1(x,y), m_2(x,y))$(Nair and Nair (1988)) defined by

$$m_1(x,y) = E(X - x|X > x, Y > y) = \frac{1}{\bar{F}(x,y)}\int_x^\infty \bar{F}(t,y)\,dt \tag{4.21}$$

and

$$m_2(x,y) = E(Y - y|X > x, Y > y) = \frac{1}{\bar{F}(x,y)} \int_y^{\infty} \bar{F}(x,t)\, dt. \tag{4.22}$$

The mean residual life vector $m_1(x,y)$ and $m_2(x,y)$ can be represented in the terms of copula as

$$M_1(u,v) = m_1(F_1^{-1}(u), F_2^{-1}(v)) = -\frac{1}{\hat{C}(u,v)} \int_0^u \hat{C}(p,v) \frac{dF_1^{-1}(p)}{dp} dp;\ p =\ F_1(t) \tag{4.23}$$

and

$$M_2(u,v) = -\frac{1}{\hat{C}(u,v)} \int_0^v \hat{C}(u,p_1) \frac{dF_2^{-1}(p_1)}{dp_1} dp_1;\ p_1 =\ F_2(t). \tag{4.24}$$

Definition 4.8. The vector $(M_1(u,v), M_2(u,v))$ given in (4.23) and (4.24) is defined as the bivariate mean residual quantile function of (X,Y).

We present some properties of $(M_1(u,v), M_2(u,v))$ in the following propositions.

Proposition 4.9. *The survival copula $\hat{C}(u,v)$ of an absolutely continuous non-negative random vector can be expressed in terms of $(M_1(u,v), M_2(u,v))$ as*

$$\hat{C}(u,v) = \frac{vM_1(1,v)}{M_1(u,v)} \exp\left[-\int_u^1 \frac{\frac{d}{dp}(p\, M_1(p,1))}{pM_1(p,v)} dp\right] \tag{4.25}$$

$$= \frac{uM_2(u,1)}{M_2(u,v)} \exp\left[-\int_v^1 \frac{\frac{d}{dp_1}(p_1\, M_2(1,p_1))}{p_1 M_2(u,p_1)} dp_1\right]. \tag{4.26}$$

Proof. Differentiating (4.23) with respect to u, we obtain

$$\hat{C}(u,v)\, \frac{\partial M_1(u,v)}{\partial u} + M_1(u,v) \frac{\partial \hat{C}(u,v)}{\partial u} = -\hat{C}(u,v) \frac{d\bar{F}_1^{-1}(u)}{du}. \tag{4.27}$$

Also

$$\begin{aligned} M_1(u,1) =& m_1(\bar{F}_1^{-1}(u),0) = \frac{1}{\bar{F}_1(x)} \int\limits_x^{\infty} \bar{F}_1(t)\, dt \\ =& -\frac{1}{u} \int\limits_0^{u} p\, \frac{d\bar{F}_1^{-1}(p)}{dp} dp, \end{aligned}$$

or

$$\frac{d\bar{F}_1^{-1}(u)}{du} = -\frac{1}{u} \frac{d(uM_1(u,1))}{du}. \tag{4.28}$$

Substituting (4.28) in (4.27), we get

$$\hat{C}(u,v)\, \frac{\partial M_1(u,v)}{\partial u} + M_1(u,v) \frac{\partial \hat{C}(u,v)}{\partial u} = \frac{\hat{C}(u,v)}{u} \frac{d(uM_1(u,1))}{du},$$

or

$$\frac{1}{M_1(u,v)} \frac{\partial M_1(u,v)}{\partial u} + \frac{1}{\hat{C}(u,v)} \frac{\partial \hat{C}(u,v)}{\partial u} = \frac{1}{uM_1(u,v)} \frac{d(uM_1(u,1))}{du}.$$

Thus

$$\frac{\partial \log \hat{C}(u,v)}{\partial u} = \frac{\frac{d}{du}(u\, M_1(u,1))}{u\, M_1(u,v)} - \frac{\partial \log M_1(u,v)}{\partial u}. \tag{4.29}$$

Similarly,

$$\frac{\partial \log \hat{C}(u,v)}{\partial v} = \frac{\frac{d}{dv}(v\, M_2(1,v))}{v\, M_2(u,v)} - \frac{\partial \log M_2(u,v)}{\partial v}. \tag{4.30}$$

From (4.29) we have

$$\int\limits_u^1 \frac{\partial \log \hat{C}(p,v)}{\partial p} dp = \int\limits_u^1 \frac{\frac{d}{dp}(p\, M_1(p,1))}{p\, M_1(p,v)} dp - \int\limits_u^1 \frac{\partial \log M_1(p,v)}{\partial p} dp$$

which yields,

$$\log \hat{C}(1,v) - \log \hat{C}(u,v) = \int_u^1 \frac{\frac{d}{dp}(p\,M_1(p,1))}{p\,M_1(p,v)} dp + \log M_1(u,v) - \log M_1(1,v).$$

Thus

$$\log v - \log \hat{C}(u,v) + \log \frac{M_1(1,v)}{M_1(u,v)} = \int_u^1 \frac{\frac{d}{dp}(p\,M_1(p,1))}{p\,M_1(p,v)} dp$$

which leads to

$$\frac{M_1(u,v)\hat{C}(u,v)}{vM_1(1,v)} = \exp\left[-\int_u^1 \frac{\frac{d}{dp}(\,p\,M_1(p,1))}{pM_1(p,v)} dp\right]$$

or

$$\hat{C}(u,v) = \frac{vM_1(1,v)}{M_1(u,v)} \exp\left[-\int_u^1 \frac{\frac{d}{dp}(\,p\,M_1(p,1))}{pM_1(p,v)} dp\right].$$

In a similar manner, (4.24) leads to (4.26). □

Remark 4.7. By virtue of formula (4.25) or (4.26), only one of the components of $(M_1(u,v), M_2(u,v))$ is required to determine the copula $\hat{C}(u,v)$. This is the advantage of $(M_1(u,v), M_2(u,v))$ over the usual mean residual life function $(m_1(x,y), m_2(x,y))$. In the case of the latter, determination of the survival function can be accomplished only when both $m_1(x,y)$ and $m_2(x,y)$ are known through the formula (Nair and Nair (1988))

$$\bar{F}(x,y) = \frac{m_1(0,0)\,m_2(x,0)}{m_1(x,0)\,m_2(x,y)} \exp\left[-\int_0^x \frac{dt}{m_1(t,0)} - \int_0^y \frac{dt}{m_2(x,t)}\right].$$

Remark 4.8. The identity (4.25) or (4.26) can be used to determine a copula through the functional form of either $M_1(u,v)$ or $M_2(u,v)$.

We present a set of sufficient conditions on $M_1(u,v)$ to determine a copula.

Proposition 4.10. *If a function $M_1 : \mathbf{I}^2 \to [0,\infty)$ is such that*

(i) $\frac{d}{du}(uM_1(u,1)) \geq u\frac{\partial M_1(u,v)}{\partial u}$

(ii) $\lim\limits_{u\to 0} \exp[-\int\limits_u^1 \frac{\frac{\partial}{\partial p}pM_1(p,1)}{pM_1(p,v)}dp] = \infty$ *and*

(iii) $B(u_2,v_2) - B(u_2,v_1) - B(u_1,v_2) + B(u_1,v_1) \geq 0$ *for* $u_2 \geq u_1, v_2 \geq v_1,$ *where*

$$B(u,v) = \frac{M_1(1,v)}{M_1(u,v)} \exp[-\int\limits_u^1 \frac{\frac{\partial}{\partial p}pM_1(p,1)}{pM_1(p,v)}dp]$$

then

$$\hat{C}(u,v) = v\frac{M_1(1,v)}{M_1(u,v)} \exp[-\int\limits_u^1 \frac{\frac{\partial}{\partial p}pM_1(p,1)}{pM_1(p,v)}dp]$$

is a copula with mean residual component $M_1(u,v)$.

Proof. Condition (i) follows from the identity

$$G_1(u,v) = \frac{\frac{d}{du}uM_1(u,1)}{M_1(u,v)} - u\frac{\partial \log M_1(u,v)}{\partial u} \geq 0.$$

Now

$$\hat{C}(u,1) = \frac{M_1(1,1)}{M_1(u,1)} \exp[-\int\limits_u^1 \frac{\partial}{\partial p} \log pM_1(p,1)dp] = u.$$

It is easy to see that $\hat{C}(1,v) = v$ and $\hat{C}(u,0) = 0$. Also

$$\hat{C}(0,v) = v\exp[-\int\limits_0^1 \frac{\frac{\partial}{\partial p}pM_1(p,1)}{pM_1(p,v)}dp] = 0$$

by (ii).

Condition (iii) implies that for $v_2 \geq v_1$

$$v_2[B(u_2,v_2) - B(u_1,v_2)] \geq v_1[B(u_2,v_1) - B(u_1,v_1)]$$

which is equivalent to the 2-increasing property of $\hat{C}(u,v)$. Thus $\hat{C}(u,v)$ is a copula.

□

Remark 4.9. Empirical evidence may suggest some simple forms for $M_1(u,v)$ or $M_2(u,v)$,which leads to the generation of new copulas.

Remark 4.10. Although the survival copula $\hat{C}(u,v)$ corresponding to a given $(M_1(u,v), M_2(u,v))$ can be evaluated, characterization of $\hat{C}(u,v)$ cannot be accomplished as the information on the form of the marginals is required for the purpose. To avoid this difficulty, we consider the vector $(L_1(u,v), L_2(u,v))$, where

$$L_1(u,v) = \frac{1}{\hat{C}(u,v)} \int_0^u \frac{\hat{C}(p,v)}{p} dp \tag{4.31}$$

and

$$L_2(u,v) = \frac{1}{\hat{C}(u,v)} \int_0^v \frac{\hat{C}(u,p_1)}{p_1} dp_1. \tag{4.32}$$

Then $\hat{C}(u,v)$ is obtained from $L_1(u,v)$ and $L_2(u,v)$ as

$$\hat{C}(u,v) = v\frac{L_1(1,v)}{L_1(u,v)} \exp\left[-\int_u^1 \frac{dp}{pL_1(p,v)}\right] \tag{4.33}$$

$$= u\frac{L_2(u,1)}{L_2(u,v)} \exp\left[-\int_v^1 \frac{dp_1}{p_1 L_2(u,p_1)}\right]. \tag{4.34}$$

Both $L_1(u,v)$ and $L_2(u,v)$ can be evaluated from $\hat{C}(u,v)$ using (4.31) and (4.32). However, we cannot find a meaningful reliability interpretation for $(L_1(u,v), L_2(u,v))$ in terms of residual life. However, the following properties of $(L_1(u,v), L_2(u,v))$ are useful for the modelling and analysis of bivariate lifetime data.

Proposition 4.11. *The form*

$$(L_1(u,v), L_2(u,v)) = (K_1^*(v), K_2^*(u)) \tag{4.35}$$

where $K_1^*(v)$ is independent of u and $K_2^*(u)$ is independent of v is satisfied for all u, v if and only if $\hat{C}(u,v)$ is Gumbel-Barnett survival copula given in (4.11).

Proof. The necessary part of the theorem follows from direct calculations, using (4.11) in (4.31) and (4.32). This gives

$$(L_1(u,v), L_2(u,v)) = (\frac{1}{1-\beta\log v}, \frac{1}{1-\beta\log u}),$$

which is of the form (4.35). We prove the converse part by assuming (4.35). Equation (4.33) yields

$$\begin{aligned} \hat{C}(u,v) &= v\exp\left[-\int_u^1 \frac{dp}{pK_1^*(v)}\right] \\ &= v\exp\left(\frac{\log u}{K_1^*(v)}\right) \end{aligned} \tag{4.36}$$

and from (4.34) we have

$$\hat{C}(u,v) = u\exp\left(\frac{\log v}{K_2^*(u)}\right). \tag{4.37}$$

On equating (4.36) and (4.37) we obtain

$$\hat{C}(u,v) = v \exp\left(\frac{\log u}{K_1^*(v)}\right) = u \exp\left(\frac{\log v}{K_2^*(u)}\right). \tag{4.38}$$

This leads to the functional equation

$$\frac{\log v}{1 - \frac{1}{K_1^*(v)}} = \frac{\log u}{1 - \frac{1}{K_2^*(u)}} \tag{4.39}$$

which is true for all u, v. This happens if and only if (4.39) is a constant, say $\dfrac{1}{\beta}$, $\beta > 0$. Thus from (4.39),we obtain

$$K_1^*(v) = \frac{1}{1 - \beta \log v}$$

and

$$K_2^*(u) = \frac{1}{1 - \beta \log u}.$$

Substituting $K_1^*(v)$ and $K_2^*(u)$ in (4.36), we obtain the Gumbel-Barnett survival copula.

There exist some identities connecting $(G_1(u,v), G_2(u,v))$, $(M_1(u,v), M_2(u,v))$ and $(L_1(u,v), L_2(u,v))$. From (4.29) and (4.30), we have

$$G_1(u,v) = [M_1(u,v)]^{-1}[\frac{d}{du} u M_1(u,1) - u \frac{\partial M_1(u,v)}{\partial u}]$$

and

$$G_2(u,v) = [M_2(u,v)]^{-1}[\frac{d}{dv} v M_2(1,v) - v \frac{\partial M_2(u,v)}{\partial v}].$$

Differentiating (4.31), we obtain

$$L_1(u,v)\frac{\partial \hat{C}(u,v)}{\partial u} + \hat{C}(u,v)\frac{\partial L_1(u,v)}{\partial u} = \frac{\hat{C}(u,v)}{u}$$

or

$$L_1(u,v)\frac{\partial \log \hat{C}(u,v)}{\partial u} + \frac{\partial L_1(u,v)}{\partial u} = \frac{1}{u}.$$

Using the definition of $G_1(u,v)$, we get

$$L_1(u,v)G_1(u,v) + u\frac{\partial L_1(u,v)}{\partial u} = 1. \tag{4.40}$$

Similarly

$$L_2(u,v)G_2(u,v) + v\frac{\partial L_2(u,v)}{\partial v} = 1. \tag{4.41}$$

Also $(L_1(u,v), L_2(u,v))$ is related to $(M_1(u,v), M_2(u,v))$ by the identities

$$\frac{1}{L_1(u,v)}\left(\frac{\partial L_1(u,v)}{\partial u} - \frac{1}{u}\right) = \frac{1}{M_1(u,v)}\left(\frac{\partial M_1(u,v)}{\partial u} - \frac{1}{u}\frac{d(uM_1(u,1))}{du}\right)$$

and

$$\frac{1}{L_2(u,v)}\left(\frac{\partial L_2(u,v)}{\partial v} - \frac{1}{v}\right) = \frac{1}{M_2(u,v)}\left(\frac{\partial M_2(u,v)}{\partial v} - \frac{1}{v}\frac{d(vM_2(1,v))}{dv}\right).$$

□

Proposition 4.12. *The relationship* $L_i(u,v)G_i(u,v) = 1,\ \ i = 1,2$ *holds for all* u, v *if and only if* $\hat{C}(u,v)$ *is the Gumbel-Barnett survival copula.*

Proof. Proof follows from Propositions 4.4 and 4.11. □

Remark 4.11. Propositions 4.11 and 4.12 are generalizations of the characterizations of the Gumbel (1960) Type I bivariate exponential distribution given in Nair and Nair (1988) to a wider class of distributions.

Bivariate life distributions are classified according to the monotonic nature of the mean residual life. Zahedi (1985) proposed four definitions of decreasing mean residual life, of which one typical definition is as follows.

We say that (X, Y) is said to have decreasing mean residual life $(DMRL(x, y))$ if for $t \geq 0$

$$m_1(x+t, y) \leq m_1(x, y)$$

and

$$m_2(x, y+t) \leq m_2(x, y).$$

When $m_1(x, y)$ and $m_2(x, y)$ are differentiable this is equivalent to $\dfrac{\partial m_1(x, y)}{\partial x} \leq 0$ for $y > 0$ and $\dfrac{\partial m_2(x, y)}{\partial y} \leq 0$ for $x > 0$.

In terms of $(M_1(u, v), M_2(u, v))$ we propose the following,

Definition 4.13. The mean residual quantile function $(M_1(u, v), M_2(u, v))$ is said to be decreasing $(DMRL^*(u, v))$ if and only if $\frac{\partial M_1(u,v)}{\partial u} \leq 0$ for $0 \leq v \leq 1$ and $\frac{\partial M_2(u,v)}{\partial v} \leq 0$ for $0 \leq u \leq 1$. Similarly $(M_1(u, v), M_2(u, v))$ is said to be increasing $(IMRL^*(u, v))$ if and only if $\frac{\partial M_1(u,v)}{\partial u} \geq 0$ for $0 \leq v \leq 1$ and $\frac{\partial M_2(u,v)}{\partial v} \geq 0$ for $0 \leq u \leq 1$.

Proposition 4.14.

$$DMRL(x, y) \Leftrightarrow IMRL^*(u, v).$$

$$IMRL(x, y) \Leftrightarrow DMRL^*(u, v).$$

The proof follows from

$$\frac{\partial m_1(x,y)}{\partial x} = \frac{\partial}{\partial u} m_1(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)) \frac{\partial u}{\partial \bar{F}_1^{-1}(u)} = \frac{\partial}{\partial u} M_1(u,v) \frac{\partial u}{\partial \bar{F}_1^{-1}(u)}$$

and the fact that $\frac{\partial \bar{F}_1^{-1}(u)}{\partial u} \leq 0$.

4.4 Analysis of bivariate exponential copulas

In this section, we apply the results obtained in the previous sections to analyse the reliability properties of the copulas of a bivariate exponential family of distributions of Nair & Sankaran (2014 b). The family has survival function of the form

$$\bar{F}(x,y) = \exp[-g(x,y)], \quad x, y > 0 \tag{4.42}$$

where

$$g(x,y) = H^{-1}(H(x) + H(y))$$

$$\frac{\partial g}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial^2 g}{\partial x \partial y} \geq 0$$

and $H(.)$ is the cumulative hazard function of a non-negative random variable Z satisfying

$$P(Z > g(x,y)|Z > x) = P(Z > y).$$

The survival copula corresponding to (4.42)

$$\hat{C}(u,v) = \exp[-g(-\log u, -\log v)] \tag{4.43}$$

is Archimedean with generator $\phi(t) = H(-\log t)$, which is taken to be convex. Some members of the family (4.42) and the corresponding copulas are given in Table 4.1. A computationally convenient aspect of (4.42) is that the marginals are unit exponential so that $\bar{F}_1^{-1}(u) = -\log u$ and $\bar{F}_2^{-1}(v) = -\log v$. The copula-hazard rates of the family are presented in Table 4.2.

In view of the expressions of $\bar{F}_1^{-1}(u)$ and $\bar{F}_2^{-1}(v)$, from the definitions it follows that for the family $A_i(u,v) = G_i(u,v)$ and $M_i(u,v) = L_i(u,v), i = 1, 2$.

