MODELLING AND ANALYSIS OF DISCRETE LIFETIME DATA

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Nidhi P Ramesh

Department of Statistics
Cochin University of Science and Technology
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CERTIFICATE

This is to certify that the thesis entitled **"MODELLING AND ANALYSIS OF DISCRETE LIFETIME DATA "** is a bonafide record of work done by Mr. Nidhi P Ramesh under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

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Dr. P. G. Sankaran

Professor and Head,

Department of Statistics,

Cochin University of

Science and Technology.

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January 2017

Dr. P. G. Sankaran
Professor and Head,
Department of Statistics,
Cochin University of
Science and Technology.

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Cochin- 22 **Nidhi P Ramesh**

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Nidhi P Ramesh

To My Mother...

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Chapter 1

Introduction

The theory of reliability has been formulated as the science of predicting, estimating and optimizing the probability of survival, the mean life and more generally the life distribution of components or systems. The problem of finding an appropriate model to represent lifetime data, that of assessing the capability of a component and deriving methods for improving the performance of a system, are of vital importance to frame up maintenance policies as well as future planning in industry. During the past decades, the development of reliability as a separate discipline has been rapid, mainly because of its applications in several branches of learning, such as Biology, Medicine, Engineering, Economics, Demography, etc.

The word reliability is used to denote the probability of a device (component, item or organism) to perform its defined purpose adequately for a specified period of time, under the operating conditions specified, and is often used as a measure of the capability of the device to operate without failure when put in service. Earlier works in reliability theory were centred around the problem of estimating reliability, based on observed data. However, recently a lot of interest has been evoked to model lifetime data and to classify the life distributions based on certain ageing properties. Accordingly, large number of research papers have come out which examine the behaviour of the life distributions based on certain criterion for ageing. In most of the discussions on reliability theory, the lifetime is treated as continuous. Barlow and Proschan [16], Kalbfleisch and Prentice [75] and Lai and Xie [85] give a comprehensive review of the topic. Comparatively much less literature on modelling is available when lifetime is discrete. However, there are some compelling reasons to consider failure times as discrete random variables taking on non-negative integer values. When a piece of equipment operates in cycles and the observation is the number of cycles completed before failure, the lifetime is clearly discrete. So also is the case when the device is monitored only in completed units of time, like how many failures have occurred at the completion of one hour, two hours etc. The lack of accuracy of the measuring devices may also generate discrete lives. There are occasions to prefer counts over clock time even when the latter is available. For reliability of weapons, the number of rounds fired is more important than the age at failure. Similar is the case with lifetime of car tyres where the number of kilometres run before it becomes out of use is preferred to the number of days before failure. Thus, there is a strong reason for studying reliability in discrete time. The concepts in continuous and discrete times are the same, but the definitions and interpretations may differ from one case to another. To derive reliability properties similar to the continuous case, occasionally continuous distributions are discretized. But it is not necessary in such cases that the distributional properties are the same nor the discretization process to always produce meaningful discrete models. Thus, there are conceptual and mathematical problems in developing discrete reliability theory. Motivated by these facts, in the present work, we intend to study the reliability properties of discrete lifetime models.

The primary concern in reliability theory is to understand the pattern in which failures occur for different devices under varying operating environments. This is often done by analysing the observed failure times or ages at failure with the help of a model that satisfactorily represents the predominant features of the data. One direct method is to find a probability distribution that provides a reasonable fit to the observations. Sometimes there may exist more than one distribution that pass an appropriate goodness of fit test. Thus, it is more desirable to find a probability model that manifest certain physical properties of the failure mechanism. In reliability theory, some basic concepts that help in the study of failure patterns have been developed. Two important aspects that necessitate the study of these concepts are (a) various functions considered in this context determine the life distribution uniquely, so that the knowledge of their functional form is equivalent to that of the distribution itself and (b) it should be easier to deal with these functions than the distribution function or probability density function of the corresponding distributions. Keeping these facts in mind, in the following, we present such basic reliability concepts and discuss their properties and inter-relationships.

1.1 Basic concepts

1.1.1 Hazard rate function

Let X be a non-negative integer valued random variable representing lifetime of a unit with probability mass function $f(x) = P[X = x]$. Let $S(x) = P[X \geq x]$ be the survival(reliability) function of X. The hazard rate $h(x)$ of X is defined as

$$h(x) = \frac{f(x)}{S(x)},\ x = 0, 1, 2, ... \tag{1.1.1}$$

It is shown that $h(x)$ uniquely determines the distribution by the identity

$$S(x) = \begin{cases} \prod_{t=0}^{x-1}(1 - h(t)) & : x \geq 1 \\ 1 & : x = 0. \end{cases} \tag{1.1.2}$$

Although, in the continuous set-up, the concept of hazard rate dates back to historical studies in human mortality, its discrete version came up much later in the works of Barlow et al. [17], Cox [38] and Kalbfleisch and Prentice [75] to mention a few.

In the discrete case, it is obvious from (1.1.1) that $0 \leq h(x) \leq 1$. For various properties of (1.1.1), we refer to Gupta [52], Shaked et al. [136], Kemp [78] and Lai and Xie [85].

The hazard rate $h(x)$ does not satisfy properties analogous to the continuous case in which the cumulative hazard rate $H(x)$ satisfies the identity $H(x) = -\log S(x)$. In view of these, Cox and Oakes [37] proposed an alternative definition of cumulative hazard rate in the form

$$H^*(x) = -\log S(x), \tag{1.1.3}$$

which means that

$$H^*(x) = -\log \prod_{t=0}^{x-1}(1 - h(t)). \tag{1.1.4}$$

If

$$H^*(x) = \sum_{t=0}^{x-1} h^*(t), \quad x = 1, 2, ...,$$

then $H^*(x)$ is a cumulative hazard rate corresponding to an alternative hazard rate defined by

$$h^*(x) = \log \frac{S(x)}{S(x+1)}, \quad x = 0, 1, 2, ... \tag{1.1.5}$$

Xie et al. [145] and Kemp [78] have studied various properties and applications of both $h(x)$ and $h^*(x)$ in the context of lifetime data analysis. The two functions $h(x)$ and $h^*(x)$ are related through

$$h(x) = 1 - \exp[-h^*(x)]. \tag{1.1.6}$$

Equation (1.1.6) shows that the monotonic behaviour of $h(x)$ and $h^*(x)$ are same.

Shaked et al. [136] gave the necessary and sufficient conditions for a sequence, $\{h(x), x \geq 0\}$ to be a failure rate as follows:

Theorem 1.1.1. A function $h : \mathcal{N} = \{0, 1, 2, ...\} \rightarrow [0, 1]$ is a failure rate function of some random variable with support in $\mathcal{N}$ if and only if

(a) $h(m) = 1$ for some $m \in \mathcal{N}$ and $h(x) = 1 \; \forall \, x > m$
or

(b) $h(x) \in [0, 1)$ for $x \in \mathcal{N}$ and $\sum_{x=0}^{\infty} h(x) = \infty$.

1.1.2 Residual life functions

The analysis of the lifetime of a device or organism after it has attained age x is of special relevance in reliability and survival analysis. Thus, if X is the original lifetime with survival function $S(x)$, the corresponding residual lifetime after age x is the random variable $X_x = (X - x | X > x)$. From the definition of conditional probability we can write the distribution of X_x as

$$S_x(t) = \frac{S(x+t+1)}{S(x+1)}; \; t = 0, 1, 2, ...; \; x = -1, 0, 1, ... \tag{1.1.7}$$

The mean residual life (MRL) of X is defined as

$$\begin{aligned} m(x) &= E[X - x | X > x] \\ &= \frac{1}{S(x+1)} \sum_{t=x+1}^{\infty} (t - x) f(t) \end{aligned}$$

$$= \frac{1}{S(x+1)} \sum_{t=x+1}^{\infty} S(t); \; x = -1, 0, 1, ... \tag{1.1.8}$$

Characterizations of the distribution of X in-terms of $m(x)$ and $h(x)$ have been studied by Nair and Hitha [96]. From (1.1.8),

$$S(x)m(x-1) - S(x+1)m(x) = S(x),$$

which leads to the identity

$$h(x) = \frac{1 + m(x) - m(x-1)}{m(x)}; \; x = 0, 1, 2, ... \tag{1.1.9}$$

From (1.1.2)

$$S(x) = \begin{cases} \prod_{t=0}^{x-1} \left(\frac{m(t-1) - 1}{m(t)} \right) & : x \geq 1 \\ 1 & : x = 0. \end{cases} \tag{1.1.10}$$

Thus, the three functions $h(x), m(x)$ and $S(x)$ determine each other uniquely. Though $h(x)$ can be determined from $m(x)$ and vice-versa, both have unique features that ensure their necessity in reliability theory. The mean residual life function may exist even when the hazard rate function does not exist and vice-versa. Nair and Hitha [96], Hitha and Nair [64] and Nair and Sudheesh [104] give characterizations based on MRL function.

The variance of the residual life X_x is studied in reliability theory in various contexts. Primarily, its role is to define ageing concepts that are weaker than some ageing criteria based on the hazard rate and the mean residual life. Secondly, variance of residual life has the same role as the usual variance, when estimators of mean residual life are discussed. It is also required for studying the coefficient of variation of residual life. Assuming that $E[X^2] < \infty$, we define the variance residual life (VRL) function as

$$\sigma^2(x) = E[(X-x)^2 | X > x] - m^2(x) \tag{1.1.11}$$

$$= \frac{2}{S(x+1)} \sum_{t=x+1}^{\infty} \sum_{u=t+1}^{\infty} S(u) + m(x), \; x = 0, 1, 2, ... \tag{1.1.12}$$

There exist inter-relationships between the reliability functions discussed so far. We have

$$\sigma^2(x+1) - \sigma^2(x) = h(x+1) \left[\sigma^2(x+1) - m(x+1)(m(x) - 1) \right]. \tag{1.1.13}$$

Unlike the hazard rate and mean residual life functions, there is no inversion formula that expresses the survival function in-terms of the variance residual life. Further, there are only a few standard distributions for which $\sigma^2(x)$ has simple tractable forms. Therefore, characterizations of life distributions involving $\sigma^2(x)$ take the form of its relationship with other concepts. For characterization results, we refer to Hitha and Nair [64] and Nair and Sudheesh [105].

1.1.3 Equilibrium distribution

Equilibrium distribution plays a vital role in the analysis of lifetime data. In the discrete set-up, the equilibrium distribution for X is defined by

$$g(x) = \frac{P[X > x]}{\mu} = \frac{S(x+1)}{\mu}, x = 0, 1, 2, ..., \tag{1.1.14}$$

where $\mu = E[X]$. Li [89] observed that when X represents the claim size, the first order equilibrium distribution can be interpreted as the distribution of the amount of first drop below the initial reserve, given that there is such a drop. He also gave some properties of the equilibrium distribution useful in connection with ruin theory.

Definition 1.1.1. The equilibrium distribution of order n of X is defined by

$$f_n(x) = \frac{S_{n-1}(x)}{\mu_{n-1}},\ n = 1, 2, 3, ...;\ x = 0, 1, 2, ... \tag{1.1.15}$$

with $\mu_{n-1} = \sum_{x=0}^{\infty} S_{n-1}(x+1) < \infty,\ S_0(x) = S(x)$ and $\mu_0 = \mu$.

For various properties of equilibrium distribution, we refer to Nair et al. [107] and their references.

1.1.4 Ageing classes

By the term ageing, we mean the phenomenon by which the life remaining to the unit is affected by its current age in some probability sense. Generally, ageing is classified into positive ageing, negative ageing and no ageing. If the residual lifetime is decreasing when the age is increasing, we say that the unit is ageing positively. For instance, the efficiency of various equipments in common use or mechanical systems tend to decrease due to wear and tear as a result of prolonged use. Naturally, the remaining lifespan decreases when the time for which they are used is increasing. In short, the effect of ageing is to decrease the reliability in the case of positive ageing. On the other hand there are situations in which the performance of a unit improves with increasing age. A classical example is that of human being whose remaining lifetime increases once they pass infancy. So also is the case of equipments that undergo efficient preventive maintenance. In all such cases, we say that the unit has negative ageing, which in technical terms is explained as the increase in residual lifetime as the age is increasing. Contrast to the above two ageing categories, there are situations in which the residual lifetime remains the same at all ages. This is described as no-ageing property. It is well-known that only geometric distribution possesses such a property among all discrete lifetime models. The basic reliability concepts such as survival (reliability) function, hazard rate, mean residual life etc. form the fabric with which the ageing characteristics are built upon.

In the discrete case, Langberg et al. [86] showed that the class of decreasing hazard rate (DHR) life distributions is a convex class and have obtained the extreme points of this class. They showed how to represent a discrete DHR distribution as a mixture of these extreme points. Fagiuoli and Pellerey [44] studied different classes of life distributions and found relationships among them. They also considered life distribution of a device subject to shocks occurring randomly according to a Poisson process and sufficient conditions for it to belong to different ageing classes were discussed. Gupta et al. [51] developed statistical tools for the determination of IHR and DHR property for a wide class of discrete distributions, making use of the ratio of two consecutive probabilities, and they applied it to various well-known families of discrete distributions. Bracquemond et al. [24] studied basic notions of ageing, such as IHR, IHRA and NBU, when the system lifetimes are discrete random variables. Kemp [78] derived the relationships between various ageing classes to which a discrete lifetime distribution and its current age distribution belong.

Here we discuss some important ageing classes that have been widely applied in the field of reliability analysis.

Definition 1.1.2. A discrete lifetime random variable X is said to have an increasing (decreasing) hazard rate (IHR/DHR) if the sequence $\{h(x)\}$ is increasing (decreasing). i.e., X is IHR (DHR) according as $h(x+1) \geq (\leq) h(x),\ \forall\, x = 0, 1, 2, ...$

It is well-known that the IHR class of distributions is not closed under the formation of coherent systems. The increasing hazard rate average concept was introduced in an effort to find the class of distributions that preserves this closure property.

In discrete time case, more than one definition for IHRA and NBU classes have been proposed in literature. We refer to Shaked et al. [136], Bracquemond et al. [24] and Roy and Gupta [127] for more details.

Definition 1.1.3. A discrete lifetime random variable X is said to have

(a) increasing (decreasing) hazard rate average-1 ($IHRA_1/DHRA_1$) property, if $[S(x)]^{\frac{1}{x}}$ is decreasing(increasing) in x or equivalently, the alternative hazard rate average

$$\frac{H^*(x)}{x} = \frac{1}{x}\sum_{t=0}^{x-1} h^*(t) \tag{1.1.16}$$

is increasing(decreasing) in x.

(b) increasing (decreasing) hazard rate average-2 ($IHRA_2/DHRA_2$) property, if the hazard rate average $\frac{H(x)}{x} = \frac{1}{x}\sum_{t=0}^{x-1} h(t)$ is increasing (decreasing) in x.

(c) decreasing (increasing) mean residual life (DMRL/IMRL) property if the sequence $\{m(x)\}$ is decreasing (increasing) in x.

(d) new better (worse) than used (NBU_1/NWU_1) property if $S(x+t) \leq (\geq) S(x)S(t)$ for all $x, t = 0, 1, 2, ...$

(e) new better (worse) than used (NBU_2/NWU_2) property if $\sum_{x=0}^{k-1} h(x) \leq (\geq) \sum_{x=j}^{j+k-1} h(x)$, $j = 0, 1, 2, ..., k = 1, 2, ...$

(f) decreasing (increasing) variance residual life (DVRL/IVRL) property if $\sigma^2(x)$ is decreasing (increasing) in x.

(g) new better(worse) than used in expectation(NBUE/NWUE) property if $m(x) \leq (\geq) E[X]$ for all $x = 1, 2, ...$

(h) harmonically new better(worse) than used in expectation(HNBUE/HNWUE) property if $\sum_{x=0}^{n} m(x)^{-1} \geq (\leq) \frac{n}{E[X]}$.
and

(i) NBU-y_0(NWU-y_0) property if $S(x + y_0) \leq (\geq) S(x)S(y_0)$ for all $x = 0, 1, 2, ...$, $y_0 = 1, 2, ...$

The NBU-y_0 (NWU-y_0) property requires that the lifetime after y_0 is smaller (larger), compared to the original one in probability sense. Instead of keeping y_0 fixed, we can think of the above behaviour beyond y_0, giving rise to the $NBU{*}y_0$ ($NWU{*}y_0$) class as given below.

Definition 1.1.4. A discrete lifetime random variable X is said to have $NBU{*}y_0$ ($NWU{*}$ y_0) property, if $S(x + t) \leq (\geq) S(x)S(t),\ x = 0, 1, 2... : t = y_0, y_0 + 1, ...$

We have the following chain of implications among these classes.

$$
\begin{array}{lllll}
\text{IHR} \Rightarrow & \text{IHRA}_1 \Rightarrow & \text{NBU}_1 \Rightarrow & \text{NBUE} \Rightarrow & \text{HNBUE} \\
\Downarrow & \Downarrow & \Downarrow & & \\
\text{DMRL} & \text{IHRA}_2 & \text{NBU}{*}y_0 & & \\
\Downarrow & \Downarrow & \Downarrow & & \\
\text{DVRL} & \text{NBU}_2 & \text{NWU-}y_0 & &
\end{array}
$$

1.1.5 Stochastic orders

Recently, researchers have focused on investigation of ageing properties using stochastic orders. Stochastic orders enable global comparison of two distributions in-terms of their characteristics. In reliability studies, researchers have reliability function, hazard rate, mean residual life, etc. for such a comparison. An excellent review on various stochastic orders and their properties is available in Shaked and Shanthikumar [135].

Definition 1.1.5. Let X_1 and X_2 be two non-negative integer valued random variables representing lifetimes of two units. Then X_1 is smaller than X_2 in

(a) usual stochastic order $(X_1 \leq_{st} X_2)$ if $S_{X_1}(x) \leq S_{X_2}(x), x = 0, 1, 2, ...$

(b) hazard rate order $(X_1 \leq_{hr} X_2)$ if $h_{X_1}(x) \geq h_{X_2}(x), x = 0, 1, 2, ...$

(c) MRL ordering $(X_1 \leq_{mrl} X_2)$ if $m_{X_1}(x) \leq m_{X_2}(x),\ x = 0, 1, 2, ...$

(d) likelihood ratio order $(X_1 \leq_{lr} X_2)$ if $\dfrac{f_{X_1}(x)}{f_{X_2}(x)}$ is non-increasing in x over the union of supports of X_1 and X_2.
and

(e) concave transform order $(X_1 \leq_c X_2)$ if $F_{X_2}^{-1}F_{X_1}(x)$ is concave in the support of X_1 where $F_{X_1}(.)$ and $F_{X_2}(.)$ are the distribution functions of X_1 and X_2 respectively.

The definitions (a) to (d) are given in Shaked and Shanthikumar [135] and the definition (e) is the discrete analogue of the continuous case discussed by them.

1.1.6 Odds function

The role of odds function and odds rate in the context of lifetime data analysis is of substantial interest in recent times. The motivation for consideration of these are- (a) easy to compute and interpret, (b) estimation of these functions are relatively simpler and (c) behaviour of other reliability functions can be ascribed through them.

Definition 1.1.6. The odds function for failure by age x for a discrete random variable X is defined as

$$\bar{\omega}(x) = \frac{F(x)}{S(x+1)};\ x = 0, 1, 2, ...; \bar{\omega}(-1) = 0. \tag{1.1.17}$$

From the definition, it follows that $\bar{\omega}(\infty) = \infty$ and $\bar{\omega}(x)$ is increasing. Odds function $\bar{\omega}(x)$ is an important tool in survival analysis for comparing a treatment group and a control group and in developing models for survival data. We refer to Collet [36] and Kirmani and Gupta [81] for further details. The role of odds function in discrete reliability theory has been studied by Nair and Sankaran [103].

It is easy to prove that

$$F(x) = \frac{\bar{\omega}(x)}{1 + \bar{\omega}(x)} \tag{1.1.18}$$

and

$$h(x) = \frac{\bar{\omega}(x) - \bar{\omega}(x-1)}{1 + \bar{\omega}(x)} \quad x = 0, 1, 2, ... \tag{1.1.19}$$

Further, $h(x)$ is increasing implies that $\bar{\omega}(x)$ is convex. For details, we refer to Nair and Sankaran [103].

1.1.7 Reversed hazard rate

Let X be a discrete random variable taking values in $S = \{0, 1, 2, ..., b\}$, where b can be ∞. We denote the probability mass function and distribution function of X by $f(x)$ and $F(x)$ respectively. Then the reversed hazard rate of X is defined as

$$\lambda(x) = P[X = x | X \leq x] = \frac{f(x)}{F(x)},\ x = 0, 1, 2, ..., b. \tag{1.1.20}$$

The distribution of X is determined uniquely by $\lambda(x)$ through the formula

$$F(x) = \prod_{t=x+1}^{b} (1 - \lambda(t)). \tag{1.1.21}$$

For more properties of the univariate reversed hazard rate, see Nair and Sankaran [99]. The definition (1.1.20), when applied to the continuous case has the form $\lambda(x) = \frac{d}{dx} \log F(x)$ implying $F(x) = \exp\{-\int\limits_{x}^{b} \lambda(t)dt\}$. This exponential representation in the continuous case contribute to the additive property of reversed hazard rate for parallel systems. But this property is not shared by (1.1.20) in the discrete case.

Thus, analogous to the alternative hazard rate due to Cox and Oakes [37], an alternative reversed hazard function can be defined as

$$\lambda^*(x) = \log \frac{F(x)}{F(x-1)},\ x = 1, 2, ..., b. \tag{1.1.22}$$

Then the distribution of X is obtained as

$$F(x) = F(0) \exp\left[\sum_{t=1}^{x} \lambda^*(t)\right],\ x = 1, 2, ..., b. \tag{1.1.23}$$

Note that the alternative reversed hazard rate in (1.1.22) possesses additivity for parallel system.

It is well-known that the lack of memory property is equivalent to constancy of the hazard rate. Analogous to the lack of memory property, Nair and Sankaran [99] defined the univariate reversed lack of memory property in the following way.

Definition 1.1.7. The random variable X is said to satisfy the reversed lack of memory property if

$$P[X \leq t|X \leq t+s] = P[X \leq 0|X \leq s] \tag{1.1.24}$$

for all t and s in S.

In the context of maintenance problems, the property (1.1.24) can be interpreted as follows. When X represents the lifetime of a device, (1.1.24) implies that its inactivity time (time since failure) is independent of the age of the device.

1.1.8 Reversed mean and variance residual lives

A second measure of interest in reversed time is the reversed mean residual life. Suppose that a device has failed before attaining age t. Then the random variable ${}_tX = t - X|X < t$ is the time elapsed since the device has failed, conditioned on the fact that its lifetime is less than t and is called the reversed residual life or inactivity time of X. It is easy to see that ${}_tX$ has the distribution function

$${}_tF(x) = \frac{F(t-1) - F(t-x-1)}{F(t-1)};\ x = 0, 1, 2, ... \tag{1.1.25}$$

The mean of this distribution is called the reversed mean residual life or mean inactivity time and is denoted by $r(x)$. We define $r(x)$ as

$$r(x) = E[x - X|X < x] = \frac{1}{F(x-1)} \sum_{t=1}^{x-1} tf(t);\ x = 1, 2, ..., \tag{1.1.26}$$

with $r(0) = 0$. Note also that $r(1) = 1$. The reversed hazard rate and reversed mean residual life function are related by

$$\lambda(x) = \frac{1 + r(x) - r(x+1)}{r(x)};\ x = 1, 2, ... \tag{1.1.27}$$

For more properties of $r(x)$, we refer to Goliforushani and Asadi [49].

Just as the mean of the reversed residual life ${}_tX$, the variance of ${}_tX$ is also an important function in reliability analysis, called the reversed variance residual life or variance inactivity time and is denoted by $\nu(x)$. In algebraic manipulations, different expressions for $\nu(x)$ are employed. They are

$$\begin{aligned}\nu(x) &= V[x - X|X < x] = V[X|X < x]\\ &= E[X^2|X < x] - E^2[X|X < x]\end{aligned} \tag{1.1.28}$$

and

$$\nu(x) = E[(x - X)^2|X < x] - r^2(x) \tag{1.1.29}$$

$$= \frac{2}{F(x-1)} \sum_{t=1}^{x} \sum_{u=1}^{t} F(u-1) - r(x)(r(x) + 1). \tag{1.1.30}$$

In the continuous case, identities that connect the three functions $\lambda(x)$, $r(x)$ and $\nu(x)$ have been established. The corresponding results in the discrete case are

$$\nu(x+1) - \nu(x) = \lambda(x)\left[r(x)(r(x+1) - 1) - \nu(x)\right] \tag{1.1.31}$$

and

$$\frac{\nu(x+1)}{r(x+1) - 1} + r(x+1) - 1 = \frac{\nu(x)}{r(x)} + r(x). \tag{1.1.32}$$

Equations (1.1.31) and (1.1.32) are employed in finding $\nu(x)$ when the others are known, especially in characterization problems and also in the discussions on the monotonicity of the reversed variance residual life function.

1.2 Multivariate reliability concepts

There are many complex devices and systems whose functioning depends on several components that may be dependent or independent. In such a scenario, the lifetime is made up of the lifetimes of the components that may depend on different physical properties. The study of system reliability can be facilitated only through the joint distribution of the component lifetimes. Thus, there is a need to extend various univariate reliability concepts into higher dimensions. The development of concepts depends on the manner in which a univariate notion is generalized to suit the multivariate case. There are several possible definitions for a particular notion depending on the property that is employed in each situation. Consequently, the resulting models will also change. It is difficult to propose a set of criteria based upon which generalization of univariate formulations have to be done in evolving multivariate concepts. However, it may be reasonable to have a multivariate definition which coincides with the existing definition for a single variable, when appropriately reduced to one dimension and the implications and chain of relationships between multivariate concepts should follow the patterns in the univariate case.