From Remark 4.5 and Proposition 4.14 the monotonic nature of the reliability functions follows.

$$IHR(x,y) \Leftrightarrow DHR^*(u,v)$$

$$DMRL(x,y) \Leftrightarrow IMRL^*(u,v).$$

The family (4.42) is flexible in the monotonic behaviour of their hazard rates. Type 2 for $\alpha > 2$; types 4, 5; type 6 for $-1 < \theta < 0$ and type 7 are $DHR^*(u,v)$ and type 1 for $\alpha = 1$, type 6 for $\theta > 0$ are $IHR^*(u,v)$ and type 3 satisfies the locally no-ageing property

$$P[X_i > x_i + y_i | X_i > x_i, X_j > x_j] = P[X_i > y_i | X_j > x_j],\ j = 3-i,\ i,j = 1,2,\ i \neq j. \tag{4.44}$$

Relationship (4.44) can also be written as

$$v\hat{C}(\bar{F}_1(\bar{F}_1^{-1}(u) + \bar{F}_1^{-1}(p)), v) = \hat{C}(p,v)\,\hat{C}(u,v)$$

and

$$u\hat{C}(\bar{F}_2(\bar{F}_2^{-1}(v) + \bar{F}_2^{-1}(p)), u) = \hat{C}(u,p)\,\hat{C}(u,v)$$

for all $0 \leq u, v, p \leq 1$.

In the case of unit exponential marginals the above equations reduce to the nice form

$$v\hat{C}(up, v) = \hat{C}(p, v)\hat{C}(u, v)$$

and

$$u\hat{C}(u, vp) = \hat{C}(u, p)\hat{C}(u, v)$$

It is easy to see that the Gumbel-Barnett copula (type 3) is characterized by the no-ageing property and also that this property is equivalent to the local constancy of $(G_1(u, v), G_2(u, v))$ and $(L_1(u, v), L_2(u, v))$.

Notions of Bivariate IHR, takes a different approach in the Bayesian framework. From Bassan & Spizzichino (1999) and Bassan et al. (2002), an exchangeable pair (X, Y) is IHR-1 (x, y) if and only if for $x \leq y$

$$L(X - x|X > x, Y > y) \geq_{st} L(Y - y|X > x, Y > y) \tag{4.45}$$

and IHR-2 (x, y) if and only if for $x \leq y$

$$L(X - x|X > x, Y > y) \geq_{hr} L(Y - y|X > x, Y > y) \tag{4.46}$$

where L represents the law, $\geq_{st}$ and $\geq_{hr}$ denotes the usual stochastic order and hazard rate order defined in Shaked & Shanthikumar (2007). Answering whether (X, Y) has IHR-1 is simple, as the copulas of the bivariate exponential family are Archimedean and such copulas are Schur-concave and Schur-concave copulas are IHR-1 (Bassan & Spizzichino (1999)). Regarding (4.46) we note that it is equivalent to saying that

$$\frac{\bar{F}(x+t,y)}{\bar{F}(y+t,x)}$$

is increasing in t for all $0 \leq x \leq y$.

For the exponential family

$$\frac{\bar{F}(x+t,y)}{\bar{F}(y+t,x)} = \frac{\hat{C}(up,v)}{\hat{C}(vp,u)}, \quad p = \bar{F}_1(t). \tag{4.47}$$

Logarithmic differentiation of (4.47) leads to the condition for $\hat{C}(u,v)$ to be IHR^*-$2(u,v)$ as

$$\frac{\partial \log \hat{C}(up,v)}{\partial p} \geq \frac{\partial \log \hat{C}(vp,u)}{\partial p}$$

for all $v \leq u$.

This is the same as

$$G_1(up,v) \geq G_2(vp,u), \;\; v \leq u. \tag{4.48}$$

Condition (4.48) holds for type 2 and type 6 and accordingly they are IHR^*-$2(u,v)$.

In the Bayesian approach, an exchangeable pair (X,Y) is said to be bivariate decreasing mean residual life ($DMRL$-$1(x,y)$) if and only if

$$m_1(x,y) \geq m_2(x,y), \;\; x < y.$$

or

$$\int_x^\infty \bar{F}(t,y)\,dt \geq \int_y^\infty \bar{F}(t,x)\,dt, \;\; x < y.$$

In the exponential case this reduces to

$$\int_0^u \frac{\hat{C}(p,v)}{p}dp \geq \int_0^v \frac{\hat{C}(p,u)}{p}dp, \quad u > v.$$

or

$$L_1(u,v) \geq L_2(u,v). \tag{4.49}$$

4.5 Application

The application of our results in a real data situation can be accomplished in two ways. The first is to find the copula function appropriate to the data and then use it to explain the failure patterns through the functions and properties described above. To illustrate this, we consider the American football league data, given in Table 2.5.

We now apply the Type 2 copula model given in Table 4.1 to the American football league data. The copula parameter α is estimated using the relation connecting the Kendall's correlation coefficient τ_C and the copula parameter α given in (1.16). For the Type 2 copula model, we have $\tau_C = 1 - \dfrac{1}{\alpha}$. The nonparametric estimate of Kendall's τ_C from the data is 0.68 and hence $\hat{\alpha} = 3.13$.

There are different methods of multivariate goodness-of-fit tests in literature (D'Agostino (1986), Berg (2009) or Genest et al. (2009)). We employ a method which is based on the copula and independent of the marginal distributions.

Let us assume that a random sample $(x_1, y_1), ..., (x_n, y_n)$ is drawn from the distribution function $F(x, y)$. If we denote

$$W_i = \#\{(x_j, y_j) : x_j < x_i, y_j < y_i\}/(n-1), \;\; 1 \leq i \leq n$$

then a nonparametric estimator of $K_C(w)$ is given by,

$$K_n(w) = \sum_{i=1}^{n} \frac{\delta(w - W_i)}{n} \tag{4.50}$$

where $\#$ represents the cardinality of a set, $\delta(t)$ is the distribution function of a point mass at the origin. Genest and Rivest (1993) have shown that the empirical distribution function $K_n(w)$ is a $\sqrt{n}$-consistent estimator of $K_C(w)$. The cumulative distribution function $K_C(w)$ and the empirical distribution function $K_n(w)$ from the data are plotted in Figure 4.1 and it can be seen that both are close to each other. Let $C(u, v)$ be the copula of a random vector (X, Y). The statistic used to test the hypothesis $H_0 : C(u, v) = C_0(u, v)$, where $C_0(u, v)$ is a known copula is

$$D_n = \sqrt{n} \sup_{0 \leq w \leq 1} |K_n(w) - K_C(w)| \tag{4.51}$$

According to Saunders and Laud (1980), D_n is exactly the classical Kolmogorov test statistic. For the level of test 0.05, L(1-0.05)=1.358, where L(r) is the limit distribution function of the classical Kolmogorov test statistic. Since D_n value of the copula discussed is $0.57 < 1.358$, we cannot reject H_0. Thus the bivariate data set (X, Y) can be fitted by the copula $\hat{C}_2(u, v)$ with $\alpha = 3.13$.

For the type 2 copula model, the expression for the bivariate hazard rate is given in Table 4.2 and the plot of $G_1(u, v)$ is given in Figure 4.2. From the plot one may observe that $G_1(u,v)$ is decreasing in u for all v for $\alpha = 3.13$. A similar plot will be obtained for $G_2(u,v)$. Thus the type 2 copula model is $DHR^*(u,v)$ for $\alpha = 3.13$.

The second method for modelling the data is to directly infer the reliability functions from the data by proposing estimators with desirable properties.

4.6 Conclusion

In the present work, we have proposed some basic definitions and results that facilitate the modelling of lifetime data through bivariate copulas, instead of the traditional approach using bivariate distributions. We presented the vector-valued hazard rate function of the survival copula $\hat{C}(u,v)$ of (X,Y) and also the bivariate mean residual quantile function, both derived from $\hat{C}(u,v)$ as alternatives to (4.2) and (4.21) in modelling survival data. The advantages of the proposed functions over the measures (4.2) and (4.21) are (i) the survival copula and hence the associated bivariate distributions can be determined from the functional forms of one of the components of the copula-based measures, whereas both components of (4.2) and (4.21) are needed to determine $\bar{F}(x,y)$ uniquely and (ii) the copula-based functions characterize the survival copula which represents a class of distributions. On the other hand $(h_1(x,y), h_2(x,y))$ or $(m_1(x,y), m_2(x,y))$ can determine only a particular distribution specified by $\bar{F}(x,y)$.

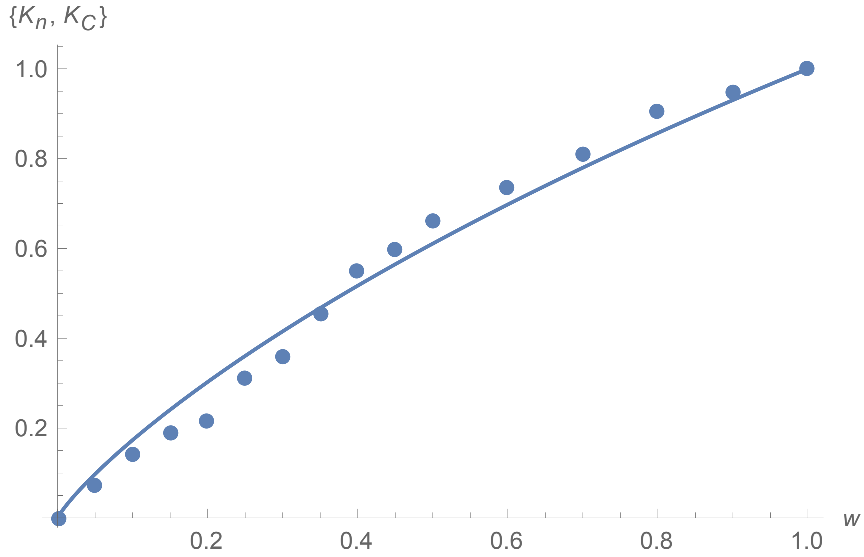

FIGURE 4.1: Graph of K_n and K_C

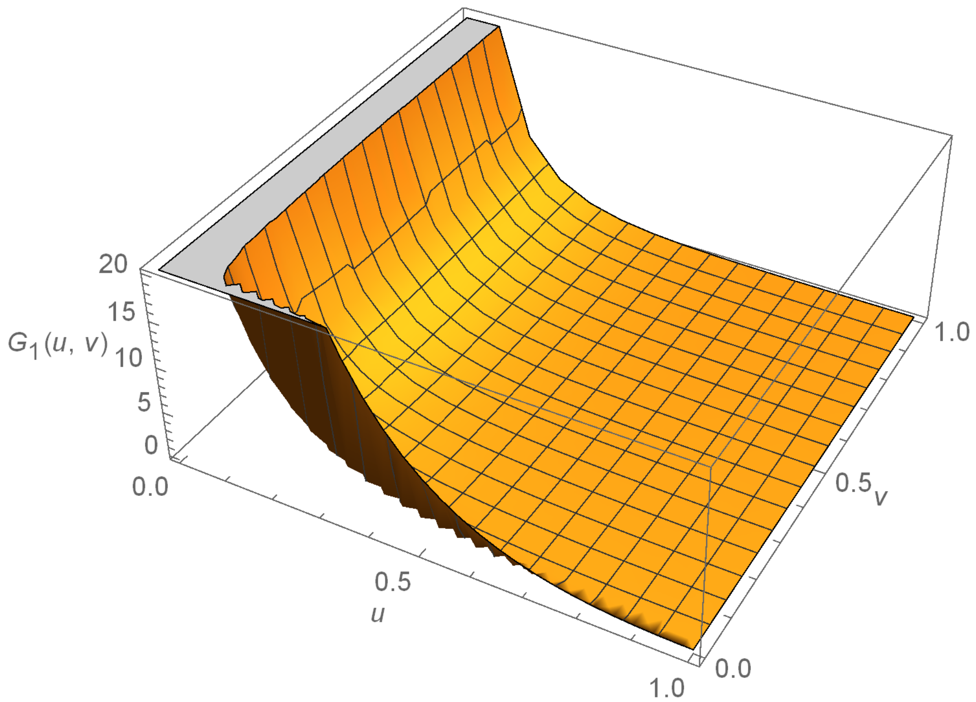

FIGURE 4.2: Plot of $G_1(u, v)$

TABLE 4.1: Survival copulas of bivariate exponential family

Type	$\phi(t)$	Survival copula
1	$-\lambda \log t$	$\hat{C}_1(u,v) = uv$
2	$\lambda^{\alpha}(-\log t)^{\alpha}$	$\hat{C}_2\,(u,v) = \exp\left[-\{(-\log u)^{\alpha} + (-\log v)^{\alpha}\}^{\frac{1}{\alpha}}\right], \alpha \geq 1$
3	$\alpha \log(1-\beta \log t)$	$\hat{C}_3(u,v) = uv \exp\left[-\beta \log u \log v\right],\ \ 0 \leq \beta \leq 1$
4	$k \log(1+(-\log t)^{c})$	$\hat{C}_4(u,v) = \exp\left[-\{(-(\log u)^{c} + (-\log v)^{c} + (-\log u)^{c}(-\log v)^{c}\}^{\frac{1}{c}}\right], c > 1$
5	$\log(\frac{1+t^{-\frac{1}{\sigma}}}{2})$	$\hat{C}_5(u,v) = \max\left[\frac{1}{2}\left(u^{\frac{-1}{\sigma}} + v^{\frac{-1}{\sigma}} + (uv)^{\frac{-1}{\sigma}} - 1\right), 0\right]^{-\sigma}, \sigma > 0$
6	$B\theta^{-1}(t^{-\theta}-1)$	$\hat{C}_6(u,v) = \left\{\max[u^{-\theta} + v^{-\theta} - 1, 0]\right\}^{-\frac{1}{\theta}},\ \theta = \log C \neq 0, -1 < \theta < \infty$
7	$\sigma \log(\frac{2}{t} - 1)$	$\hat{C}_7\,(u,v) = \frac{uv}{1+(1-u)(1-v)}$
8	$\log(\frac{t^{-\lambda}-q}{p})$	$\hat{C}_8(u,v) = \left[q + \frac{(u^{-\lambda}-q)(v^{-\lambda}-q)}{p}\right]^{\frac{1}{\lambda}}, p+q = 1; \lambda > 0$

TABLE 4.2: Bivariate hazard rates of the exponential family

Type	$(G_1(u,v), G_2(u,v))$
1	$(1,1)$
2	$[(-\log\, u)^{\alpha} + (-\log v)^{\alpha}]^{\frac{1}{\alpha}}\,((-\log\, u)^{\alpha-1}, (-\log\, v)^{\alpha-1})$
3	$(1-\beta\log v, 1-\beta\log u)$
4	$[(-\log u)^c + (-\log\, v)^c + (-\log u)^c(1-\log\, v)^c]\,((-\log u)^{c-1}(1+(-\log\, v)^c), (-\log v)^{c-1}(1+(-\log\, u)^c))$
5	$\left(u^{\frac{-1}{\sigma}} + v^{\frac{-1}{\sigma}} + (u\,v)^{\frac{-1}{\sigma}} - 1\right)^{-1}\,(u^{\frac{-1}{\sigma}}(1+v)^{\frac{-1}{\sigma}}, v^{\frac{-1}{\sigma}}(1+u)^{\frac{-1}{\sigma}})$
6	$\left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}\,(u^{-\theta}, v^{-\theta})$
7	$(1+(1-u)(1-v))^{-1}\,(2+2u\,v-u, 2+2\,u\,v-v)$
8	$[p\,q + (u^{-\lambda}-q)(v^{-\lambda}-q)]^{-1}((v^{-\lambda}-q)u^{-\lambda}, (u^{-\lambda}-q)v^{-\lambda})$

Chapter 5

Modelling and analysis of negative dependent Archimedean copulas

5.1 Introduction

The modelling and analysis of statistical data using copula has been extensively studied in literature. One could refer to Joe (1997), Nelsen (2006), Salvadori et al. (2007), Schweizer and Sklar (2011) and McNeil et al. (2015) . A question that usually arises in the study is the choice of the functional form of the copula. The selection of the copula for a given data set depends on the range of dependence among the variables. Copulas like Clayton copula and Frank copula incorporate strong positive dependence, independence and strong negative dependence. However Gumbel copula can only incorporate positive association and independence . In many practical situations, we may come across with large number of data sets with negative dependence as in the case of gross domestic product and infant mortality rate of developed countries. The analysis of such data sets is done using negative dependent copulas. We now discuss two one-parameter families of Archimedean copulas among the twenty-two families given in Nelsen (2006) which are suitable for

[1]The results in this chapter have been communicated as entitled "Negative dependent Archimedean copulas"(see Preethi and Sankaran (2017 a))

modelling negative dependent data sets. The distributional as well as dependence properties of the chosen copulas are not studied in detail in literature. Motivated by this fact, the properties of the corresponding copulas are studied in detail and we have fitted the copulas to a real data set.

The rest of the chapter is organized as follows. In Section 5.2, we discuss a class of Archimedean copulas and study their distributional properties. Section 5.3 deals with the dependence structure of the distributions using copula theory. In Section 5.4, different bivariate distributions useful in reliability analysis are introduced. In Section 5.5, we discuss the inference procedure of the models and then apply the copula models to a real data set. Finally, Section 5.6 summarizes major conclusions of the study.

5.2 The copula models

Consider two survival copula models given by

$$\hat{C}_1(u,v) = \frac{uv}{\left(1+(1-u^{\frac{1}{\beta}})(1-v^{\frac{1}{\beta}})\right)^{\beta}} \;;\; 0 \le u,v \le 1, \beta \ge 1 \tag{5.1}$$

and

$$\hat{C}_2(u,v) = \exp\left(1 - \left((1-\log(u))^{\theta} + (1-\log(v))^{\theta} - 1\right)^{1/\theta}\right); \theta > 0. \tag{5.2}$$

Note that (5.1) and (5.2) are one-parameter families of Archimedean copulas which can be used for modelling negative dependent data.

The generator of the Archimedean copulas (5.1) and (5.2) are respectively,

$$\phi_1(t) = \log(2t^{\frac{-1}{\beta}} - 1), \tag{5.3}$$

and

$$\phi_2(t) = (1 - \log(t))^{\theta} - 1. \tag{5.4}$$

The conditional copula densities of (5.1) and (5.2) are obtained as

$$\frac{\partial \hat{C}_1(u,v)}{\partial u} = -v(-2 + v^{\frac{1}{\beta}})(2 - v^{\frac{1}{\beta}} + u^{\frac{1}{\beta}}(-1 + v^{\frac{1}{\beta}}))^{-1-\beta}; \;\; \beta \geq 1; \;\; 0 < v < 1,$$

$$\frac{\partial \hat{C}_1(u,v)}{\partial v} = -u(-2 + u^{\frac{1}{\beta}})(2 - v^{\frac{1}{\beta}} + u^{\frac{1}{\beta}}(-1 + v^{\frac{1}{\beta}}))^{-1-\beta}; \;\; \beta \geq 1; \;\; 0 < u < 1,$$

$$\frac{\partial \hat{C}_2(u,v)}{\partial u} = \frac{1}{u}\{(1 - \log(u))^{\theta-1}\left((1 - \log(u))^{\theta} + (1 - \log(v))^{\theta} - 1\right)^{\frac{1}{\theta}-1} \exp\left(1 - \left((1 - \log(u))^{\theta} + (1 - \log(v))^{\theta} - 1\right)^{1/\theta}\right)\}; \;\; \theta > 0; \;\; 0 < v < 1,$$

and

$$\frac{\partial \hat{C}_2(u,v)}{\partial v} = \frac{1}{v}\{(1 - \log(v))^{\theta-1}\left((1 - \log(u))^{\theta} + (1 - \log(v))^{\theta} - 1\right)^{\frac{1}{\theta}-1} \exp\left(1 - \left((1 - \log(u))^{\theta} + (1 - \log(v))^{\theta} - 1\right)^{1/\theta}\right)\} \; \theta > 0, 0 < u < 1.$$

We have already discussed in Remark 1.2 that the distribution function of the random variable $W^* = \hat{C}(U,V)$ can also be derived using the generator given in (5.3) and (5.4).