Let $\mathbf{X} = (X_1, X_2, ..., X_p)'$ be a random vector taking values in $\mathbf{N^p}$ with distribution function $\mathbf{F}(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x})$, where $\mathbf{N} = \{0, 1, 2, ...\}$ and $\mathbf{x} = (x_1, x_2, ..., x_p)'$. The notation $(\mathbf{X} \leq \mathbf{x})$ means $(X_1 \leq x_1,\ X_2 \leq x_2,...X_p \leq x_p)$. The survival function of $\mathbf{X}$ is defined as

$$\mathbf{S}(\mathbf{x}) = P(\mathbf{X} \geq \mathbf{x}) = P(X_1 \geq x_1, X_2 \geq x_2, ..., X_p \geq x_p) \tag{1.2.1}$$

and the probability mass function is given by

$$\mathbf{f}(\mathbf{x}) = P[\mathbf{X} = \mathbf{x}]. \tag{1.2.2}$$

As in the continuous case, several definitions of the hazard rate are possible in the discrete case also. The scalar hazard rate of $\mathbf{X}$ is defined as (Nair and Sankaran [102])

$$a(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{\mathbf{S}(\mathbf{x})}. \tag{1.2.3}$$

The scalar hazard rate fails to determine the underlying distribution uniquely. Nair and Sankaran [102] derived the conditions under which the distribution is uniquely determined by the scalar hazard rate. A more popular version of the multivariate hazard rate is the vector hazard rate due to Nair and Asha [95], which is a generalization of the bivariate vector hazard rate proposed by Nair and Nair [97], defined as

$$\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}), ..., c_p(\mathbf{x}))' \tag{1.2.4}$$

where

$$\begin{aligned} c_i(\mathbf{x}) &= P[X_i = x_i | \mathbf{X} \geq \mathbf{x}] \\ &= 1 - \frac{S(\mathbf{x}_{(i)}, x_i + 1)}{S(\mathbf{x})} \end{aligned} \tag{1.2.5}$$

with $\mathbf{x}_{(i)} = \mathbf{x} - \{x_i\}$. For the definitions and properties of these two hazard rates, we refer to Nair and Sankaran [102] and Nair and Asha [95]. In the bivariate case, the two hazard rates are related through

$$\begin{aligned} a(x_1, x_2) &= c_1(x_1, x_2) - c_1(x_1, x_2 + 1) + c_1(x_1, x_2 + 1)c_2(x_1, x_2) \\ &= c_2(x_1, x_2) - c_2(x_1 + 1, x_2) + c_1(x_1, x_2)c_2(x_1 + 1, x_2). \end{aligned} \tag{1.2.6}$$

Ageing classes based on these hazard rates can be seen in Nair and Sankaran [102] and Nair and Asha [95].

There are several definitions for the mean residual life function in higher dimensions. These depend on the manner in which the extension of the univariate concept is carried over in the multivariate case. In all cases, the interest is on the lifetime of a p-component system after the i th component has attained age x_i; $i = 1, 2, ..., p$.
Buchanan and Singapurwalla [27] proposed a version of multivariate mean residual life in the continuous case. In the discrete case, we can analogously define

$$m^*(\mathbf{x}) = \frac{\sum_{t_1=0}^{\infty} \cdots \sum_{t_p=0}^{\infty} S(x_1 + t_1, ..., x_p + t_p)}{S(x_1, x_2, ..., x_p)}. \tag{1.2.7}$$

However, (1.2.7) lacks any physical interpretation and moreover it fails to determine the corresponding lifetime distribution uniquely. In order to overcome the drawbacks of (1.2.7), the vector MRL function is defined using the random vector

$$\mathbf{X_x} = [\mathbf{X} - \mathbf{x} | \mathbf{X} > \mathbf{x}] \tag{1.2.8}$$

which is the residual life of the system after it has survived age $\mathbf{x}$. The survival function of $\mathbf{X_x}$ is

$$\mathbf{S_x}(\mathbf{t}) = \frac{\mathbf{S}(\mathbf{x}+\mathbf{t}+\mathbf{e})}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \tag{1.2.9}$$

where $\mathbf{t} = (t_1, t_2, ..., t_p)'$ and $\mathbf{e} = (1, 1, ..., 1)'$ and $t_i = 0, 1, 2, ...$ for all $i = 1, 2, ..., p$. The popular definition of the multivariate mean residual life function conceives it as the mean of the distribution (1.2.9). Thus, the multivariate mean residual life (MMRL) function is defined as the vector

$$\mathbf{m}(\mathbf{x}) = (m_1(\mathbf{x}), m_2(\mathbf{x}), ..., m_p(\mathbf{x}))' \tag{1.2.10}$$

where

$$m_i(\mathbf{x}) = E[X_i - x_i | \mathbf{X} > \mathbf{x}],\ i = 1, 2, ..., p;\ x_i = -1, 0, 1, ... \tag{1.2.11}$$

From (1.2.11), we have

$$\begin{aligned} m_i(\mathbf{x}) &= \frac{1}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_1=x_1+1}^{\infty} \cdots \sum_{t_p=x_p+1}^{\infty} (t_i - x_i)\mathbf{f}(\mathbf{x}) \\ &= \frac{1}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} \mathbf{S}(x_1+1, ..., x_{i-1}+1, t, x_{i+1}+1, ..., x_p+1). \end{aligned} \tag{1.2.12}$$

Also,

$$\frac{\mathbf{S}(\mathbf{x_{(i)}}+\mathbf{e}, x_i+2)}{\mathbf{S}(\mathbf{x}+\mathbf{e})} = \frac{m_i(\mathbf{x}) - 1}{m_i(\mathbf{x}_{(i)}, x_i+1)};\ i = 1, 2, ..., p, \tag{1.2.13}$$

arising from (1.2.12). In the bivariate case, (1.2.10) reduces to $(m_1(x_1, x_2), m_2(x_1, x_2))'$ where

$$m_1(x_1, x_2) = \frac{1}{\mathbf{S}(x_1+1, x_2+1)} \sum_{t=x_1+1}^{\infty} \mathbf{S}(t, x_2+1) = E[X_1 - x_1 | X_1 > x_1, X_2 > x_2] \tag{1.2.14}$$

and

$$m_2(x_1, x_2) = \frac{1}{\mathbf{S}(x_1+1, x_2+1)} \sum_{t=x_2+1}^{\infty} \mathbf{S}(x_1+1, t) = E[X_2 - x_2 | X_1 > x_1, X_2 > x_2]. \tag{1.2.15}$$

A detailed study of the mean residual life vector will be carried out in Chapter 5. The multivariate variance residual life will be introduced and studied in Chapter 6.

1.3 Present study

The discussions in the above sections reveal the necessity of studying reliability concepts in discrete time. In the present study, our main aim is a systematic study on modelling and analysis of lifetime data in discrete time. The thesis is organized into ten chapters. The chapter-wise summary of work is as follows.

Chapter 1 discusses the relevance and scope of the study. We also present basic reliability concepts such as survival function, hazard rate function, mean residual life function, etc. We give definitions of some of the basic ageing classes and stochastic orders. Existing literature on reversed time reliability concepts are discussed. Finally, the multivariate extensions of the univariate reliability concepts are presented.

In Chapter 2, we study some properties of univariate hazard rate-based ageing classes. To be specific, we study the ageing classes namely IHR, IHR(2), IHRA, NBUHR and NBUHRA. In many practical situations, the reliability functions exhibit non-monotone behaviour. The important among them are distributions where the hazard rates are bathtub-shaped (BT) and upside-down bathtub-shaped (UBT). We study the closure properties of discrete distributions with BT and UBT hazard rate functions. In our study, we show that BT(UBT) hazard rate classes are not closed under the reliability operations such as formation of mixtures, convolution, coherent system, etc. A result on the convergence of BT(UBT) distributions is also presented. We study the existence of bounds on reliability functions and moment properties. Various ageing criteria discussed in this chapter play a fundamental role in the development of reliability theory and practice.

The processes that manifest different types of ageing behaviour necessitate models that can accommodate and explain the special characteristics of the given data. The limited number of discrete BT and UBT models, proposed in the literature, are unlikely to meet the requirements of modelling for the large number of BT(UBT) data sets that need to be analysed using reliability concepts in discrete time (See Lai and Xie [85] for examples). This points out to the need for criteria for determination of the shape of the hazard rates and also for developing new models. Chapter 3 focuses on this vital problem and investigates some general conditions for assessing the shape of the hazard rates. Our results also help in generating new distributions that have simple hazard rate forms. We discuss the general methods for constructing discrete BT and UBT distributions. Some new models thus obtained are found to be useful in real life situations. We illustrate the applicability of the newly derived models by fitting them to real datasets.

Often, situations arise where one has to compare the reliabilities of more than one device. For example, when the same kind of device is produced by several manufactures, the choice has to be made with reference to the ageing patterns of the competing devices. Relative ageing concepts specify which of two device age faster by comparing the two on the basis of some ageing criterion. Also, there are models in which the nature of ageing depends on the parameter values. Thus, an analysis that reveals the relationship between the ageing property and the model parameters is required. This becomes more important in cases where there are critical values of the parameters which partitions one ageing feature into another which is distinctly different. At present, it appears that there is no study concerning the relative ageing of two devices in the discrete time domain. The objective of Chapter 4 is to fill this gap by presenting some concepts and results that help the comparison of the intensity of ageing among competing devices, when the lifetime is discrete. In this chapter, we introduce specific ageing factor, relative ageing factor and ageing intensity function into the discrete domain. The present chapter includes some results which have no continuous counterparts, and a discussion of stochastic orders for comparing discrete life distributions in-terms of ageing concepts.

In view of the importance and applications of the multivariate reliability concepts, rest of the thesis is devoted to the study of multivariate reliability concepts in the discrete domain. In Chapter 5, we study the mean residual life function in the multivariate discrete domain. We study the properties of multivariate mean residual life (MMRL) function and propose ageing classes based on it. Inter-relationships between these ageing classes are explored. Characterization results using these ageing classes are obtained.

Chapter 6 presents a theoretical exposition of the properties of the multivariate discrete variance residual life. It includes properties of the variance residual life, characterization of life distributions and classes of life distributions based on the monotonic properties of the variance residual life. As a by-product, we also get some properties, that do not seem to have been obtained in the univariate case, by specializing our results.

In Chapter 7, the covariance residual life and its properties are discussed. A measure of association for bivariate discrete data is proposed. Relationships with other association measures are discussed. The application of the theoretical results for real data is provided, by way of illustration, using the multiple tumour recurrence data of patients with bladder cancer in Andrews and Herzberg [10].

The reversed hazard rate function has not been studied in the multivariate discrete set-up. Chapter 8 introduces multivariate reversed hazard rate functions in the discrete case. We present four versions of the multivariate reversed hazard rate in the discrete domain, namely scalar reversed hazard rate, vector reversed hazard rate, alternative reversed hazard rate and conditional reversed hazard rate. Properties of these multivariate reversed hazard rate functions are discussed. Multivariate discrete distributions are characterized based on them.

The Schur-constancy property is studied in the bivariate discrete domain in Chapter 9. The aim of the present chapter is to investigate various properties of discrete Schur-constant models. Specifically, we express the bivariate reliability functions such as bivariate scalar hazard rate, bivariate vector hazard rate, bivariate mean residual life, etc. as functions of the univariate reliability concepts corresponding to the baseline distribution. Using these relationships, we study ageing phenomenon of bivariate Schur-constant models based on univariate ageing concepts. We also study time-dependent measures in the context of discrete Schur-constant models and it is shown that the dependence structure can be studied using univariate ageing properties. The study of Schur-constant models in the bivariate discrete domain is of interest because it helps us to describe the multivariate ageing concepts in-terms of the well-studied univariate ageing concepts.

Finally, Chapter 10 presents major conclusions of the study along with a brief report on the future work to be done.

Chapter 2

Univariate Ageing Classes Based on Hazard Rate

2.1 Introduction

A considerable amount of literature in reliability theory is dedicated to the study of ageing concepts, their properties, implications and applications. The term ageing represents the phenomenon by which the residual life of a unit is affected by its current age in some probability sense. Generally, ageing is classified into positive ageing, negative ageing and no ageing according to whether the residual lifetime decreases, increases or remains the same as age advances.

The basic reliability concepts such as reliability function, hazard rate, mean residual life, etc. are usually employed for developing various ageing classes. Using these concepts, around forty different ageing classes and their properties have been studied in the literature of continuous lifetime. Lai and Xie [85] and Nair et al. [108] provide a comprehensive review on this topic.

Although the relevance of lifetime in discrete units is well established, there is not as

Results in this chapter have been published in the journals "Research & Reviews: Journal of Statistics"and "Communications in Statistics-Theory and Methods"(See Sankaran et al. [130] and Nair et al. [111])

much work as in the continuous case that deals with discrete ageing classes in the literature. Motivated by this, in the present chapter, we study different ageing classes in discrete time and derive some new properties. While discussing different ageing classes, one can group together those based on hazard rate, functions of residual life, reliability function and reliability functions in reversed time, etc. In the present work, we study various ageing classes based on hazard rate.

The rest of the chapter is organized as follows. The ageing classes using monotone hazard rate are studied in Section 2.2. The inter-relationships among various stochastic orders and the ageing classes are established. In Section 2.3, we study increasing (decreasing) hazard rate of order 2 classes. The increasing (decreasing) hazard average classes are studied in Section 2.4. Section 2.5 deals with new better(worse) than used in hazard rate classes. Bathtub and upside-down bathtub hazard rate classes are discussed in Section 2.6. Section 2.7 presents the major conclusions of the study.

2.2 Monotone hazard rate classes

These classes of life distributions are defined by the nature of the monotonicity of the hazard rate. In the sequel, we use the term increasing (decreasing) in the sense of non-decreasing (non-increasing). Recall from Definition 1.1.2 that a discrete lifetime random variable X is said to have IHR (DHR) property if the sequence $\{h(x)\}$ is increasing (decreasing) in x. There are several equivalent conditions for X to be IHR (DHR). The following conditions are direct to verify (Bracquemond et al. [24]).

Proposition 2.2.1. X has IHR (DHR) property if and only if one of the statements holds.

(a) $S_x(t) = \dfrac{S(x+t+1)}{S(x+1)}$ is decreasing (increasing) in t, $\forall\, x = 0, 1, 2, ...$

(b) The sequence $\{\log S(x)\}$ is concave (convex) or $\{S(x)\}$ is log concave (log convex).

(c) $(S(x+1))^2 \geq (\leq)\, S(x)S(x+2),\ \ \forall\, x = 0, 1, 2, ...$

Remark 2.2.1. The condition (a) implies that the residual life distribution is stochastically decreasing (increasing) in t.

Now, we prove a proposition connecting monotonicity of $h(x)$ and various stochastic orders given in Definition 1.1.5.

Proposition 2.2.2. X is IHR (DHR) if and only if one of the following orders holds.

(a) $X \geq_{hr} (\leq_{hr})(X - x|X > x),\ \forall\, x = 0, 1, 2, ...$

(b) $X \geq_{c} (\leq_{c})Y$, where Y is a geometric random variable with distribution function $F_Y(x) = 1 - q^x,\ x = 0, 1, 2, ...$

(c) $(X - x|X > x) \geq_{st} (\leq_{st})(X - y|X > y),\ \forall\, x, y = 0, 1, 2, ... : x \leq y.$

(d) $(X - x|X > x) \geq_{hr} (\leq_{hr})(X - y|X > y),\ \forall\, x, y = 0, 1, 2, ... : x \leq y.$

Proof. We have,

$$
\begin{aligned}
X \geq_{hr} (\leq_{hr})(X - x|X > x) &\iff h(t) \leq (\geq) h(t + x + 1),\ \forall\, t, x = 0, 1, 2, ... \\
&\iff X \text{ is IHR(DHR)}.
\end{aligned}
$$

When Y is geometric,

$$F_Y^{-1}(u) = \frac{\log(1-u)}{\log q}.$$

Thus,

$$
\begin{aligned}
X \geq_c Y &\iff F_Y^{-1} F_X(x) \text{ is convex.} \\
&\iff \frac{\log(1 - F_X(x))}{\log q} \text{ is convex.} \\
&\iff \log(1 - F_X(x)) - \log(1 - F_X(x+1)) \text{ is increasing in } x. \\
&\iff \log \frac{S(x+2)}{S(x+1)} \text{ is decreasing in } x. \\
&\iff X \text{ is IHR.}
\end{aligned}
$$

The proof for DHR class is similar. The proof for (c) is obvious from Proposition 2.2.1. To prove (d), we consider probability mass function of $(X - x|X > x)$, given by

$$\frac{f(t + x + 1)}{S(x + 1)}, t = 0, 1, 2, ...$$

and the survival function
$$\frac{S(t+x+1)}{S(x+1)}, t=0,1,2,....$$
Accordingly, the hazard rate function of $(X-x|X>x)$ becomes

$$h_x(t)=\frac{f(t+x+1)}{S(t+x+1)}=h(t+x+1).$$

Similarly, for $(X-y|X>y)$, the hazard rate function is

$$h_y(t)=h(t+y+1).$$

Then,

$$\begin{aligned}(X-x|X>x)\geq_{hr}(X-y|X>y) &\iff h(t+y+1)\geq h(t+x+1)\,,x\leq y\\ &\iff X \text{ is IHR.}\end{aligned}$$

The proof for DHR class is similar. ■

We now illustrate the above result with the help of two distributions, one possessing IHR property and the other has DHR property.

Example 2.2.1. Let X be distributed as Waring distribution(Nair et al. [107]) with survival function specified by

$$S(x)=\frac{(m)_x}{(m+n)_x};\ m,n>0;\ x=0,1,2,... \tag{2.2.1}$$

and $(b)_x=b(b+1)(b+2)...(b+x-1)$, denotes the Pochhammer symbol.
It is easy to verify that the hazard rate function given by

$$h(x)=\frac{n}{m+n+x};\ x=0,1,2,... \tag{2.2.2}$$

is decreasing. Thus, the distribution is DHR. From (1.1.7), we obtain the survival function of residual life random variable $X_x=[X-x|X>x]$ as

$$S_x(t)=\frac{\Gamma(x+m+n+1)\Gamma(x+t+m+1)}{\Gamma(x+m+1)\Gamma(x+t+m+n+1)};\ x,t=0,1,2,... \tag{2.2.3}$$

and the hazard rate function as

$$h_x(t) = \frac{n}{x+t+m+n+1};\ x, t = 0, 1, 2, ... \tag{2.2.4}$$

From (2.2.2) and (2.2.4), we get

$$\frac{h_x(t)}{h(t)} = \frac{t+m+n}{x+t+m+n+1} < 1.$$

Thus, $X \leq_{hr} (X - x|X > x)$. Now consider

$$\log S(x+1) - \log S(x+2) = \log\left[\frac{(m)_{x+1}}{(m+n)_{x+1}}\frac{(m+n)_{x+2}}{(m)_{x+2}}\right] = \log\left[\frac{x+m+n+1}{x+m+1}\right],$$

which is decreasing in x. Thus, $\dfrac{\log[S(x)]}{\log q}$ is concave implying that $X \leq_c Y$.
Similarly,

$$\frac{S_{x+1}(t)}{S_x(t)} = \frac{(x+m+n+1)(x+t+m+1)}{(x+m+1)(x+t+m+n+1)} \geq 1.$$

The last inequality holds since $\dfrac{x+m+n+1}{x+m+1}$ is decreasing in x. Thus, $(X-x|x > x) \leq_{st} (X - y|X > y)$.
Finally,

$$\frac{h_{x+1}(t)}{h_x(t)} = 1 - \frac{1}{x+m+n+t+2} < 1,$$

yielding $(X - x|X > x) \leq_{hr} (X - y|X > y)$.

Example 2.2.2. Let X follow the negative hyper-geometric distribution with survival function

$$S(x) = \frac{\begin{pmatrix} k+n-x \\ n-x \end{pmatrix}}{\begin{pmatrix} k+n \\ n \end{pmatrix}};\ x = 0, 1, 2, ..., n;\ k > 0. \tag{2.2.5}$$

The hazard rate function of X is given by

$$h(x) = \frac{k}{k+n-x},\quad x = 0, 1, 2, ..., n \tag{2.2.6}$$

which is increasing in x. Hence the distribution is IHR. The survival function and hazard rate function of the residual life are calculated as

$$S_x(t) = \frac{\Gamma(n-x)\Gamma(k+n-t-x)}{\Gamma(k+n-x)\Gamma(n-t-x)};\ x = 0,1,2,...,n;\ t = 0,1,2,...,n-x-1 \quad (2.2.7)$$

and

$$h_x(t) = \frac{k}{k+n-t-x-1};\ x = 0,1,2,...,n;\ t = 0,1,2,...,n-x-1. \quad (2.2.8)$$

Thus, we obtain

$$\frac{h_x(t)}{h(t)} = \frac{k+n-t}{k+n-t-x-1} > 1$$

implying $(X-x|X>x) \leq_{hr} X$.
Now consider

$$\begin{aligned}\log S(x+1) - \log S(x+2) &= \log\left[\binom{k+n-x-1}{n-x-1}\binom{k+n-x-2}{n-x-2}\right]\\ &= \log\left[1+\frac{k}{n-x-1}\right],\end{aligned}$$

which is increasing. Thus, $X \geq_c Y$. As in the previous example, it is easy to verify that

$$\frac{S_{x+1}(t)}{S_x(t)} = \frac{(k+n-x-1)(n-t-x-1)}{(n-x-1)(k+n-t-x-1)} \leq 1$$

and

$$\frac{h_{x+1}(t)}{h_x(t)} = \frac{1}{k+n-t-x-2} + 1 > 1,$$

implying $(X-x|x>x) \geq_{st} (X-y|X>y)$ and $(X-x|X>x) \geq_{hr} (X-y|X>y)$.

Next result is helpful in finding the monotonicity of $h(x)$ based on the nature of equilibrium random variable X_E.

Proposition 2.2.3. The random variable X has IHR (DHR) property if and only if one of the following statements holds;

(a) $X \geq_{lr} (\leq_{lr}) X_E$.

(b) $X \geq_{lr} (\leq_{lr})(X_E - x|X_E > x),\ \forall\, x = 0,1,2,...$

(c) $(X - x|X > x) \geq_{lr} (\leq_{lr})(X_E - y|X_E > y),\ \forall\, x, y = 0, 1, 2, ... : x \leq y.$

(d) $X_E \geq_{lr} (\leq_{lr})(X_E - x|X_E > x),\ \forall\, x = 0, 1, 2, ...$, where X_E is the equilibrium random variable corresponding to X.

Proof. We now prove the result for IHR class. The proof for DHR class is similar. From the definition of probability mass function of X_E, we have

$$\begin{aligned} X \geq_{lr} X_E &\iff \frac{S(x+1)}{\mu f(x)} \text{is decreasing in x.} \\ &\iff \frac{S(x+1)}{S(x)} \frac{S(x)}{f(x)} \text{ is decreasing in } x. \\ &\iff \frac{1 - h(x)}{h(x)} \text{ is decreasing in } x. \\ &\iff h(x) \text{ is increasing in } x. \\ &\iff X \text{ is IHR.} \end{aligned}$$

To prove (b), the survival function of X_E is

$$S_E(x) = \sum_{u=x}^{\infty} \frac{S(u+1)}{\mu}, x = 0, 1, 2, ...$$

Then the survival function of $X_E - x|X > x$ is

$$S_{E,x}(t) = \frac{\sum\limits_{u=t+x+1}^{\infty} S(u+1)}{\sum\limits_{u=x+1}^{\infty} S(u+1)}; t, x = 0, 1, 2, ...$$

which gives the probability mass function

$$f_{E,x}(t) = \frac{S(x+t+2)}{\sum\limits_{u=x+1}^{\infty} S(u+1)}.$$

Now,
$X \geq_{lr} (X_E - x|X_E > x)$

$$\begin{aligned}
&\Longleftrightarrow \frac{f_{E,x}(t)}{f_X(t)} \text{ is decreasing in } t. \\
&\Longleftrightarrow \frac{S(x+t+2)}{S(t)-S(t+1)}, \\
&\quad \text{is decreasing in } t. \\
&\Longleftrightarrow h(t) \text{ is increasing in } t. \\
&\Longleftrightarrow X \text{is IHR }.
\end{aligned}$$

Now, to prove (c), we have,

$(X - x|X > x) \geq_{lr} (X_E - y|X_E > y)$

$$\begin{aligned}
&\Longleftrightarrow \frac{S(x+1)}{f(t+x+1)} \frac{S(t+y+2)}{\sum\limits_{u=y+1}^{\infty} S(u+1)}, \\
&\quad \text{is decreasing in } t. \\
&\Longleftrightarrow \frac{S(t+y+2)}{f(t+x+1)}, \\
&\quad \text{is decreasing in } t. \\
&\Longleftrightarrow \prod_{u=t+x+1}^{t+y+1} \frac{(1-h(u))}{h(t+x+1)}, \qquad (2.2.9) \\
&\quad \text{is decreasing in } t.
\end{aligned}$$

Now using (2.2.9), we get,

$$\frac{1-h(t+y+2)}{h(t+x+2)} < \frac{1-h(t+x+1)}{h(t+x+1)},$$

and hence,

$$\frac{h(t+x+1)-h(t+x+2)}{h(t+x+2)h(t+x+1)} < \frac{h(t+y+2)-h(t+x+2)}{h(t+x+2)}. \qquad (2.2.10)$$

The above inequality holds only when X has IHR property. To prove this, if possible suppose that (2.2.10) is valid and X is DHR. Then the left hand side of (2.2.10) is positive, but the right hand side will be negative. This is not possible. Hence our assumption that X

is DHR is wrong. Thus, X is IHR. To prove (d), we have,
$X_E \geq_{lr} (X_E - x | X_E > x)$

$$\iff \frac{\left(\dfrac{S(x+t+2)}{\sum\limits_{u=x+1}^{\infty} S(u)} \right)}{S(t+1)} \text{ is decreasing in } t.$$

$$\iff \frac{S(x+t+2)}{S(t+1)} \text{is decreasing in } t.$$
$$\iff X \text{ is IHR }.$$

■

Example 2.2.3. Consider the Waring distribution in Example 2.2.1. We have seen that the distribution is DHR. From(2.2.1), we calculate the probability mass function as

$$f(x) = S(x) - S(x+1) = \frac{n\,(m)_x}{(m+n)_{x+1}};\ x = 0, 1, 2, ... \tag{2.2.11}$$

It can be easily verified that

$$\frac{S(x+1)}{f(x)} = \frac{m+x}{n},$$

is increasing in x. Hence $X \leq_{lr} X_E$. Again, consider

$$\frac{S(x+t+2)}{f(x)} = \frac{\Gamma(m+n+t+1)\Gamma(m+t+x+2)}{n\Gamma(m+t)\Gamma(m+n+t+x+2)},$$

which is increasing in t, implying $X \leq_{lr} (X_E - x | X_E > x)$. Now, to show that $(X - x | X > x) \leq_{lr} (X_E - y | X_E > y)$, consider

$$\frac{S(t+y+2)}{f(t+x+1)} = \frac{\Gamma(m+n+t+x+2)\Gamma(m+t+y+2)}{n\Gamma(m+n+t+y+2)\Gamma(m+t+x+1)},$$

which is increasing in t.

Lastly, to show that $X_E \leq_{lr} (X_E - x|X_E > x)$, we evaluate

$$\frac{S(x+t+2)}{S(t+1)} = \frac{\Gamma(m+n+t+1)\Gamma(m+t+x+2)}{\Gamma(m+t+1)\Gamma(m+n+t+x+2)},$$

which is increasing in t.