The distribution function of W^* for the two copula models are given by,

$$K_{C_1}(w) = w\left(1 + \frac{\beta(2 - w^{\frac{1}{\beta}})\log(2w^{\frac{-1}{\beta}} - 1)}{2}\right); \quad \beta \geq 1, 0 < w < 1 \tag{5.5}$$

and

$$K_{C_2}(w) = w\left(1 + \frac{w\left((1 - \log(w))^{\theta} - 1\right)(1 - \log(w))^{1-\theta}}{\theta}\right); \theta > 0, 0 < w < 1. \tag{5.6}$$

Conway (1979) suggested a method for graphical representation of copula using a contour diagram, which is the plot of the sets in $\mathbf{I}^2$ given by $\hat{C}(u, v) = k^*$, where k^* is a constant(see Figures 5.1 and 5.2).

Remark 5.1. The diagonal section of copula is the function δ_C from $\mathbf{I}$ to $\mathbf{I}$ defined by $\delta_C(t) = C(t, t)$, which is non-decreasing and uniformly continuous on $\mathbf{I}$.

The diagonal section of the copulas (5.1) and (5.2) are given by,

$$\delta_{C_1}(t) = t^2\left(1 + \left(1 - t^{\frac{1}{\beta}}\right)^2\right)^{-\beta}, 0 \leq t \leq 1, \beta \geq 1 \tag{5.7}$$

and

$$\delta_{C_2}(t) = e^{1-\left(2(1-\log(t))^{\theta}-1\right)^{1/\theta}}, 0 \leq t \leq 1, \theta \geq 1. \tag{5.8}$$

The plot of the diagonal section of the copulas are given in Figure 5.3 and Figure 5.4 respectively. It can be noted that the diagonal section of the copulas are non-decreasing and uniformly continuous on $\mathbf{I}$.

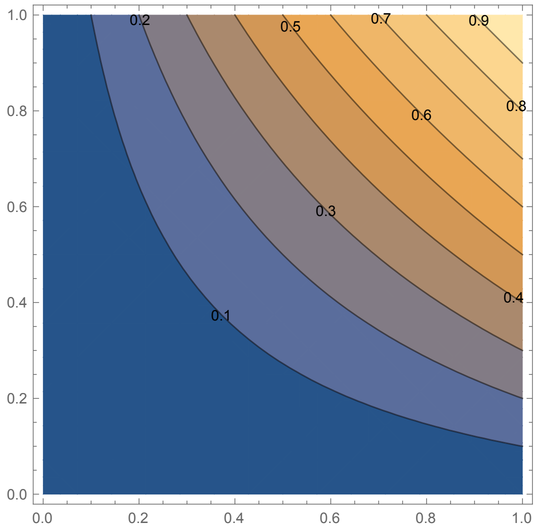

FIGURE 5.1: Contour diagram of $\hat{C}_1(u, v)$

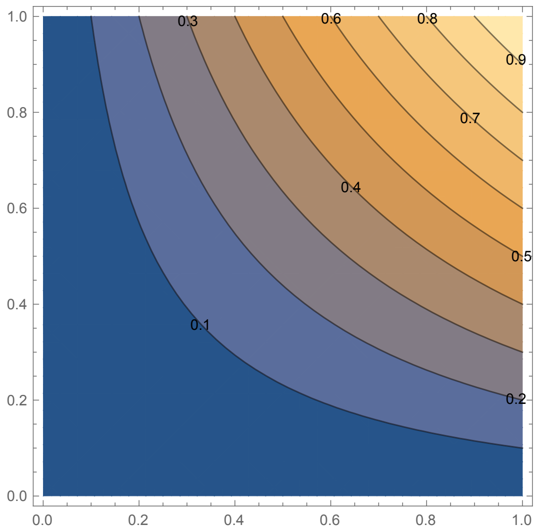

FIGURE 5.2: Contour diagram of $\hat{C}_2(u, v)$

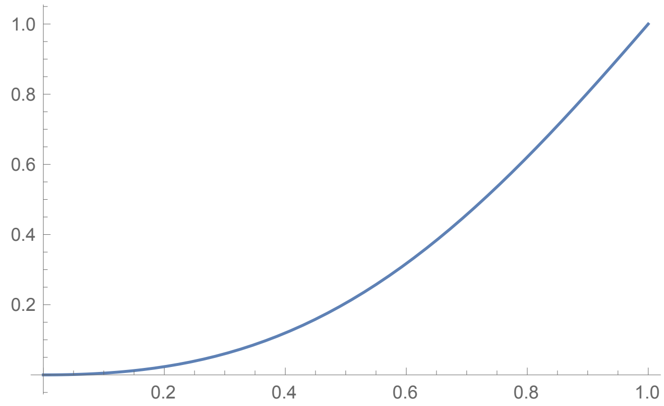

FIGURE 5.3: Plot of diagonal section of $\hat{C}_1(u, v)$

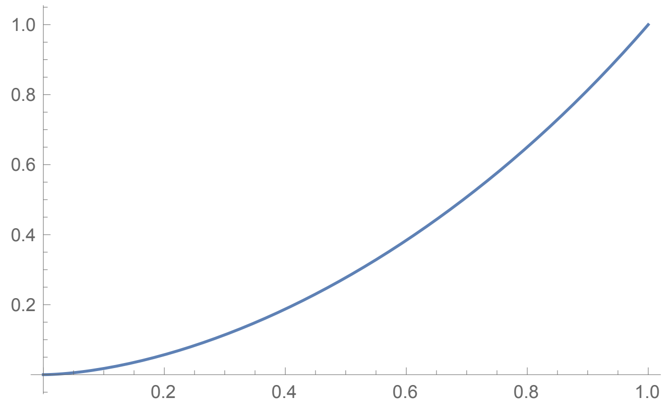

FIGURE 5.4: Plot of diagonal section of $\hat{C}_2(u, v)$

5.3 Dependence

The various global, tail and local dependence measures of the copula models are discussed below:

5.3.1 Spearman's rho and Kendall's tau.

The correlation between two random variables U and V can be assessed using Spearman's rho and Kendall's tau (see Nelsen (2006) and Sklar (1959)).

Theorem 5.1. *The Spearman's rho for the copula in* (5.1) *is given by*

$$\rho_{C_1} = 3\left[{}_3F_2(1,1,\beta;1+2\beta,1+2\beta;-1)-1\right] \tag{5.9}$$

where

$${}_pF_q(a_1,...,a_p;b_1,...,b_q;x) = \sum_{r=0}^{\infty}\frac{(a_1)_r...(a_p)_r x^r}{(b_1)_r...(b_q)_r r!};$$

$(a)_r = a(a+1)...(a+r-1), (a)_0 = 1$ *and* $a_1,...,a_p,b_1,...,b_q$ *are parameters.*

Proof. From the definition of Spearman's rho, we have

$$\begin{aligned}
\rho_{C_1} &= 12\int_0^1\int_0^1 \hat{C}(u,v)dudv - 3\\
&= 12\int_0^1\int_0^1 \frac{uv}{\left(1+(1-u^{\frac{1}{\beta}})(1-v^{\frac{1}{\beta}})\right)^{\beta}}dudv - 3.
\end{aligned}$$

By substituting $1-u^{\frac{1}{\beta}} = s, 1-v^{\frac{1}{\beta}} = t$, we get

$$\begin{aligned}
\rho_{C_1} &= 12\beta^2\int_0^1\int_0^1 \frac{(1-s)^{2\beta-1}(1-t)^{2\beta-1}}{(1+st)^{\beta}}dsdt - 3\\
&= 12\beta^2\int_0^1 \frac{(1-t)^{2\beta-1}}{2\beta}{}_2F_1(1,\beta;1+2\beta;-t)dt - 3\\
&= 3\left[{}_3F_2(1,1,\beta;1+2\beta,1+2\beta;-1)-1\right].
\end{aligned}$$

□

It can be seen from Figure 5.5 that ρ_{C_1} increases in β for $\beta \geq 1$ and tends to zero as $\beta \to \infty$. Since the Spearman's ρ_{C_1} is negative, the pair (U,V) is NQD for all values of $\beta \geq 1$. It may be noted that $\rho_{C_1} \in [-0.271, 0)$.

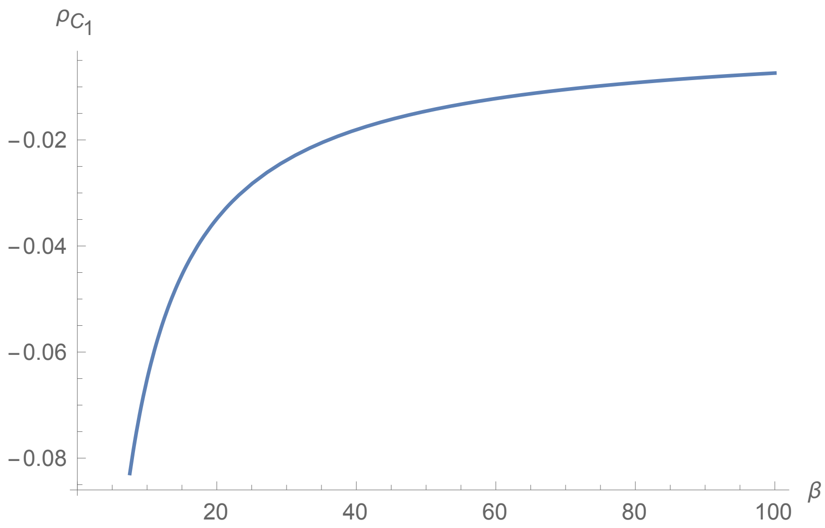

FIGURE 5.5: Plot of Spearman's ρ_{C_1}

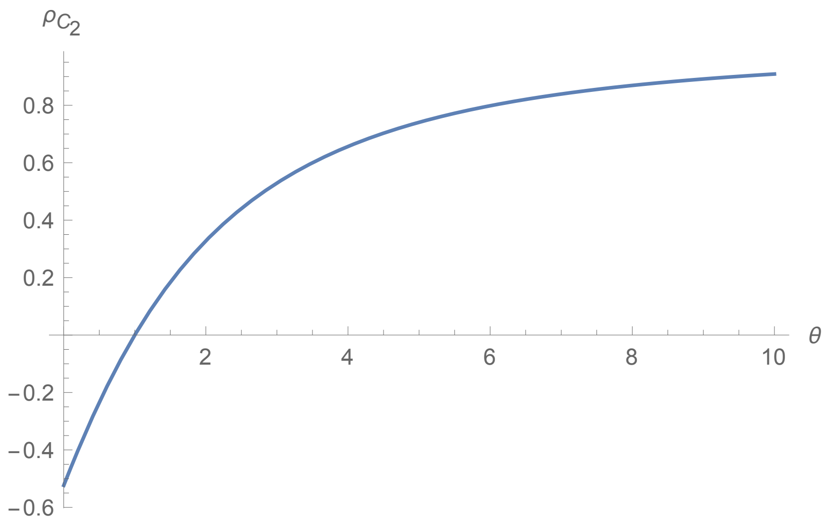

FIGURE 5.6: Plot of Spearman's ρ_{C_2}

For the copula model (5.2), we have no closed form analytical expression for Spearman's rho. The plot of ρ_{C_2} is given in Figure 5.6. One may note that the pair (U, V) is NQD for all values of $0 < \theta < 1$ as ρ_{C_2} is negative for all values of $0 < \theta < 1$ and the pair (U, V) is PQD for all values of $\theta \geq 1$.

Let U and V be random variables with $W^* = \hat{C}(U, V)$. Now $K_C(w)$ is related to the population value of Kendall's tau via

$$\tau_C = 4E(W^*) - 1 \tag{5.10}$$

and

$$E(W^*) = \int_0^1 \{1 - K_C(w)\}dw$$

paves the way for the estimation and goodness-of-fit procedures for different classes of copulas using an empirical version of $K_C(w)$.

Theorem 5.2. *The Kendall's tau for the copula* (5.1) *is given by,*

$$\tau_{C_1} = \frac{-2\beta}{(2\beta+1)^2} {}_2F_1(1,1;2\beta+2;-1). \tag{5.11}$$

Proof. We have

$$E(W^*) = \frac{1}{4} - \frac{\beta}{2(2\beta+1)^2} {}_2F_1(1,1;2\beta+2;-1).$$

Therefore from (5.10), Kendall's measure of association between U and V is given by,

$$\tau_{C_1} = \frac{-2\beta}{(2\beta+1)^2} {}_2F_1(1,1;2\beta+2;-1); \quad \beta \geq 1.$$

□

From Figure 5.7, it follows that τ_{C_1} increases in β for $\beta \geq 1$. Since the Kendall's τ_{C_1} is negative, the pair (U, V) is NQD for all values of $\beta \geq 1$. It may be noted that $\tau_{C_1} \in [-0.182, 0)$.

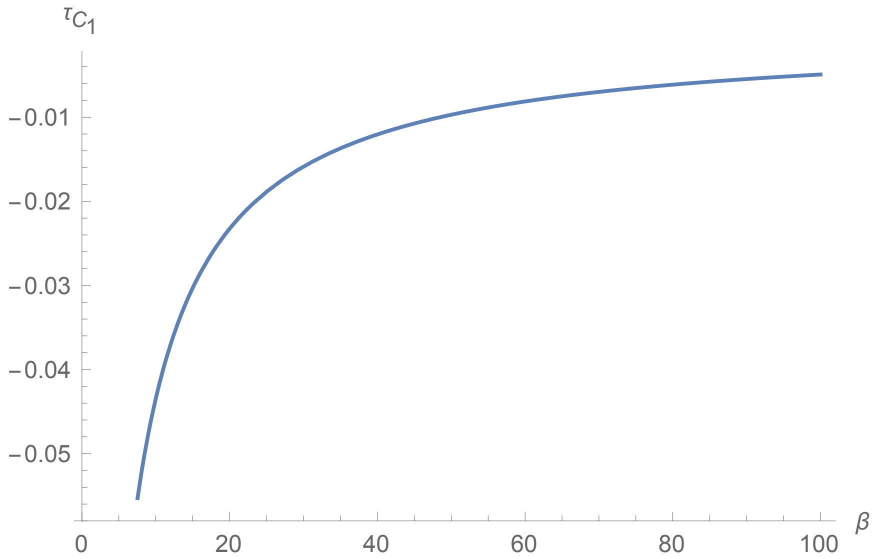

FIGURE 5.7: Plot of Kendall's τ_{C_1}

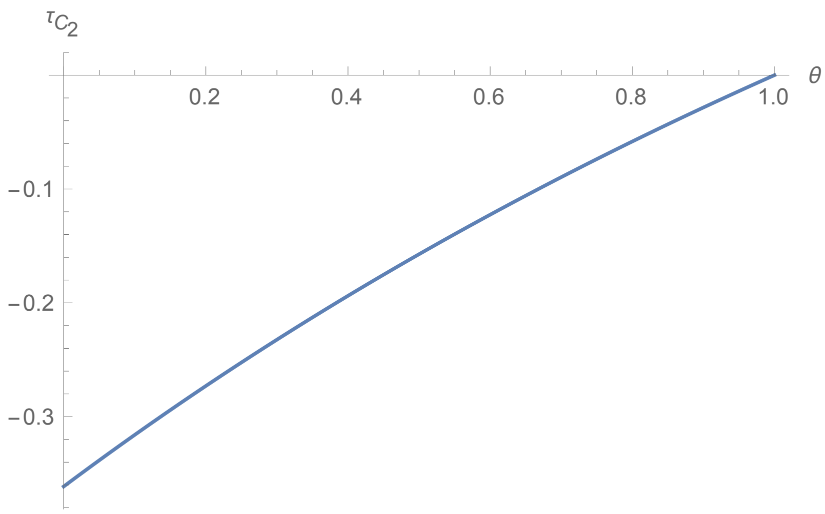

FIGURE 5.8: Plot of Kendall's τ_{C_2}

Theorem 5.3. *The Kendall's tau for the copula (5.2) is*

$$\tau_{C_2} = -\frac{(\theta-1)\left(2e^2 E_\theta(2)-1\right)}{\theta} \tag{5.12}$$

where $E_n(z)$ is the exponential integral function discussed in Abramowitz and Stegun (1966).

From Figure 5.8, we may notice that the Kendall's tau, τ_{C_2} is negative for all values of $0 < \theta < 1$ and hence the pair (U, V) is NQD for all values of $0 < \theta < 1$.

5.3.2 Measure based on Blomqvist's β.

Blomqvist (1950) proposed and studied a measure using population medians, the measure often called as medial correlation coefficient, denoted by β_C , given by,

$$\beta_C = 4\hat{C}(1/2, 1/2) - 1.$$

For the copulas (5.1) and (5.2), we have

$$\beta_{C_1} = 4(1 - 2e^{\frac{0.6931}{\beta}} + 2e^{\frac{1.38629}{\beta}})^{-\beta};\ \ \beta \geq 1$$

and

$$\beta_{C_2} = 4e^{1-\left(2(1+\log(2))^{\theta}-1\right)^{1/\theta}} - 1;\ \ \theta > 0.$$

Although Blomqvist's β depends on the copula only through its value at the center of $\mathbf{I}^2$, it can nevertheless often provide an accurate approximation to Spearman's rho and Kendall's tau.

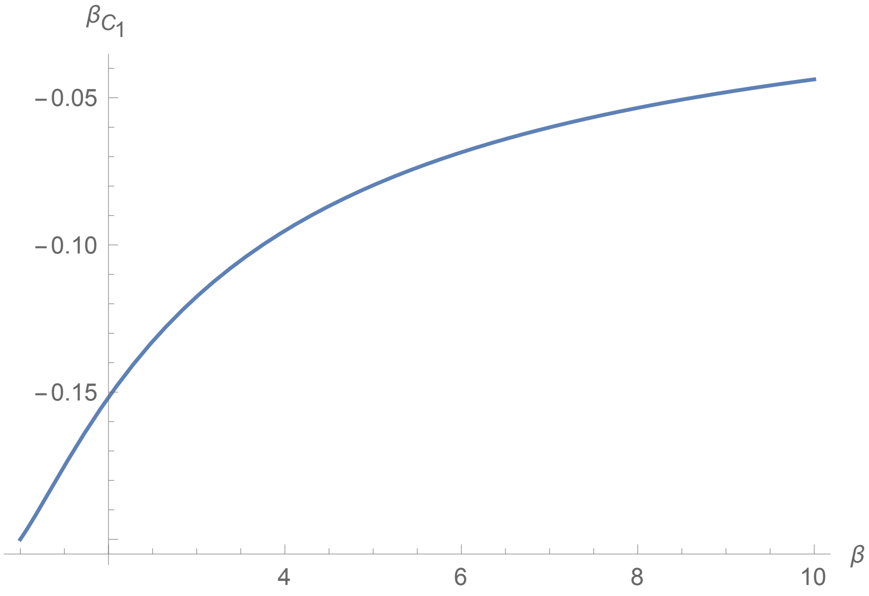

FIGURE 5.9: Plot of Blomqvist's β_{C_1}

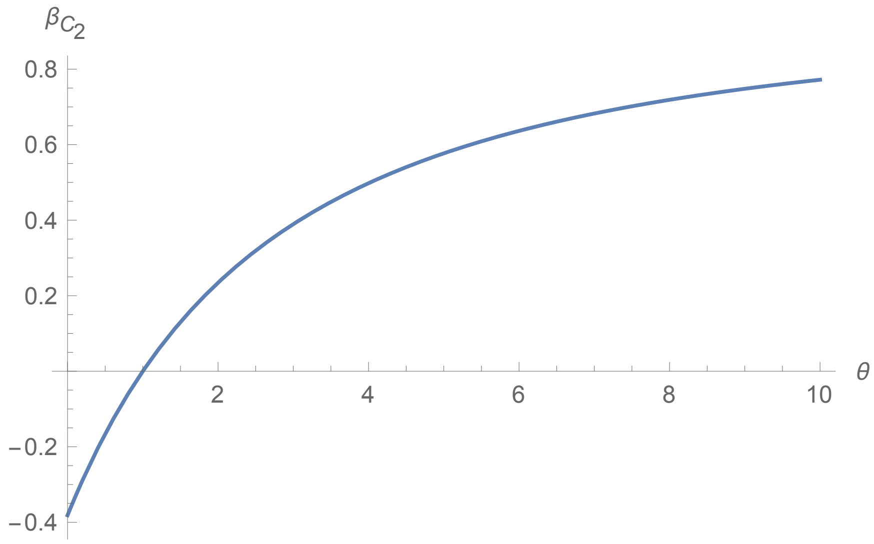

FIGURE 5.10: Plot of Blomqvist's β_{C_2}

We observe that for the copula model (5.1), β_{C_1} is negative for all values of $\beta \geq 1 \Rightarrow (U, V)$ is NQD. For the copula model (5.2), β_{C_2} is negative for $0 < \theta < 1$ and positive for $\theta \geq 1$, therefore the pair (U, V) is NQD for $0 < \theta < 1$ and PQD for $\theta \geq 1$.

5.3.3 Tail dependence properties

The notion of tail dependence is interesting in the analysis of bivariate data as it describes the limiting proportion that one margin exceeds a certain threshold given that other margin has already exceeded that threshold.

From Proposition 1.19, we have,

$$\lambda_L = \lim_{u \to 0} \frac{C(u,u)}{u} = 0$$

and

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1 - u} = 0$$

for both the copula models (5.1) and (5.2). Therefore the copulas has neither lower nor upper tail dependence ($\lambda_L = \lambda_U = 0$) and hence suitable for modelling data characterized by weak tail dependence.

5.3.4 Local dependence measures

We discuss two local dependence measures, such as ψ-measure and the Clayton-Oakes association measure (θ- measure) for the copulas which measures the dependence structure at specific values of u and v.

5.3.4.1 ψ -measure

The association measure $\psi(x, y)$ is defined by Anderson et al. (1992) as

$$\psi(x, y) = \frac{P(X > x|Y > y)}{P(X > x)} = \frac{\bar{F}(x, y)}{\bar{F}_1(x)\bar{F}_2(y)}.$$

When X and Y are independent $\psi(x, y) = 1$ and large (small) values of $\psi(x, y)$ indicates positive (negative) association. In terms of the copula we have,

$$\zeta(u, v) = \psi(F_1^{-1}(u), F_2^{-1}(v)) = \frac{\hat{C}(u, v)}{uv}.$$

Consequently $\zeta(u, v) = 1$ indicates independence and increasing (decreasing) values of $\zeta(u, v)$ for u, v imply positive (negative) dependence. For the copula in (5.1),

$$\zeta_1(u, v) = \Big(1 + (1 - u^{\frac{1}{\beta}})(1 - v^{\frac{1}{\beta}})\Big)^{-\beta}; \beta \geq 1.$$

Note that $\zeta_1(u, v) < 1$ for all finite values of $\beta \geq 1$ and hence (U, V) is NQD. Similarly for the copula in (5.2), we have

$$\zeta_2(u, v) = \frac{2 \exp\Big(1 - \big((1 + \log(2))^\theta + (1 - \log(u))^\theta - 1\big)^{1/\theta}\Big)}{u}; \quad \theta \geq 0$$

and we may note that for $0 < \theta < 1$, $\zeta_2(u, v) < 1$ which implies negative association; for $\theta = 1$, we have $\zeta_2(u, v) = 1$ which implies independence and for $\theta > 1$, $\zeta_2(u, v) > 1$ which implies positive association.

5.3.4.2 θ -measure

Clayton (1978) and Oakes (1989) defined the associated measure $\theta(x, y)$ as given in (2.25). In terms of the copula we have,

$$\xi(u, v) = \theta(F_1^{-1}(u), F_2^{-1}(v)) = \frac{\hat{C}(u, v)\frac{\partial^2 \hat{C}(u,v)}{\partial u \partial v}}{\frac{\partial \hat{C}(u,v)}{\partial u}\frac{\partial \hat{C}(u,v)}{\partial v}}.$$

We can see that U and V are positive (negative) if $\xi(u, v) > (<)1$ and independent if $\xi(u, v) = 1$.

For the copula (5.1),

$$\xi_1(u, v) = \frac{-u^{\frac{1}{\beta}} v^{\frac{1}{\beta}} + \beta(-2 + u^{\frac{1}{\beta}})(-2 + v^{\frac{1}{\beta}})}{\beta(-2 + u^{\frac{1}{\beta}})(-2 + v^{\frac{1}{\beta}})} = 1 - \frac{u^{\frac{1}{\beta}} v^{\frac{1}{\beta}}}{\beta(-2 + u^{\frac{1}{\beta}})(-2 + v^{\frac{1}{\beta}})} \tag{5.13}$$

where $\xi(u, v) < 1$ for all finite $\beta \geq 1$. Hence U and V are negatively associated.

For the copula (5.2),

$$\xi_2(u, v) = (\theta - 1)\left((1 - \log(u))^{\theta} + (1 - \log(v))^{\theta} - 1\right)^{-1/\theta} + 1$$

we have negative association for $0 < \theta < 1$ as in this case $\xi_2(u, v) < 1$; for $\theta = 1$, we have $\xi_2(u, v) = 1$ which implies independence and positive association for $\theta > 1$ as $\xi_2(u, v) > 1$ in this range.

Remark 5.2. For the copula (5.1), we have $\xi(u, v) < 1$ and $\frac{\partial^2}{\partial u \partial v}\zeta(u, v) < 0$, therefore (U, V) is hazard negative dependent (HND) for $\beta \geq 1$ (see Asadian et al. (2009)).

5.4 Distributions with various marginals

Copula techniques can be used for the construction of various bivariate distributions by assigning different marginals in $\bar{F}(x, y) = C(\bar{F}_1(x), \bar{F}_2(y))$.

5.4.1 Distributions with Pareto marginals

Suppose that the marginal distributions of X and Y are Pareto I distributions with $\bar{F}_1(x) = x^{-a} = u$ and $\bar{F}_2(y) = y^{-b} = v;\ x > 1, y > 1, a, b > 0$. By substituting for u and v in (5.1) and (5.2) we have the survival functions with Pareto marginals.

$$\bar{F}_\beta(x, y) = x^{-a} y^{-b} \left(\left(\left(x^{-a}\right)^{1/\beta} - 1 \right) \left(\left(y^{-b}\right)^{1/\beta} - 1 \right) + 1 \right)^{-\beta};\ \ \beta \geq 1 \tag{5.14}$$

and

$$\bar{F}_\theta(x, y) = \exp\left(1 - \left(\left(1 - \log\left(x^{-a}\right)\right)^\theta + \left(1 - \log\left(y^{-b}\right)\right)^\theta - 1 \right)^{1/\theta} \right);\ \theta > 0. \tag{5.15}$$

5.4.2 Distributions with Weibull marginals

Assume that the marginal distributions of X and Y are Weibull distributions with $\bar{F}_1(x) = e^{-\left(\frac{x}{\eta_1}\right)^{b_1}}$ and $\bar{F}_2(y) = e^{-\left(\frac{y}{\eta_2}\right)^{b_2}}$; $x, y > 0, b > 0, \eta_i > 0$ for $i = 1, 2$. Then we have the survival functions with Weibull marginals as

$$\bar{F}_\beta(x, y) = e^{-\left(\frac{x}{\eta_1}\right)^{b_1} - \left(\frac{y}{\eta_2}\right)^{b_2}} \left(\left(\left(e^{-\left(\frac{x}{\eta_1}\right)^{b_1}} \right)^{1/\beta} - 1 \right) \left(\left(e^{-\left(\frac{y}{\eta_2}\right)^{b_2}} \right)^{1/\beta} - 1 \right) + 1 \right)^{-\beta};\beta \geq 1 \tag{5.16}$$

and

$$\bar{F}_\theta(x,y)=\exp\left(1-\left(\left(1-\log\left(e^{-\left(\frac{x}{\eta_1}\right)^{b_1}}\right)\right)^{\theta}+\left(1-\log\left(e^{-\left(\frac{y}{\eta_2}\right)^{b_2}}\right)\right)^{\theta}-1\right)^{1/\theta}\right);\ \theta>0. \tag{5.17}$$

5.4.3 Distributions with exponential marginals

Let X and Y be exponentially distributed with $\bar{F}_1(x)=e^{-\lambda x}$ and $\bar{F}_2(y)=e^{-\lambda y}$; $x,y>0,\lambda>0$. If we substitute $u=e^{-\lambda x}$ and $v=e^{-\lambda y}$ in (5.1) and (5.2), we obtain survival functions with exponential marginals as

$$\bar{F}_\beta(x,y)=e^{-\lambda(x+y)}\left(\left(\left(e^{-\lambda x}\right)^{1/\beta}-1\right)\left(\left(e^{-\lambda y}\right)^{1/\beta}-1\right)+1\right)^{-\beta};\beta\geq 1 \tag{5.18}$$

and

$$\bar{F}_\theta(x,y)=\exp\left(1-\left(\left(1-\log\left(e^{-\lambda x}\right)\right)^{\theta}+\left(1-\log\left(e^{-\lambda y}\right)\right)^{\theta}-1\right)^{1/\theta}\right);\ \theta>0. \tag{5.19}$$

5.5 Applications

To illustrate the applicability of the copula models, we consider the Iris flower data set which consists of 50 samples from each of three species of Iris . Four features were measured from each sample: the length and the width of the sepals and petals, in centimetres. The data are available on http://www.math.uah.edu/stat/data/Fisher.html. We have considered the variables sepal length and sepal width of the species.

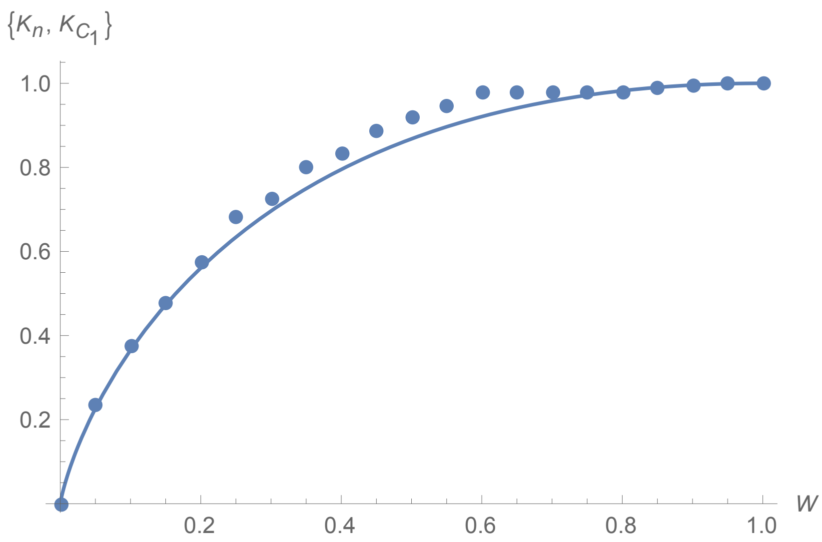

FIGURE 5.11: Plots of K_n and K_{C_1}

The marginals of the data set are fitted by *Normal* [3.057, 0.43] and *Normal*[5.84, 0.82] with Kolmogorov-Smirnov test statistic values 0.106 and 0.089 respectively. The copula parameter β of (5.1) and θ of (5.2) are estimated from Kendall's tau using Theorem 5.2 and Theorem 5.3. The estimated value of Kendall's tau is -0.077 and therefore $\beta = 4.9$ and $\theta = 0.75$.

Now we discuss the goodness-of-fit of our copula models to the given data. The cumulative distribution function of the copulas and the empirical distribution function $K_n(w)$ from the data are plotted in Figure 5.11 and Figure 5.12 respectively. It can be seen that in both cases the cumulative distribution function and the empirical distribution function are almost identical.

Let $C(u, v)$ be the copula of a random vector (X, Y). To test the hypothesis $H_0 : C(u, v) = C_0(u, v)$, where $C_0(u, v)$ is a known copula, we use the statistic given in (4.51). Under H_0, for the level of test 0.05, $L(1 - 0.05) = 1.358$, where $L(r)$ is the limit distribution function of the classical Kolmogorov test statistic. We have fitted both the copula models given in (5.1) and (5.2) for the given data. Since D_n

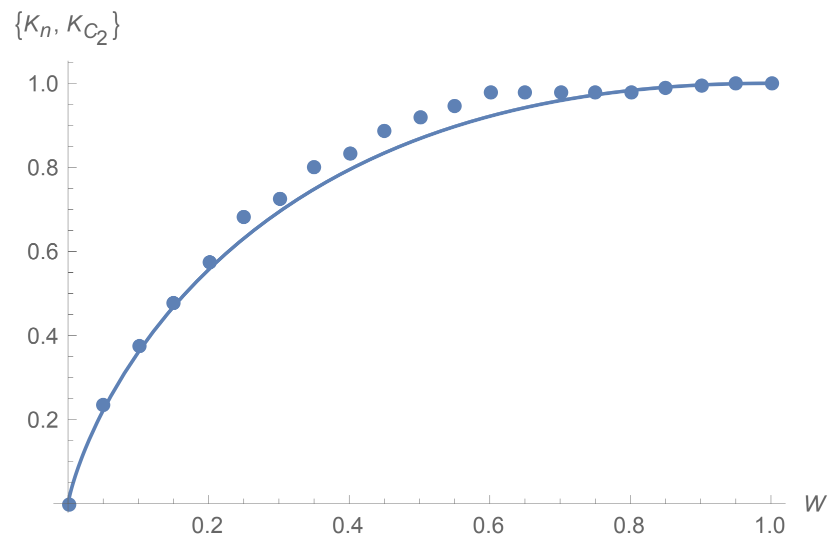

FIGURE 5.12: Plots of K_n and K_{C_2}

in both cases is less than 1.358, we cannot reject H_0 and the copula of (X, Y) can be approximated by $\hat{C}_1(u, v)$ with $\beta = 4.9$ or by $\hat{C}_2(u, v)$ with $\theta = 0.75$. Now to make a conclusion which model suits more appropriate for our data, we use the AIC criteria given in Akaike (1987). The AIC values of (5.1) and (5.2) are -3.9 and -3.1 which indicates that the copula model (5.1) is more appropriate to our data.

5.6 Conclusion

In the present chapter, we have considered two Archimedean copulas suitable for modelling negative dependent data. The distributional properties as well as the dependence structure of the copulas are studied. The models can be extended directly to multivariate set up. The applicability of the models are illustrated with a data set.

Chapter 6

Modelling and analysis of a positive dependent Archimedean copula

6.1 Introduction

In real life situations we come across a large number of data sets with positive dependence and modelling of these positive dependent data sets using many well known copulas have been discussed in literature. We now discuss a copula model which is a member of one-parameter families of Archimedean copulas given in Nelsen (2006). The motivation of choosing this particular copula among the twenty-two families is that this family of Archimedean copula possesses both upper and lower tail dependence and hence it can be used for modelling data with positive dependency. Moreover, the dependence properties of this particular family are not studied in literature. Our objective is to study such properties of the copula in detail. In addition, we apply this copula model to a real data set.

The rest of the chapter is organized as follows. In Section 6.2, we discuss a class of Archimedean copula and have derived the distribution function using the generator

[1]The results in this chapter have been accepted for publication in Journal of Applied Mathematics and Statistics(see Preethi and Sankaran (2016))

function. Section 6.3 deals with the dependence structure as well as the tail monotonicity of the copula. We have derived the Kendall's measure and also a measure based on Blomqvist's β. In Section 6.4, we introduce different bivariate distributions useful in reliability analysis. Section 6.5 discusses the inference procedure of the copula model. We then apply the proposed model to a real data set. Finally, Section 6.6 summarizes the major conclusions of the study.

6.2 The copula model

As we have already defined in (1.8), a copula $C(u, v)$ is said to be Archimedean if there exists a representation of the form

$$C(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)) \tag{6.1}$$

where ϕ is a continuous, strictly decreasing function from $\mathbf{I}$ to $[0, \infty)$ such that $\phi(1) = 0$ and $\phi^{[-1]}$ is the pseudo-inverse of ϕ. ϕ is called the generator of $C(u, v)$. From (1.10), we have that, for every Archimedean copula with generator ϕ, there exists $F^*(t) = 1 - \phi^{-1}(t)$.

If we choose the generator as

$$\phi_\theta(t) = \left(\frac{1}{t} - 1\right)^\theta, \quad \theta \in [1, \infty) \tag{6.2}$$

then the univariate cumulative distribution function is obtained as

$$F_\theta^*(t) = \frac{1}{1 + t^{-\frac{1}{\theta}}}, t > 0.$$

From (6.2), we obtain the one-parameter Archimedean copula as,

$$C_\theta(u,v) = \left(1 + \left(\left(\frac{1}{u}-1\right)^\theta + \left(\frac{1}{v}-1\right)^\theta\right)^{\frac{1}{\theta}}\right)^{-1}; \quad \theta \in [1,\infty). \tag{6.3}$$

The copula density is

$$c_\theta(u,v) = \begin{cases} \dfrac{\left(\frac{1}{u}-1\right)^\theta\left(\frac{1}{v}-1\right)^\theta\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{\frac{1}{\theta}-2}\left(\theta+(\theta+1)\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{1/\theta}-1\right)}{(u-1)u(v-1)v\left(\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{1/\theta}+1\right)^3} & ;0<u,v<1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4}$$

The conditional copula densities are given by,

$$\begin{aligned} c_{v|u}(u,v) &= \frac{\partial C(u,v)}{\partial u} \\ &= \frac{\left(\frac{1}{u}-1\right)^{\theta-1}\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{\frac{1}{\theta}-1}}{u^2\left(\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{1/\theta}+1\right)^2}; \quad 0<v<1 \end{aligned}$$

and

$$\begin{aligned} c_{u|v}(u,v) &= \frac{\partial C(u,v)}{\partial v} \\ &= \frac{\left(\frac{1}{v}-1\right)^{\theta-1}\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{\frac{1}{\theta}-1}}{v^2\left(\left(\left(\frac{1}{u}-1\right)^\theta+\left(\frac{1}{v}-1\right)^\theta\right)^{1/\theta}+1\right)^2}; \quad 0<u<1. \end{aligned}$$

The survival copula is given by,

$$\hat{C}(u,v) = u+v-1+C_\theta(1-u,1-v) = \frac{1}{\left(\left(\frac{u}{1-u}\right)^\theta+\left(\frac{v}{1-v}\right)^\theta\right)^{1/\theta}+1}+u+v-1; \quad 0<u,v<1.$$

The distribution function of $W^* = C(U,V)$ is given by;

$$\begin{aligned} K_C(w) &= w - \frac{\phi(w)}{\phi'(w)} \\ &= \frac{\left(\frac{1}{w}-1\right)w^2}{\theta} + w \\ &= \frac{w(\theta - w + 1)}{\theta}; \quad 0 < w < 1. \end{aligned}$$

The density function of W^* is;

$$k(w) = \frac{d}{dw}K_C(w) = \frac{\theta - 2w + 1}{\theta}; \quad 0 < w < 1.$$

This function can be employed to check whether the proposed copula model is a good fit for the data by plotting the cumulative distribution function $K_C(w)$ and the empirical distribution function $K_n(w)$ from the data, as seen in Section 6.5.