Example 2.2.4. The probability mass function of negative hyper-geometric distribution in Example 2.2.2 is given by

$$f(x) = \frac{\binom{-1}{x}\binom{-k}{n-x}}{\binom{-1-k}{n}};\ x = 0, 1, 2, ..., n. \tag{2.2.12}$$

We have

$$\frac{S(x+1)}{f(x)} = \frac{n-x}{k},$$

which is a decreasing function in x implying $X \geq_{lr} X_E$. Similarly,

$$\frac{S(x+t+2)}{f(x)} = \frac{\binom{k+n-t-x-2}{n-t-x-2}}{\binom{k+n-x}{n-x} - \binom{k+n-x-1}{n-x-1}},$$

$$\frac{S(t+y+2)}{f(t+x+1)} = \frac{\binom{k+n-t-y-2}{n-t-y-2}}{\binom{k+n-t-x-1}{n-t-x-1} - \binom{k+n-t-x-2}{n-t-x-2}}$$

and

$$\frac{S(x+t+2)}{S(t+1)} = \frac{\binom{k+n-t-x-2}{n-t-x-2}}{\binom{k+n-t-1}{n-t-1}}$$

are all decreasing functions in t implying $X \geq_{lr} (X_E - x|X_E > x)$, $(X - x|X > x) \geq_{lr} (X_E - y|X_E > y)$ and $X_E \geq_{lr} (X_E - x|X_E > x)$.

The concepts based on residual life can be expressed in-terms of odds function. The residual odds function is given by

$$\bar{\omega}_x(t) = \frac{1}{S_x(t+1)} - 1; x, t = 0, 1, 2, ... \tag{2.2.13}$$

which gives

$$S_x(t+1) = \frac{1}{1+\bar{\omega}_x(t)}. \tag{2.2.14}$$

We can easily prove that

$$\bar{\omega}_x(t) = \frac{1+\bar{\omega}(x)}{\bar{\omega}(x+t+1)-\bar{\omega}(x)}. \tag{2.2.15}$$

Now we prove a result connecting the odds function and hazard rate.

Proposition 2.2.4. The distribution of X has IHR (DHR) property if and only if the odds function of the residual life $\bar{\omega}_x(t)$ is increasing (decreasing) in t.

Proof. The proof follows from the identity

$$\bar{\omega}_x(t) = \frac{1}{S_x(t+1)} - 1$$

and Proposition 2.2.1 above. ■

The IHR (DHR) family of distributions possesses several interesting properties as listed below.

(a) The $h(x)$ and $h^*(x)$ are related by (1.1.6). It follows that $h(x)$ is increasing (decreasing) if and only if $h^*(x)$ is increasing (decreasing). Thus, monotonic property of $h(x)$ and $h^*(x)$ are equivalent.

(b) If X is DHR, then its distribution function is concave. This follows from the fact that the cumulative hazard rate is concave whenever X is DHR and $S(x) = e^{-H^*(x)}$ is decreasing in x. Also, if $H^*(x)$ is concave, so is $F(x)$.

(c) If $F(x)$ is concave, then it is log concave. Further if $f(x)$ is log convex, the distribution is DHR and hence $F(x)$ is log concave. Log concavity of $S(x)$ corresponds to IHR class. It becomes apparent that the log concave class of distributions contain the concave class which in turn contains the DHR distributions.

(d) We say that X has increasing (decreasing) likelihood ratio property, denoted by ILR (DLR) if $f(x)$ is log concave (log convex). We can easily see that ILR (DLR)$\Rightarrow$ IHR (DHR).

To prove this, we note that X is ILR implies $\dfrac{f(x+1)}{f(x)}$ is decreasing or $g(x)-g(x+1)\geq 0$, where $g(x)=\dfrac{f(x+1)}{f(x)}$.

Now,

$$\begin{aligned}\frac{1}{h(x+1)}-\frac{1}{h(x)} &= \frac{S(x+1)}{f(x+1)}-\frac{S(x)}{f(x)}.\\ &= g(x+1)-g(x)+g(x+1)[g(x+2)-g(x)]+...\\ &\leq 0,\end{aligned}$$

whenever X is ILR. Thus, $h(x+1)\geq h(x)$ and X is IHR. The proof for DLR is similar. However, the converse of the above need not be true.

(e) Sometimes, it is enough to know the values of hazard rates at time zero to ascertain whether X is IHR (DHR) by employing the following result that involves a new ordering.

Definition 2.2.1. We say that X is less than Y in the initial hazard rate $(X \leq_{h(0)} Y)$ if $h_X(0)\geq h_Y(0)$.

Then we have X is IHR (DHR) $\iff X_{x_2}\leq_{h(0)} X_{x_1}; 0\leq x_1\leq x_2$.

(f) If X has IHR property, then the residual life X_x has also IHR property. This is obvious from the relation between hazard rates of X and X_x.

(g) Two important properties associated with discrete models in insurance, finance, reliability, queuing, etc. are convolution and mixing. Pavlova et al. [119] and Hu et al. [67] have discussed these aspects with necessary proofs and counter examples.

(h) Gupta et al. [51] studied IHR (DHR) property using the concept

$$\eta(x)=\frac{f(x)-f(x+1)}{f(x)}.$$

They have shown that if $\Delta\eta(x)>(<)0$, then X is IHR (DHR) and if $\Delta\eta(x)=0$, X is geometric or uniform with $f(x)=f(0)$ or as

$$f(x)=\frac{c^x}{1+c+c^2+...+c^m}: x=0,1,2,...,m.$$

In the last two cases X is IHR.

(i) Let Y_n be the random variable possessing equilibrium distribution of order n of X . Then if Y_n is IHR (DHR), then Y_{n+1} is IHR (DHR) for every n. But the converse need not be true (see Nair et al. [107]). Consequently, $Y_0 = X_E$ is IHR (DHR) whenever X is IHR (DHR).

The reliability bounds for IHR and DHR classes in discrete set-up are discussed in Sengupta et al. [134].

2.3 Increasing (decreasing) hazard rate of order 2

The concept of increasing (decreasing) hazard rate of order 2 (IHR(2)/DHR(2)) was introduced in the continuous case using the notion of stochastic dominance. In the discrete case, Fagiuoli and Pellerey [44] proposed (IHR(2)/DHR(2)) classes, following the notations of Abouammoh and Ahmed [3]. In this section, we study some properties of IHR(2) (DHR(2)) in discrete time.

Analogous to the continuous case, we say that among two discrete random variables X_1 and X_2 with distribution functions $F_1(x)$ and $F_2(x)$, X_1 has stochastic dominance of the first order over X_2, denoted by $X_1 \leq_{SD_1} X_2$, if,

$$F_1(x) \leq F_2(x),\ \forall\, x = 0, 1, 2, ... \tag{2.3.1}$$

Similarly, stochastic dominance of order 2 $(X_1 \leq_{SD_2} X_2)$ is defined as

$$\sum_{t=0}^{x} F_1(t) \leq \sum_{t=0}^{x} F_2(t),\ \forall\, x = 0, 1, 2, ... \tag{2.3.2}$$

and that of order 3 $(X_1 \leq_{SD_3} X_2)$ as

$$\sum_{t=0}^{x} \sum_{u=0}^{t} F_1(u) \leq \sum_{t=0}^{x} \sum_{u=0}^{t} F_2(u),\ \forall\, x = 0, 1, 2, ... \tag{2.3.3}$$

Sometimes, instead of using the distribution function, the survival function is used to define $X_1 \leq_{\bar{SD}_2} X_2$ as

$$\sum_{t=x}^{\infty} S_1(t) \geq \sum_{t=x}^{\infty} S_2(t) \tag{2.3.4}$$

and $X_1 \leq_{\bar{SD_3}} X_2$ as

$$\sum_{t=x}^{\infty}\sum_{u=0}^{t} S_1(u) \geq \sum_{t=x}^{\infty}\sum_{u=0}^{t} S_2(u), \tag{2.3.5}$$

where $S_1(x)$ and $S_2(x)$ are the survival functions of X_1 and X_2 respectively. Since SD_1 is equivalent to $S_1(x) \geq S_2(x)$, it is easy to see that

$$SD_1 \Rightarrow SD_2 \Rightarrow SD_3. \tag{2.3.6}$$
$$SD_1 \Rightarrow \bar{SD_2} \Rightarrow \bar{SD_3}.$$

In the sequel, we denote the various types of dominance relations by $X_1 \leq_{SD_1} X_2$, $X_1 \leq_{SD_2} X_2$, $X_1 \leq_{SD_3} X_2$, etc. In defining IHR(2) (DHR(2)), the requirement is that the residual life X_{x_2} has stochastic dominance of order 2 over X_{x_1}.

Definition 2.3.1. The random variable X is said to have increasing(decreasing) hazard rate of order 2 if for every fixed x, the sum

$$a_x(t) = \sum_{u=x}^{x+t} \frac{S(u)}{S(x)}, \tag{2.3.7}$$

is decreasing (increasing) in t.

From Proposition 2.2.2,

$$(X - x|X > x) \geq_{st} (X - y|X > y),\ \forall\, x, y = 0, 1, 2, ...; x \leq y \iff X \text{ is IHR}.$$

$$X_1 \leq_{SD_1} X_2 \iff X_1 \geq_{st} X_2$$

and

$$X_1 \leq_{SD_1} X_2 \Rightarrow X_1 \leq_{SD_2} X_2.$$

Now, we have X is IHR $\Rightarrow$ X is IHR(2). Similarly, we can prove that X is DHR$\Rightarrow$ X is DHR(2). Thus, the notion of IHR(2) (DHR(2)) is weaker than that of IHR(DHR). The properties of the IHR(2) class and its application are open problems.

2.4 Increasing(decreasing) hazard rate average

As mentioned in Section 1.1, the hazard rate average class can be defined in two ways (Definition 1.1.3). From Definition 1.1.3, we have the following result for $\text{IHRA}_1(\text{DHRA}_1)$ classes.

Definition 2.4.1. Let X_1 and X_2 be random variables with distribution functions $F_{X_1}(x)$ and $F_{X_2}(x)$ respectively. We say that X_1 is smaller than X_2 in star order, denoted by $X_1 \leq_* X_2$, if $F_{X_2}^{-1}F_{X_1}(x)$ is star shaped in x, or $\dfrac{F_{X_2}^{-1}F_{X_1}(x)}{x}$ is decreasing in $x = 0, 1, 2, ...$

Proposition 2.4.1. X is IHRA_1 if and only if $X \leq_* Y$ where Y is a geometric random variable with parameter q and $\leq_*$ is the star shaped order.

Proof. The distribution function of Y is $F_Y(x) = 1 - q^x$. Hence,

$$F_Y^{-1}(x) = \frac{\log(1-x)}{\log q}.$$

$$\begin{aligned}
X \leq_* Y \quad &\Longleftrightarrow \quad \frac{\log(1-F(x))}{x \log q} \text{ is decreasing in } x. \\
&\Longleftrightarrow \quad \frac{-H^*(x)}{x} \text{ is decreasing in } x. \\
&\Longleftrightarrow \quad \frac{H^*(x)}{x} \text{ is increasing in } x. \\
&\Longleftrightarrow \quad X \text{ is IHRA}_1.
\end{aligned}$$

■

The following properties of $\text{IHRA}_1(\text{DHRA}_1)$ classes can be seen in Roy and Gupta [126].

(a) X is IHR(DHR)$\Rightarrow$ X is $\text{IHRA}_1(\text{DHRA}_1)$.

(b) $\text{IHRA}_1(\text{DHRA}_1) \Rightarrow \text{IHRA}_2(\text{DHRA}_2)$. But the converse is not true.

For more properties of IHRA(DHRA) classes, we refer to Roy and Gupta [126], Khalique [79] and Bracquemond and Gaudoin [23]. Brown and Rao [25] have discussed the relation between IHRA_1 class of distributions and the first passage time distribution of a

Markov chain. The bounds of survival function for IHRA$_2$(DHRA$_2$) classes were studied by Sengupta and Jammalamadaka [133].

2.5 New better than used in hazard rate

The new better (worse) than used in hazard rate NBUHR (NWUHR) class arises when the hazard rate of a new item is less(more) than that of a used one at any age.

Definition 2.5.1. A discrete random variable X is NBUHR(NWUHR) if

$$h(x) \geq (\leq) h(0), \; \forall\, x = 0, 1, 2, ... \tag{2.5.1}$$

The following observations are direct from definition 2.5.1.

(a) NBUHR is equivalent to the NBUFR class given in Abouammoh and Ahmed [3], for which $S(x+1) \leq S(x)S(1)$, since

$$\frac{S(x+1)}{S(x)} \leq 1 - f(0) \iff 1 - h(x) \leq 1 - h(0).$$

Note that X is NBUHR $\iff S(x+1) \geq S(x)S(1)$.

(b) $h(x) \geq (\leq) h(0) \iff h^*(x) \geq (\leq) h^*(0)$. Hence both $h(x)$ and $h^*(x)$ produce the same NBUHR(NWUHR) class of distributions.

(c) IHRA(DHRA)$\Rightarrow$ NBUHR(NWUHR).

We have the following results for NBUHR(NWUHR) class of distributions.

Proposition 2.5.1. The mixtures of NWUHR class of distributions are NWUHR.

Proof. To prove this, let X be distributed with survival function $S(x;\theta)$, where θ is continuous with distribution function $G(\theta)$. The mixture distribution has survival function $S(x)$ where $S(x) = \int_\theta S(x|\theta) dG(\theta)$. When X is NWUHR, we have

$$\begin{aligned} S(x)S(1) &= \int_\theta S(x|\theta)dG(\theta) \int_\theta S(1|\theta)dG(\theta). \\ &\leq \int_\theta S(x|\theta)S(1|\theta)dG(\theta). \\ &= \int_\theta S(x+1|\theta)dG(\theta) = S(x+1). \end{aligned}$$

So that $S(x)$ is NWUHR. ■

There are two different ways in which NBUHR can be introduced as a partial order. The first one uses the following definition.

Definition 2.5.2. Two discrete random variables X_1 and X_2 are ordered with respect to initial failure rate $h(0)$, $(X_1 \leq_{h(0)} X_2)$ if $h_{X_1}(0) \geq h_{X_2}(0)$.

The above definition reads as $X_x \leq_{h(0)} X$, which means that $h_x(0) \geq h_X(0)$ or $h(t) \geq h(0),\ \forall\, t = 1, 2, ...$ since $h_x(t) = h(x+t)$. Thus, $X_x \leq_{h(0)} X \iff X$ is NBUHR.

The second is in-terms of the s-order (Abouammoh [2]) which says that $X_1 \leq_s X_2$ if $\Delta F^{-1}_{X_2}(F_{X_1}(x)) \leq \Delta F^{-1}_{X_2}(F_{X_1}(y))|_{y=0}$. When $F_{X_2}(.)$ is geometric, we can prove that $X_1 \leq_s X_2 \iff X_1$ is NBUHR. Thus, $\leq_{h(0)}$ offers a comparison between the hazard rate of residual life distribution with the original distribution, while the second compares the baseline distribution with the geometric law. Reversing the inequalities, we have $X_x \geq_{h(0)} X \iff X$ is NWUHR and $X_1 \geq_s X_2 \iff X_1$ is NWUHR.

A weaker class of life distributions can be obtained if we use the averages of the hazard rate.

Definition 2.5.3. The lifetime random variable X is said to have a new better (worse) than used in hazard rate average (NBUHRA/NWUHRA) if,

$$h(0) \leq (\geq) \frac{1}{x} \sum_{t=0}^{x-1} h(t). \tag{2.5.2}$$

Remark 2.5.1. X is NBUHRA (NWUHRA) $\iff H(0) \leq (\geq) \dfrac{H(x)}{x}$.

When $h^*(x)$ is used, we have a different class of life distributions.

Definition 2.5.4. (Khalique [79]) A discrete random variable X is said to have new better (worse) than used in hazard rate average-1(NBUHRA-1/NWUHRA-1) if

$$h^*(0) \leq (\geq) \frac{1}{x} \sum_{t=0}^{x-1} h^*(t) \tag{2.5.3}$$

or equivalently

$$S(1)^x \geq (\leq) S(x), \; \forall \, x = 1, 2, ... \tag{2.5.4}$$

Proposition 2.5.2. X has NBUHRA-1$\Rightarrow$ X has NBUHRA.

Proof. X is NBUHRA-1 $\iff$ $h^*(0) - \dfrac{H^*(x)}{x} \leq 0.$
Since the arithmetic mean is not less than the geometric mean, we have

$$\log \left[\frac{1}{x} \sum_{t=0}^{x-1} (1 - h(t)) \right] \geq \frac{1}{x} \sum_{t=0}^{x-1} \log(1 - h^*(t))$$

$$\Rightarrow \log(1 - \frac{H(x)}{x}) \geq -\frac{H^*(x)}{x}.$$

Also,

$$\log(1 - h(0)) = -h^*(0).$$

So that

$$\log(1 - h(0)) - \log(1 - \frac{H(x)}{x}) \geq \frac{H^*(x)}{x} - h^*(0) \geq 0.$$

The last inequality means that $\dfrac{H^*(x)}{x} \geq h^*(0)$ and hence X is NBUHRA-1. By way of implication, we note that

NBUHR$\Rightarrow$NBUHRA and IHRA$_1$ $\Rightarrow$NBUHRA$_1$ $\Rightarrow$NBUHRA.

■

Example 2.5.1. Consider the Waring distribution in Example 2.2.1. We have seen that the hazard rate function given by,

$$h(x) = \frac{n}{m + n + x}; \; x = 0, 1, 2, ...; \; m, n > 0, \tag{2.5.5}$$

is decreasing and hence $h(0)$ is greater than $h(x)$ for $x = 1, 2, ...$ Thus, X is NWUHR. Now

$$\frac{1}{x}\sum_{t=0}^{x-1} h(t) - h(0) = -\frac{n\left(x - \sum\limits_{t=0}^{x-1} \dfrac{m+n}{m+n+x}\right)}{(m+n)x} \leq 0$$

for $x = 1, 2, ...$ Hence X is NWUHRA.

Example 2.5.2. Let X be distributed as the discrete Weibull II distribution (Stein and Dattero [137]) with hazard rate function

$$h(x) = \left(\frac{x}{m}\right)^{\beta-1}; \ x = 0, 1, 2, ..., m. \tag{2.5.6}$$

The distribution is IHR for $\beta > 1$. Clearly, X is NBUHR. Now

$$\frac{1}{x}\sum_{t=0}^{x-1} h(t) - h(0) = \frac{\sum_{t=0}^{x-1} \left(\frac{t}{m}\right)^{b-1}}{x} \geq 0$$

for $x = 0, 1, 2, ..., m$. Hence X is NBUHRA also.

2.6 BT and UBT hazard rate classes

Among non-monotone hazard rate functions, those with bathtub-shape (BT) and upside-down bathtub-shape (UBT) have been extensively discussed for continuous lifetime in the review on the subject in Lai and Xie [85] and Nair et al. [108]. Compared to this, the work on BT and UBT hazard rate models for discrete time is much less.

BT models represent hazard rates which is decreasing initially, then remain constant and thereafter increasing. Illustration of how such models arise in practice and their justification in the case of biological organisms and mechanical devices are provided in Marshall and Olkin [91]. Jiang [71], Noughabi et al. [117], Noughabi et al. [118], Bebbington et al. [19] and Almalki and Nadarajah [7] present a variety of discrete life distributions that represent real data possessing BT hazard rate functions. On the other hand, UBT models have an increasing hazard rate initially, followed by one with constant and then a decreasing hazard rate. Component hardening in mechanical systems and age selection for health may influence the occurrence of decreasing hazard rate at longer lives. Jazi et al. [70]

and Hussain and Ahmad [68] discuss discrete models that adequately describe lifetime datasets with UBT hazard rates. The existence and the usefulness of discrete models in describing BT and UBT hazard rates point out to the need of a detailed study of such classes of distributions. To the best of our knowledge, various properties of the BT and UBT hazard rate classes of discrete distributions relevant to reliability analysis have not been investigated so far. The present section makes an attempt in this direction.

We now focus on the monotonicity of hazard rate and MRL to study the properties of BT and UBT hazard rate classes of distributions. To be more specific, we investigate whether BT (UBT) hazard rates, describing the pattern of ageing, is closed with respect to the reliability operations such as convolutions, formation of series and parallel systems, mixtures, residual life and equilibrium distributions. We also study existence of moments and obtain bounds on reliability functions and moments of BT (UBT) hazard rate models, which are of special interest.

Let X be a discrete random variable taking values in $\{0, 1, 2, ...b,\ b \leq \infty\}$ with probability mass function $f(x)$, survival function $S(x)$ and the hazard rate function $h(x)$ as defined in (1.1.1).

Definition 2.6.1. We say that the hazard rate of X is BT (UBT) or equivalently, X has a BT(UBT) distribution if there exist integers $1 \leq x_0 \leq x_1 < \infty$, such that $h(x)$ is decreasing (increasing) in $[0, x_0] = (0, 1, 2...x_0)$, a constant in $[x_0, x_1] = (x_0, x_0+1, ..., x_1)$ and increasing (decreasing) in $[x_1, b] = (x_1, x_1 + 1, ..., b)$. The points x_0 and x_1 at which $h(x)$ changes its shape are called the change points of $h(x)$.

Frequently, it is enough to consider models in which there is only one change point. i.e., the case when $x_0 = x_1$. In this case, we have only one change point x_0. Throughout our study, we consider BT(UBT) distributions with single change point only. By virtue of (1.1.6), if X is BT (UBT) in-terms of $h(x)$, it is BT (UBT) in the sense of $h^*(x)$ also, but with a different change point obtained from (1.1.6).

2.6.1 Closure properties

Now, we examine whether the BT or UBT property of hazard rate function is preserved under various reliability operations mentioned in the introduction.

2.6.1.1 Convolution

In reliability analysis, convolution of lifetimes has important applications. While maintaining the working of a system, it is customary to replace a failed component by a spare. Then the lifetime of the component is obtained by adding the lifetimes of the original and spare. Such an operation is an essential activity in the formulation of maintenance policies.

Proposition 2.6.1. The convolution of two BT hazard rate distributions need not be a BT hazard rate distribution.

Proof. Let X be distributed with survival function

$$S(x) = q^{ax+\frac{b}{2}x^2+\frac{c}{3}x^3},\ x = 0, 1, 2, ...;\ q = e^{-1};\ b < 0;\ a,\ c > 0;\ b \geq -2\sqrt{ac}. \quad (2.6.1)$$

The corresponding hazard function is

$$h(x) = 1 - \frac{S(x+1)}{S(x)} = 1 - q^{a+\frac{b}{2}(2x+1)+\frac{c}{3}(3x^2+3x+1)} \quad (2.6.2)$$

and accordingly $h(x+1) - h(x)$ has a zero at

$$x_0 = \left(-\frac{b}{2c} - 1\right),\ b < 0,\ c > 0. \quad (2.6.3)$$

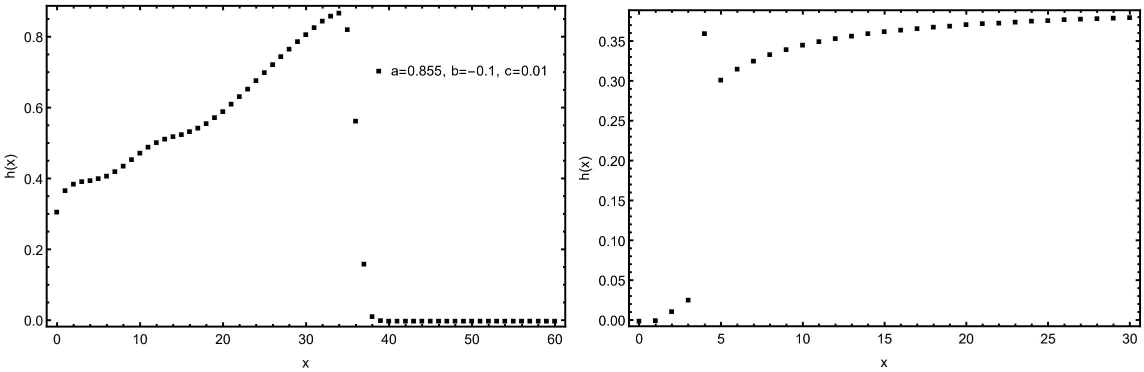
We take the change point as the integer part of x_0. Since

$$h(x_0 + 1) - h(x_0) = 1 - q > 0$$

and

$$h(x_0) - h(x_0 - 1) = 1 - q^{-2c} < 0,$$

$h(x)$ is BT. (For the parameter values $a = 0.855,\ b = -0.1$ and $c = 0.01$, we have the BT



(a) Convolution of BT distributions

(b) Convolution of UBT distributions

Figure 2.1: Hazard rate functions of convolutions

shape for (2.6.2) with change point $x_0 = 4$).

Consider the convolution of X with itself. For the same parameter values, the hazard rate function of the convolution is UBT as seen in Figure 2.1a. The survival function and hence the hazard rate of the convolution is algebraically too complex to prove the result analytically. ■

Proposition 2.6.2. The convolution of two UBT hazard rate distributions need not be UBT hazard rate distribution.

Proof. Consider a sequence of hazard rates,

$$h(0) = 0.01,\ h(1) = 0.02,\ h(2) = 0.6 \text{ and } h(x) = 0.4,\ x \geq 3.$$

Obviously, the corresponding distribution has UBT hazard rate with change point $x = 2$. The hazard rate of the convolution of the distribution with itself is not UBT as seen in Figure 2.1b.

■

2.6.1.2 Mixtures

Consider two discrete lifetimes X_1 and X_2 with survival functions $S_1(x)$ and $S_2(x)$ and hazard rate functions $k_1(x)$ and $k_2(x)$. Then the mixture of X_1 and X_2 has survival function

$$S(x) = \alpha S_1(x) + (1-\alpha)S_2(x),\ 0 \le \alpha \le 1 \tag{2.6.4}$$

and hazard rate

$$h(x) = p(x)k_1(x) + (1-p(x))k_2(x), \tag{2.6.5}$$

where

$$p(x) = \frac{\alpha S_1(x)}{\alpha S_1(x) + (1-\alpha)S_2(x)}.$$

Proposition 2.6.3. The mixture of two BT (UBT) hazard rate distributions need not be BT (UBT) hazard rate distribution.

Proof. Take $S_1(x)$ as (2.6.1) which has BT hazard rate at the parameter values $a = 0.855, b = -0.1$ and $c = 0.01$. Also, let

$$S_2(x) = \frac{\exp\left[-\frac{a}{2}x^2\right]}{(1+bx)^{\frac{c}{b}}},\ x = 0, 1, 2, ...;\ a,\ b,\ c > 0; a + c > 0, \tag{2.6.6}$$

which is the discretized version of the Hjorth [65] model. It has hazard rate function of the form

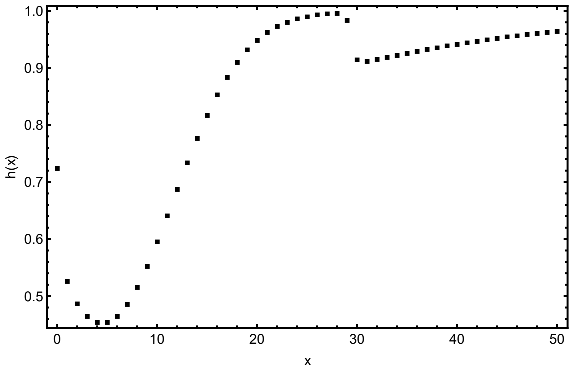
$$k_2(x) = 1 - \left(\frac{1+bx}{1+b+bx}\right)^{\frac{c}{b}} \exp\left[-\frac{a}{2}(2x+1)\right]. \tag{2.6.7}$$

At $a = 0.06,\ b = 0.23$ and $c = 4.54$, we have a BT form for $k_2(x)$.