Let $C_{\theta_1}(u,v)$ and $C_{\theta_2}(u,v)$ be members of family (6.3) with parameters θ_1 and θ_2 and with generators $\phi_{\theta_1}(t)$ and $\phi_{\theta_2}(t)$ respectively. Note that $\dfrac{\phi_{\theta_1}(t)}{\phi_{\theta_2}(t)} = \left(\frac{1}{t}-1\right)^{\theta_1-\theta_2}$. If $\theta_1 \leq \theta_2$, then we have $\dfrac{\phi_{\theta_1}(t)}{\phi_{\theta_2}(t)}$ is non-decreasing in t on (0,1) and hence $C_1 \prec C_2$. Hence this family is positively ordered (see Remark 1.3).

The diagonal section is,

$$\delta(t) = \frac{1}{2^{1/\theta}\left(\left(\frac{1}{t}-1\right)^{\theta}\right)^{1/\theta}+1}, \quad 0 < t < 1.$$

The contour diagram and the plot of the diagonal section of copula are given in Figure 6.1 and in Figure 6.2 respectively.

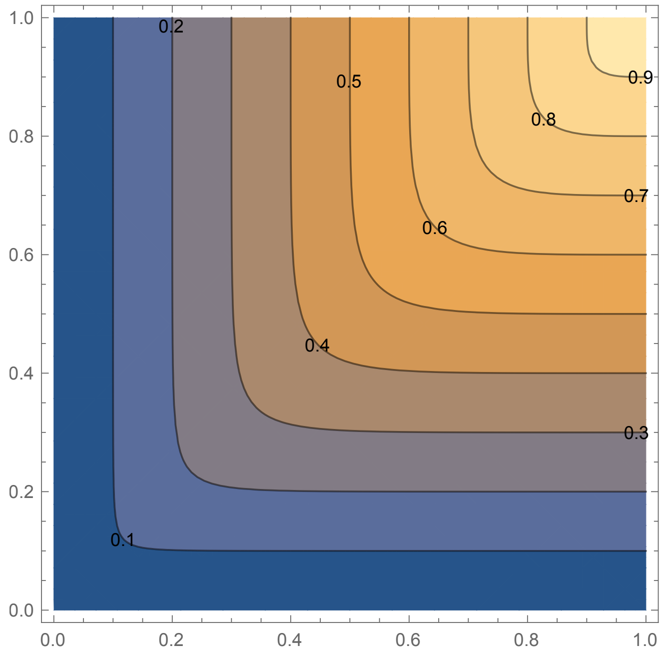

FIGURE 6.1: Contour diagram of $C_\theta(u,v)$

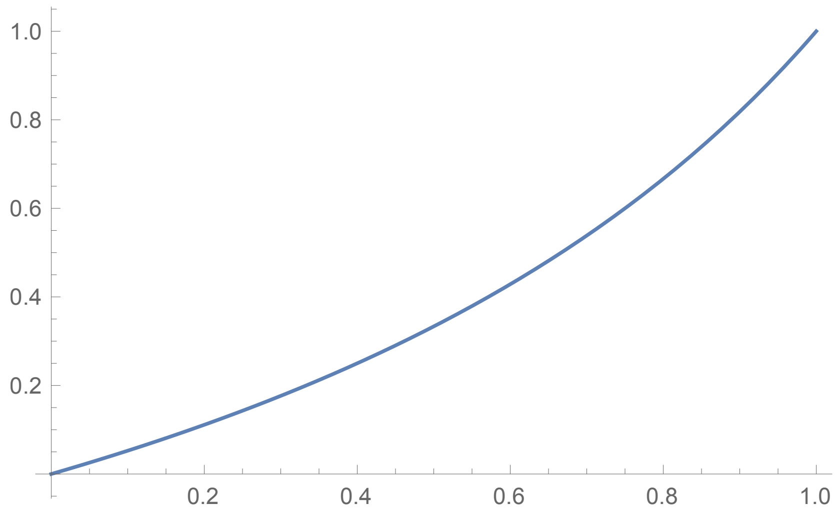

FIGURE 6.2: Plot of diagonal section of $C_\theta(u,v)$

6.3 Dependence concepts

To measure the dependence or association between random variables there are variety of methods available and it is the copula which can capture the distribution-free or scale-invariant nature of the association between random variables. Here we discuss the tail dependence properties, tail monotonicity and Kendall's tau for the copula model (6.3).

6.3.1 Tail dependence properties

For the copula (6.3), the lower and upper tail dependence are respectively obtained as:

$$\lambda_L = \lim_{u\to 0} \frac{C(u,u)}{u} = \lim_{u\to 0} \frac{1}{u\left(2^{1/\theta}\left(\left(\frac{1}{u}-1\right)^{\theta}\right)^{1/\theta}+1\right)} = 2^{-1/\theta}$$

and

$$\lambda_U = \lim_{u\to 1} \frac{1-2u+C(u,u)}{1-u} = \lim_{u\to 1} \frac{\frac{1}{2^{1/\theta}\left(\left(\frac{1}{u}-1\right)^{\theta}\right)^{1/\theta}+1} - 2u + 1}{1-u} = 2 - 2^{1/\theta}.$$

The copula possess both upper and lower tail dependence. The values of λ_L and λ_U are special cases of the Clayton-Gumbel survival (BB1) copula family given in Joe and Hu (1996).

6.3.2 Tail monotonicity

Let X and Y be continuous random variables with the copula given in (6.3). Then we have

$$\frac{C(u,v)}{u} = \frac{1}{u\left(\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+1\right)}, \tag{6.5}$$

$$\frac{v-C(u,v)}{1-u} = \frac{v-\dfrac{1}{\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+1}}{1-u}, \tag{6.6}$$

and

$$\frac{\partial c(v|u)}{\partial u} = \begin{array}{c} \dfrac{(1-u)^{\theta-2}\left(\frac{1}{u}\right)^{\theta+2}\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{\frac{1}{\theta}-2}\left(2\left(\frac{1}{u}-1\right)^{\theta}\left(u\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+u-1\right)\right)}{\left(\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+1\right)^{3}} - \\ \dfrac{(1-u)^{\theta-2}\left(\frac{1}{u}\right)^{\theta+2}(\theta-2u+1)\left(\frac{1}{v}-1\right)^{\theta}\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{\frac{1}{\theta}-2}\left(\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+1\right)}{\left(\left(\left(\frac{1}{u}-1\right)^{\theta}+\left(\frac{1}{v}-1\right)^{\theta}\right)^{1/\theta}+1\right)^{3}} \end{array}. \tag{6.7}$$

We observe that $\dfrac{\partial C(u,v)}{\partial u} \leq \dfrac{C(u,v)}{u}$ for almost all u and hence $LTD(Y|X)$, and similarly we can prove $LTD(X|Y)$. It can also be observed that $\dfrac{\partial C(u,v)}{\partial u} \geq \dfrac{v-C(u,v)}{1-u}$ for almost all u which implies $RTI(Y|X)$ and it can also be shown that $\hat{C}(u,v)$ is $RTI(X|Y)$. Thus $\hat{C}(u,v)$ is RTI and $C(u,v)$ is LTD(refer Nelsen (2006)).

It may be noted that $\dfrac{\partial c(v|u)}{\partial u} < 0$ for all values of $\theta \in [1,\infty)$, therefore $C(u,v)$ in (6.3) is positively regression dependent.

Remark 6.1. As the copula described in (6.3) is left tail decreasing (LTD), the cumulative distribution function $F_{\theta}^{*}(.)$ is DFR (refer Avérous and Dortet-Bernadet (2004)).

6.3.3 Kendall's tau

As already mentioned, Spearman's rho and Kendall's tau are considered to be the measures of degree of monotonic dependence between U and V, whereas Pearson's correlation measures only the linear dependence.

Theorem 6.1. *The Kendall's tau for the copula in (6.3) is given by,*

$$\tau_{C_\theta} = 1 - \frac{2}{3\theta}. \tag{6.8}$$

Proof. Let U and V be random variables with $W^* = C_\theta(U, V)$. We have

$$E(W^*) = \frac{1}{2} - \frac{1}{6\theta}.$$

Therefore from (5.10), Kendall's measure of association between U and V is given by,

$$\tau_{C_\theta} = 1 - \frac{2}{3\theta}.$$

$\square$

Since the Kendall's τ_{C_θ} is positive, the pair (U, V) is PQD for all values of $\theta \in [1, \infty)$. The τ_{C_θ} value lies in the interval [0.33,1) which means that the copula in (6.3) accounts for both weaker and strong positive dependence. The plot of Kendall's τ_{C_θ} is given in Figure 6.3.

For the copula in (6.3), the measure based on Blomqvist's β is obtained as

$$\beta_C = \frac{4}{2^{1/\theta} + 1} - 1.$$

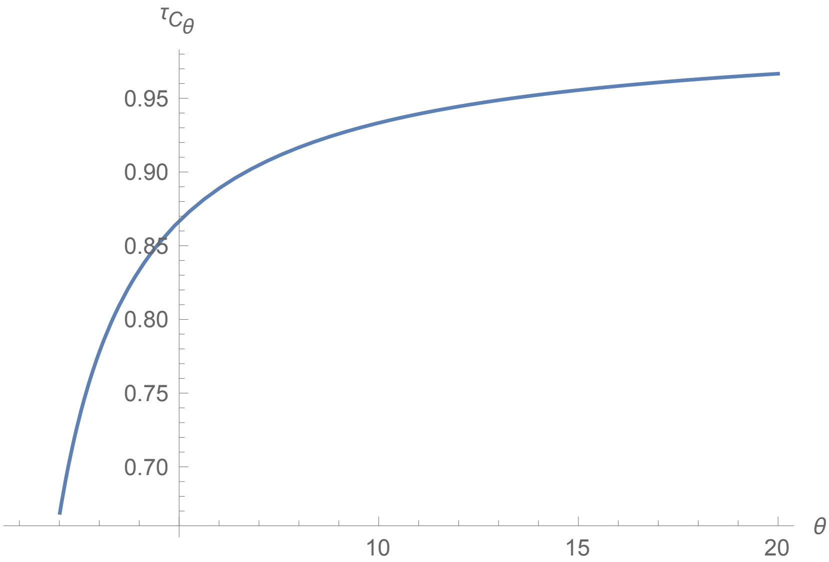

FIGURE 6.3: Plot of Kendall's τ_{C_θ}

The values of β_C is positive $\Rightarrow (U, V)$ is PQD.

Remark 6.2. As the copula described in (6.3) is PQD, the cumulative distribution function $F_\theta^*(.)$ is new worse than used (NWU) (refer Avérous and Dortet-Bernadet (2004)).

We have no closed form analytical expression for Spearman's ρ_{C_θ}. The plot of ρ_{C_θ} is given in Figure 6.4 and the plot of the Archimedean copula is given in Figure 6.5.

6.4 Distributions with various marginals

Copula techniques can be used for the construction of various bivariate distributions with given marginals. We now construct bivariate distribution functions with Pareto, Weibull, exponential and Weibull-Logistic marginals.

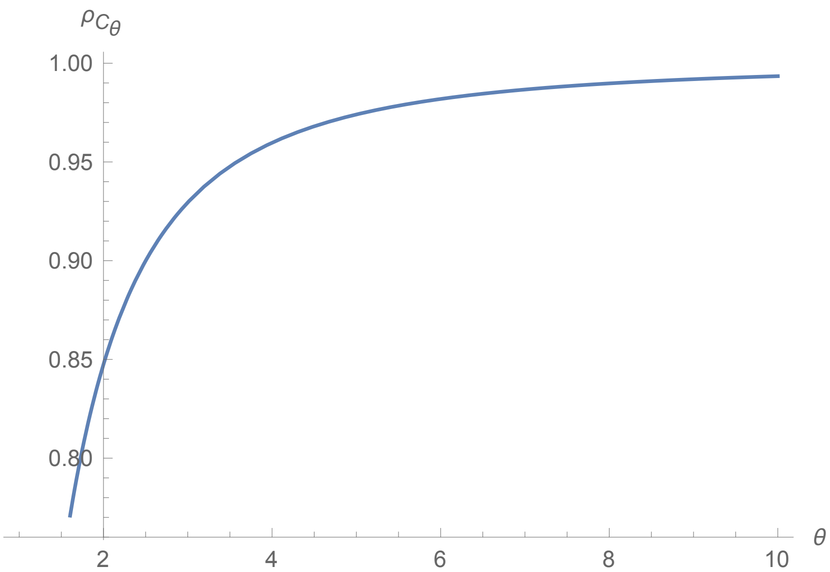

FIGURE 6.4: Plot of Spearman's ρ_{C_θ}

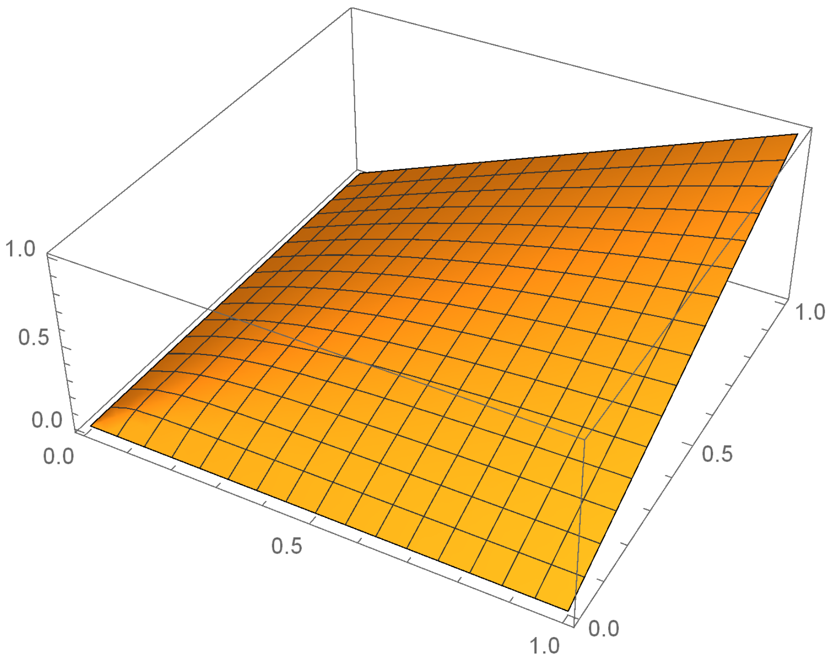

FIGURE 6.5: Plot of Archimedean copula $C_\theta(u, v)$

6.4.1 Distribution with Pareto marginals

If the marginal distributions of X and Y are Pareto I distributions with $F_1(x) = 1 - x^{-a} = u$ and $F_2(y) = 1 - y^{-b} = v;\ x > 1, y > 1, a, b > 0$, then the distribution function of (X, Y) is given by,

$$F(x,y) = \frac{1}{\left(\left(\frac{1}{x^a-1}\right)^\theta + \left(\frac{1}{y^b-1}\right)^\theta\right)^{1/\theta} + 1}; \quad \theta \in [1, \infty). \tag{6.9}$$

6.4.2 Distribution with Weibull marginals

Assume that the marginal distributions of X and Y are Weibull distributions with $F_1(x) = 1 - e^{-\left(\frac{x}{\eta_1}\right)^{b_1}}$ and $F_2(y) = 1 - e^{-\left(\frac{y}{\eta_2}\right)^{b_2}};\ x, y > 0, b > 0, \eta_i > 0$ for $i = 1, 2$. Substituting $u = 1-e^{-\left(\frac{x}{\eta_1}\right)^{b_1}}$ and $v = 1-e^{-\left(\frac{y}{\eta_2}\right)^{b_2}}$ in (6.3), we get a class of bivariate distributions with Weibull marginals.

The distribution function of (X, Y) is given by

$$F(x,y) = \frac{1}{\left(\left(\frac{1}{e^{\left(\frac{x}{\eta_1}\right)^{b_1}}-1}\right)^\theta + \left(\frac{1}{e^{\left(\frac{y}{\eta_2}\right)^{b_2}}-1}\right)^\theta\right)^{1/\theta} + 1}; \quad \theta \in [1, \infty). \tag{6.10}$$

6.4.3 Distribution with exponential marginals

Let X and Y be exponentially distributed with $F_1(x) = 1 - e^{-\lambda x}$ and $F_2(y) = 1 - e^{-\lambda y};\ x, y > 0, \lambda > 0$. Then we obtain a class of bivariate distributions with exponential marginals as

$$F(x,y) = \frac{1}{\left(\left(\frac{1}{e^{\lambda x}-1}\right)^{\theta} + \left(\frac{1}{e^{\lambda y}-1}\right)^{\theta}\right)^{1/\theta} + 1}; \quad \theta \in [1,\infty). \tag{6.11}$$

6.4.4 Distribution with Weibull-Logistic marginals

Let X be distributed as Weibull with $F_1(x) = 1 - e^{-\left(\frac{x}{q}\right)^n}$, $x, q, n > 0$ and Y be distributed as logistic with $F_2(y) = \frac{1}{e^{-\frac{y-s}{h}}+1}$, $-\infty < y < \infty$, $h > 0$. We obtain a class of bivariate distributions with Weibull-Logistic marginals as

$$F(x,y) = \frac{1}{\left(\left(e^{-\frac{y-s}{h}}\right)^{\theta} + \left(\frac{1}{1-e^{-\left(\frac{x}{q}\right)^n}} - 1\right)^{\theta}\right)^{1/\theta} + 1}; \quad \theta \in [1,\infty). \tag{6.12}$$

6.5 Applications

We now apply the proposed copula model (6.3) to a real life data from American football league, given in Table 2.5. The copula parameters are estimated using the relation connecting the Kendall's correlation coefficient τ_{C_θ} and the copula parameter given in (6.8). The estimated value of Kendall's τ_{C_θ} is 0.68 and hence the copula parameter value θ =2.09. The confidence limits for Kendall's τ_{C_θ} and for the copula parameter θ based on 1000 bootstrap re-samples are $[0.505, 0.821]$ and $[1.35, 3.75]$ respectively.

The cumulative distribution function $K_{C_\theta}(w)$ and the empirical distribution function $K_n(w)$ from the data are plotted in Figure 6.6. We can see that both are

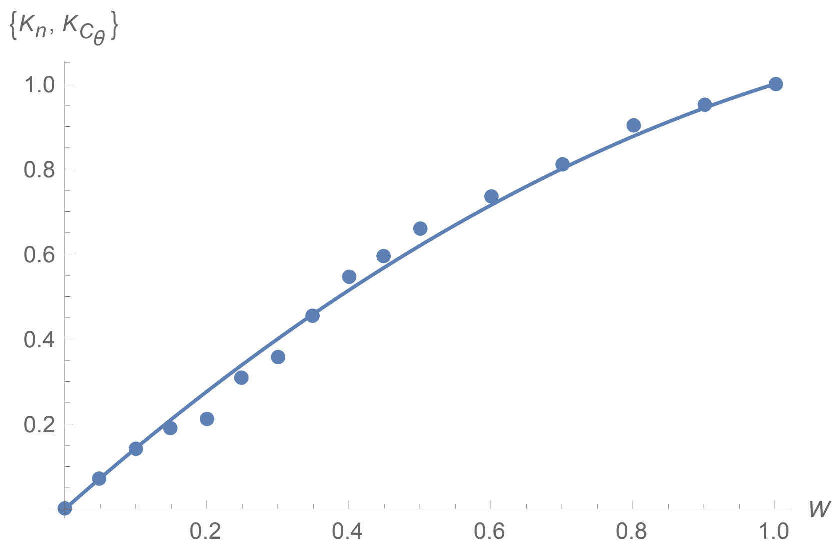

FIGURE 6.6: Plot of K_n and K_{C_θ}

almost identical. Let $C(u, v)$ be the copula of a random vector (X, Y). To test the hypothesis $H_0 : C(u, v) = C_\theta(u, v)$, we use the statistic given in (4.51).