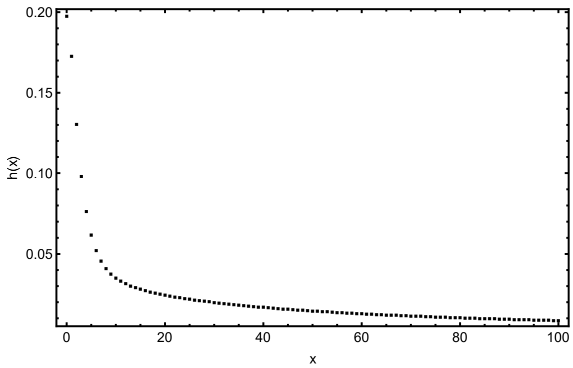
When a mixture of the form

$$S(x) = 0.6S_1(x) + 0.4S_2(x) \tag{2.6.8}$$

is taken, the corresponding $h(x)$ (calculated from (2.6.5)) at the above parameter values, where $S_1(x)$ and $S_2(x)$ are survival functions of BT hazard rate distributions, is initially decreasing, then increasing, again decreasing and finally increasing as shown in Figure 2.2a. Thus, the hazard rate of $S(x)$ is not BT- shaped.



(a) Mixture of BT distributions



(b) Mixture of UBT distributions

Figure 2.2: Hazard rate functions of mixtures

To prove the result for UBT hazard rate models, we consider

$$S_1(x) = \left[1 - c\left(\frac{x}{a}\right)^b\right]^{\frac{1}{c}}, \; x = 0, 1, 2, ...; \; a, \; b > 0; c < 0. \tag{2.6.9}$$

With $a = 1.81, \; b = 1.184$ and $c = -0.35$, the corresponding hazard rate function

$$k_1(x) = 1 - \left[\frac{1 + c(\frac{x+1}{a})^b}{1 + c\left(\frac{x}{a}\right)^b}\right]^{\frac{1}{c}} \tag{2.6.10}$$

is UBT.

As a second UBT hazard rate distribution, we take

$$S_2(x) = \frac{1}{1 + cx^{\alpha_1}}, x = 0, 1, 2, ...; c, \alpha_1 > 0. \tag{2.6.11}$$

The hazard function

$$k_2(x) = 1 - \frac{1 + c(x+1)^{\alpha_1}}{1 + cx^{\alpha_1}} \tag{2.6.12}$$

is UBT-shaped at the parameter values $\alpha_1 = 1.1$ and $c = 0.03$. Forming the mixture by taking $\alpha = 0.5$, the hazard function is decreasing as seen in Figure 2.2b. Hence it is not UBT-shaped. ■

2.6.1.3 Series and parallel systems

Consider a series system with n components whose lifetimes are independent with survival functions $S_{X_1}(x), S_{X_2}(x), ...S_{X_n}(x)$.

Proposition 2.6.4. If the component hazard rates of a series system are BT (UBT)-shaped with a common change point x_0, the system hazard rate is also BT (UBT)-shaped with the same change point.

Proof. The survival function of the system is

$$S(x) = S_{X_1}(x)S_{X_2}(x)...S_{X_n}(x).$$

Then

$$\frac{S(x+1)}{S(x)} = \frac{S_{X_1}(x+1)}{S_{X_1}(x)}...\frac{S_{X_n}(x+1)}{S_{X_n}(x)}.$$

Hence the alternative hazard rate in (1.1.15) obeys the relationship

$$h^*(x) = h^*_{X_1}(x) + h^*_{X_2}(x) + ... + h^*_{X_n}(x).$$

So that

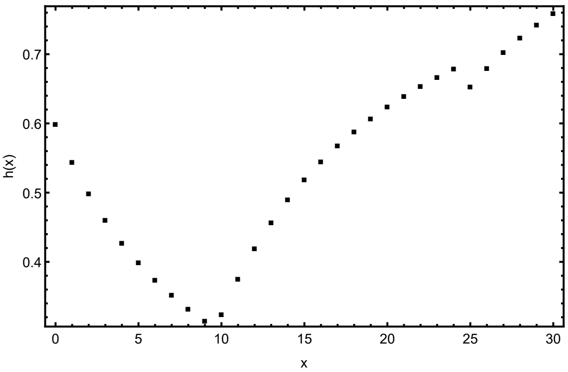
$$h^*(x+1) - h^*(x) = [h^*_{X_1}(x+1) - h^*_{X_1}(x)] + ... + [h^*_{X_n}(x+1) - h^*_{X_n}(x)].$$

Since the components have BT (UBT) hazard rate with same change point x_0, x_0 is a zero of each of the terms on the right and so is of $h^*(x+1) - h^*(x)$. This proves the proposition. ■

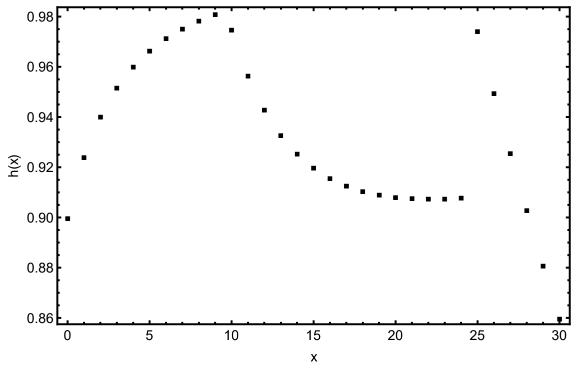
Remark 2.6.1. If the component hazard rates of a series system have BT(UBT) hazard rate distribution with different change points, the system hazard rate need not be BT(UBT)-shaped.

Example 2.6.1. Consider two lifetime random variables X_1 and X_2 with hazard rate functions defined as

$$h_{X_1}(x) = \begin{cases} \dfrac{5}{x+10} & : x = 0, 1, 2, ..., 24 \\ 1 - \dfrac{24}{x} & : x = 25, 26, ... \end{cases}$$



(a) Series system with BT distributions



(b) Series system with UBT distributions

Figure 2.3: Hazard rate functions of series system

and

$$h_{X_2}(x) = \begin{cases} \dfrac{1}{x+5} & : x = 0, 1, 2, ..., 9 \\ 1 - \dfrac{9}{x} & : x = 10, 11, ... \end{cases}$$

Clearly $h_{X_1}(x)$ and $h_{X_2}(x)$ are BT-shaped with respective change points 25 and 10. Using (1.1.2), we evaluate the survival functions $S_{X_1}(x)$ and $S_{X_2}(x)$ of X_1 and X_2 respectively. The survival function corresponding to $\min(X_1, X_2)$ is given by

$$S(x) = S_{X_1}(x)S_{X_2}(x),$$

from which we can evaluate the hazard rate function numerically. Figure 2.3a shows the hazard rate function. From the figure, we see that the hazard rate function is not BT-shaped. Thus, the BT property of hazard rate is not closed under the formation of series system when change points of the component-hazard rates are different.
For the UBT case, consider the hazard rate functions as

$$h_{X_1}(x) = \begin{cases} 1 - \dfrac{5}{x+10} & : x = 0, 1, 2, ..., 24 \\ \dfrac{24}{x} & : x = 25, 26, ... \end{cases}$$

and

$$h_{X_2}(x) = \begin{cases} 1 - \dfrac{1}{x+5} & : x = 0, 1, 2, ..., 9 \\ \dfrac{9}{x} & : x = 10, 11, ... \end{cases}$$

The above hazard rates are UBT-shaped with respective change points 25 and 10. The hazard rate of $\min(X_1, X_2)$ is shown in Figure 2.3b. It is not UBT-shaped. Thus, the UBT shape of hazard rate is not closed under formation of series system when change points of the component-hazard rates are different.

Proposition 2.6.5. If the component hazard rates of a parallel system are BT-shaped, the system hazard rate need not be BT-shaped.

Proof. Consider a parallel system consisting of two independent components with common hazard function

$$h(x) = 1 - \exp[-(a + bx + cx^2)],\ x = 0, 1, 2, ...; a, c > 0; b \geq -2\sqrt{ac}. \tag{2.6.13}$$

The corresponding survival function is

$$S(x) = \exp\left(-\frac{1}{6}x(6a + (x-1)(3b + c(2x-1)))\right), x = 0, 1, 2, ... \tag{2.6.14}$$

For the parameter values

$$a = 1.32,\ b = -4.09,\ \text{and } c = 3.795,$$

the hazard function is BT-shaped.

Now, the survival function of the system is given by

$$\begin{aligned} S_2(x) = &\ 2S(x) - S^2(x) \\ = &\ 2\exp\left(-\tfrac{1}{6}x(6a + (x-1)(3b + c(2x-1)))\right) \\ &- \exp\left(-\tfrac{1}{3}x(6a + (x-1)(3b + c(2x-1)))\right) \end{aligned}$$

and the corresponding hazard function is given by

$$h(x) = \frac{e^{-2(a+x(b+cx))}\left(1 - 2\exp\left(\frac{1}{6}(x+1)(6a + x(3b + 2cx + c))\right)\right)}{2\exp\left(\frac{1}{6}x(6a + (x-1)(3b + 2cx - c))\right) - 1} + 1 \tag{2.6.15}$$

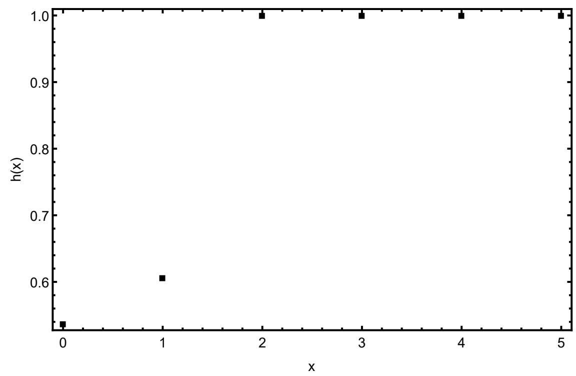


Figure 2.4: Hazard rate function of parallel system

Table 2.1: Hazard rate function of a parallel system with UBT hazard rate components.

x	0	1	2	3	4	5	6
System hazard rate	0.16	0.3928	0.5063	0.4819	0.3916	0.3950	.3970

For the parameter values

$$a = 1.32,\ b = -4.09,\ \text{and}\ c = 3.795,$$

the system hazard function is increasing as seen in Figure 2.4. ■

Proposition 2.6.6. If the component hazard rates of a parallel system are UBT-shaped, the system hazard rate need not be UBT-shaped.

Proof. Consider a parallel system of two components with a common sequence of hazard rates as

$$h(0) = 0.4, h(1) = 0.5,\ h(2) = 0.55,\ h(3) = 0.5,\ \text{and}\ h(x) = 0.4 \text{ for } x = 4, 5, ...$$

Clearly, the hazard rate sequence is UBT-shaped. Now consider the hazard function corresponding to the system. Values of the hazard rate function are tabulated in Table 2.1, for some values of x. From the table, we see that the system hazard rate is not UBT-shaped as the function is again increasing slightly from the point 4. ■

Table 2.2 presents the closure properties of discrete BT(UBT) hazard rate distributions.

Table 2.2: Reliability operations on BT and UBT hazard rate classes

Domain of distribution	Distribution class	Reliability operation		
		Coherent system	Convolution	Mixture
Continuous	BT hazard rate	Not Closed	Not Closed	Not Closed
Discrete	BT hazard rate	Not Closed	Not Closed	Not Closed
Discrete	UBT hazard rate	Not Closed	Not Closed	Not Closed

The continuous analogues of BT hazard rate distributions are given. The closure properties of continuous UBT distributions are yet to be studied.

2.6.2 Residual life distribution

In the following, we discuss how the property of BT(UBT) hazard rate, of a distribution is carried to its residual life distribution.

Proposition 2.6.7. If X is BT (UBT) with change point x_0, then the distribution of $X_t = X - t|X > t$ is also BT (UBT) with change point $x_0 - t$, provided $x_0 > t$.

Proof. The proof follows from the fact that the hazard rate of X_t is $h(x + t + 1)$. ■

2.6.3 Equilibrium distribution

The following proposition gives a necessary and sufficient condition for the BT (UBT) hazard rate of an equilibrium distribution, in-terms of mean residual life.

Proposition 2.6.8. If Y_n is the random variable representing the equilibrium distribution of order n in (1.1.15) of the baseline random variable X, then Y_n is BT (UBT) with change point x_0 if and only if the mean residual life of Y_{n-1} is BT (UBT)-shaped with the same change point.

Proof. When Y_n is the equilibrium random variable of order n of X, then the hazard rate $h_n(x)$ of Y_n and the mean residual life $m_{n-1}(x)$ of Y_{n-1} are related by (Nair et al. [107])

$$h_n(x) = \frac{1}{m_{n-1}(x)} \tag{2.6.16}$$

Thus,

$$h_n(x+1) - h_n(x) = \frac{m_{n-1}(x) - m_{n-1}(x+1)}{m_n(x)m_{n-1}(x)}$$

and the result follows from the fact that the zeroes of the left and right sides are identical and they have the same behaviour. ■

2.6.4 Bounds and moments

In this section, we derive some bounds on the reliability function and moments of BT and UBT distributions.

Proposition 2.6.9. If X is BT (UBT) then $X \leq_{st} (\geq_{st}) Y$, where Y is a geometric random variable with parameter $q = 1 - h(x_0)$ where $h(x)$ is the hazard rate of X and x_0 is the change point of the hazard rate.

Proof. Assume that X is BT with change point x_0. Then $h(x) \geq h(x_0)$ for all x in the support of X. This gives

$$\begin{aligned} S_X(x) &= \prod_{t=0}^{x-1}(1-h(t)) \leq \prod_{t=0}^{x-1}(1-h(x_0)) \\ &= q^x, \; q = 1 - h(x_0) \end{aligned}$$

Thus, $X \leq_{st} Y$. The proof of the UBT case is obtained by reversing the inequalities. ■

Proposition 2.6.10. For a BT distribution, moments of all order exist.

Proof.

$$X \leq_{st} Y \iff E[X^{(r)}] \leq E[Y^{(r)}],$$

where $X^{(r)} = X(X-1)...(X-r+1)$. Thus,

$$E[X^{(r)}] \leq r! \left(\frac{1-h(x_0)}{h(x_0)}\right)^r < \infty$$

for every r. ■

Remark 2.6.2. Proposition 2.6.9 gives an upper bound (lower bound) to the reliability of a BT (UBT) distribution, while Proposition 2.6.10 gives the upper bound to the moments when X is BT. Geometric interpretation of Proposition 2.6.9 is that every survival function of a BT (UBT) law lies below (above) that of a geometric distribution with mean $\dfrac{1-h(x_0)}{h(x_0)}$ with x_0 as the change point.

Proposition 2.6.11. Let $\{S_n(x)\},\ n = 1, 2, ...$ be a sequence of a survival functions with BT(UBT) hazard rates with monotone sequence of change points $\{y_n\},\ n = 1, 2, ...$ If $S_n(x)$ converges in distribution to $S(x)$ then $S(x)$ is also BT (UBT).

Proof. First observe that from the definition of the hazard rate

$$h(x) = 1 - \frac{S(x+1)}{S(x)} \tag{2.6.17}$$

and therefore

$$\begin{aligned} h(x+1) - h(x) &= \frac{S(x+1)}{S(x)} - \frac{S(x+2)}{S(x+1)} \\ &= \frac{S^2(x+1) - S(x)S(x+2)}{S(x)S(x+1)} \end{aligned}$$

Thus, $h(x)$ is increasing (decreasing) is equivalent to $S^2(x+1) - S(x)S(x+2) \geq (\leq) 0$ and BT or UBT is equivalent to $S^2(x+1) - S(x)S(x+2)$ has a zero $x_0 > 0$ and $x_0 < \infty$.

Now assume that $\{S_n(x)\}$ converges to $S(x)$. The sequence of change points $\{y_n\}$ is monotonic, bounded and hence convergent. This means that we can find a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that given $t > 0$, there exists a k_0 satisfying

$$|y_{n_k} - x_0| < \epsilon \text{ for } k \geq k_0.$$

Here x_0 stands for the points to which $\{y_n\}$ converges. Since $y_{n_k} \in (x_0 - \epsilon, x_0 + \epsilon)$ for a point $x > x_0 + \epsilon$, we can write

$$\begin{aligned} S^2(x+1) - S(x)S(x+2) &= \lim_{k \to \infty} \left(S^2_{n_k}(x+1) - S_{n_k}(x) S_{n_k}(x+2) \right) \\ &\geq 0 \end{aligned}$$

as the hazard rate is increasing in $[x_0, \infty)$. Similarly, by considering a point $x < x_0 - \epsilon$,

$$S^2(x+1) - S(x)S(x+2) \leq 0.$$

As $\epsilon \to 0+$, we must have

$$S^2(x+1) - S(x)S(x+2) = 0$$

which is equivalent to $h(x+1) - h(x) = 0$ at $x = x_0$. Then if $S_n(x)$ are BT distributions, $S(x)$ also is a BT distribution. The proof of the UBT case is similar. ■

We give an example to illustrate Proposition 2.6.11.

Example 2.6.2. Let

$$S_n(x) = \exp\left[-ax - \frac{1}{2}(b+\frac{1}{n})x(x-1) - \frac{c}{6}x(x-1)(2x-1)\right]; x = 0, 1, 2, ...$$

$a, c > 0, 4ac \geq (b + \frac{1}{n})^2$, with hazard rate function

$$h_n(x) = 1 - \exp\left[-a - (b+\frac{1}{n})x - cx^2\right].$$

Then $h_n(x)$ is BT-shaped. In this case $S_n(x)$ converges in distribution to

$$S(x) = \exp\left[-ax - \frac{1}{2}bx(x-1) - \frac{c}{6}x(x-1)(2x-1)\right].$$

The hazard rate of $S(x)$ is

$$h(x) = 1 - \exp\left[-a - bx - cx^2\right]$$

which is again BT with change point $x_0 = \frac{1}{2}\left(-\frac{b}{c} - 1\right)$ for $-b > (c+1)$.

2.7 Conclusion

In the present chapter, we have studied ageing classes for discrete life distributions using two different versions of the hazard rate. The relationships among these ageing classes were derived. It may be noted that properties of various ageing classes based on the hazard function, in the continuous set-up, are not directly transformed into discrete set-up. Various ageing criteria discussed in this chapter play a fundamental role in the development of reliability theory and practice. We have established some properties of the class of distributions with BT or UBT hazard rates, which could be useful in reliability practice. Being general results, they can readily be applied in finding bounds for the reliability.

Chapter 3

Discrete Bathtub and Upside-Down Bathtub Distributions

3.1 Introduction

There is enormous literature on bathtub distributions when lifetime is treated as a continuous random variable, as can be seen from the reviews and references in Rajarshi and Rajarshi [120], Lai and Xie [85] and Nair et al. [108]. Two main topics discussed in this connection are methods to identify and construct bathtub distributions and models that provide bathtub-shaped hazard rates. Compared to the continuous case, there are only a few papers dealing with this subject when lifetime is discrete and that too of recent origin, presenting a few models. The discrete inverse Weibull (Jazi et al. [70]), competing risks model (Jiang [71]), modified Weibull (Noughabi et al. [117]), additive Weibull (Bebbington et al. [19]), modified Weibull extension (Noughabi et al. [118]), the reduced modified Weibull extension (Almalki and Nadarajah [7]) and the discrete inverse Rayleigh model (Hussain and Ahmad [68]) appear to exhaust the current works on discrete bathtub distributions. The work on general conditions that enable the identification of bathtub models initiated by Glaser [47] and subsequently developed by Gupta and Warren [57]. Works of Ghitany

Results in this chapter have been accepted for publication in the journals “International Journal of Reliability, Quality and Safety Engineering”and “South African Statistical Journal”(See Nair et al. [109] and Nair et al. [110])

[46] and Marshall and Olkin [91] in the continuous case have no counterparts in the discrete case.

There is a large number of data sets that need to be analyzed using reliability concepts in discrete time (See Lai and Xie [85] for examples and applications). The limited number of models mentioned above are unlikely to meet the requirements of modeling. This points out to the need for criteria for determination of the shape of the hazard rates and also for developing new models. The present chapter is focused on this vital problem and investigates some general conditions for assessing the shape of the hazard rates. Our results also help in generating new distributions that have simple hazard rate forms.

A brief outline of the chapter is as follows. In Section 3.2, we present some definitions and results that are needed in the sequel. Following this, Section 3.3, presents theorems that enable the determination of the hazard rate shape. In Section 3.4, we demonstrate how life distributions can be constructed with the help of the results in Section 3.3 and also some associated results. We discuss the general procedures for construction of BT and UBT distributions in Section 3.5. In Section 3.6, we study the newly proposed discretized quadratic hazard model in detail. The chapter ends with the conclusions of study in Section 3.7.

3.2 Basic results

Let X be a discrete random variable taking values in $S = (0, 1, 2, ..., b)$, where b can be finite or infinite, with probability mass function $f(x)$ and survival function, $S(x)$. Then the hazard rate of X is given by (1.1.1). When dealing with discretized versions of continuous distributions, generally the survival function or distribution function has a tractable form, as in the case of bathtub distributions reviewed earlier. It is also convenient to find the solution of the equation,

$$S^2(x + 1) - S(x)S(x + 2) = 0, \tag{3.2.1}$$

which will give the same change point obtained by solving the equation in $h(x)$. However, there are many distributions for which $S(x)$ may not be analytically tractable. Therefore, as in the case of continuous random variables, where the score function $\frac{-g^{'}(x)}{g(x)}$, where $g(x)$ is the probability density function, is employed, it is easier to deal with its discrete analogue

$$\eta(x) = \frac{f(x) - f(x+1)}{f(x)}, \tag{3.2.2}$$

called the score function. Thus, the ratios of the probability mass function is the only requirement in determining the shape of $h(x)$. Gupta et al. [51] have demonstrated the simplicity of using $\eta(x)$ for several distributions.

Instead of using the classical definition (1.1.1), one can also make use of the alternative hazard rate proposed by Cox and Oakes [37] as given in (1.1.2). Since $h^*(x)$ and $h(x)$ are related through (1.1.6), the shapes of the hazard rates will be identical in both cases. But the change points will be different as seen from (1.1.6).

3.3 Main results

With the notations and terminology of the previous section, we present the following theorems. The first theorem (Theorem 3.3.1) deals with the case of unbounded support. The case of distributions with bounded support which needed special attention is discussed in Theorem 3.3.2.

In the sequel, we use D (I) for decreasing (increasing).

Theorem 3.3.1. Let X be a discrete random variable with support $\mathbf{N}$, where $b = \infty$ and $f(\infty) = 0$. Then,

(i) if $\eta(x)$ is decreasing (D), then $h(x)$ is decreasing (D),

(ii) if $\eta(x)$ is increasing (I), then $h(x)$ is increasing (I),

(iii) if $\eta(x)$ is bathtub-shaped (BT) and $f(0) = 0$ $(f(0) \neq 0, \frac{\eta(0)}{f(1)}(1 - f(0)) > 1)$, then $h(x)$ is $I(BT)$ and

(iv) if $\eta(x)$ is upside-down bathtub-shaped (UBT) and $f(0) = 0$ $(f(0) \neq 0, \frac{\eta(0)}{f(1)}(1 - f(0)) > 1)$, then $h(x)$ is $D(UBT)$.

Proof. Writing $C(x) = \dfrac{1}{h(x)}$, we have

$$\Delta C(x) = C(x+1) - C(x) = \frac{h(x) - h(x+1)}{h(x)h(x+1)}. \tag{3.3.1}$$

Also,

$$\begin{aligned}
\Delta C(x) &= \frac{S(x+1)}{f(x+1)} - \frac{S(x)}{f(x)} \\
&= \frac{S(x+1)}{f(x+1)} - \frac{S(x+1) + f(x)}{f(x)} \\
&= \frac{S(x+1)}{f(x+1)} \left(\frac{f(x) - f(x+1)}{f(x)} \right) - 1 \\
&= \frac{\eta(x)}{f(x+1)} \sum_{t=x+1}^{b} f(t) - 1 \\
&= \frac{1}{f(x+1)} \left[\sum_{t=x+1}^{b} \left(\eta(x) - \eta(t)\right) f(t) + \sum_{t=x+1}^{b} f(t)\eta(t) \right] - 1 \\
&= \frac{1}{f(x+1)} \left[\sum_{t=x+1}^{b} \left(\eta(x) - \eta(t)\right) f(t) - f(b) \right].
\end{aligned} \tag{3.3.2}$$

Equation (3.3.2) holds irrespective of whether b is finite or infinite. Further discussion is with regard to the shapes of $\eta(x)$.

(a) When $\eta(x)$ is D, $\eta(x) < \eta(t),\ t < x$ and $f(b) = f(\infty) = 0$. Hence from (3.3.2), $\Delta C(x) > 0$ which in turn implies $h(x) > h(x+1)$ and so $h(x)$ is D, proving (i). The proof of (ii) is obtained by changing D to I and then reversing the inequalities.

(b) Assume that $\eta(x)$ is bathtub-shaped and has exactly one zero, say x_0.

Since $\eta(x)$ is I in $[x_0, b)$, we must have

$$\eta(x) - \eta(t) < 0,\ x_0 \le x < t < b$$

and therefore,

$$\Delta C(x) < 0 \Rightarrow h(x) < h(x+1).$$

Thus, $h(x)$ is I in $[x_0, b)$. This fact will be referred to as property P. When $f(0) = 0,\ \eta(x)$ being bathtub-shaped, it is D in $[0, x_0)$.

Now write,

$$\begin{aligned} a(x) &= [\Delta C(x)] f(x+1) \\ &= \sum_{t=x+1}^{b} (\eta(x) - \eta(t)) f(t) - f(b), \text{ from (3.3.2).} \\ &= \eta(x) S(x+1) - [f(x+1) - f(b)] - f(b) \\ &= \eta(x) S(x+1) - f(x+1), \end{aligned} \tag{3.3.3}$$

irrespective of whether b is finite or infinite.