For the level of test 0.05, $L(1-0.05) = 1.358$. Since D_n value of the copula discussed is $0.403 < 1.358$, we cannot reject H_0 and the copula of (X, Y) can be approximated by $C_\theta(u, v)$. The marginals of the data set can be fitted by $Weibull(1.39, 9.98)$ and $Weibull(1.16, 14.18)$ with Kolmogorov-Smirnov test statistic values 0.083 and 0.115 respectively.

To illustrate the applicability of the model in (6.12) we consider the Iris flower data set which consists of 50 samples. The data are available on http://www.math.uah.edu /stat/data/Fisher.html. We have considered the variables sepal length and petal width of the species. The marginals of the data set follow $Weibull(7.45, 6.208)$ and $Logistic(1.22, 0.46)$. The copula parameter θ is estimated as 1.6. We use the bivariate version of Kolmogrov-Smirnov (K.S.) test given in Justel et al. (1997) to test the goodness of fit. Since the K.S. statistic value D^* is 0.15 which is less than the tabled value 0.16 at first percentile, the model (6.12) is fit for the given positive dependent data.

Let us consider a one-parameter Archimedean copula of the form,

$$C_\beta(u,v) = \frac{\beta}{\log\left(-e^\beta + e^{\beta/u} + e^{\beta/v}\right)}; \quad \beta \in (0,\infty) \tag{6.13}$$

with the generator

$$\phi_\beta(t) = e^{\beta/t} - e^\beta. \tag{6.14}$$

The univariate cumulative distribution function $F^*_\beta(t)$ that corresponds to ϕ is given by,

$$F^*_\beta(t) = 1 - \log(e^\beta + t)^{-\beta}.$$

We have, the ratio of derivatives

$$\frac{\phi'_{\beta_1}(t)}{\phi'_{\beta_2}(t)} = \frac{\beta_1}{\beta_2} exp\left(\frac{\beta_1 - \beta_2}{t}\right).$$

It can be seen that if $\beta_1 \le \beta_2$, $\dfrac{\phi'_{\beta_1}}{\phi'_{\beta_2}}$ is non-decreasing on (0,1) and hence $C_1 \prec C_2$. The copula family (6.13) is also positively ordered.

The distribution function of W^* is given by,

$$\begin{aligned} K_{C_\beta}(w) &= w - \frac{\phi(w)}{\phi'(w)} \\ &= \frac{w\left(\beta + w\left(-e^{\beta - \frac{\beta}{w}}\right) + w\right)}{\beta}; \quad 0 < w < 1. \end{aligned}$$

Theorem 6.2. *The Kendall's tau for the copula in* (6.13) *is given by,*

$$\tau_{C_\beta} = \frac{1}{3}\left(1 + 2\beta + 2e^\beta\beta^2\left(CoshIntegral\,[\beta] - SinhIntegral\,[\beta]\right)\right). \tag{6.15}$$

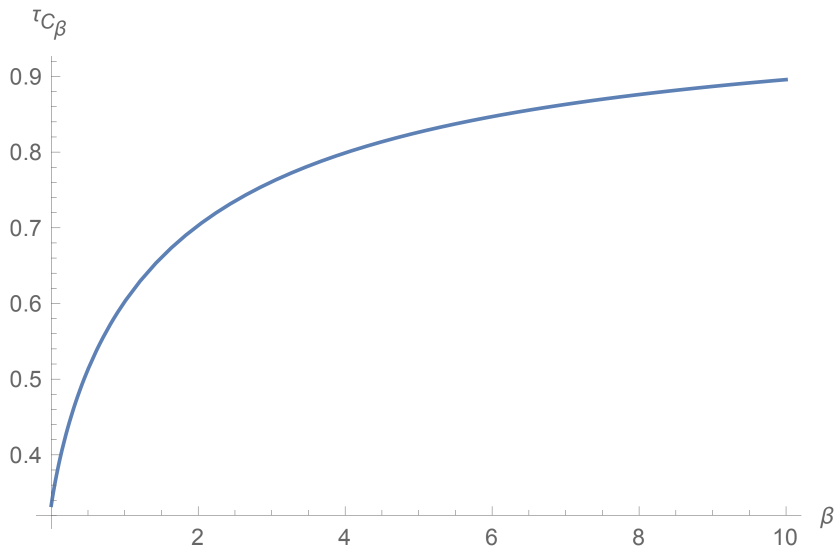

FIGURE 6.7: Plot of Kendall's τ_{C_β}

where $SinhIntegral\,[z]$ *gives the hyperbolic sine integral function and similarly* $CoshIntegral\,[z]$ *gives the hyperbolic cosine integral function.*

Since the Kendall's τ_{C_β} is positive, the pair (U, V) is PQD for all values of $\beta \in (0, \infty)$. The plot is given in Figure 6.7.

The copula parameter β is estimated as 1.7 using the relation connecting the Kendall's τ_{C_β} and the copula parameter. The cumulative distribution function $K_{C_\beta}(w)$ and the empirical distribution function $K_n(w)$ in Figure 6.8 can be seen as almost identical and the D_n value is $0.394 < 1.358$. The copula of (X, Y) can also be approximated by $C_\beta(u, v)$.

Now we compare the above two models to make a conclusion among the models which suits more appropriate for our data. The choice of our copula which suits best for the above positive dependent data can be made by using Akaike Information Criterion (AIC) given in Akaike (1987). A smaller relative AIC represents a better model fit. We have computed the AIC values of two models and they are -36.52 and

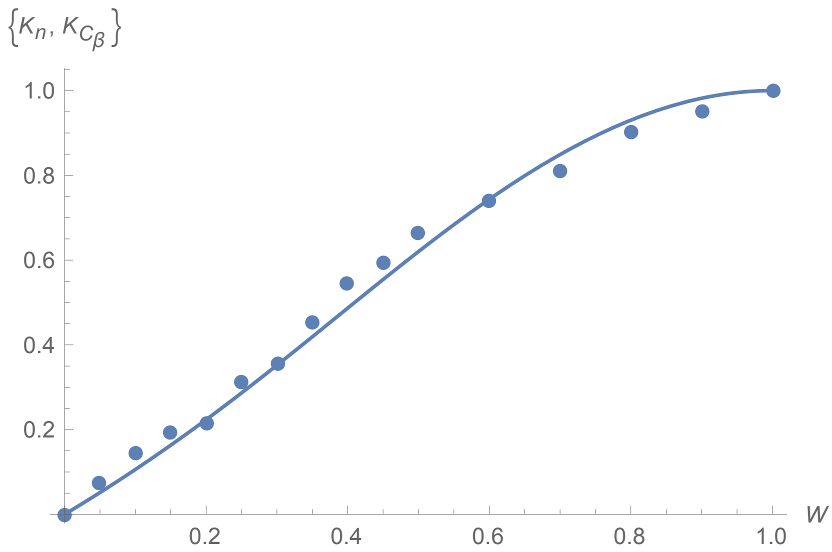

FIGURE 6.8: Plots of K_n and K_{C_β}

-24.011 respectively. This indicates that the copula model given in (6.3) is more appropriate for the data.

6.6 Conclusion

In this chapter we have considered a one-parameter family of Archimedean copula. The dependence structure of the model has been studied. It is shown that the copula suits for a positive dependent data set. The model can be extended directly to multivariate set up and the applicability of the model is illustrated with a data set.

Chapter 7

A class of bivariate Weibull distributions and their copulas

7.1 Introduction

The Weibull distribution has been used successfully in many applications due to its flexible shape and ability to model a wide range of failure time data. The model can be derived theoretically as a form of extreme value distribution, governing the time to occurrence of the "weakest link" of many competing failure processes. For more properties and applications of bivariate Weibull distributions one could refer to the extensive literature discussed in the introductory chapter.

As discussed in Chapter 1, if any one of the distributions discussed in literature fails to pass the test of adequacy, the whole process needs to be initiated afresh with another model. Thus it is appropriate to start with a flexible family of distributions having enough members that can accommodate different data situations. Further, when there is very little prior information about the data generating mechanism, it is appropriate to begin with a family of distributions which is quite flexible in

[1] The results in this chapter are published in American Journal of Mathematical and Management Sciences "(see Preethi & Sankaran (2017))

the desired characteristics. Motivated by this fact, we introduce a class of bivariate Weibull distributions in the sense that the marginals are univariate Weibull distributions.

The rest of the article is organized as follows. In Section 7.2, we introduce a family of bivariate Weibull distributions. Various members belonging to the family are identified. The distributional properties of the family are discussed in Section 7.3. In Section 7.4, we study the dependence structure of the family of distributions. In Section 7.5 we discuss the inference procedure and apply the proposed class of models to real data sets. Finally, Section 7.6 summarizes the major conclusions of the study.

7.2 Bivariate Weibull family

Suppose (X, Y) is a non-negative random vector having absolutely continuous survival function $\bar{F}(x, y) = P(X > x, Y > y)$. In order to construct the proposed family of bivariate Weibull distributions we require the following characterization theorem.

Theorem 7.1. *Assume that Z is a non-negative random variable with continuous and strictly decreasing survival function $\bar{G}(z)$ and cumulative hazard function $H(z) = -\log \bar{G}(z)$. Then the random variable Z satisfies the property*

$$P(Z > g(x, y)^{\alpha} | Z > x^{\alpha}) = P(Z > y^{\alpha}) \tag{7.1}$$

for all $x, y, \alpha > 0$ *if and only if*

$$g(x,y) = [H^{-1}(H(x^\alpha) + H(y^\alpha))]^{\frac{1}{\alpha}}. \tag{7.2}$$

Proof. By assuming (7.2), we obtain,

$$\begin{aligned} P(Z > g(x,y)^\alpha | Z > x^\alpha) &= \frac{P(Z > g(x,y)^\alpha)}{P(Z > x^\alpha)} \\ &= \frac{\bar{G}(g(x,y)^\alpha)}{\bar{G}(x^\alpha)} \\ &= \frac{\bar{G}\left[H^{-1}(H(x^\alpha) + H(y^\alpha))\right]}{\bar{G}(x^\alpha)}. \end{aligned} \tag{7.3}$$

Note, that the relationship $H(z) = -\log \bar{G}(z)$ provides,

$$H^{-1}(t) = \bar{G}^{-1}(e^{-t}), t > 0.$$

Thus,

$$\begin{aligned} H^{-1}(H(x^\alpha) + H(y^\alpha)) &= \bar{G}^{-1}(\exp[-(H(x^\alpha) + H(y^\alpha))]) \\ &= \bar{G}^{-1}(\bar{G}(x^\alpha).\bar{G}(y^\alpha)) \end{aligned} \tag{7.4}$$

or

$$\bar{G}[H^{-1}(H(x^\alpha) + H(y^\alpha))] = \bar{G}(x^\alpha).\bar{G}(y^\alpha). \tag{7.5}$$

By substituting (7.5) into (7.3), we have (7.1).

To prove the converse part we assume (7.1). This is equivalent to,

$$\begin{aligned}\bar{G}(g(x,y)^{\alpha}) &= \bar{G}(x^{\alpha}).\bar{G}(y^{\alpha}) \\ &= \bar{G}\,[H^{-1}(H(x^{\alpha})+H(y^{\alpha}))].\end{aligned} \tag{7.6}$$

Using (7.4) and (7.5), (7.6) leads to (7.2). This completes the proof. □

Theorem 7.2. *The distribution of the random vector* (X,Y) *taking values on* $\mathbf{R}^{2+}=\{(x,y)|x>0,y>0\}$ *is bivariate Weibull with survival function of the form,*

$$\bar{F}(x,y)=\exp[-g(x,y)^{\alpha}], x>0, y>0, \alpha>0 \tag{7.7}$$

if and only if there exists a bivariate function $g(x,y)$ *satisfying,*

$$g(x,y)=[H^{-1}(H(x^{\alpha})+H(y^{\alpha}))]^{\frac{1}{\alpha}} \tag{7.8}$$

and

$$\frac{(1-\alpha)+\alpha g(x,y)^{\alpha}}{g(x,y)}\frac{\partial g}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial^2 g}{\partial x\partial y}\geq 0 \tag{7.9}$$

where $H(.)$ *is the cumulative hazard function of non-negative random variable* Z *satisfying* (7.1).

Proof. To prove if part, we assume (7.8). As $H(x)$ is strictly increasing, $H^{-1}(x)$ is also strictly increasing and, therefore, $g(x,y)$ is strictly increasing and also continuous in each of the variables x and y. Further, $H(0)=0$ and $H(\infty)=\lim_{x\to\infty}H(x)=\infty$, give $\lim_{(x,y)\to(0,0)} g(x,y)=0$ and $\lim_{(x,y)\to(\infty,\infty)} g(x,y)=\infty$. Thus,

$$\lim_{x\to 0} g(x,y)=[H^{-1}(H(y^{\alpha}))]^{\frac{1}{\alpha}}=y$$

and

$$\lim_{y \to 0} g(x, y) = [H^{-1}(H(x^\alpha))]^{\frac{1}{\alpha}} = x.$$

Thus we obtain,

$$\bar{F}(x, 0) = \bar{F}_1(x) = e^{-[H^{-1}(H(x^\alpha))]} = e^{-x^\alpha} \text{ and}$$
$$\bar{F}(0, y) = \bar{F}_2(y) = e^{-[H^{-1}(H(y^\alpha))]} = e^{-y^\alpha}.$$

Conversely, let (X, Y) be distributed as in (7.7). The time-transformed exponential model is given by the following.

$$\bar{F}(x, y) = \overline{W}[R(x) + R(y)], x, y > 0 \tag{7.10}$$

where $\overline{W}(.)$ is a continuous strictly positive and decreasing convex survival function, and $R : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing function satisfying $R(0) = 0$ and $R(\infty) = \infty$. From (7.7) and (7.8), it can be proved that (7.7) is a time transformed exponential model with $R(z) = H(z^\alpha)$, the cumulative hazard function of some random variable Z and $\overline{W}(z) = \exp[-H^{-1}(z)]$ is a survival function. Substituting $R(z) = H(z^\alpha)$ and $\overline{W}(z) = \exp[-H^{-1}(z)]$ in (7.10), we have,

$$\bar{F}(x, y) = \exp[-H^{-1}(H(x^\alpha) + H(y^\alpha))] = \exp[-g(x, y)^\alpha].$$

□

Let $\boldsymbol{A^*}$ be the class of univariate distributions satisfying the property (7.1). By choosing different functional forms of $g(x, y)$ satisfying (7.2), we get a class of bivariate distributions which is denoted by $\boldsymbol{B^*}$. Since the marginal distributions of all bivariate distributions are Weibull, we refer the class $\boldsymbol{B}^*$ as a family of bivariate Weibull distributions. The family of Archimedean copulas in our discussion also satisfies all the properties of the time-transformed exponential model.

The marginal survival functions of (7.7) are identical, given by $\bar{F}_1(x) = e^{-x^\alpha}$ and $\bar{F}_2(y) = e^{-y^\alpha}$. The survival copula corresponding to (7.7) is as follows.

$$\hat{C}(\bar{F}_1(x), \bar{F}_2(y)) = \hat{C}(u, v) = \exp[-g(-(\log u)^{\frac{1}{\alpha}}, -(\log v)^{\frac{1}{\alpha}})^\alpha] \tag{7.11}$$

with

$$g((-\log u)^{\frac{1}{\alpha}}, (-\log v)^{\frac{1}{\alpha}}) = [H^{-1}(H(-\log u) + H(-\log v))]^{\frac{1}{\alpha}}.$$

Then,

$$H(-\log \hat{C}(u, v)) = H(-\log u) + H(-\log v).$$

If we set $\phi(t) = H(-\log t)$, then,

$$\hat{C}(u, v)) = \phi^{-1}(\phi(u) + \phi(v)), 0 < u, v < 1, \tag{7.12}$$

where $\phi(.)$ is a continuous, strictly decreasing function and $\phi(1) = H(0) = 0$. If we take $\phi(.)$ to be convex, then we have an Archimedean survival copula with generator $\phi(.)$. From the above conclusions, we have the following result.

Theorem 7.3. *Let $\bar{G}(.)$ be a continuous strictly decreasing survival function on $(0, \infty)$ with cumulative hazard function $H(.)$. If we choose the bivariate survival* *function as* $\exp[-g(x,y)^\alpha]$ *having Weibull marginals, then the survival copula so obtained is Archimedean with generator,*

$$\phi(t) = -\log \bar{G}(-\log t) = H(-\log t)$$

provided $\phi(.)$ *is convex. Conversely, a survival function can be generated by* $\bar{G}(.)$ *via. a bivariate Archimedean copula where* $\bar{G}(.)$ *is such that* ϕ *is convex and if the copula represents a bivariate Weibull law, then the survival function is of the form* $\exp[-g(x,y)^\alpha]$.

We designate $\bar{G}(z)$ as the baseline distribution corresponding to $\bar{F}(x,y)$. The different choices of the baseline distribution $\bar{G}(.)$ as exponential, Weibull, Lomax, Gompertz and Burr provide bivariate Weibull distributions with Weibull marginals. The copulas satisfy the conditions given in equations (7.8) and (7.9) with $\phi(t) = H(-\log t)$ as convex. We now list various bivariate distributions in class $\boldsymbol{B^*}$ by choosing $H(z)$ and $g(x,y)$.

Type I

Let $\bar{G}_1(z) = \exp[-\lambda z], z > 0, \lambda > 0$, then $H(z) = \lambda z$, we have $g(x,y) = (x^\alpha + y^\alpha)^{\frac{1}{\alpha}}$. The bivariate distribution is the independent Weibull distribution,

$$\bar{F}^{(1)}(x,y) = \exp[-x^\alpha - y^\alpha], x, y > 0. \tag{7.13}$$

The survival copula is the product copula

$$\hat{C}_1(u,v) = uv, 0 \le u, v \le 1.$$

Type II

When $\bar{G}_2(z) = \exp\left[-(\lambda z)^a\right],\ \ z > 0, a, \lambda > 0$, represents the Weibull distribution, then, $H\left(z\right) = \left(\lambda z\right)^a$ and $g\left(x, y\right) = \left(x^{a\alpha} + y^{a\alpha}\right)^{\frac{1}{a\alpha}}$ giving,

$$\bar{F}^{(2)}\left(x, y\right) = \exp\left[-\left(x^{a\alpha} + y^{a\alpha}\right)^{\frac{1}{a}}\right]. \tag{7.14}$$

The following survival copula is valid for $a \geq 1$.

$$\hat{C}_2\left(u, v\right) = \exp\left[-\{(-\log u)^a + (-\log v)^a\}^{\frac{1}{a}}\right].$$

Type III

When Z has Lomax distribution $\bar{G}_3(z) = (1 + \beta z)^{-a}, z > 0, a, \beta > 0,$ we have $H(z) = a\ \log(1 + \beta z)$ and $g(x, y) = x + y + \beta xy$. Then,

$$\bar{F}^{(3)}(x, y) = \exp\left[-x^\alpha - y^\alpha - \beta x^\alpha y^\alpha\right],\ \ 0 \leq \beta \leq 1 \tag{7.15}$$

and the copula is,

$$\hat{C}_3(u, v) = uv\exp\left[-\beta\log u\log v\right],$$

usually referred to as the Gumbel-Barnett family in literature.