Also,

$$\begin{aligned} \Delta a(x) = \eta(x+1)[S(x+1) - f(x+1)] - \eta(x) S(x+1) - [f(x+2) - f(x+1)] \\ = [\Delta \eta(x)] S(x+1). \end{aligned} \tag{3.3.4}$$

Since $\eta(x)$ is D in $[0, x_0)$, by virtue of (3.3.4), $a(x)$ is also D in the same interval. Since $f(0) = 0$ and $f(1) > 0$, we have $a(0) < 0$ and this together with $a(x)$ is D in $[0, x_0)$ shows that $a(x) < 0$ in $[0, x_0)$.

Further, from (3.3.4),

$$\begin{aligned} \Delta h(x) &= (C(x) - C(x+1))\, h(x) h(x+1) \\ &= \frac{-[\Delta C(x)]}{C(x) C(x+1)} = \frac{-a(x)}{f(x+1) C(x) C(x+1)}. \end{aligned} \tag{3.3.5}$$

Hence $a(x) < 0$ in $[0, x_0)$ implies $\Delta h(x) > 0$ in $(0, x_0)$. Combining this with the property P, $h(x)$ is I in $[0, b)$ and the first part of (iii) is proved.

Now, we use the condition $f(0) \neq 0$ and $\dfrac{\eta(0)}{f(1)}(1 - f(0)) > 1$ to infer that $\Delta C(0) > 0$. Hence from (3.3.3), $a(0) > 0$. There are two cases to distinguish; $a(x_0) > 0$ or $a(x_0) < 0$.

Taking $a(x_0) > 0$, $a(x) > 0$ since $a(x)$ is D in $[0, x_0)$. Thus, $\Delta h(x) < 0$ or $h(x)$ is D

in $[x, x_0)$. This along with P means that $h(x)$ is BT.
On the other hand, $a(x_0) < 0$, there exists a point x_1 in $[0, x_0)$ such that $a(x) > 0$ in $[0, x_1)$, $a(x_1) = 0$ and $a(x) < 0$ in $(x_1, x_0]$. Hence from (3.3.5), $h(x)$ is BT in $[0, x_0]$ and this combined with P shows that $h(x)$ is BT in $[0, b)$. Part (iv) is similarly proved. ∎

The following theorem addresses the case when b is finite.

Theorem 3.3.2. Let X be the random variable in Theorem 3.3.1 with $b < \infty$. Then,

(i) if $\eta(x)$ is D and $f(b) = 0$, then $h(x)$ is D,

(ii) if $\eta(x)$ is I, then $h(x)$ is I,

(iii) if $\eta(x)$ is BT and $f(0) = 0(f(0) \neq 0, \dfrac{\eta(0)}{f(1)}(1 - f(0)) > 1)$, then $h(x)$ is $I(BT)$ and

(iv) if $\eta(x)$ is UBT, $f(0) = 0(f(0) \neq 0, \dfrac{\eta(0)}{f(1)}(1 - f(0)) > 1)$ and $f(b) = 0$, then $h(x)$ is $UBT(I)$.

Remark 3.3.1. Theorems 3.3.1 and 3.3.2 extend the results of Gupta et al. [51] to the cases when $h(x)$ is BT and UBT. Their results address the case of monotonic hazard rates only.

The main problem in applying the above theorems in practice to locate BT and UBT cases is that the computation of $\Delta\eta(x)$ and its solution becomes necessary. This results in much involved algebra in many cases. To overcome this, we have the following result which utilizes only $\eta(x)$ and not its difference in the calculations. It appears that there is no counterpart in the continuous case for this result.

Theorem 3.3.3. The random variable X has,

(i) increasing (decreasing) hazard rate if and only if $h(x + 1) \geq (\leq)\eta(x)$ and

(ii) a bathtub (upside-down bathtub) shape if and only if $h(x + 1) - \eta(x)$ has a unique zero $x_0 > 0$ such that $h(x - 1) \geq (\leq)h(x)$ in $[0, x_0)$ and $h(x - 1) \leq (\geq)h(x)$ in $[x_0, b)$.

Proof. We have

$$\begin{aligned} h(x+1)-h(x) &= \frac{f(x+1)}{S(x+1)} - \frac{f(x)}{S(x)} \\ &= \frac{f(x+1)(S(x+1)+f(x)) - f(x)S(x+1)}{S(x)S(x+1)} \\ &= -\eta(x)h(x) + h(x)h(x+1). \end{aligned}$$

Hence,

$$\frac{h(x+1)-h(x)}{h(x)} = h(x+1) - \eta(x). \tag{3.3.6}$$

The identity (3.3.6) shows that $h(x+1) \geq (\leq) h(x)$ if and only if $h(x+1) \geq (\leq) \eta(x)$. This proves (i). Also, $h(x+1) - h(x)$ has the same zero as $h(x+1) - \eta(x)$ which proves (ii). ■

Remark 3.3.2. Theorems 3.3.1 and 3.3.2 provide only sufficient conditions for the shape of $h(x)$, but Theorem 3.3.3 gives a necessary and sufficient condition. Thus, when more than one model fits the observations, Theorem 3.3.3 provides a criteria to choose the correct model by empirically examining the form of $h(x+1) - \eta(x)$ in each case.

Remark 3.3.3. When X is a continuous random variable with probability density function $g(x)$ and survival function $\bar{G}(x)$, (3.3.6) takes the form,

$$\frac{1}{h(x)} \frac{dh(x)}{dx} = h(x) - \eta(x),$$

where $h(x) = \dfrac{g(x)}{\bar{G}(x)}$ and $\eta(x) = \dfrac{-f'(x)}{f(x)}$. Similar conclusions can be derived from (3.3.6) about the nature of $h(x)$.

Remark 3.3.4. The hazard rate function $h(x)$ appearing in (3.3.6) is not an additional input needed to apply the theorem, since $h(x)$ can be expressed in-terms of $\eta(x)$ as,

$$h(x) = \frac{\prod_{t=0}^{x-1} (1-\eta(t)) f(0)}{1 - f(0) \sum_{t=1}^{x} (1-\eta(1))(1-\eta(2))...(1-\eta(t-1))}, \; x = 1, 2, ..., b \tag{3.3.7}$$

3.4 Some applications of the results

In this section, we demonstrate the usefulness of the above results in determining the behaviour of the hazard function and also in deriving new models. The methodology is to extract some simple relationships between $\eta(x)$ and $h(x+1)$. We give three examples, one each for distributions having monotone, BT and UBT hazard rates.

Example 3.4.1. Assume that $\eta(x) = c\,h(x+1)$, where c is a constant.
Substituting in (3.3.6), we arrive at the recurrence relation,

$$h(x+1) = \frac{h(x)}{1 - c\,h(x)}, \quad x = 0, 1, 2, ...$$

Successive reduction yields the solution,

$$\begin{aligned} h(x) &= \frac{h(0)}{1 + (c-1)h(0)x} \\ &= \frac{1}{\alpha x + \beta},\ \beta = \frac{1}{h(0)} > 0 \text{ and } \alpha = (c-1),\ \text{real.} \end{aligned} \tag{3.4.1}$$

From Xekalaki [143], the hazard rate function is of the form (3.4.1) if and only if the distribution of X is,

(i) geometric with $f(x) = \frac{1}{\beta}\left(\frac{\beta - 1}{\beta}\right)^x$, $x = 0, 1, 2, ...\ (c = 1)$.

(ii) Waring with $f(x) = \frac{\left(\frac{\beta-1}{\alpha}\right)_x}{\beta\left(\frac{\beta}{\alpha}+1\right)_x}$, $x = 0, 1, 2, ...\ (c > 1)$, where $(t)_x = t(t+1)...(t+x-1)$, and

(iii) negative hyper-geometric with $f(x) = \frac{\binom{-1}{x}\binom{\frac{1}{\alpha}}{n-x}}{\binom{-1+\frac{1}{\alpha}}{n}}$, $x = 0, 1, 2, ..., n$, $n = \frac{1-\beta}{\alpha}$, integer $(c < 1)$.

Further special cases are when $\alpha = 1$ in (iii), X has uniform distribution on $(0, 1, ...\beta - 1)$ and when $\beta = \alpha + 1$ in (ii), the Yule distribution results. The distributions involved have monotone hazard rates.

Example 3.4.2. Consider the relationship,

$$h(x+1) - \eta(x) = \frac{\alpha x + \beta}{h(x)}.$$

Again from (3.3.6), we have,

$$h(x+1) = h(x) + \alpha x + \beta,$$

leading to

$$h(x) = \frac{\alpha x(x-1)}{2} + \beta(x-1) + h(0). \tag{3.4.2}$$

Since (3.4.2) can be written as,

$$h(x) = a_1 x^2 + b_1 x + c_1,$$

we have a quadratic hazard rate in which the conditions $b_1 < 0$ and $b_1^2 - 4a_1c_1 > 0$ are imposed. Then $h(x)$ provides a bathtub shape with change point $x_0 = -\frac{1}{2}\left(1 + \frac{b_1}{a_1}\right)$, where x_0 is taken as the positive integer part of the right side. The random variable X has survival function,

$$S(x) = \prod_{t=0}^{x-1} \left[1 - (a_1 t^2 + b_1 t + c_1)\right], x = 0, 1, 2, ... \tag{3.4.3}$$

We call (3.4.3) the quadratic hazard rate distribution, which does not seem to have appeared in literature. Also, (3.4.3) represents a family of distributions consisting of the geometric when $a_1 = b_1 = 0$, the linear hazard rate distribution when $a_1 = 0,\ b_1 > 0, c_1 > 0$ and linear hazard rate distribution with bounded support on $(0, 1, ..., -\frac{c_1}{b_1})$, where $-\frac{c_1}{b_1}$ is a positive integer. The converse part using the relationship between $\eta(x)$ and $h(x+1)$, can be easily verified from (3.4.3).

To examine whether the model is useful in practice, we have applied it to the data in Aarset [1] pertaining to 50 lifetimes of devices by taking the first two observations 0.1 and 0.2 as zeros. The method of least squares is employed to estimate the parameters by

Table 3.1: χ^2-test for Example 3.4.2.

Class	$0-7$	$7-30$	$30-65$	$65-71$	>71
Obs. frequencies	11	8	11	4	16
Exp. frequencies	13	8	9	5	15

minimizing

$$L(a_1, b_1, c_1) = \sum_x \left(\sum_{i=0}^{x} a_1 i^2 + b_1 i + c_1 - \sum_{i=0}^{x} \frac{\hat{S}(i) - \hat{S}(i+1)}{\hat{S}(i)} \right)^2,$$

where $\hat{S}(x)$ is the empirical survival function. The estimates obtained were,

$$\hat{a_1} = 379 \times 10^{-7},\ \hat{b_1} = -259 \times 10^{-5},\ \hat{c_1} = 443 \times 10^{-4},$$

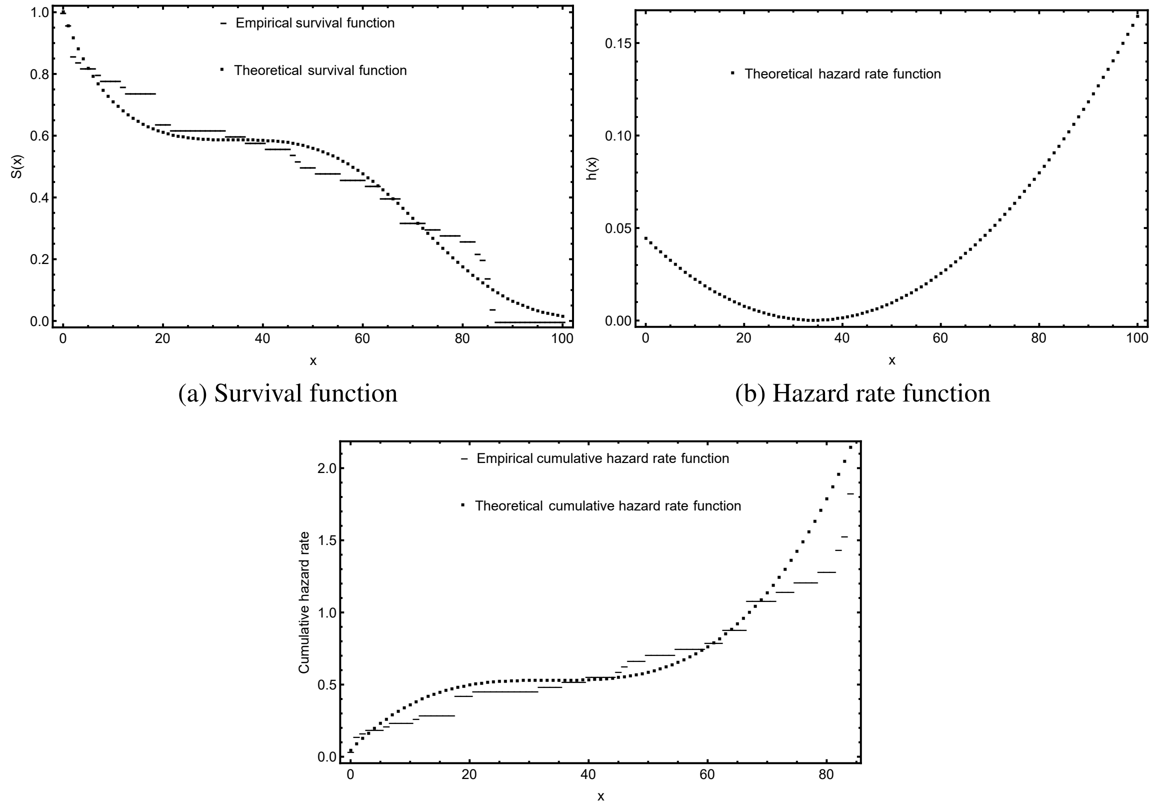
and the error in estimation is $L_{min} = 2.4989$. The model adequacy is checked using χ^2 goodness of fit. The observed and expected frequencies are given in Table 3.1. The value of χ^2 statistic is 1.01 with one degree of freedom and the corresponding p-value is 0.60. Thus, the distribution is a good fit and provides a bathtub-shaped hazard rate function with change point $x_0 = 33$. The plots of the survival function, hazard rate function and cumulative hazard rate function are exhibited in Figures 3.1a-3.1c. From Figure 3.1a, it is clear that the model fits the data well.

Example 3.4.3. Let X be distributed with survival function

$$S(x) = \left(1 + cx^2\right)^{-1},\ x = 0, 1, 2, ...;\ c > 0.$$

Then,

$$\begin{aligned} f(x) &= (1 + cx^2)^{-1} - (1 + c(x+1)^2)^{-1}, \\ h(x) &= 1 - (1 + cx^2)(1 + c(x+1)^2)^{-1}, x = 0, 1, 2, ... \\ \eta(x) &= 1 - \frac{[1 + c(x+1)^2]^{-1} - [1 + c(x+2)^2]^{-1}}{[1 + cx^2]^{-1} - [1 + c(x+1)^2]^{-1}}. \end{aligned}$$



(a) Survival function

(b) Hazard rate function

(c) Cumulative hazard rate function

Figure 3.1: Survival, hazard rate and cumulative hazard rate functions for the data in Example 3.4.2.

To assess the behaviour of $h(x)$, we calculate,

$$\eta(x) - h(x+1) = \frac{2x+3}{2x+1}\left[\frac{1+c(x+1)^2}{1+c(x+2)^2}\right] - 1,$$

leading to the equation,

$$2cx^2 + 4cx + c - 2 = 0,$$

whose zeros are,

$$x = -1 \pm \frac{1}{2}\left(4 - 2\left(1 - \frac{2}{c}\right)\right)^{\frac{1}{2}}.$$

A positive solution greater than zero exists only when $c \leq 2$. As an illustration, when $c = \frac{2}{31}$, the change point of $h(x)$ is $x_0 = 3$. Further $h(x)$ is increasing for $x = 0, 1, 2$ and decreasing for $x = 3, 4, ...$ Thus, $h(x)$ has an upside-down bathtub-shape when $c < 2$ and X has decreasing hazard rate in $[2, \infty)$.

As a slight point of departure, utility of the score function $\eta(x)$ in comparing life distributions in-terms of the basic reliability functions will also be pointed out. Recall that among two lifetime random variables X_1 and X_2, X_1 is smaller than X_2 in likelihood ratio order, $X_1 \leq_{lr} X_2$, if and only if $\frac{f_{X_2}(x)}{f_{X_1}(x)}$ is increasing in x (Shaked and Shanthikumar [135]). Let $\eta_{X_1}(x)$ and $\eta_{X_2}(x)$ be the score functions of X_1 and X_2 respectively.

Theorem 3.4.1.

$$\eta_{X_1}(x) \leq \eta_{X_2}(x) \text{ for } x = 0, 1, 2, ... \iff X_1 \leq_{lr} X_2$$

Proof.

$$\begin{aligned} \eta_{X_1}(x) \leq \eta_{X_2}(x) \quad &\iff 1 - \frac{f_{X_1}(x+1)}{f_{X_1}(x)} \geq 1 - \frac{f_{X_2}(x+1)}{f_{X_2}(x)} \\ &\iff \frac{f_{X_2}(x+1)}{f_{X_1}(x+1)} \geq \frac{f_{X_2}(x)}{f_{X_1}(x)} \end{aligned},$$

so that $\frac{f_{X_2}(x)}{f_{X_1}(x)}$ is increasing in x and therefore $X_1 \leq_{lr} X_2$. ■

Remark 3.4.1. Since the likelihood ratio order implies both hazard rate order and reversed hazard rate order and these two imply the mean residual life order and reversed mean residual life order, Theorem 3.4.1 gives a sufficient condition for comparing life distributions using $\eta_X(x)$ based on all these concepts.

3.5 Construction of discrete BT and UBT models

The origin of distributions with bathtub-shaped hazard rates can be traced back to the attempts to model data on bird populations (Deevey [42]) and to bus motor failure data in Davis [41], where monotone hazard rate distributions failed to provide reasonable fits. Since then, there has been a continuous flow of literature on various types of bathtub distributions. In most of the work on this topic, lifetime is treated as a continuous random variable. For a review of the literature, discussion and references on bathtub models we refer to Rajarshi and Rajarshi [120], Lai and Xie [85] and Nair et al. [108].

Unlike the voluminous literature in the continuous case, only a limited number of investigations have been carried out when the lifetime X is treated as a discrete random variable.

Lai and Wang [84] proposed a discrete power distribution with

$$f(x) = \frac{x^{\alpha}}{\sum_{0}^{b} x^{\alpha}},\ x = 0, 1, 2, ..., b;\ \alpha \in \mathbf{R} \tag{3.5.1}$$

for lifetime random variables and it was proved that $h(x)$ is BT for $\alpha < 0$. The rest of the BT models are of recent origin. The discretized version of the inverse Weibull law was considered in Jazi et al. [70],

$$S(x) = 1 - q^{(x-1)^{\beta}},\ x = 1, 2, ...;\ 0 < q < 1;\ \beta > 0 \tag{3.5.2}$$

which is UBT. A special case when $\beta = 2$ is discussed in Hussain and Ahmad [68] called inverse Rayleigh, whose hazard rate can also be UBT with change point at $x = 1$ or 2 for $0 < q < 0.75$ and change point $x_0 = 2$ as $q \to 0$. A competing risks model with hazard rate of the form,

$$h(x) = p + (1 - p)r_1(x), \tag{3.5.3}$$

where $r_1(x)$ is the hazard rate of an exponential Poisson law, was shown to have BT shape in Jiang [71]. The discretized version of the modified Weibull distribution with reliability function,

$$S_1(x) = q^{x^{\beta}} c^{x},\ x = 0, 1, 2, ...,\ 0 < q < 1,\ \beta > 0,\ c \geq 1 \tag{3.5.4}$$

discussed in Noughabi et al. [117] possess a BT hazard rate. Another Weibull related distribution is the discrete additive Weibull with

$$S_2(x) = q_1^{x^{\alpha}} q_2^{x^{\beta}},\ x = 0, 1, 2, ...;\ \alpha, \beta > 0, \tag{3.5.5}$$

where $q_1 = e^{-\lambda_1},\ q_2 = e^{-\lambda_2},\ \lambda_1, \lambda_2 > 0$ presented by Bebbington et al. [19]. They have studied the shape of the hazard rate and found that if $\alpha < 1 < \beta$, $h(x)$ is BT with minimum achieved at one of the three points $[t_{\alpha,\beta}]$, $1 + [t_{\alpha,\beta}]$ and $2 + [t_{\alpha,\beta}]$, where $[t_{\alpha,\beta}]$ is the largest integer contained in

$$t_{\alpha,\beta} = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(\beta-1)\lambda_2}\right)^{\frac{1}{\beta-\alpha}}. \tag{3.5.6}$$

A similar conclusion holds for $\beta < 1 < \alpha$. Later, Noughabi et al. [118] proposed the discrete modified Weibull extension,

$$S_3(x) = q^{\alpha\left(e^{(\frac{x}{\alpha})^{\beta}}-1\right)},\ x = 0, 1, 2, ...,\ 0 < q < 1,\ \beta, \alpha > 0, \tag{3.5.7}$$

as a bathtub distribution. Yet another Weibull extension is the reduced modified Weibull family discussed in Almalki and Nadarajah [7] with reliability function,

$$S_4(x) = q^{\sqrt{x}(1+bc^x)},\ x = 0, 1, 2, ...,\ 0 < q < 1,\ b > 0,\ c \geq 1. \tag{3.5.8}$$

The hazard rate of this distribution is increasing if $bc(c-\sqrt{2}) < \sqrt{2}-1$ and has BT shape otherwise.

The limited number of prevailing BT distributions reviewed above appears to be insufficient to model a wide variety of data sets. If the stochastic mechanism that generates the data is known, we need a model that is appropriate to it. Further, the observations may sometimes suggest a BT shape through the empirical hazard rate with a known shape that would require a distribution satisfying this particular shape. All these point out to the need for evolving some methods of arriving at BT distributions, which do not appear to have been considered so far. This motivates the present investigation. There is a huge literature on the methods of such constructions in the continuous case to adopt them for discrete lifetime as well. While this is the case in some of the methods we propose, there are some methods for which there is no counterpart in the continuous case.

3.5.1 Method using score function

In Section (3.4), we have already seen the method of construction of BT models by giving different functional forms to the score function $\eta(x)$. Here we give one more example, which gives rise to a new model that is applicable in many real life data sets.

Example 3.5.1. Consider the identity,

$$h(x+1) - \eta(x) = \left(\alpha - \frac{\frac{\theta\beta}{(1+\beta x)(1+\beta(x+1))}}{h(x)}\right). \tag{3.5.9}$$

$$h(x+1) - h(x) = \alpha + \frac{\theta}{1+\beta(x+1)} - \frac{\theta}{1+\beta x},$$

leaving the solution,

$$h(x) = \alpha x + \frac{\theta}{1+\beta x},\ \theta = h(0). \tag{3.5.10}$$

In order that (3.5.10) is a hazard rate, one should have $\alpha,\ \beta > 0,\ 0 < \theta < 1$. The reliability function is

$$S(x) = \begin{cases} \prod_{t=0}^{x-1} \left(1 - \alpha t - \frac{\theta}{1+\beta t}\right) & : x = 1, 2, 3, ... \\ 1 & : x = 0 \end{cases} \tag{3.5.11}$$

It is seen that $h(x)$ is BT when $0 < \alpha < \frac{\theta\beta}{1+\beta}$. The form of the hazard rate is similar to the one in the continuous case obtained by Hjorth [65]. However, the reliability function (3.5.11) is not the discretized version of the Hjorth model.

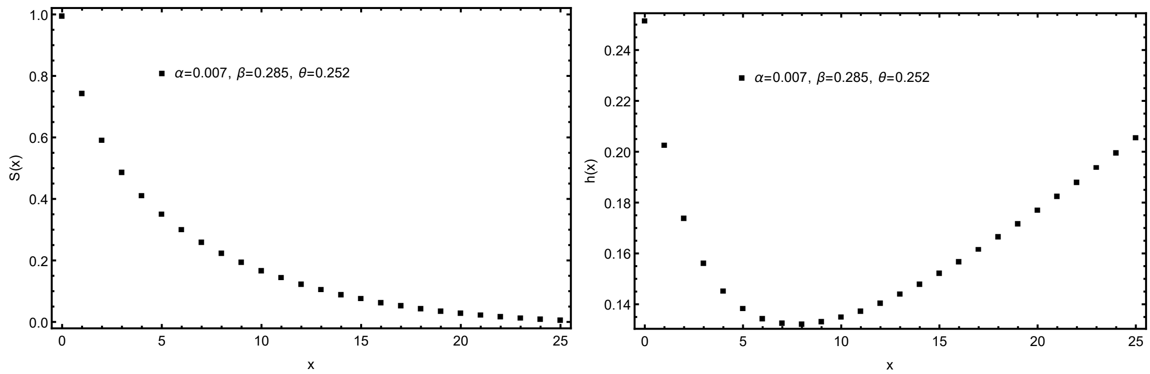
The expression (3.5.10) is the sum of the hazard rates of the linear hazard rate distribution and the Waring distribution. It is known that by taking the sum of hazard rates, one of which is decreasing and another is increasing, we may have BT hazard rate. This is also suggested as a method of deriving a new BT model. The above example can also be seen in this context. There are several continuous distributions based on hazard rates having such a structure, see for example Murthy et al. [93], Jaisingh et al. [69], Canfield and Borgman [29], Xie and Lai [144], Jiang and Murthy [72], Usgaonkar and Mariappan [141] and Wang [142]. The method of this section can be considered to these cases as well by appropriately choosing $a(x)$. From (3.5.10), it is easy to see that the change point x_0 is the solution of the quadratic equation

$$\beta x^2 + \alpha x + \theta,$$

provided that $4\beta\theta < \alpha^2$ and $x_0 > 0$. The parameters of the model are estimated by minimising the discrepancy

$$\sum \left(\alpha x + \frac{\theta}{1+\beta x} - \frac{\hat{S}(x) - \hat{S}(x+1)}{\hat{S}(x)}\right)^2$$

between the model and the empirical hazard rates. Since $\theta = h(0)$, we take it as the observed value of $h(0)$. Thus, the only parameters to be estimated are α and β. The hazard rate function and the reliability function can be seen in Figures 3.2a-3.2b.



(a) Survival function (b) Hazard function

Figure 3.2: Survival and hazard rate functions for the model in Example 3.5.1.

3.5.2 Discretizing continuous bathtub distribution

Let Y be a continuous lifetime random variable with reliability function $\bar{F}(x) = P[Y \geq x]$. If time is recorded at unit intervals, the discrete random variable $X = [Y]$, the largest integer contained in Y, has the reliability function $S(x) = \bar{F}(x)$, $x = 0, 1, 2, ...$ and probability mass function,

$$f(x) = S(x) - S(x+1). \tag{3.5.12}$$

When Y has a bathtub hazard rate, it may turn out that X also has a BT hazard rate. The reliability functions $S_1(x)$ through $S_4(x)$ discussed earlier were obtained in this way. We shall further illustrate this method with two examples, one of which renders BT and the other UBT.