Type IV

When $\bar{G}_4(z) = \exp[-(e^{\frac{z}{\theta}} - 1)^\theta];\ \ z > 0, \theta \geq 1$, giving $H(z) = (e^{\frac{z}{\theta}} - 1)^\theta;\ \ \theta \geq 1$, then,

$$\bar{F}^{(4)}(x, y) = \left(1 + \left(\left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{-1/\theta}\right)^\theta + \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{-1/\theta}\right)^\theta\right)^{\frac{1}{\theta}}\right)^{-\theta};\ \ \theta \geq 1 \tag{7.16}$$

and the copula

$$\hat{C}_4(u,v) = \left(1 + \left(\left(u^{\frac{-1}{\theta}} - 1\right)^{\theta} + \left(v^{\frac{-1}{\theta}} - 1\right)^{\theta}\right)^{\frac{1}{\theta}}\right)^{-\theta}; \quad \theta \geq 1.$$

TypeV

When $\bar{G}_5(z) = 1 - (1 - \exp(-z))^{\theta}; \ z > 0, \ \theta \geq 1$ and $H(z) = -\log[1 - (1 - \exp(-z))^{\theta}]; \ \theta \geq 1$, then the corresponding bivariate survival function is as follows.

$$\bar{F}^{(5)}(x,y) = 1 - \left(\left(1 - \mathrm{e}^{-x^{\alpha}}\right)^{\theta} + \left(1 - \mathrm{e}^{-y^{\alpha}}\right)^{\theta} - \left(1 - \mathrm{e}^{-x^{\alpha}}\right)^{\theta}\left(1 - \mathrm{e}^{-y^{\alpha}}\right)^{\theta}\right)^{\frac{1}{\theta}}; \ \theta \geq 1 \tag{7.17}$$

and the copula is

$$\hat{C}_5(u,v) = 1 - \left((1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right)^{\frac{1}{\theta}}; \ \theta \geq 1.$$

Type VI

If Z follows the Gompertz distribution $\bar{G}_6(z) = \exp\left[-B\frac{(C^z-1)}{\log C}\right], z \geq 0, B, C > 0$, then $H(z) = \frac{B(C^z-1)}{\log C}$, so that,

$$\bar{F}^{(6)}(x,y) = (e^{\theta x^{\alpha}} + e^{\theta y^{\alpha}} - 1)^{-\frac{1}{\theta}} \tag{7.18}$$

which is the gamma fraility model derived by Bjarnason and Hougaard (2000) with the parameter $\lambda = 1$. The survival copula is

$$\hat{C}_6(u,v) = \max[(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, 0]$$

where $\theta = \log C \neq 0$ and $-1 < \theta < \infty$, which is the well known Clayton survival copula used in survival analysis.

Type VII

Let $\bar{G}_7(z) = \exp\left[-\theta\left((z+1)^a - 1\right)\right],\ \ z \geq 0, a > 0, \theta > 0$. Then we have $H(z) = \theta\left[(z+1)^a - 1\right]$ which leads to the bivariate Weibull,

$$\bar{F}^{(7)}(x,y) = \exp\left[1 - \left\{(x^\alpha + 1)^a + (y^\alpha + 1)^a - 1\right\}^{\frac{1}{a}}\right] \tag{7.19}$$

with copula

$$\hat{C}_7(u,v) = \max\left[1 - ((1-\log u)^a + (1-\log v)^a - 1)^{\frac{1}{a}}, 0\right].$$

Type VIII

If $\bar{G}_8(z) = (2e^z - 1)^{-\sigma}, \sigma > 0$, we have $H(z) = \sigma\log(2e^z - 1)$. Thus,

$$\bar{F}^{(8)}(x,y) = 2[1 + (2e^{x^\alpha} - 1)(2e^{y^\alpha} - 1)]^{-1} \tag{7.20}$$

and

$$\hat{C}_8(u,v) = \frac{uv}{1 + (1-u)(1-v)}.$$

It is known from (1.10) that for every Archimedean copula with generator ϕ, there exists $\bar{F}^*(z) = \phi^{-1}(z)$, a univariate survival function taking values in $[0,\infty)$ with mode at 0. From our earlier relationship $\phi(z) = H(-\log z)$ we have,

$$\bar{F}^*(z) = exp[-H^{-1}(z)].$$

The survival function $\bar{F}^*(z)$ corresponding to the various types of models discussed, and their generators are given in Table 7.1.

7.3 Properties of the class of bivariate Weibull distributions

The marginal distributions of the family of distributions are univariate Weibull distributions. The joint density functions of the various models in $\boldsymbol{B}^*$ are presented in Table 7.2.

7.3.1 Conditional distributions

The conditional densities $f_1^*(x|y) = \frac{f(x,y)}{a_2(y)}$ and $f_2^*(y|x) = \frac{f(x,y)}{a_1(x)}$, where $f(x,y)$ is the joint density function and $a_1(x)$ and $a_2(y)$ are, respectively, the marginal density functions of X and Y are exhibited in Table 7.3. The conditional survival functions $P(X > x|Y > y)$ and $P(Y > y|X > x)$ are given in Table 7.4.

7.3.2 Hazard rate function

As already discussed in Chapter 4, one of the basic concepts useful for the analysis of lifetime data is hazard rates, which provides more information than the survival function about the pattern of failure. We use the bivariate vector-valued hazard rate given in (4.1).

The random vector (X, Y) has increasing (decreasing) hazard rate, IHR (DHR) if $-\dfrac{\partial \log \bar{F}(x, y)}{\partial x}$ is an increasing (a decreasing) function of x. We observe that,

$$h_1(x, y) = \frac{\partial(g(x, y)^{\alpha})}{\partial x} \tag{7.21}$$

and

$$h_2(x, y) = \frac{\partial(g(x, y)^{\alpha})}{\partial y}. \tag{7.22}$$

The hazard rates for the bivariate Weibull models are given in Table 7.5. From Table 7.5, it follows that the members of the family have an IHR or DHR property depending on the parameter values.

The copula hazard rates as defined in (4.6) and (4.7) for the $\boldsymbol{B^*}$ class of distributions are given in Table 7.6.

7.4 Dependence structure

To study the association between the two random variables, we use the numerical measure Kendall's tau and the local dependence measure say the Clayton-Oakes measure, which give the dependence structure at specific values of x and y and, also, the tail dependence properties of the family $\boldsymbol{B^*}$.

7.4.1 Kendall's tau

We present the Kendall's tau in terms of copula as it is computationally easy to obtain this dependence measure. The Kendall's tau of the bivariate distributions are given in Table 7.7. For type I model, the variables are independent, but for types III and VIII models the measure is negative. It can be noticed in Table 7.7 that for type II model, the measure is positive. However, for types VI and VII models, the association measure is positive, negative or zero depending on the parameter values.

We have the following proposition discussing reliability properties.

Proposition 7.4. *A distribution in $\boldsymbol{B}^*$ or the corresponding survival copula is PQD ($LTD(Y|X)$,PKD, stochastically increasing) if and only if $\bar{G}(.)$ is new better than used (increasing failure rate, increasing failure rate average,strongly unimodal).*

The proof follows from Averous and Dortet-Bernadit (2005).

The type II bivariate Weibull distribution is PQD as the baseline Weibull distribution is new better than used (NBU) for $\alpha > 1$.

Remark 7.1. Type VII distribution is $RTI(Y|X) \Leftrightarrow h_1(x,y) \leq 1 \Leftrightarrow$ Z is IHR for $\theta > 0, \;\; a \geq 1, \;\;\; 0 < \alpha \leq 1$.

7.4.2 Clayton measure

The values of $\theta(x,y)$ and the nature of dependence for various models are presented in Table 7.8. From the table it can be seen that for type I model, the variables X and Y are independent. For types II and IV models, X and Y have positive association and for type VIII model, the variables are negatively associated. It is to be noted that for types VI and VII models, the variables X and Y are positively or negatively associated or independent based on the parameter values.

TABLE 7.1: Generators and induced distributions

Model type	$\phi(z)$	$\bar{F}^*(z)$
I	$-\lambda \log z$	$\exp(-\frac{z}{\lambda}), \lambda > 0, z > 0.$
II	$\lambda^a(-\log z)^a$	$\exp(-\frac{z^{\frac{1}{a}}}{\lambda}), a \geq 1, \lambda > 0, z > 0.$
III	$a \log(1 - \beta \log z)$	$\exp[\frac{1-e^{z/a}}{\beta}], 0 \leq \beta \leq 1, a > 0, z > 0.$
IV	$(z^{\frac{-1}{\theta}} - 1)^\theta$	$(1 + z^{\frac{1}{\theta}})^{-\theta}, \theta \geq 1, k > 0, z > 0.$
V	$-\log[1 - (1 - z)^\theta]$	$1 - (1 - e^{-z})^{\frac{1}{\theta}}, \theta \geq 1, z > 0.$
VI	$B\theta^{-1}(z^{-\theta} - 1)$	$(1 + \frac{\theta z}{B})^{-\frac{1}{\theta}} \;\; \theta \geq -1, \theta \neq 0, z > 0.$
VII	$\theta[(1 - \log z)^a - 1]$	$\exp[1 - (\frac{z}{\theta} + 1)^{\frac{1}{a}}], a > 0, z > 0.$
VIII	$\sigma \log(\frac{2}{z} - 1)$	$\frac{2}{1+e^{\frac{z}{\sigma}}}, \sigma > 0, z > 0.$

TABLE 7.2: Joint density function $f(x, y)$ for bivariate Weibull models

Model type	$f(x,y)$
I	$\alpha^2 x^{\alpha-1} y^{\alpha-1} e^{-x^\alpha - y^\alpha}; x, y, \alpha > 0.$
II	$\alpha^2 x^{\alpha a-1} y^{\alpha a-1} e^{(-x^{\alpha a}-y^{\alpha a})^{1/a}} \left(-x^{\alpha a} - y^{\alpha a}\right)^{\frac{1}{a}-2} \left(\left(-x^{\alpha a} - y^{\alpha a}\right)^{1/a} - a + 1\right); x, y, \alpha, a > 0.$
III	$\alpha^2 x^{\alpha-1} y^{\alpha-1} e^{x^\alpha(-(\beta y^\alpha+1))-y^\alpha} \left(\beta \left(x^\alpha \left(\beta y^\alpha + 1\right) + y^\alpha - 1\right) + 1\right); x, y, \alpha > 0, 0 \leq \beta \leq 1.$
IV	$\frac{x^{-1+\alpha} y^{-1+\alpha} \alpha^2}{\left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{\frac{1}{\theta}}\right) \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{\frac{1}{\theta}}\right) \theta} \left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{-1/\theta}\right)^\theta \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{-1/\theta}\right)^\theta$ $\left(\left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{-1/\theta}\right)^\theta + \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{-1/\theta}\right)^\theta\right)^{-2+\frac{1}{\theta}}$ $\left(1 + \left(\left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{-1/\theta}\right)^\theta + \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{-1/\theta}\right)^\theta\right)^{\frac{1}{\theta}}\right)^{-2-\theta}$ $\left(-1 + \theta + 2\left(\left(-1 + \left(\mathrm{e}^{-x^\alpha}\right)^{-1/\theta}\right)^\theta + \left(-1 + \left(\mathrm{e}^{-y^\alpha}\right)^{-1/\theta}\right)^\theta\right)^{\frac{1}{\theta}} \theta\right); x, y, \alpha > 0, \theta \geq 1.$
V	$-\frac{\alpha^2 x^{\alpha-1} y^{\alpha-1} \left(1-e^{-x^\alpha}\right)^\theta \left(1-e^{-y^\alpha}\right)^\theta \left(\left(1-e^{-x^\alpha}\right)^\theta - \left(\left(1-e^{-x^\alpha}\right)^\theta - 1\right)\left(1-e^{-y^\alpha}\right)^\theta\right)^{\frac{1}{\theta}-2} \left(\left(\left(1-e^{-x^\alpha}\right)^\theta - 1\right)\left(\left(1-e^{-y^\alpha}\right)^\theta - 1\right) - \theta\right)}{\left(e^{x^\alpha}-1\right)\left(e^{y^\alpha}-1\right)};$ $x, y, \alpha > 0, \theta \geq 1.$
VI	$\alpha^2(\theta+1) x^{\alpha-1} y^{\alpha-1} e^{\theta(x^\alpha+y^\alpha)} \left(e^{\theta x^\alpha} + e^{\theta y^\alpha} - 1\right)^{-\frac{1}{\theta}-2}; x, y, \alpha > 0, -1 < \theta < \infty.$
VII	$\alpha^2 x^{\alpha-1} y^{\alpha-1} \left(x^\alpha + 1\right)^{a-1} \left(y^\alpha + 1\right)^{a-1} e^{1-((x^\alpha+1)^a+(y^\alpha+1)^a-1)^{1/a}}$ $\left(\left(x^\alpha+1\right)^a + \left(y^\alpha+1\right)^a - 1\right)^{\frac{1}{a}-2} \left(\left(\left(x^\alpha+1\right)^a + \left(y^\alpha+1\right)^a - 1\right)^{1/a} + a - 1\right); x, y, a, \alpha > 0.$
VIII	$-\frac{2\alpha^2 x^{\alpha-1} y^{\alpha-1} e^{x^\alpha+y^\alpha} \left(\left(2e^{x^\alpha} - 1\right) e^{y^\alpha} - e^{x^\alpha}\right)}{\left(-2e^{x^\alpha+y^\alpha} + e^{x^\alpha} + e^{y^\alpha} - 1\right)^3}; x, y, \alpha > 0.$

TABLE 7.3: Conditional densities $f_1^*(x|y)$ and $f_2^*(y|x)$ for bivariate Weibull models

Model type	$f_1^*(x\|y)$	$f_2^*(y\|x)$
I	$-\alpha e^{-x^\alpha} x^{\alpha-1}$	$-\alpha e^{-y^\alpha} y^{\alpha-1}$
II	$(-\alpha x^{\alpha a-1} y^{\alpha(a-1)} e^{(-x^{\alpha a}-y^{\alpha a})^{1/a}+y^\alpha} \left(-x^{\alpha a}-y^{\alpha a}\right)^{\frac{1}{a}-2} \left(\left(-x^{\alpha a}-y^{\alpha a}\right)^{1/a}-a+1\right)$	$-\alpha x^{\alpha(a-1)} y^{\alpha a-1} e^{(-x^{\alpha a}-y^{\alpha a})^{1/a}+x^\alpha} \left(-x^{\alpha a}-y^{\alpha a}\right)^{\frac{1}{a}-2} \left(\left(-x^{\alpha a}-y^{\alpha a}\right)^{1/a}-a+1\right)$
III	$-\alpha x^{\alpha-1} e^{-x^\alpha(\beta y^\alpha+1)} \left(\beta\left(x^\alpha\left(\beta y^\alpha+1\right)+y^\alpha-1\right)+1\right)$	$-\alpha y^{\alpha-1} e^{y^\alpha(-(\beta x^\alpha+1))} \left(\beta\left(x^\alpha\left(\beta y^\alpha+1\right)+y^\alpha-1\right)+1\right)$
V	$\frac{\mathrm{e}^{y^\alpha} x^{-1+\alpha}\alpha}{\left(-1+\mathrm{e}^{x^\alpha}\right)\left(-1+\mathrm{e}^{y^\alpha}\right)}\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta \left(\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta-\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta\left(-1+\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\right)\right)^{-2+\frac{1}{\theta}} \left(\left(-1+\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\right)\left(-1+\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta\right)-\theta\right)$	$\frac{\mathrm{e}^{x^\alpha} y^{-1+\alpha}\alpha}{\left(-1+\mathrm{e}^{x^\alpha}\right)\left(-1+\mathrm{e}^{y^\alpha}\right)}\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta \left(\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta-\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta\left(-1+\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\right)\right)^{-2+\frac{1}{\theta}} \left(\left(-1+\left(1-\mathrm{e}^{-x^\alpha}\right)^\theta\right)\left(-1+\left(1-\mathrm{e}^{-y^\alpha}\right)^\theta\right)-\theta\right)$
VI	$-\alpha(\theta+1)x^{\alpha-1}e^{\theta(x^\alpha+y^\alpha)+y^\alpha}\left(e^{\theta x^\alpha}+e^{\theta y^\alpha}-1\right)^{-\frac{1}{\theta}-2}$	$-\alpha(\theta+1)y^{\alpha-1}e^{(\theta+1)x^\alpha+\theta y^\alpha}\left(e^{\theta x^\alpha}+e^{\theta y^\alpha}-1\right)^{-\frac{1}{\theta}-2}$
VIII	$\frac{2\alpha x^{\alpha-1}e^{x^\alpha+2y^\alpha}\left(\left(2e^{x^\alpha}-1\right)e^{y^\alpha}-e^{x^\alpha}\right)}{\left(-2e^{x^\alpha+y^\alpha}+e^{x^\alpha}+e^{y^\alpha}-1\right)^3}$	$-\frac{2\alpha y^{\alpha-1}e^{2x^\alpha+y^\alpha}\left(-2e^{x^\alpha+y^\alpha}+e^{x^\alpha}+e^{y^\alpha}\right)}{\left(-2e^{x^\alpha+y^\alpha}+e^{x^\alpha}+e^{y^\alpha}-1\right)^3}$

TABLE 7.4: Conditional survival functions for bivariate Weibull models

Model type	$P(X > x \| Y > y)$	$P(Y > y \| X > x)$
I	$e^{-x^{\alpha}}$	$e^{-y^{\alpha}}$
II	$e^{(-x^{a\alpha}-y^{a\alpha})^{1/a}+y^{\alpha}}$	$e^{(-x^{a\alpha}-y^{a\alpha})^{1/a}+x^{\alpha}}$
III	$e^{-x^{\alpha}(\beta y^{\alpha}+1)}$	$e^{y^{\alpha}(-(\beta x^{\alpha}+1))}$
VI	$e^{y^{\alpha}}\left(e^{\theta x^{\alpha}}+e^{\theta y^{\alpha}}-1\right)^{-1/\theta}$	$e^{x^{\alpha}}\left(e^{\theta x^{\alpha}}+e^{\theta y^{\alpha}}-1\right)^{-1/\theta}$
VII	$e^{-((x^{\alpha}+1)^{a}+(y^{\alpha}+1)^{a}-1)^{1/a}+y^{\alpha}+1}$	$e^{-\left((x^{\alpha}+1)^{b}+(y^{\alpha}+1)^{b}-1\right)^{1/b}+x^{\alpha}+1}$
VIII	$\dfrac{2e^{y^{\alpha}}}{\left(2e^{x^{\alpha}}-1\right)\left(2e^{y^{\alpha}}-1\right)+1}$	$\dfrac{2e^{x^{\alpha}}}{\left(2e^{x^{\alpha}}-1\right)\left(2e^{y^{\alpha}}-1\right)+1}$