Example 3.5.2. One of the earliest bathtub models introduced by Bain [12] and Bain and Englehardt [11] was the quadratic hazard rate model with

$$\bar{F}(x) = \exp\left[-ax - \frac{bx^2}{2} - \frac{cx^3}{3}\right],\ x > 0,\ c > 0,\ b \geq -(2ac)^{\frac{1}{2}}. \tag{3.5.13}$$

The reliability function and probability mass function of the corresponding discrete model is given by

$$S(x) = q^{\left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)},\ q = e^{-1}; x = 0, 1, 2, ... \tag{3.5.14}$$

and

$$f(x) = q^{\left(ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right)}\left[1-q^{a+\frac{b}{2}(2x+1)+\frac{c}{3}(3x^2+3x+1)}\right]. \tag{3.5.15}$$

Accordingly,

$$h(x) = 1-q^{a+\frac{b}{2}(2x+1)+\frac{c}{3}(3x^2+3x+1)}. \tag{3.5.16}$$

The model introduced in (3.5.14) will be called discretized quadratic hazard model and it will be denoted using DQHM(a,b,c). We study the model in detail in Section 3.6. In Section 3.6, we show that DQHM possesses BT hazard rate for specified values of parameters.

Example 3.5.3. The log logistic distribution (Gupta et al. [59]) of a continuous random variable Y is specified by the reliability function

$$\bar{F}(x) = P[Y > x] = \frac{1}{1+cx^{\alpha}},\ x \geq 0,\ c, \alpha > 0. \tag{3.5.17}$$

It is known that this distribution has a decreasing (upside-down bathtub-shaped) hazard function, when $\alpha \leq (>)1$. In the UBT case, the change point is given by,

$$x_0 = \left(\frac{\alpha-1}{c}\right)^{\frac{1}{\alpha}}.$$

The application of the distribution in analysing survival data has been pointed out by several authors. We refer to Gupta et al. [59] and their references for details. The integer part X of Y has reliability function,

$$S(x) = \frac{1}{1+cx^{\alpha}},\ x = 0, 1, 2, ...,\ c, \alpha > 0, \tag{3.5.18}$$

probability mass function

$$f(x) = \frac{1}{1+cx^{\alpha}} - \frac{1}{1+c(x+1)^{\alpha}} \tag{3.5.19}$$

and hazard function

$$h(x) = 1-\frac{1+cx^{\alpha}}{1+c(x+1)^{\alpha}}. \tag{3.5.20}$$

To ascertain the use of the model, we apply it to the data on the times from remission to relapse of 84 patients with acute non-lymphoblastic leukaemia reported in Glucksberg et al. [48]. For the present analysis, the censored observations are omitted and the rest of 51

Table 3.2: χ^2-test for leukaemia data.

Class	$0-100$	$100-150$	$151-200$	$201-250$	$251-300$	$301-400$	>400
Obs. frequencies	6	7	7	7	5	7	12
Exp. frequencies	10	5	6	5	8	6	11

observations are only utilized. We minimize the squared distance between $S(x)$ and $\hat{S}(x)$ to estimate the parameters of the model. This gives the estimates

$$\hat{\alpha} = 2.33009 \text{ and } \hat{c} = 2.78614 \times 10^{-6}$$

and the error between the fitted values and observed survival probabilities is 0.573 for the above $\hat{\alpha}$ and $\hat{c}$. The model adequacy is checked through the χ^2-test. The observed and expected frequencies are exhibited in Table 3.2 and the graphs of survival, hazard rate and cumulative hazard rate functions are given as Figures 3.3a-3.3c. The χ^2-value of 5.97 at 4 degrees of freedom yields a p-value of 0.20.

3.5.3 Modifying decreasing hazard rate functions

A third method that may result in BT distributions is to consider

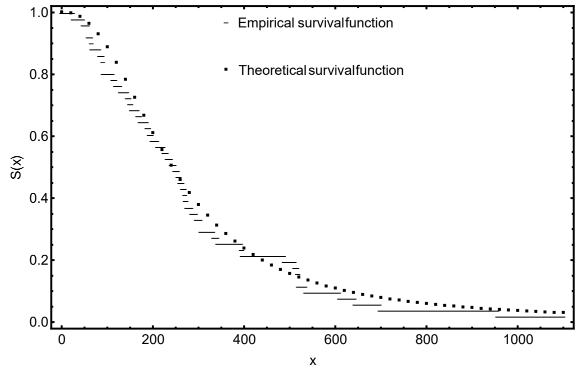
$$h_*(x) = \frac{h(x)}{S(x)},\ h(x) \leq S(x), \tag{3.5.21}$$

where $h(x)$ is a decreasing hazard rate with reliability function $S(x)$. Under the given conditions, $0 \leq h_*(x) \leq 1$ and

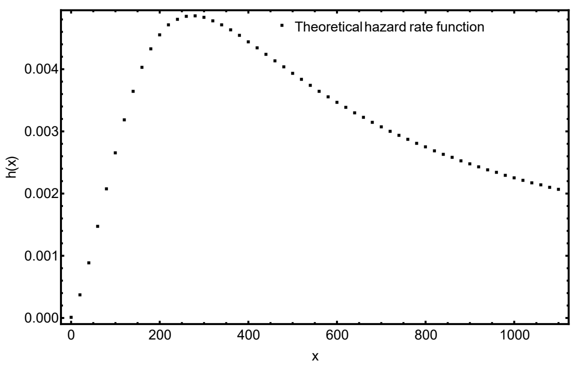
$$\sum_{x=0}^{\infty} h_*(x) = \sum_{x=0}^{\infty} \frac{h(x)}{S(x)} \geq \sum_{x=0}^{\infty} h(x) = \infty,$$

so that $h_*(x)$ is a hazard rate with reliability function $S_*(x)$. Now consider,

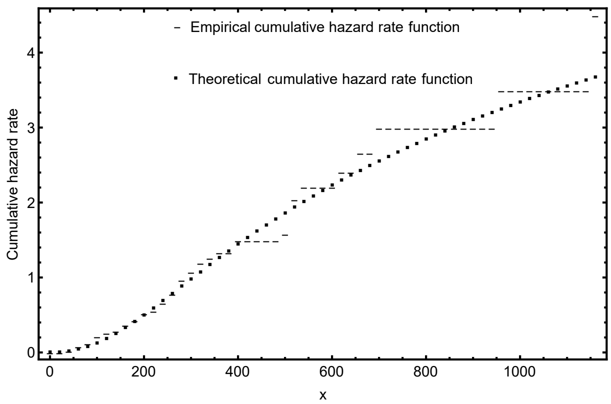
$$\begin{aligned} h_*(x+1) - h_*(x) &= \frac{h(x+1)S(x) - h(x)S(x+1)}{S(x)S(x+1)} \\ &= \frac{h(x+1)}{S(x+1)} - \frac{h(x)}{S(x)} \end{aligned}$$



(a) Survival function



(b) Hazard function



(c) Cumulative hazard function

Figure 3.3: Survival, hazard rate and cumulative hazard rate functions for the leukaemia data in Example 3.5.3.

For $h_*(x)$ to be BT, the right of above expression must be zero for a unique $x_0 > 0$. But,

$$\frac{h(x+1)}{S(x+1)} = \frac{h(x)}{S(x)}$$

implies

$$\frac{h(x+1)}{h(x)} = \frac{S(x+1)}{S(x)}$$

or

$$\frac{h(x+1)}{h(x)} = 1 - h(x).$$

Thus, for $h_*(x)$ to be BT or UBT,

$$h(x+1) = h(x)(1 - h(x)) \tag{3.5.22}$$

must have a unique solution $x_0 > 0$. The idea behind the modification (3.5.21) is that initially $S(x)$ has values close to unity to keep the decreasing nature of $h(x)$ and hence that of $h_*(x)$. But as x increases, $S(x)$ becomes closer to zero to increase the value of $h(x)$ that may transform $h_*(x)$ to an increasing function, so that the overall shape of $h_*(x)$ may be BT. If $h_*(x)$ does not produce a BT, then the process can be repeated with $h_{**}(x) = \dfrac{h_*(x)}{S_*(x)}$, provided $h_*(x) \leq S_*(x)$ and so on. We give two examples that illustrate how the method works in practice.

Example 3.5.4. A good share of continuous bathtub distributions are related to the Weibull distribution as can be seen from Chapter 5 of Lai and Xie [85]. In this example, we apply the above method to generate a BT model from the discretized Weibull I distribution.

$$S(x) = q^{x^\beta},\ x = 0, 1, 2, ...,\ 0 < q < 1,\ \beta > 0 \tag{3.5.23}$$

of Nakagawa and Osaki [113]. In this case,

$$h(x) = 1 - \frac{q^{(x+1)^\beta}}{q^{x^\beta}}, \tag{3.5.24}$$

so that

$$h_*(x) = \frac{q^{x^\beta} - q^{(x+1)^\beta}}{q^{2x^\beta}} \tag{3.5.25}$$

and $h(x) \leq S(x)$. From Figure 3.4 representing the graph of $h_*(x)$, it is seen that $h_*(x)$ can be BT.

Example 3.5.5. Let X follow the Waring (Nair et al. [107]) distribution,

$$S(x) = \frac{(m)_x}{(m+n)_x},\ x = 0, 1, 2...;\ m, n > 0. \tag{3.5.26}$$

Then,

$$h(x) = \frac{n}{m+n+x},$$

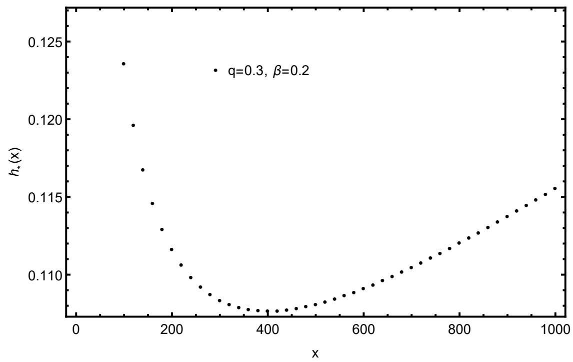


Figure 3.4: Hazard rate function for the model in Example 3.5.4.

which is clearly decreasing.

$$h_*(x) = \frac{n(m+n)_x}{(m+n+x)(m)_x}.$$

By virtue of the Waring expansion,

$$\frac{1}{(x-a)} = \frac{1}{x} + \frac{a}{x(x+1)} + \frac{a(a+1)}{x(x+1)(x+2)} + ...,$$

we can write

$$S(x) = \frac{(m)_x}{(m+n)_x} = n\frac{(m)_x}{(m+n)_x}\left[\frac{1}{m+n+x} + \frac{m+x}{(m+n+x)(m+n+x+1)} + ...\right].$$

From this, it can be seen that $h(x) \leq S(x)$. Also,

$$h(x+1) - h(x)(1-h(x)) = 0$$

leads to

$$(m+n)^2 + (m+n)x - mx - x - m(m+n) - m = 0.$$

The unique solution is,

$$x_0 = \frac{m(m+n) + m - (m+n)^2}{n-1}$$

which will give a change point provided $(m+n-1) < m < \frac{(m+n)^2}{m+n+1}$. As an illustration, taking $m = 0.46,\ n = 0.86$, we have $x_0 = 4.8$ which is taken as 4. Figure 3.5 shows the

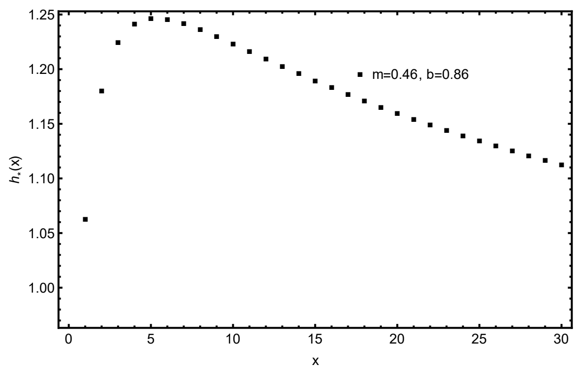


Figure 3.5: Hazard rate function for the model in Example 3.5.5

plotted hazard rate function and clearly it is UBT-shaped. The reliability function $S_*(x)$ is derived from (1.1.2).

3.5.4 Other methods

In this section, we discuss certain methods borrowed from the continuous case. The mixture of a distribution with increasing hazard rate and a distribution with decreasing hazard rate may produce a BT distribution. Let $f_1(x)(S_1(x))$ and $f_2(x)(S_2(x))$ be the probability mass(survival) functions of two discrete lifetimes X_1 and X_2. Then the two-component mixture of $f_1(x)$ and $f_2(x)$,

$$f(x) = \alpha f_1(x) + (1-\alpha) f_2(x),\ 0 \leq \alpha \leq 1$$

has a hazard rate of the form

$$h(x) = p(x)h_1(x) + (1-p(x))h_2(x), \tag{3.5.27}$$

where $h_1(x)$ and $h_2(x)$ are hazard rate functions of X_1 and X_2 and

$$p(x) = \frac{\alpha S_1(x)}{\alpha S_1(x) + (1-\alpha) S_2(x)}.$$

Although the expression (3.5.27) looks compact, it is difficult to prove analytically that $h(x)$ has a maximum or minimum.

Table 3.3: Observed and expected frequencies for Aarset data.

Class	$0-4$	$5-18$	$19-50$	$51-67$	$68-84$	>84
Observed	9	9	8	8	9	7
Expected	7	10	7	8	10	8

A strictly convex function which satisfies $0 \le h(x) \le 1$ and $\sum_{t=0}^{\infty} h(t) = \infty$ for non-negative integer values can be a candidate hazard rate function that is bathtub-shaped.

Example 3.5.6. The function

$$h(x) = 1 - e^{-(ax^2+bx+c)};\ a, c > 0,\ ac \ge \frac{b^2}{4}. \tag{3.5.28}$$

satisfies the above conditions. The parameters are estimated by regression of $-\log(1 - \hat{h}(x))$, where $\hat{h}(x) = \dfrac{\hat{S}(x) - \hat{S}(x+1)}{\hat{S}(x)}$ is the empirical hazard rate, on a quadratic function. This method was applied to the analysis of data in Aarset [1] pertaining to 50 lifetimes of devices by taking the first two observations 0.1 and 0.2 as zeros to obtain the estimates

$$\hat{a} = 227.975 \times 10^{-7},\ \hat{b} = -156.645 \times 10^{-5},\ \hat{c} = 326.186 \times 10^{-4}$$

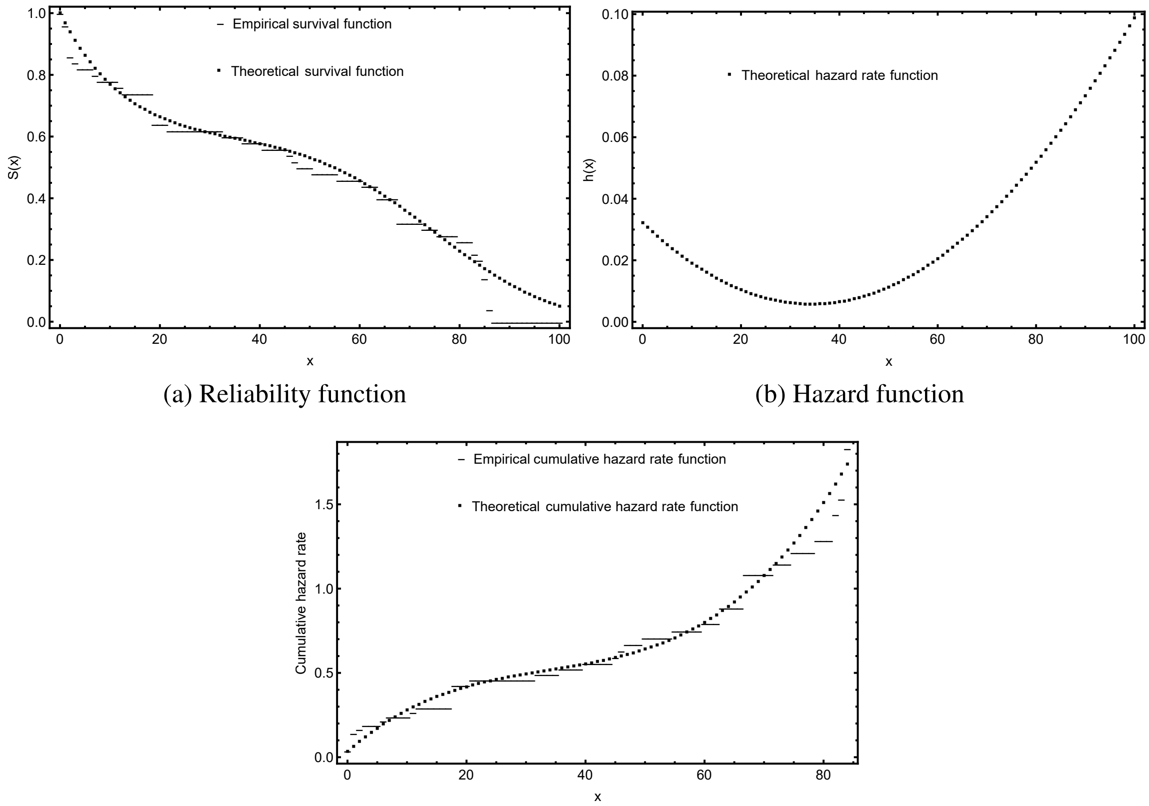
The sum of squares of the errors between the model and empirical values is 0.041. Applying the χ^2-test, we have the observed and expected frequencies as in Table 3.3. The χ^2-value of 1.03 at 2 degrees of freedom gives a p-value of 0.59. The change point is

$$x_0 = \left[-\frac{(b+a)}{2a}\right] = 33,$$

the integer part of x_0. See Figures 3.6a-3.6c for the reliability, hazard rate and cumulative hazard rate functions. From (1.1.2) and (3.5.28), we arrive at a nice form for the reliability function as

$$S(x) = q^{\frac{ax^3}{3} + \frac{a-b}{2}x^2 + \frac{(a-3b-6c)}{6}x}, x = 0, 1, 2, ...$$

The quadratic hazard rate family of Example 3.5.2 is another distribution that obeys the above criterion.



(a) Reliability function (b) Hazard function

(c) Cumulative hazard function

Figure 3.6: Survival, hazard rate and cumulative hazard rate functions for the model in Example 3.5.6.

3.6 Discretized quadratic hazard model

The discretized quadratic hazard model (3.5.14), introduced in Section 3.5.2, deserves a separate study because of its interesting reliability properties. In this section, we study the reliability properties of DQHM and propose a two stage procedure for estimating the parameters. A real dataset has been analysed using this estimation procedure and we can see that the model performs well.

For the model,

$$h(x+1) - h(x) = q^{a+\frac{b}{2}(2x+1)+\frac{c}{3}(3x^2+3x+1)}\left(1 - q^{b+2c(x+1)}\right), \quad (3.6.1)$$

has a unique zero

$$[x_0] = \left[-\frac{b}{2c} - 1\right],\ b < 0,\ c > 0, \tag{3.6.2}$$

where $x_0 > 0$ and $[x_0]$ is the integer part of x_0. For x_0 to be non-negative, we need $-b > 2c$. Further,

$$h(x_0 + 1) - h(x_0) = 1 - q > 0,$$

and

$$h(x_0) - h(x_0 - 1) = 1 - q^{-2c} < 0,$$

showing that $h(x)$ is decreasing in $[0, x_0)$ and increasing in $[x_0, \infty)$ yielding a BT shape.

Thus, the hazard rate function is BT when $-b > 2c$ and is increasing for $b > 0$. The DQHM(a,b,c) has a non-zero hazard rate at the point 0, which is not common.

The following particular cases are applicable for DQHM(a, b, c).

(a) When $b = c = 0$, the model reduces to geometric distribution with parameter $\theta = q^a$.

(b) When $c = 0$ and $b > 0$, it has the hazard rate function

$$h(x) = 1 - q^{-(a+\frac{b}{2}+bx)}, \tag{3.6.3}$$

which is increasing in x.

(c) When $a = c = 0$, we have the discretized version of Rayleigh distribution.

Theorem 3.6.1. Consider a series system consisting of n components. Let the component lifetimes be independently distributed as DQHM(a_i, b_i, c_i), $i = 1, 2, ..., n$. Then the system lifetime is distributed as DQHM($\sum_{i=1}^n a_i, \sum_{i=1}^n b_i, \sum_{i=1}^n c_i$).

The proof is direct.

3.6.1 Residual life

The following theorem gives the closure property of the residual life random variable of DQHM(a, b, c).

Theorem 3.6.2. For DQHM(a, b, c), the residual life variable X_x is distributed as DQHM (a_1, b_1, c_1) with survival function

$$S_x(t) = q^{a_1 t + \frac{b_1}{2}t^2 + \frac{c_1}{3}t^3} \tag{3.6.4}$$

where, $a_1 = \frac{1}{6}\left(6a + 6bx + 6b + 6c(x+1)^2\right)$, $b_1 = \frac{2}{6}(3b + 6c(x+1))$ and $c_1 = c$.

The proof of the theorem follows directly from the definition of $S_x(t)$.

3.6.2 Transformation

In this section, we study the behaviour of DQHM distribution under scale transformation. The following theorem shows the closure of DQHM under change of scale, whose proof is direct.

Theorem 3.6.3. Let X be a non-negative integer valued random variable and $k > 0$ be a constant. Then $Y = kX$ is distributed as DQHM(a_1, b_1, c_1) if and only if X follows DQHM(a,b,c), where $a_1 = \frac{a}{k}$, $b_1 = \frac{b}{k^2}$ and $c_1 = \frac{c}{k^3}$.

3.6.3 Estimation of parameters

Suppose $X_1, X_2, ..., X_n$ be a random sample from (3.5.14). We apply the maximum likelihood procedure for estimating the parameters. In the present set-up, the likelihood is very complicated and there is a possibility of multiple roots for the score equation. The convergence of the estimates depends on the initial value we give. So we consider a two stage estimation procedure, which consists of estimating the initial values of the parameters by least square fit and then maximizing the likelihood with these estimates as starting points. Based on the random sample from the DQHM(a,b,c), the log likelihood is given by

$$\begin{aligned} l[\mathbf{x}, a, b, c] &= \sum_{i=1}^{n} \log[e^{-ax_i - \frac{bx_i^2}{2} - \frac{cx_i^3}{3}}(1 - e^{-(a + \frac{b}{2} + \frac{c}{3} + (b+c)x_i + cx_i^2)})] \\ &= \sum_{i=1}^{n}(-ax_i - \frac{bx_i^2}{2} - \frac{cx_i^3}{3}) + \log(1 - e^{-(a + \frac{b}{2} + \frac{c}{3} + (b+c)x_i + cx_i^2)}) \end{aligned} \tag{3.6.5}$$

The score equations are

$$\frac{\delta}{\delta a} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} \left[\frac{x_i \left(-e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} \right) + x_i + 1}{e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1} \right] = 0 \quad (3.6.6)$$

$$\frac{\delta}{\delta b} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} \frac{x_i^2 \left(-\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right) \right) + 2x_i + 1}{2 \left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right)} = 0 \quad (3.6.7)$$

$$\frac{\delta}{\delta c} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} \frac{x_i^3 \left(-\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right) \right) + 3x_i^2 + 3x_i + 1}{3 \left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right)} = 0. \quad (3.6.8)$$

The second order derivatives are given by

$$\frac{\delta^2}{\delta a^2} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} \frac{1}{2 - 2\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right)} \quad (3.6.9)$$

$$\frac{\delta^2}{\delta b^2} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} -\frac{(2x_i + 1)^2}{8\left(\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right) - 1\right)} \quad (3.6.10)$$

$$\frac{\delta^2}{\delta c^2} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} -\frac{(3x_i(x_i + 1) + 1)^2}{18\left(\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right) - 1\right)} \quad (3.6.11)$$

$$\frac{\delta^2}{\delta ab} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} -\frac{2x_i + 1}{4\left(\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right) - 1\right)} \quad (3.6.12)$$

$$\frac{\delta^2}{\delta ac} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} -\frac{3x_i(x_i + 1) + 1}{6\left(\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right) - 1\right)} \quad (3.6.13)$$

$$\frac{\delta^2}{\delta bc} l[\mathbf{x}, a, b, c] = \sum_{i=1}^{n} -\frac{(2x_i + 1)(3x_i(x_i + 1) + 1)}{12\left(\cosh\left(a + b\left(x_i + \frac{1}{2}\right) + c\left(x_i^2 + x_i + \frac{1}{3}\right)\right) - 1\right)}. \quad (3.6.14)$$

We can see that the score equations are non-linear in a, b and c. We need to use numerical methods to solve it. As mentioned before, we need appropriate initial values to use numerical methods effectively. To obtain these initial values, we proceed as follows. Let $\hat{S}(x)$ be the empirical survival function calculated from the sample. We propose a linear regression model

$$-\,log(\hat{S}(x_i)) = ax_i + \frac{bx_i^2}{2} + \frac{cx_i^3}{3} + \epsilon_i,\ i = 1, 2, ..., n, \quad (3.6.15)$$

where ϵ_i's are independent and identically distributed random variables with mean 0 and variance σ^2.

The model in (3.6.15) can be rewritten in matrix form as

$$\mathbf{y} = \mathbf{M}\boldsymbol{\theta} + \boldsymbol{\epsilon}, \tag{3.6.16}$$

where $\mathbf{y} = [-log(\hat{S}(x_i))]'$, $i = 1, ..., n$, $\boldsymbol{\theta} = [a, b, c]'$, and $\boldsymbol{\epsilon} = [\epsilon_1, ..., \epsilon_n]'$. The design matrix corresponding to the above model is given by

$$\mathbf{M} = \begin{bmatrix} x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{3} \\ x_2 & \frac{x_2^2}{2} & \frac{x_2^3}{3} \\ ... & ... & ... \\ x_n & \frac{x_n^2}{2} & \frac{x_n^3}{3} \end{bmatrix} \tag{3.6.17}$$

Estimate of $\boldsymbol{\theta}$ is obtained by ordinary least square method as

$$\hat{\boldsymbol{\theta}} = [\hat{a}, \hat{b}, \hat{c}] = (\mathbf{M}'\mathbf{M})^{-1}\mathbf{M}'\mathbf{y}. \tag{3.6.18}$$

We use these estimates as initial values for the maximization of the log-likelihood.

It is easy to see that the probability mass function satisfies the regularity condition given by Cramér [39]. Thus, by Cramér-Huzurbazar theorem (see Lehmann and Casella [88]), we can see that $\hat{\boldsymbol{\theta}}$ is consistent and $\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is asymptotically normal with mean vector $\mathbf{0}$ and dispersion matrix $\frac{1}{\sqrt{n}}\mathbf{I}^{-1}(\boldsymbol{\theta})$, where $\mathbf{I}(\boldsymbol{\theta})$ is the Fisher information matrix. From (3.6.9)-(3.6.14) we can evaluate the observed Fisher information matrix numerically, which gives estimate of $\mathbf{I}(\boldsymbol{\theta})$.

We now illustrate the method with one real dataset. We compare the model performance with other existing models.

Example 3.6.1. We consider a dataset consisting of the lifetimes of 18 electronic components, reported in Wang [142], which was recently analysed by Almalki and Nadarajah [7] using the discretized reduced modified Weibull(DRMW) distribution. To obtain the least square estimates, we form the design matrix as

$$\mathbf{M} = \begin{bmatrix} x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{3} \\ x_2 & \frac{x_2^2}{2} & \frac{x_2^3}{3} \\ \cdots & \cdots & \cdots \\ x_{18} & \frac{x_{18}^2}{2} & \frac{x_{18}^3}{3} \end{bmatrix}$$

and propose the model,

$$-\,log(\hat{S}(x_i)) = ax_i + \frac{bx_i^2}{2} + \frac{cx_i^3}{3} + \epsilon_i, \;\; i = 1, 2, ..., 18, \tag{3.6.19}$$

where $\hat{S}(x)$ is the empirical survival function.

The least square estimates are

$$\hat{a} = 532.272 \times 10^{-5}, \; \hat{b} = -303.786 \times 10^{-7} \text{ and } \hat{c} = 1.372 \times 10^{-7}$$

Using these as initial estimates, the log likelihood is numerically maximized. The maximum of l is obtained as

$$l_{max} = -108.213$$

for the values

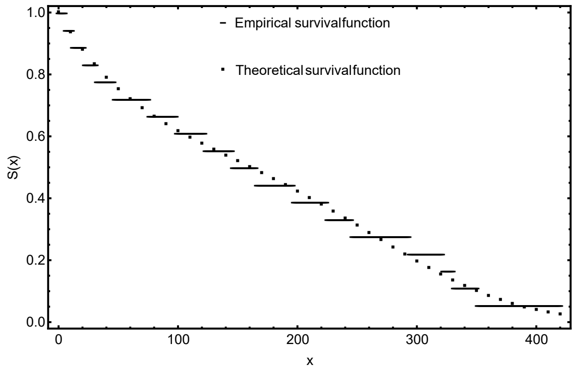
$$\hat{a} = 695.067 \times 10^{-5}, \hat{b} = -585.678 \times 10^{-7} \text{ and } \hat{c} = 2.421 \times 10^{-7}.$$

The survival function, hazard rate function and cumulative hazard rate function are plotted in Figures 3.7a-3.7c.

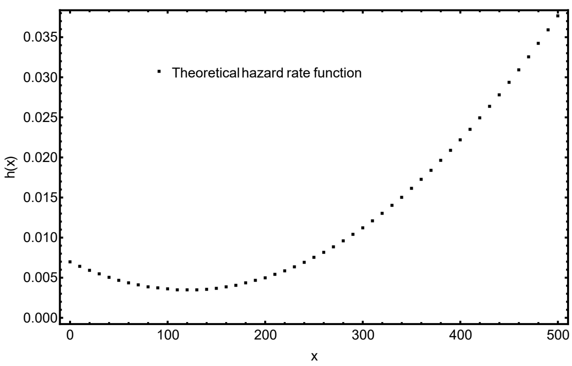
Now, to compare the performance of DQHM with the existing models, we calculate the Kolmogorov Smirnov distance, Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC).
We have,

$$AIC = 2k - 2l[\hat{\boldsymbol{\theta}}, \mathrm{x}] \tag{3.6.20}$$

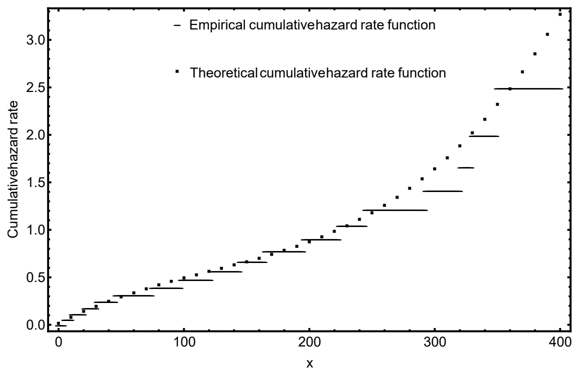
where, k is the dimension of vector $\boldsymbol{\theta}$ and $l[\hat{\boldsymbol{\theta}}, \mathrm{x}]$ is the log likelihood at $\hat{\boldsymbol{\theta}}$.



(a) Survival function



(b) Hazard function



(c) Cumulative hazard function

Figure 3.7: Survival, hazard rate and cumulative hazard rate functions for the data in Example 3.6.1.

Also,

$$BIC = k \log n - 2l[\hat{\boldsymbol{\theta}}, \mathbf{x}] \tag{3.6.21}$$

and

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}. \tag{3.6.22}$$

Table 3.4 provides the measures for model adequacy. From Table 3.4, we see that the DQHM outperforms the other four models, namely discrete reduced modified Weibull (DRMW) due to Almalki and Nadarajah [7], discrete modified Weibull(DMW) due to Noughabi et al. [117], discrete additive Weibull (DAddW) due to Bebbington et al. [19] and discrete Weibull (DW) due to Nakagawa and Osaki [113].

Table 3.4: Model adequacy of the data in Example 3.6.1

Model	AIC	BIC	CAIC	K-S
DRMW	223.9	226.5	225.6	.084
DMW	225.6	228.3	227.3	.092
DAddW	227.9	231.4	230.9	.099
DW	226.1	227.9	226.9	.137
DQHM	222.426	225.097	224.14	0.0702

3.7 Conclusion

In this chapter, we have presented some theorems that help in detecting the shape of the hazard rate function when lifetime is treated as discrete. All the results will work out when the probability mass function alone is known. Following this, we have discussed various methods of construction of discrete bathtub and upside-down bathtub distributions. We have provided examples in which the models were applied to real data and we have studied the properties of discretized quadratic hazard model in detail. They supplement the existing list of BT models in literature.

Chapter 4

Quantification of Relative Ageing

4.1 Introduction

As mentioned in Introduction chapter, the study of relative ageing concepts is inevitable in situations where one has to compare reliabilities of more than one device. At present, it appears that there is no study concerning the relative ageing of two devices in the discrete time domain. The objective of the present chapter is to fill this gap by presenting some concepts and results that help the comparison of the intensity of ageing among competing devices, when the lifetime is discrete. The present chapter includes some results which have no continuous counterparts, and a discussion of stochastic orders for comparing discrete life distributions in-terms of ageing concepts.

The rest of the chapter is organized as follows. In Section 4.2, we introduce stochastic ordering by ageing concepts when lifetime is treated as a discrete random variable. The concept of specific ageing factor is discussed in Section 4.3. The relative ageing concepts are discussed in Section 4.4. Characterizations of ageing concepts using these concepts are presented. Section 4.5 discusses ageing intensity function in the context of discrete lifetime data analysis. The chapter ends with a brief conclusion in Section 4.6.

Results in this chapter have been accepted for publication in "Metron". (See Nair et al. [112])

4.2 Stochastic orders

To discuss relative ageing, let X_1 and X_2 denote the discrete lifetimes of two devices whose hazard rates are $h_{X_1}(x)$ and $h_{X_2}(x)$ and alternative hazard rates are $h^*_{X_1}(x)$ and $h^*_{X_2}(x)$ respectively. Notice that there are some results specific to discrete case owing to alternative ways in which various concepts can be defined.

Definition 4.2.1. The random variable X_1 is ageing faster than X_2 in hazard rate, written as $X_1 \leq_{IHR} X_2$, if $\dfrac{h_{X_1}(x)}{h_{X_2}(x)}$ is increasing in x, provided $h_{X_2}(x) \neq 0$.

Definition 4.2.2. The random variable X_1 is ageing faster than X_2 in alternative hazard rate, written as $X_1 \leq_{IHR^*} X_2$, if $\dfrac{h^*_{X_1}(x)}{h^*_{X_2}(x)}$ is increasing in x.

When we look at the binary relation $\leq_{IHR}$ more closely, it says that the lifetime X_1 is less than lifetime X_2, in the sense of the hazard rate, if X_1 is more IHR than X_2. In other words, if the hazard rates of X_1 and X_2 are such that $\dfrac{h_{X_1}(x)}{h_{X_2}(x)}$ is a constant (increasing /decreasing), then the device with lifetime X_1 ages at the same rate (faster/ slower) than the device with lifetime X_2.

Looking at the properties of the stochastic order, the different cases that can occur are: (a) $h_{X_1}(x) \geq h_{X_2}(x),$ for all $x = 0, 1, 2, ...$ (b) $h_{X_1}(x)$ crosses $h_{X_2}(x)$ from below. Then, it is not essential that $X_1 \leq_{IHR} X_2$ should imply a corresponding ordering of $h_{X_1}(x)$ and $h_{X_2}(x)$. This is illustrated in the following examples.

Example 4.2.1. Suppose that X_1 has geometric distribution with $p = \dfrac{3}{4}$ and X_2 has Waring distribution(Nair et al. [107]) with probability mass function

$$f(x) = \frac{n(m)_x}{(m+n)_{x+1}},\ x = 0, 1, 2, ...;\ m, n > 0,$$

where, $(m)_x$ denotes the Pochhammer symbol. When $m = 1$ and $n = 2$, the hazard rate is $h_{X_2}(x) = \dfrac{2}{3+x},\ x = 0, 1, 2, ...$ Then $X_1 \leq_{IHR} X_2$ and $h_{X_1}(x) \geq h_{X_2}(x)$.

Example 4.2.2. When $h_{X_1}(x) = \dfrac{1}{6-x}$, corresponding to the uniform distribution in $[1, 5]$, and $h_{X_2}(x) = \dfrac{2}{3+x}$ as in Example 4.2.1, we have $h_{X_1}(x)$ and $h_{X_2}(x)$ crossing at $x = 3$ and $X_1 \leq_{IHR} X_2$.

Theorem 4.2.1. $X_1 \leq_{IHR} X_2$ and $X_1 \leq_{IHR^*} X_2$ are not equivalent.

Proof. To prove the assertion, take

$$h_{X_1}(x) = 1 - e^{-2(x+1)-1} \text{ and } h_{X_2}(x) = 1 - e^{-4(x+1)-1}, x = 0, 1, 2, ...$$

Then $X_1 \leq_{IHR} X_2$. But using (1.1.6),

$$\frac{h^*_{X_1}(x)}{h^*_{X_2}(x)} = \frac{2x+3}{4x+5}$$

is decreasing in x. Thus, $X_1 \geq_{IHR^*} X_2$. ■

Some properties of the order $\leq_{IHR}$ are given below.

(a) From Definition 4.2.1, we have

$$X_1 \leq_{IHR} X_2 \iff \frac{h_{X_1}(x+1)}{h_{X_2}(x+1)} \geq \frac{h_{X_1}(x)}{h_{X_2}(x)} \iff \frac{h_{X_1}(x+1)}{h_{X_1}(x)} \geq \frac{h_{X_2}(x+1)}{h_{X_2}(x)}$$

and so,

$$X_1 \leq_{IHR} X_2 \text{ and } X_2 \leq_{IHR} Z \Rightarrow \frac{h_{X_1}(x+1)}{h_{X_1}(x)} \geq \frac{h_{X_2}(x+1)}{h_{X_2}(x)} \geq \frac{h_Z(x+1)}{h_Z(x)}$$

$$\Rightarrow X_1 \leq_{IHR} Z.$$

Thus, the ordering $\leq_{IHR}$ is defined as partial order among equivalence classes generated by the ratio $\dfrac{h_{X_2}(x)}{h_{X_1}(x)}$.

(b) Also, we obtain

$$X_1 \leq_{IHR} X_2 \text{ and } X_2 \leq_{IHR} X_1 \Rightarrow h_{X_1}(x) = c.h_{X_2}(x)$$

for some constant $c > 0$. Thus, both $X_1 \leq_{IHR} X_2$ and $X_2 \leq_{IHR} X_1$ correspond to the equivalence class in which the proportional hazard rates model for discrete lifetimes hold.

(c) The partial order $\leq_{IHR}$ can be used to define the IHR concept. Moreover, X is IHR $\iff X \leq_{IHR} X_G$, where X_G is the geometric random variable with probability mass function $f(x) = q^x p;\ x = 0, 1, 2, ...$

Remark 4.2.1. It can be seen that Definition 4.2.1 and the ordering with X_G, given above are analogous to Proposition 2.2(iii) and Proposition 2.1(iii), respectively in Sengupta and Deshpande [132] in the continuous case. But the main difference is that neither Proposition 2.2(i),(ii) nor Definition 1, of Sengupta and Deshpande [132], holds in the discrete case since $-\log S(x)$ is not the cumulative hazard rate in the discrete case.

When the hazard rate is replaced by other reliability functions, we can have similar stochastic orders representing relative ageing. We consider only some such important representations here.

Definition 4.2.3. A discrete random lifetime X_1 is ageing faster than another random lifetime X_2 in hazard rate average-1 $(X_1 \leq_{IHRA_1} X_2)$ if $\dfrac{\log S_{X_1}(x)}{\log S_{X_2}(x)}$ is increasing in $x = 1, 2, ...$

Definition 4.2.4. A discrete random lifetime X_1 is ageing faster than another random lifetime X_2 in hazard rate average-2 $(X_1 \leq_{IHRA_2} X_2)$ if $\dfrac{\sum_{0}^{x-1} h_{X_1}(t)}{\sum_{0}^{x-1} h_{X_2}(t)}$ is increasing in $x = 0, 1, 2, ...$

We interpret $X_1 \leq_{IHRA_1} X_2$ $(X_1 \leq_{IHRA_2} X_2)$ by saying that a device with lifetime X_1 has lesser life than the device with lifetime X_2 if X_1 is more $IHRA_1(IHRA_2)$ than X_2. Since X_1 has greater $IHRA_1(IHRA_2)$, it ages more positively than X_2. The stochastic orderings in Definitions 4.2.3 and 4.2.4 satisfy the following properties.

(i) X is $IHRA_1 \iff X \leq_{IHRA_1} X_G$ and
X is $IHRA_2 \iff X \leq_{IHRA_2} X_G$.

(ii) $X_1 \leq_{IHR} X_2 \Rightarrow X_1 \leq_{IHRA_2} X_2$.

(iii) $X_1 \leq_{IHR^*} X_2 \Rightarrow X_1 \leq_{IHRA_1} X_2$,
since

$$\frac{\log S_{X_1}(x)}{\log S_{X_2}(x)} = \frac{\sum_{t=0}^{x-1} h^*_{X_1}(t)}{\sum_{t=0}^{x-1} h^*_{X_2}(t)}.$$

(iv) $X_1 \leq_{IHR} X_2 \nRightarrow X_1 \leq_{IHRA_1} X_2$.
To prove this, let $h_{X_1}(x) = 1 - e^{-2(x+1)-1}$ and $h_{X_2}(x) = 1 - e^{-4(x+1)-1}$. Then, $\dfrac{h_{X_1}(x)}{h_{X_2}(x)}$ is increasing in x and so $X_1 \leq_{IHR} Y$. But $\dfrac{\log S_{X_1}(x)}{\log S_{X_2}(x)} = \dfrac{x+2}{2x+3}$ is decreasing in x.

Remark 4.2.2. Definition 4.2.3 is analogous to Proposition 2.3(iii) in Sengupta and Deshpande [132], but Definitions 4.2.3 and 4.2.4 are not equivalent because $\sum_0^{x-1} h(t) \neq -\log S(x)$ in the discrete case.

Other stochastic orders like $\leq_{DMRL}$, $\leq_{NBU}$ etc. may be considered as in Kochar and Wiens [83] and Kochar [82].

4.3 Specific ageing factor

Consider two devices or systems, whose lifetimes follow the same distribution with survival function $S(x)$. One of the devices is new and the other is aged x_1 units. Then, $S(x_2)$ is the probability that the new device survives age x_2 and $\dfrac{S(x_1+x_2)}{S(x_1)}$ is the probability that the older device aged x_1 survives the same duration x_2. Following Bryson and Siddiqui [26] in the continuous case, we define the specific ageing factor in discrete case as,

$$A(x_1, x_2) = \frac{S(x_1)S(x_2)}{S(x_1+x_2)}, \tag{4.3.1}$$

which compares the survival probabilities of the two units. When $A(x_1, x_2) \geq 1 (\leq 1)$, $P[X \geq x_1 + x_2 | X \geq x_1] < (>) P[X \geq x_2]$, which means that a device of age x_1 surviving for an additional lifetime x_2 has lesser (greater) probability of surviving than a new unit to survive the same lifetime x_2. Thus, $A(x_1, x_2)$ provides a measure of relative ageing of an older unit in comparison with a new one. In other words, it gives the impact of having survived x_1 units of age, in the future life of the old unit. It seems that the potential of the specific ageing factor in determining ageing patterns and relative ageing have not been exploited in the continuous case. We give some properties of $A(x_1, x_2)$.

Theorem 4.3.1. The random variable X has increasing (decreasing) hazard rate if and only if $A(x_1, x_2)$ is increasing (decreasing) in x_1.

Proof. $A(x_1, x_2)$ is increasing in x_1

$$\begin{aligned}
&\iff A(x_1+1, x_2) - A(x_1, x_2) \geq 0. \\
&\iff S(x_1+1)S(x_1+x_2) \geq S(x_1)S(x_1+x_2+1) \\
&\iff \frac{S(x_1+1)}{S(x_1)} \geq \frac{S(x_1+x_2+1)}{S(x_1+x_2)} \\
&\iff 1-h(x_1) \geq 1-h(x_1+x_2) \\
&\iff X \text{ is } IHR.
\end{aligned}$$

The proof for DHR case is similar. ∎

Theorem 4.3.2. (i) X is $NBU_1(NWU_1) \iff A(x_1, x_2) \geq (\leq)1$ for all $x_1, x_2 = 0, 1, 2, ...$

(ii) X is NBU-$y_0 \iff A(y_0, x_2) \geq (\leq)1$ for all $x_2 = 0, 1, 2, ...; y_0 = 1, 2, ...$

(iii) X is NBU*y_0 (NWU*y_0) $\iff A(x_1, x_2) \geq (\leq)1$ for all $x_1 = y_0, y_0+1, ... : x_2 = 0, 1, 2, ...$

The proof is a direct consequence of Definition 1.1.3: (d), (i) and Definition 1.1.4.

Remark 4.3.1. Since $IHR \Rightarrow IHRA_1 \Rightarrow IHRA_2$, it follows that

$$A(x_1, x_2) \text{ is increasing in } x_1 \Rightarrow X \text{ is } IHRA_1 \Rightarrow X \text{ is } IHRA_2.$$

Now since $NBU_1 \Rightarrow NBUE \Rightarrow HNBUE$,

$$A(x_1, x_2) \geq 1 \Rightarrow X \text{ is } NBU_1 \Rightarrow X \text{ is } NBUE \Rightarrow X \text{ is } HNBUE.$$

We now establish the application of the measure $A(x_1, x_2)$ to the data on the failure times of 50 devices in Aarset [1], by taking the first two observations as zeros. The discretized quadratic hazard model (DQHM) introduced in Section 3.6 is a model that fits the data with parameter values,

$$\hat{a} = 379.89 \times 10^{-4},\ \hat{b} = -19.02 \times 10^{-4},\ \text{and } \hat{c} = 27.41 \times 10^{-6}.$$

Table 4.1: Goodness of fit for Aarset data.

Class	Observed frequency	Expected frequency
$0-4$	9	8
$5-18$	9	10
$19-50$	8	7
$51-67$	8	8
$68-84$	9	10
>84	7	7
χ^2 value		0.47

Table 4.2: Values of $A(x_1, x_2)$ for different x_1 and x_2.

x_1 \ x_2	5	10	20	30	35	45	60
4	0.9674	0.9411	0.9052	0.8901	0.8899	0.9042	0.9651
8	0.9400	0.8934	0.8339	0.8134	0.8166	0.8506	0.9818
12	0.9174	0.8556	0.7819	0.7631	0.7727	0.8323	1.0527
24	0.8755	0.7922	0.7159	0.7379	0.7871	0.9883	1.7799
36	0.8692	0.7938	0.7676	0.9043	1.0570	1.6743	4.8331
50	0.9060	0.8791	1.0167	1.5466	2.1142	4.8523	28.2104
72	1.0779	1.2825	2.4410	6.8951	13.4378	68.6268	1660.4809

The parameters are estimated by minimizing the sum of squares of deviations between empirical survival function and $S(x)$. The chi-square goodness of fit test gives a value of $\chi^2 = 0.468$ and p-value 0.79 when grouped into 6 classes as shown in Table 4.1. The data yields a bathtub-shaped hazard rate function and the change point is obtained as 34 by considering the minimum of $h(x)$ with the estimated values. The values of $A(x_1, x_2)$ for different x_1 and x_2 are shown in Table 4.2. From the table, we can see that when both x_1 and x_2 are small, the values of $A(x_1, x_2)'$s are less than one implying that the older device is more reliable. But as x_1 or x_2 becomes large, the values of $A(x_1, x_2)'$s are greater than one implying that the new device is more reliable.

4.4 Relative ageing factor

An alternate approach to quantify relative ageing of a new device and an old device with the same life distribution is to consider the identity,

$$P[X \geq g(x_1, x_2) + x_1 | X \geq x_1] = P[X \geq x_2], \text{ for all } x_1, x_2 = 0, 1, 2, ... \qquad (4.4.1)$$

In (4.4.1), x_1 is the current age of an old device and x_2 is the proposed survival time of the old and new units. $g(x_1, x_2)$ is the lifetime beyond x_1 of the old device such that the probability of survival of both the old and new units are same. We can write (4.4.1) as ,

$$S(x_1 + g(x_1, x_2)) = S(x_1)S(x_2)$$

or,

$$H^*(x_1 + g(x_1, x_2)) = H^*(x_1) + H^*(x_2), \qquad (4.4.2)$$

where $H^*(.)$ is the alternative cumulative hazard rate (Cox and Oakes [37]) given in (1.1.3). For example, when X is distributed as discrete Weibull I with survival function

$$S(x) = q^{x^\beta};\ x = 0, 1, 2, ...;\ \beta > 0,$$

the identity (4.4.1) is satisfied if

$$g(x_1, x_2) = \left(x_1^\beta + x_2^\beta\right)^{\frac{1}{\beta}} - x_1.$$

It is assumed that the form of $S(x)$ is determined from the data before calculating the ageing factor in a practical situation.
We now write,

$$M(x_1, x_2) = \frac{g(x_1, x_2)}{x_2}, \qquad (4.4.3)$$

and interpret it as the rate at which an old unit is losing or gaining life in relation to a new unit with identical life distribution. Thus, $M(x_1, x_2)$ provides a measure of the effect of ageing of a device. We call $M(x_1, x_2)$ as the relative ageing factor. The expressions of $M(x_1, x_2)$ of some discrete life distributions are presented in Table 4.3.

Table 4.3: Relative ageing factor for some discrete life distributions

Distribution	$S(x)$	$g(x_1, x_2)$	$M(x_1, x_2)$
Geometric	$q^x; x = 0, 1, 2, ...;\ 0 < q < 1$	x_2	1
Weibull-I [113]	$q^{x^\beta}; x = 0, 1, 2, \ldots; 0 < q < 1;\ \beta > 0$	$\left({x_1}^\beta + {x_2}^\beta\right)^{\frac{1}{\beta}} - x_1$	$\dfrac{\left({x_1}^\beta + {x_2}^\beta\right)^{\frac{1}{\beta}} - x_1}{x_2}$
Lomax [90]	$\left(1 + \frac{x}{\alpha}\right)^c,\ x = 0, 1, ...; \alpha > 0; c \in \mathbf{R}^-$	$x_2\left(1 + \frac{x_1}{\alpha}\right)$	$1 + \frac{x_1}{\alpha}$
Burr [147]	$\left(1 + x^\lambda\right)^{-1},\ x = 0, 1, ...; \lambda > 0$	$\left({x_1}^\lambda + {x_2}^\lambda + {x_1}^\lambda {x_2}^\lambda\right)^{\frac{1}{\lambda}} - x_1$	$\dfrac{\left({x_1}^\lambda + {x_2}^\lambda + {x_1}^\lambda {x_2}^\lambda\right)^{\frac{1}{\lambda}} - x_1}{x_2}$
half-logistic [13]	$2\left(1 + e^{\frac{x}{\sigma}}\right)^{-1},\ x = 0, 1, ..., \sigma > 0$	$\sigma \log\left[\frac{1}{x_1}(1 + e^{\frac{x_1}{\sigma}})(1 + e^{\frac{x_2}{\sigma}}) - 1\right]$	${x_2}^{\frac{-1}{x_1}} \log\left[\frac{1}{x_1}(1 + e^{\frac{x_1}{\sigma}})(1 + e^{\frac{x_2}{\sigma}}) - 1\right] - x_1$
re-scaled beta [91]	$\left(1 - \frac{x}{R}\right)^c,\ x = 0, 1, ...R; c > 0$	$x_2\left(1 - \frac{x_1}{R}\right)$	$\left(1 - \frac{x_1}{R}\right)$
power [91] (uniform $\alpha = 1$)	$1 - \left(\dfrac{x}{\beta}\right)^\alpha,\ x = 1, 2...\beta; \alpha > 0$	$\left[{x_1}^\alpha + {x_2}^\alpha - \frac{{x_1}^\alpha {x_2}^\alpha}{\beta^\alpha}\right]^{\frac{1}{\alpha}}$	$\left[\left({x_1}^\alpha + {x_2}^\alpha - \frac{{x_1}^\alpha {x_2}^\alpha}{\beta^\alpha}\right)^{\frac{1}{\alpha}} - x_1\right] {x_2}^{-1}$

We now discuss the properties of $M(x_1, x_2)$ and its role as a measure of relative ageing.

(i) Since $H^*(0) = 0$,

$$H^*(x_1 + g(x_1, 0)) = H^*(x_1) + H^*(0)$$

$$H^*(g(0, x_2)) = H^*(x_2) + H^*(0)$$

implies $g(x_1, 0) = x_1$, $g(0, x_2) = x_2$ and $g(0, 0) = 0$.

(ii) When $M(x_1, x_2) = 1$, $g(x_1, x_2) = x_2$ so that (4.4.1) becomes the lack of memory property. Thus, $M(x_1, x_2) = 1$ represents the no-ageing property which characterizes the geometric law. Larger values of $M(x_1, x_2)$ indicate more positive ageing.

(iii) $M(x_1, x_2)$ provides a measure of the extent of ageing of a device at different ages. For example, assume that X has discrete Weibull I distribution in Table 4.3 with shape parameter $\beta = 2$ and that a device with this distribution has survived a lifetime of 3 units. To compare the extent of ageing of this device with a new one with the same life distribution for a time period of 4 units, we have $x_1 = 3$ and $x_2 = 4$ to yield $g(x_1, x_2) = 2$ and $M(x_1, x_2) = \frac{1}{2}$. Thus, the old unit has to work twice the time compared to the new one to produce the same output. The former is clearly wearing out and is therefore subject to positive ageing. In general $g(x_1, x_2) \leq (\geq) x_1 + x_2$ for any x_1 and fixed x_2 indicates the effect of spent life.