Table 7.5: Bivariate hazard rates for bivariate Weibull models

Model type	$h_1(x,y)$	Behaviour of hazard rates
I	$\alpha x^{-1+\alpha}$	increasing in x for $\alpha > 1$, decreasing in x for $0 < \alpha \leq 1$.
II	$\alpha x^{-1+a\alpha}(x^{a\alpha}+y^{a\alpha})^{-1+\frac{1}{a}}$	decreasing in x for $\alpha \leq (>)1, a < 1$, increasing in x for $a \geq 1, \alpha = 1$.
III	$\alpha x^{-1+\alpha}(1+\beta y^{\alpha})$	increasing in x for $\alpha > 1$, decreasing in x for $0 < \alpha < 1$; $0 \leq \beta \leq 1$.
IV	$\dfrac{\alpha x^{\alpha-1}\left(e^{-x^{\alpha}}\right)^{-1/\theta}\left(\left(e^{-x^{\alpha}}\right)^{-1/\theta}-1\right)^{\theta-1}\left(\left(\left(e^{-x^{\alpha}}\right)^{-1/\theta}-1\right)^{\theta}+\left(\left(e^{-y^{\alpha}}\right)^{-1/\theta}-1\right)^{\theta}\right)^{\frac{1}{\theta}-1}}{\left(\left(\left(e^{-x^{\alpha}}\right)^{-1/\theta}-1\right)^{\theta}+\left(\left(e^{-y^{\alpha}}\right)^{-1/\theta}-1\right)^{\theta}\right)^{1/\theta}+1}$	decreasing in x for $\alpha \leq 0.5, \theta \geq 1$, increasing in x for $\alpha > 0.5, \theta \geq 1$.
V	$\dfrac{\alpha x^{\alpha-1}\left(1-e^{-x^{\alpha}}\right)^{\theta}\left(\left(1-e^{-y^{\alpha}}\right)^{\theta}-1\right)\left(\left(1-e^{-x^{\alpha}}\right)^{\theta}-\left(\left(1-e^{-x^{\alpha}}\right)^{\theta}-1\right)\left(1-e^{-y^{\alpha}}\right)^{\theta}\right)^{\frac{1}{\theta}-1}}{\left(e^{x^{\alpha}}-1\right)\left(\left(\left(1-e^{-x^{\alpha}}\right)^{\theta}-\left(\left(1-e^{-x^{\alpha}}\right)^{\theta}-1\right)\left(1-e^{-y^{\alpha}}\right)^{\theta}\right)^{1/\theta}-1\right)}$	decreasing in x for $\alpha < 1, \theta \geq 1$, increasing in x for $\alpha \geq 1, \theta \geq 1$.
VI	$\dfrac{\alpha e^{\theta x^{\alpha}}\log\left(-1+e^{\theta x^{\alpha}}+e^{\theta y^{\alpha}}\right)^{\frac{-(1+\theta)}{\theta}}x^{-1+\alpha}}{-1+e^{\theta x^{\alpha}}+e^{\theta y^{\alpha}}}$	decreasing in x for $\alpha > 0, \theta > 0$.
VII	$\alpha x^{-1+\alpha}(1+x^{\alpha})^{-1+a}(-1+(1+x^{\alpha})^{a}+(1+y^{\alpha})^{a})^{-1+\frac{1}{a}}$	decreasing in x for $\alpha < 1, a > 0$, increasing in x for $\alpha > 1, a > 0$.
VIII	$\dfrac{2\alpha e^{x^{\alpha}}(-1+2e^{y^{\alpha}})x^{-1+\alpha}}{1+(-1+2e^{x^{\alpha}})(-1+2e^{y^{\alpha}})}$	decreasing in x for $0 < \alpha \leq 1$, increasing in x for $\alpha > 1$.

Table 7.6: Copula hazard rates for the family $\boldsymbol{B^*}$

Model type	$(G_1(u,v), G_2(u,v))$
I	$(1,1)$
II	$((-\log u)^{a}+(-\log v)^{a})^{\frac{1}{a}}((-\log u)^{a-1},(-\log v)^{a-1})$
III	$(1-\beta\log u, 1-\beta\log v)$
IV	$\left(\left(u^{-1/\theta}-1\right)^{\theta}+\left(v^{-1/\theta}-1\right)^{\theta}\right)^{\frac{1}{\theta}-1}\left(\left(\left(u^{-1/\theta}-1\right)^{\theta}+\left(v^{-1/\theta}-1\right)^{\theta}\right)^{1/\theta}+1\right)^{-\theta-1}\left(u^{1-\frac{\theta+1}{\theta}}\left(u^{-1/\theta}-1\right)^{\theta-1}, v^{1-\frac{\theta+1}{\theta}}\left(v^{-1/\theta}-1\right)^{\theta-1}\right)$
V	$\left((1-u)^{\theta}-\left((1-u)^{\theta}-1\right)(1-v)^{\theta}\right)^{\frac{1}{\theta}-1}\left(u\left(-(1-u)^{\theta-1}\right)\left((1-v)^{\theta}-1\right), v\left(-\left((1-u)^{\theta}-1\right)\right)(1-v)^{\theta-1}\right)$
VI	$(u^{-\theta}+v^{-\theta}-1)^{-1}(u^{-\theta},v^{-\theta})$
VIII	$(1+(1-u)(1-v))^{-1}(2+2uv-v, 2+2uv-u)$

TABLE 7.7: Kendall's tau for the copula models

Model type	$\phi(t)$	Kendall's tau	Dependence
I	$-\lambda \log t$	0	Independent.
II	$\lambda^a(-\log(t))^a$	$1-\dfrac{1}{a}$	Positive .
III	$a\log(1-\beta\log(t))$	$e^{2/\beta}E_i\left(-\frac{2}{\beta}\right)$	Negative .
VI	$\dfrac{B\left(t^{-\theta}-1\right)}{\theta}$	$\dfrac{\theta}{\theta+2}$	Positive for $\theta>0$, Negative for $-1<\theta<0$.
VII	$\theta\left((1-\log(t))^a-1\right)$	$-\dfrac{(a-1)\left(2e^2E_a(2)-1\right)}{a}$	Positive for $a>1$, Negative for $a<1$.
VIII	$\sigma\log\left(\frac{2}{t}-1\right)$	$1+\frac{2}{3}(1-4\log(2))=-0.182$	Negative.

TABLE 7.8: Clayton measure for bivariate Weibull models

Model type	$\theta(x,y)$	Dependence
I	1	Independent
II	$e^{-(-x^{a\alpha}-y^{a\alpha})^{1/a}}-(a-1)\left(-x^{a\alpha}-y^{a\alpha}\right)^{-1/a}$	Positive
III	$1-\dfrac{\beta}{(\beta x^\alpha+1)(\beta y^\alpha+1)}$	Negative
IV	$2+\dfrac{(\theta-1)\left(\left(\left(e^{-x^\alpha}\right)^{-1/\theta}-1\right)^\theta+\left(\left(e^{-y^\alpha}\right)^{-1/\theta}-1\right)^\theta\right)^{-1/\theta}}{\theta}$	Positive
VI	$1+\theta$	Positive for $\theta>0$, Independent for $\theta=0$, Negative for $-1<\theta<0$.
VII	$1+(a-1)\left((x^\alpha+1)^a+(y^\alpha+1)^a-1\right)^{-1/a}$	Positive for $a>1,\alpha\le(\ge)1$, Independent for $a=1,\alpha\le(\ge 1)$, Negative for $\alpha<1,a\le 1(\ge 1)$.
VIII	$1-\dfrac{1}{(2e^{x^\alpha}-1)(2e^{y^\alpha}-1)}$	Negative

7.4.3 Tail dependence measure

The tail dependent measures of the members of the family are given in Table 7.9. We can observe that model IV exhibits both upper and lower tail dependence, whereas model V has upper tail dependence and model VI has lower tail dependence. The rest of the models have neither lower nor upper tail dependence.

TABLE 7.9: Tail dependent measures for the copula models

Model type	λ_U	λ_L
I	0	0
II	0	0
III	0	0
IV	$2-2^{\frac{1}{\theta}}$	$\frac{1}{2}$
V	$2-2^{\frac{1}{\theta}}$	0
VI	0	$2^{-\frac{1}{\theta}}$

7.5 Inference and data analysis

To apply the proposed models to real life data, the inference procedures of the models have to be studied. The unknown parameters of the model are estimated using the method of maximum likelihood. Based on a random sample of n pairs $(x_i, y_i), i = 1, ..., n$ the estimates of the parameters are obtained by maximising the likelihood function. We now apply the proposed family of distributions to two real life data sets.

Example 7.1. The models (7.14),(7.16), (7.17), (7.18) and (7.19) are applied to the data set from Meintanis (2007). The data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all goals together will be called as kick goals) by any team have been considered. The variable X represents the time in minutes of the first kick goal scored by any team, and Y represents the first goal of any type scored by the home team.

The marginals of the data set follow $Weibull(2.12, 1)$ and $Weibull(1.42, 1)$ with Kolmogorov-Smirnov test statistic values 0.083 and 0.105, respectively. Since analytically closed form expressions are not available for the estimators, one has to use the numerical method. The maximum likelihood estimates of the parameters of the models (7.14),(7.16), (7.17), (7.18), and (7.19) are given in Table 7.10. From the estimated values of the parameters given in Table 7.10, we make a conclusion that the variables X and Y are positively correlated for the models (7.14),(7.16), (7.17), (7.18), and (7.19).

We use the bivariate version of Kolmogrov-Smirnov (K.S.) test given in Justel et al. (1997) to test the goodness of fit. The K.S. statistic values D^* of the models (7.14),(7.16), (7.17), (7.18) and (7.19) are given in Table 7.10. Since the D^* values are less than the tabled value at 25th percentile, the models (7.14),(7.16), (7.17), (7.18), and (7.19) fit for the given positive dependent data.

Now we compare the above models to make a conclusion which suits more appropriate for our data. The choice of our model for the above positive dependent data can be made by using Akaike Information Criteria (AIC), given in Akaike (1987). We have computed the AIC values which is given in Table 7.10. From Table 7.10, we conclude that the model given in (7.17) is more appropriate for the data as a smaller relative AIC represents a better model fit.

Example 7.2. We then consider the Iris flower data set available on http://www.math.uah.edu/stat/data/Fisher.html. We have considered the variables sepal length and sepal width of the three species. We made a suitable transformation to the data points. We have also done the goodness of fit test for the marginals, and the marginals of the transformed data set can be fitted by $Weibull(7.22, 1)$ and $Weibull(7.45, 1)$ with Kolmogorov-Smirnov test statistic values 0.122 and 0.081, respectively.

The maximum likelihood estimates of the parameters of the models (7.15) and (7.20), and the AIC values along with the test statistic values, are given in Table 7.11. Since the test statistic values are less than the tabled value at 5th percentile, we conclude that the models (7.15) and (7.20) fit for the given data set. As the model given in (7.20) has less AIC value, it is more appropriate for the given data. From the estimated values of the parameters given in Table 7.11, we conclude that the variables X and Y are negatively correlated for the models (7.15) and (7.20).

7.6 Conclusion

We have derived a class of bivariate Weibull distributions and have studied its distributional properties. The proposed family includes some existing bivariate models, as well as new bivariate distributions. The family of distributions is useful for modelling both negative and positive dependent data structures. The proposed class of distributions has been applied to two real life data sets.

TABLE 7.10: Parameter estimates of the models using Soccer data

Model type	Parameter estimates	Statistic value (D^*)	AIC value
II	$\hat{\alpha} = 1.705$ and $\hat{a} = 1.36$	0.125	102.57
IV	$\hat{\alpha} = 1.58$ and $\hat{\theta} = 1.145$	0.140	102.89
V	$\hat{\alpha} = 1.69$ and $\hat{\theta} = 1.45$	0.143	102.34
VI	$\hat{\theta} = 0.787$ and $\hat{\alpha} = 1.63$	0.142	104.57
VII	$\hat{a} = 2.50$ and $\hat{\alpha} = 1.66$	0.139	104.39

TABLE 7.11: Parameter estimates of the models using Fisher Iris data

Model type	Parameter estimates	Statistic value (D^*)	AIC value
III	$\hat{\alpha} = 7.32$ and $\hat{\beta} = 0.07$	0.136	-324.29
VIII	$\hat{\alpha} = 7.28$	0.137	-328.47

Chapter 8

Summary and future work

8.1 Summary

In the analysis of statistical data, a fundamental problem that emerges is the identification of an appropriate model that can describe the real situation. Once we recognize the correct model the original problem can be analysed with lesser effort, as the properties of the model comes handy to the analyst in drawing inferences and decisions. Owing to the availability of a large number of probability distributions at disposal, very often the selection of a particular one in a specific situation turns out to be difficult, unless one has a reasonable basis or criteria that justifies the choice. The identification of the appropriate distribution can be accomplished in more than one way. A major limitation of the probability models discussed in literature are that they are individual in nature, each based on specified properties so that they lack a uniform framework. The models have low flexibility in the sense that they cannot conform to different real data situation warranting inspection of each model separately and also a different family is needed for each marginal distribution. Copulas provide generalized approaches for modelling joint distributions and dependence aspects, in the sense that copulas contain several bivariate distributions by changing the form of the marginal distributions. Thus copula models have more flexibility in terms of forms of distributions, dependence relationships, number of parameters, and relatively easier practical implementation. They have the advantage that the constituent marginal distributions and the copula function can be modelled and estimated separately. Further, copulas remain invariant under increasing and continuous transformations. There are occasions when information about the marginal distributions are known or captured from the data on joint lifetimes. In such cases the only problem of interest to the analyst is to understand how the marginals are tied together in the joint form. Measures and concepts of dependence assist in distinguishing the appropriate form, for which nonparametric measures are readily available for copulas. In situations where it is difficult to derive a model, families of distributions provide functionally simple approximations. Motivated by these facts, in the present thesis, we have derived various families of distributions useful in different data situations.

In Chapter 2, we have introduced a family of bivariate Pareto distributions using a generalized version of dullness property. Some important bivariate Pareto distributions were derived as special cases. We have studied the distributional as well as dependence properties of the family. Finally, the family of distributions was applied to two real life data situations.

In Chapter 3, the characterization properties of a family of bivariate Pareto distributions introduced in Chapter 2 were studied. Two measures of income inequality namely income gap ratio and mean left proportional residual income were defined in the bivariate case. We have also introduced bivariate generalized failure rate useful in reliability analysis. Characterizations, using the above concepts, for various members of the family of bivariate Pareto distributions were also derived.

In Chapter 4, a variant approach was proposed by defining reliability functions directly from the copula rather than using the distribution-based measures in modelling survival data. The advantages of the proposed functions over the reliability measures were discussed. Characterizations of some well known copulas using the proposed measures were also discussed.

In Chapter 5, we discussed a one-parameter family of Archimedean copulas which suits for a negative dependent data. The distributional properties as well as the dependence measures such as tail dependence, tail monotonicity, Kendall's tau, Spearman's rho and measure based on Blomqvist's β were discussed. The local dependence measures such as ψ-measure and the Clayton-Oakes association measure (θ- measure) for the copulas were discussed. We have applied the copulas to a real data set.

In Chapter 6, we discussed a positive dependent Archimedean copula useful for modelling positive dependent data sets. Various properties of the model like the dependence structure, tail monotonicity, Kendall's measure and measure based on Blomqvist's β were discussed. We have also introduced different bivariate distributions useful in reliability analysis. The proposed model was fitted to a real data. A comparison with other positive dependent Archimedean copula was done using AIC measure.

In Chapter 7, we have discussed a class of bivariate Weibull distributions. This class includes some of the existing models as members. Our choice of the marginal distributions as Weibull lead to a copula for the proposed family. The general form of the copula was Archimedean. The dependency structure of the family was investigated. Finally, the family of distributions was applied to two real life data sets and we have done comparison among the models using AIC measure to make a conclusion, which suits more appropriate for our data sets.

8.2 Future work

In Chapter 2, a family of bivariate Pareto distributions was derived using a generalized version of dullness property. The copulas of the member distributions were also discussed. The dependence measures of these distributions as well as the reliability properties of the models using copulas is an area to be studied. Various characterizations of the family of bivariate Pareto distributions discussed in Chapter 3 can be extended to the multivariate set up.

We have expressed the copula-based reliability concepts such as hazard rate and mean residual life function in the bivariate case in Chapter 4. The methodology can be extended to the general p-variate case, in the following way.

Let $\underline{X} = (X_1, ..., X_p)$ be a vector of lifetimes with survival function

$$\bar{F}(\underline{x}) = P[X_1 > x_1, ..., X_p > x_p]$$

which is continuous with strictly decreasing and continuous one-dimensional marginals $\bar{F}_i(x_i), i = 1, 2, ..., p$. The hazard rate function of $\underline{X}$ is

$$(h_1(\underline{x}), h_2(\underline{x}), ..., h_p(\underline{x}))$$

where

$$h_i(\underline{x}) = \frac{-\partial \log \bar{F}(\underline{x})}{\partial x_i}, i = 1, 2, ..., p. \tag{8.1}$$

Using (8.1), we obtain the multivariate copula-based hazard rate as the vector with components

$$G_1(\underline{u}) = \frac{A_1(\underline{u})}{A_1(u_1, 1, ..., 1)}, G_2(\underline{u}) = \frac{A_2(\underline{u})}{A_2(1, u_2, 1, ..., 1)}, ..., G_p(\underline{u}) = \frac{A_p(\underline{u})}{A_p(1, 1, ...1, u_p)}$$

where $\underline{u} = (u_1, u_2, ..., u_p)$ and $A_i(\underline{u}) = h_i(\bar{F}_1^{-1}(u_1), ..., \bar{F}_p^{-1}(u_p))$.

From

$$G_i(\underline{u}) = u_i \frac{\partial \log \hat{C}(\underline{u})}{\partial u_i}, i = 1, 2, ..., p$$

we arrive at

$$\hat{C}(\underline{u}) = u_p \exp\left[-\int_{u_1}^{1} \frac{G_1(p, u_2, ..., u_p)}{p} dp - \int_{u_2}^{1} \frac{G_2(1, p, u_3, ..., u_p)}{p} dp - \dots \right.$$
$$\left. - \int_{u_{p-1}}^{1} \frac{G_p(1, 1, ..., p, u_p)}{p} dp \right].$$

The $(p-1)$ components of the copula hazard rate are enough to determine $\hat{C}(\underline{u})$. By similar calculations we can work out the result in the case of the mean residual life function which also requires only $(p-1)$ components to evaluate $\hat{C}(\underline{u})$. This idea can be employed to derive other reliability concepts such as reversed hazard rate, mean activity time etc. in terms of copulas.

The methodology discussed in Chapter 7 for the bivariate Weibull models can be generalized to the p-variate set up by choosing

$$g(x_1, x_2, ..., x_p) = [H^{-1}(H(x_1^{\alpha}) + H(x_2^{\alpha}) + ...H(x_p^{\alpha}))]^{\frac{1}{\alpha}}, \quad x_i > 0, \alpha > 0$$

which provides the joint survival function as

$$\bar{F}(x_1, x_2, ..., x_p) = \exp[-g(x_1, x_2, ..., x_p)^{\alpha}], \quad x_i > 0, \alpha > 0. \tag{8.2}$$

Copula models have become one of the most widely used tools in the applied modelling of multivariate data. Bayesian methods are popular in recent times to obtain efficient estimates of the parameters. However, there has been only limited works in Bayesian techniques in the formulation and estimation of copula models. This is an area of research that remains to be explored.

List of Published/Accepted papers

1. Sankaran, P. G., Nair N.U., and Preethi, John. (2014). A family of bivariate Pareto distributions. *Statistica*, LXXIV(2) , 199–215.

2. Sankaran, P. G., Nair N.U., and Preethi, John. (2015). Characterizations of a family of bivariate Pareto distributions. *Statistica* , LXXV(3), 275–290.

3. Preethi, J., and Sankaran, P. G. (2016). A positive dependent Archimedean copula. *Accepted for publication in Journal of Applied Mathematics and Statistics*.

4. Preethi, J., & Sankaran, P. G. (2017). A Bivariate Weibull Family with Applications. *American Journal of Mathematical and Management Sciences*, *36*, 162–175.

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