(iv) There are some properties of $M(x_1, x_2)$ that relate to the ageing concepts as shown below.

Theorem 4.4.1. The lifetime X is $NBU_1(NWU_1)$ if and only if $M(x_1, x_2) \leq (\geq) 1$.

Proof. Since $H^*(x)$ is an increasing function,

$$g(x_1, x_2) + x_1 \leq x_1 + x_2 \Rightarrow H^*(g(x_1, x_2) + x_1) \leq H^*(x_1 + x_2)$$

So form (4.4.2),

$$H^*(x_1) + H^*(x_2) \leq H^*(x_1 + x_2),$$

which is a necessary and sufficient condition for X to be NBU_1. Also,

$$g(x_1, x_2) \leq x_2 \iff M(x_1, x_2) \leq 1.$$

The proof of NWU_1 case is obtained by reversing the inequalities. ■

Remark 4.4.1. As in Theorem 4.3.2, we have

(a) $M(y_0, x_2) \le (\ge) 1 \iff X$ is NBU-y_0(NWU-y_0) for all $x_2 = 0, 1, 2, ...$: $y_0 = 1, 2, ...$ and

(b) $M(x_1, x_2) \le (\ge) 1 \iff X$ is NBU*y_0(NWU*y_0) for all $x_1 = y_0, y_0 + 1, ...$: $x_2 = 0, 1, 2, ...$

(v) $M(x_1, x_2)$ does not depend on the lifetime spent, in the case of the geometric, discretized version of the Lomax and re-scaled beta (see Table 4.3).

Since the specific ageing factor $A(x_1, x_2)$ and relative ageing factor $M(x_1, x_2)$ are similar in their purposes, a comparison of the two is required. First we observe that,

$$\begin{aligned} \log A(x_1, x_2) = & -\log S(x_1 + x_2) + \log S(x_1) + \log S(x_2) \\ = & \; H^*(x_1 + x_2) - H^*(x_1) - H^*(x_2), \end{aligned}$$

which is different from $M(x_1, x_2)$. For instance, the discrete Weibull distribution has,

$$M(x_1, x_2) = \frac{\left(x_1^\beta + x_2^\beta\right)^{\frac{1}{\beta}} - x_1}{x_2}, \tag{4.4.4}$$

whereas,

$$A(x_1, x_2) = \frac{q^{\left(x_1^\beta + x_2^\beta\right)}}{q^{(x_1 + x_2)^\beta}}. \tag{4.4.5}$$

Generally, the expressions for $A(x_1, x_2)$ are algebraically more complex than that of $M(x_1, x_2)$.

Theorem 4.4.2. $M(x_1, x_2)$ is decreasing (increasing) in x_1

$\iff X$ is $IHR(DHR)$.
$\iff H^*(x)$ is convex (concave).
$\iff A(x_1, x_2)$ is increasing (decreasing) in x_1, for all $x_2 = 0, 1, 2, ...$

Proof. Define,

$$H_*^{-1}(t) = \inf \{t : H^*(t) \ge t\},$$

and take the smallest integer contained in t as $H_*^{-1}(t)$. Since $H^*(x)$ is monotone, the inverse is unique. Also, note that $g(x_1, x_2) = H_*^{-1}(H^*(x_1) + H^*(x_2))$.
$M(x_1, x_2)$ is decreasing in x_1

$$\begin{aligned}
&\iff g(x_1, x_2) \text{ is decreasing in} x_1. \\
&\iff H_*^{-1}(H^*(x_1) + H^*(x_2)) - x_1 \text{ is decreasing in } x_1, \text{ from (4.4.2).} \\
&\iff H_*^{-1}(t+s) - H_*^{-1}(t) \text{ is decreasing in } t \text{ for all } s = H^*(x_2). \\
&\iff H_*^{-1}(t) \text{ is concave.} \\
&\iff H^*(x) \text{ is convex} \iff X \text{ is } IHR.
\end{aligned}$$

The rest of the implications follows from Theorem 4.3.1. ■

The $M(x_1, x_2)$ function has applications in ordering life distributions on the basis of the NBU_1 property.

Definition 4.4.1. The random variable X_1 is less NBU_1 than the random variable X_2, denoted by $X_1 \leq_M X_2$ if $M_{X_1}(x_1, x_2) \leq M_{X_2}(x_1, x_2)$ for all x_1 and a fixed x_2.

We now relate the order $\leq_M$ and super additive order in $H^*_{X_1}(H^{-1}_{*,X_2}(x))$, where $H^*_{X_1}(.)$ is the alternative cumulative hazard rate of X_1.

Theorem 4.4.3.

$$X_1 \leq_M X_2 \iff H^*_{X_1}(H^{-1}_{*,X_2}(x_1 + x_2)) \geq H^*_{X_1}(H^{-1}_{*,X_2}(x_1)) + H^*_{X_1}(H^{-1}_{*,X_2}(x_2)) \quad (4.4.6)$$

Proof.

$$\begin{aligned}
X_1 \leq_M X_2 &\iff M_{X_1}(x_1, x_2) \leq M_{X_2}(x, y) \\
&\iff g_{X_1}(x_1, x_2) \leq g_{X_2}(x_1, x_2) \\
&\iff H^{-1}_{*,X_1}(H^*_{X_1}(x_1) + H^*_{X_1}(x_2)) \leq H^{-1}_{*,X_2}(H^*_{X_2}(x_1) + H^*_{X_2}(x_2)) \\
&\iff H^*_{X_1}(x_1) + H^*_{X_1}(x_2) \leq H^*_{X_1}[H^{-1}_{*,X_2}(H^*_{X_2}(x_1) + H^*_{X_2}(x_2))] \\
&\iff H^*_{X_1}(H^{-1}_{*,X_2}(t)) + H^*_{X_1}(H^*_{X_2}(s)) \leq H^*_{X_1}(H^{-1}_{*,X_2}(t+s))
\end{aligned}$$

which proves the result. ■

Sengupta and Deshpande [132] established that a continuous random variable Y_1 is ageing faster than Y_2 in the super additive sense, if the inequality of the right hand side of (4.4.6) is satisfied. Thus, the function $M(x_1, x_2)$ provides a stochastic order that is the discrete analogue of the ordering given in Sengupta and Deshpande [132].

Remark 4.4.2. If X_2 is NBU_1, $X_1 \leq_M X_2 \Rightarrow X_1$ is NBU_1.

Finally, the relative ageing factor is useful in comparing life distributions by saying which is more positive ageing (negative ageing by reversing the inequality in (4.4.6), so that $X_1 \geq_M X_2$ is equivalent to the sub additivity of $H^*_{X_1} H^{-1}_{*,X_2}(.)$, leading to NWU_1 class) in the sense of NBU_1. For example, the discrete Weibull I distribution is less NBU_1 than the discrete Lomax distribution, since,

$$x_1^\beta + x_2^\beta \leq \left(x_1 + x_2 + \frac{x_1 x_2}{\alpha}\right)^\beta \text{ for } \beta > 0$$

Thus, the discrete Lomax distribution has more positive ageing than the discrete Weibull I distribution.

4.5 Ageing intensity function

In order to evaluate the ageing phenomenon quantitatively, Jiang et al. [73] proposed the ageing intensity function (AI) for continuous lifetime data. It is the ratio of the hazard rate to a baseline hazard rate, which was chosen by the authors as the average of $h(x)$ in $[0, x]$. In the discrete case, we can define the ageing intensity function in two different ways, based on $h(x)$ and $h^*(x)$. We first define ageing intensity function as,

$$A^*(x) = \frac{h^*(x)}{H^*(x)/x} = \frac{xh^*(x)}{H^*(x)}, \tag{4.5.1}$$

where $h^*(x)$ is the alternative hazard rate. The intensity function is unity when $h^*(x)$ is a constant (geometric case), $A^*(x) > 1$ if X is IHR and $A^*(x) < 1$ when X is DHR. Thus, large (small) values of $A^*(x)$ indicate positive (negative) ageing. When $h(x)$ is used to define the hazard rate, one can obtain a slightly different form

$$A(x) = \frac{xh(x)}{H(x)}, \tag{4.5.2}$$

where $H(x) = \sum\limits_{u=0}^{x-1} h(u)$.
Writing $A^*(x)$ in-terms of $h(x)$,

$$A^*(x) = \frac{x\log(1-h(x))}{\sum\limits_{0}^{x-1}\log(1-H(t)+H(t-1))}. \tag{4.5.3}$$

Reducing (4.5.1) recursively, after writing it as,

$$A^*(x) = x\left(\frac{\log S(x+1)}{\log S(x)} - 1\right),$$

leads to

$$S(x) = S(1)^{(A^*(1)+1)(A^*(2)+1)...(\frac{A^*(x-1)}{x-1}-1)},$$

$S(1)$ being determined by using $\sum\limits_{x=0}^{\infty} f(x) = 1$. Thus, the ageing intensity function determines the distribution uniquely. This enables comparison of distributions using their ageing intensity functions.

The monotonicity of the hazard rate function is not, in general, transferred to the AI function. Moreover $A(x)$ and $A^*(x)$ need not be equal in general. In the following examples, we illustrate this using the AI function defined in (4.5.2)

Example 4.5.1. Let X has S distribution (Bracquemond and Gaudoin [23]) with hazard rate

$$h(x) = p(1-\pi^x),\ 0 < p < 1,\ 0 \le \pi < 1,\ x = 1, 2, ...$$

Then,

$$H(x) = \sum_{1}^{x} h(t) = px - \frac{p\pi}{1-\pi}(1-\pi^x),$$

so that,

$$A^*(x) = \frac{x\log\left(q + p\pi^x\right)}{\log\prod\limits_{i=1}^{x}(q+\pi^i)}$$

and

$$A(x) = \frac{(1-\pi)(1-\pi^x)xp}{(1-\pi)xp - p\pi(1-\pi^x)},\ \text{where, } q = 1-p.$$

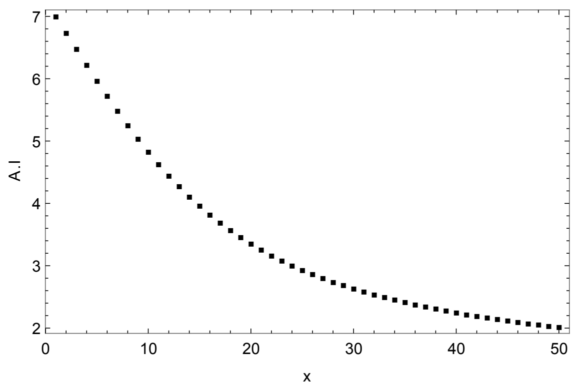


Figure 4.1: Graph of AI function.

Example 4.5.2. Let X follow negative binomial distribution with probability mass function

$$\frac{\Gamma(x+n)}{\Gamma(x+1)\Gamma(n)}p^n(1-p)^x,\ x=0,1,2,...;\ 0<p<1.$$

Then X is IHR for $n>1$. In particular, for $n=10$ and $p=0.3$, the AI function is decreasing as shown in Figure 4.1.

Definition 4.5.1. A random variable X_1 is ageing faster than X_2 in ageing intensity function, written as, $X_1 \leq_{AIF} X_2$ if, $\dfrac{A_{X_1}(x)}{A_{X_2}(x)}$ is increasing in x, provided $A_{X_2}(x) \neq 0$.

Example 4.5.3. Let X_1 has the survival function

$$S_{X_1}(x) = e^{-4x}(4x+1),\ x=0,1,2,... \tag{4.5.4}$$

and hazard function,

$$h_{X_1}(x) = 1 - \frac{e^{4x-4(x+1)}(4(x+1)+1)}{4x+1},\ x=0,1,2,... \tag{4.5.5}$$

Assume that X_2 has the survival function,

$$S_{X_2}(x) = e^{-2x}(2x+1),\ x=0,1,2,.... \tag{4.5.6}$$

and hazard function

$$h_{X_2}(x) = 1 - \frac{e^{2x-2(x+1)}(2(x+1)+1)}{2x+1},\ x=0,1,2,... \tag{4.5.7}$$

Then, $\dfrac{A_{X_1}(x)}{A_{X_2}(x)}$ is increasing in x. Hence, $X_1 \leq_{AIF} X_2$ and X_1 is ageing faster than X_2.

The AIF concept and various ageing criteria do not have any natural implications, although both are related to discrete ageing.

Theorem 4.5.1.

(i) $X_1 \leq_{AIF} X_2 \nRightarrow X_1 \leq_{IHR} X_2$ and $X_1 \leq_{IHR} X_2 \nRightarrow X_1 \leq_{AIF} X_2$

(ii) $X_1 \leq_{AIF} X_2 \nRightarrow X_1 \leq_{IHRA_1} X_2$

(iii) $X_1 \leq_{AIF} X_2 \nRightarrow X_1 \leq_{IHRA_2} X_2$

To prove the assertions, note that in Example 4.5.3, we have $X_1 \leq_{AIF} X_2$. But, $\dfrac{h_{X_1}(x)}{h_{X_2}(x)}$, $\dfrac{H_{X_1}(x)}{H_{X_2}(x)}$ and $\dfrac{\log S_{X_1}(x)}{\log S_{X_2}(x)}$ are decreasing in x. Using the hazard rate functions of X_1 and X_2 in the proof of Theorem 4.2.1,

$$\frac{h_{X_1}(x)}{h_{X_2}(x)} = \frac{1 - e^{-2(x+1)-1}}{1 - e^{-4(x+1)-1}} \tag{4.5.8}$$

is increasing. Hence,$X_1 \leq_{IHR} X_2$ and also $X_1 \leq_{IHRA_2} X_2$. Now the function,

$$\frac{A_{X_1}(x)}{A_{X_2}(x)} = \frac{e^2 \left(e^{2x+3} - 1\right) \left(e^{4x} \left(e \left(e^4 - 1\right) x - 1\right) + 1\right)}{\left(1 + e^2\right) \left(e^{4x+5} - 1\right) \left(e^{2x} \left(e \left(e^2 - 1\right) x - 1\right) + 1\right)}, \tag{4.5.9}$$

is decreasing in x. Hence $X_1 \nleq_{AIF} X_2$.

4.6 Conclusion

The role of relative ageing concepts is either to compare the ageing patterns of two devices at a fixed time or to investigate whether the same device is ageing more positively (negatively) at different points of time. In this chapter, we have presented some concepts and results that lead to a quantitative assessment of which of two devices is ageing faster. Also, the impact of spent life of a device on its residual life can also be numerically evaluated. It was proved that the relative ageing concepts are related to the well-known ageing classes such as IHR, NBU, etc.

Chapter 5

Multivariate Mean Residual Life

5.1 Introduction

Among different multivariate ageing concepts, the multivariate mean residual life plays an important role while devising techniques for obtaining optimal burn-in time. The study of multivariate mean residual life functions in the continuous domain has been extensively done by researchers, whereas their discrete analogues are not given much recognition. Roy [123] defined a bivariate version of MRL in discrete case and proved that the underlying distribution is uniquely determined by the bivariate MRL function. Nair and Nair [94] derived a characterization result for bivariate geometric distribution, based on the local constancy of the bivariate MRL function. However, there is lack of a systematic study on multivariate mean residual life(MMRL) in discrete time. It may be observed from the literature on continuous lifetime that many of the results obtained in the continuous set-up need to be modified for the discrete counterparts. Motivated by this fact, in the present chapter, we study the properties of (MMRL) in discrete time. We define ageing classes based on the multivariate MRL and study the inter-relationships between them.

The rest of the chapter is organized as follows. In Section 5.2, we give the definitions and basic results. It is shown that MMRL determines the underlying distribution uniquely. This is followed by the study of ageing classes in Section 5.3. We characterize multivariate lifetime distributions, based on the monotonicity of the MMRL function. Section 5.4 is devoted to study the inter-relationships between different ageing classes. The chapter ends with a brief conclusion in Section 5.5.

5.2 Basic results

As mentioned in Section 1.2, let $\mathbf{X} = (X_1, X_2, ..., X_p)'$ be a discrete random vector taking values in $\mathbf{N}^p$. The multivariate mean residual life(MMRL) function $\mathbf{m}(\mathbf{x})$ of the random vector $\mathbf{X}$ is defined as in 1.2.10.

Example 5.2.1. When $\mathbf{X}$ is bivariate geometric (Nair and Nair [94]) with survival function

$$\mathbf{S}_2(x_1, x_2) = q_1^{x_1} q_2^{x_2} \theta^{x_1 x_2};\ 0 < q_i < 1;\ 1 + q_1 q_2 \theta \geq q_1 + q_2;\ x_i = 0, 1, ...; i = 1, 2 \quad (5.2.1)$$

we obtain

$$m_1(x_1, x_2) = \frac{1}{1 - q_1 \theta^{x_2+1}}$$

and

$$m_2(x_1, x_2) = \left(1 - q_1 \theta^{x_2+1}\right).$$

Note that $m_1(\mathbf{x})$ is independent of x_2 and $m_2(\mathbf{x})$ is independent of x_1.

Now, we discuss some basic properties of the MMRL function. The MMRL function satisfies the following properties.

(i) $m_i(-1, -1, ..., -1) = 1 + \mu_i$ where $\mu_i = E(X_i),\ i = 1, 2, ...p.$

(ii) The mean residual life functions of the marginal distributions of $X_i;\ i = 1, 2, ..., p$ are derived as

$$m_{X_i}(x_i) = E[X_i - x_i | X_i > x_i] = m_i(-1, -1, ..., x_i, -1, ..., -1).$$

Further, if X_i and X_j are independent then $m_{X_i}(\mathbf{x}) = m_i(x_1, x_2, ..., x_{j-1}, -1, x_{j+1}..., x_p)$ and $m_{X_j}(\mathbf{x}) = m_j(x_1, x_2, ..., x_{i-1}, -1, x_{i+1}, ..., x_p)$.

(iii) The vector hazard rate and the MMRL function are related as

$$c_i(x_1+1, x_2+1, ..., x_p+1) = 1 - \frac{m_i(\mathbf{x}) - 1}{m_i(\mathbf{x}_{(i)}, x_i+1)};\ i = 1, 2, ..., p. \quad (5.2.2)$$

Since the univariate mean residual life function determines the corresponding distribution uniquely, it is of interest to know if the same is true in the multivariate case.

Theorem 5.2.1. Given the multivariate mean residual life function $\mathbf{m}(\mathbf{x})$, the survival function is uniquely determined by the identity

$$\mathbf{S}(\mathbf{x}) = \prod_{r=0}^{x_1-1} \frac{m_1(r-1, x_2, ..., x_p) - 1}{m_1(r, x_2, ..., x_p)} \prod_{r=0}^{x_2-1} \frac{m_2(0, r-1, x_3, ..., x_p) - 1}{m_2(0, r, ..., x_p)}$$
$$... \prod_{r=0}^{x_p-1} \frac{m_p(0, 0, ..., 0, r-1) - 1}{m_p(0, 0, ..., r)} \quad (5.2.3)$$

Proof. From (1.2.12), we get

$$m_i(\mathbf{x})\mathbf{S}(\mathbf{x}+\mathbf{e}) = \sum_{t_i=x_i+1}^{\infty} \mathbf{S}(x_1+1, ..., x_{i-1}+1, t, x_{i+1}+1, ..., x_p+1);\ i = 1, 2, ..., p \quad (5.2.4)$$

so that for $i = 1$

$$m_1(x_1-1, x_2, x_3, ..., x_p)\mathbf{S}(x_1, x_2+1, x_3+1, ..., x_p+1) - m_1(\mathbf{x})\mathbf{S}(\mathbf{x}+\mathbf{e})$$
$$= \mathbf{S}(x_1, x_2+1, x_3+1, ..., x_p+1). \quad (5.2.5)$$

This gives a recurrence relation in $\mathbf{S}(\mathbf{x})$ as

$$\mathbf{S}(\mathbf{x}+\mathbf{e}) = \frac{m_1(x_1-1, x_2, ..., x_p) - 1}{m(\mathbf{x})}\mathbf{S}(x_1, x_2+1, x_3+1, ..., x_p+1). \quad (5.2.6)$$

Reducing x_1 successively, we obtain

$$\mathbf{S}(\mathbf{x}) = \prod_{r=0}^{x_1-1} \frac{m_1(r-1, x_2, x_3, ..., x_p) - 1}{m_1(r, x_2, x_3, ..., x_p)}\mathbf{S}(0, x_2, x_3, ..., x_p) \quad (5.2.7)$$

Similarly from (5.2.4), for $i = 2$ we get

$$\mathbf{S}(\mathbf{x}) = \prod_{r=0}^{x_2-1} \frac{m_2(x_1, r-1, x_3, ..., x_p) - 1}{m_2(x_1, r, x_3, ..., x_p)} \mathbf{S}(x_1, 0, x_3, ..., x_p). \quad (5.2.8)$$

Setting $x_1 = 0$ in (5.2.8), we get $\mathbf{S}(0, x_2, ..., x_p)$ and substituting in (5.2.7) we get

$$\mathbf{S}(\mathbf{x}) = \prod_{r=0}^{x_1-1} \frac{m_1(r-1, x_2, x_3, ..., x_p) - 1}{m_1(r, x_2, x_3, ..., x_p)} \prod_{r=0}^{x_2-1} \frac{m_2(0, r-1, x_3, ..., x_p) - 1}{m_2(0, r, x_3, ..., x_p)} \mathbf{S}(0, 0, x_3, ..., x_p). \quad (5.2.9)$$

Now repeating the above procedure by giving values for $i = 3, 4, ..., p$ in (5.2.4), we get the desired result. ■

Remark 5.2.1. Equation (5.2.3) is only one of the several ways in which $\mathbf{S}(\mathbf{x})$ can be written in-terms of the $m_i(\mathbf{x})$ expressions. In (5.2.3), we have started with the definition of $m_1(\mathbf{x})$ in-terms of $\mathbf{S}(\mathbf{x})$. On the other hand, if we begin with $m_2(\mathbf{x})$,

$$\mathbf{S}(\mathbf{x}) = \prod_{r=0}^{x_2-1} \frac{m_2(x_1, r-1, x_3, ..., x_p) - 1}{m_2(x_1, r, x_3, ..., x_p)} \prod_{r=0}^{x_2-1} \frac{m_3(x_1, 0, r-1, x_4, ..., x_p) - 1}{m_3(x_1, 0, r, ..., x_p)} ... \prod_{r=0}^{x_1-1} \frac{m_1(r-1, 0, ..., 0) - 1}{m_1(r, 0, ..., 0)}. \quad (5.2.10)$$

Thus, on the whole, there can be $p!$ expressions which are all equal to one another.
The following theorem is a characterization result in Nair and Nair [97].

Theorem 5.2.2. The multivariate mean residual life components take the form

$$m_i(\mathbf{x}) = k_i;\ k_i > 1;\ i = 1, 2, ..., p$$

if and only if the underlying distribution is multivariate geometric with survival function

$$\mathbf{S}(\mathbf{x}) = q_1^{x_1} q_2^{x_2} ... q_p^{x_p};\ 0 < q_i < 1;\ x_i = 0, 1, 2, ...;\ i = 1, 2, ..., p \quad (5.2.11)$$

where $q_i = \dfrac{k_i - 1}{k_i}$; $i = 1, 2, ..., p$.

Proof. The proof for 'if' part is direct and the proof of 'only if' part follows from (5.2.3).

■

With these preliminary concepts, in the following section, we define and study ageing classes based on multivariate mean residual life.

5.3 Ageing classes

In the univariate case, the monotonicity of the mean residual life function reflects the nature of ageing of a device. It has been successfully employed in replacement problems, comparison of life distributions, selection of models for data analysis, etc. The present section is devoted to the study of extension of these notions in the multivariate framework. Buchanan and Singapurwalla [27] and Zahedi [146] have proposed various approaches to the generalization of multivariate monotone mean residual life function when the lifetimes are continuous. The discrete versions of the Buchanan and Singapurwalla [27] definitions are as follows.

Definition 5.3.1. A p- dimensional random vector $\mathbf{X}$ defined on $\mathbf{N}^p$ is said to have

(i) multivariate decreasing(increasing) mean residual life-very weak (MDMRL/ MIMRL-VW) if $\sum_{t=0}^{\infty} \frac{\mathbf{S}(x+t, x+t, ..., x+t)}{\mathbf{S}(x, x, ..., x)}$ is decreasing (increasing) in x.

(ii) multivariate decreasing(increasing) mean residual life-weak (MDMRL/ MIMRL-W) if $\sum_{t_1=0}^{\infty} \sum_{t_2=0}^{\infty} \cdots \sum_{t_p=0}^{\infty} \frac{\mathbf{S}(x+t_1, x+t_2, ..., x+t_p)}{\mathbf{S}(x, x, ..., x)}$ is decreasing(increasing) in x.

(iii) multivariate decreasing(increasing) mean residual life-strong (MDMRL/ MIMRL-S) if $\sum_{t=0}^{\infty} \frac{\mathbf{S}(x_1+t, x_2+t, ..., x_p+t)}{\mathbf{S}(x_1, x_2, ..., x_p)}$ is decreasing(increasing) in x_1, x_2,..., x_p. and

(iv) multivariate decreasing(increasing) mean residual life-very strong (MDMRL/ MIMRL-VS) if $\sum_{t_1=0}^{\infty} \sum_{t_2=0}^{\infty} \cdots \sum_{t_p=0}^{\infty} \frac{\mathbf{S}(x_1+t_1, x_2+t_2, ..., x_p+x_p)}{\mathbf{S}(x_1, x_2, ..., x_p)}$ is decreasing(increasing) in x_1, x_2,..., x_p.

It is easy to show that there exists a chain of implication between these four classes given by

$$MDMRL-VS \Rightarrow\ MDMRL-S \Rightarrow\ MDMRL-W \Rightarrow MDMRL-VW$$

Zahedi [146] remarked that the above definitions are not necessarily the natural extensions of the univariate case and these do not possess the intuitive physical interpretation of the univariate definitions. We conclude the study of these ageing classes by mentioning the member of boundary class. A member of the boundary class of MDMRL(MIMRL)-VS classes should satisfy

$$\sum_{t_1}\sum_{t_2}...\sum_{t_p}\frac{\mathbf{S}(\mathbf{x}+\mathbf{t}+\boldsymbol{\Delta})}{\mathbf{S}(\mathbf{x}+\boldsymbol{\Delta})}-\sum_{t_1}\sum_{t_2}...\sum_{t_p}\frac{\mathbf{S}(\mathbf{x}+\mathbf{t})}{\mathbf{S}(\mathbf{x})}=0, \tag{5.3.1}$$

where $\boldsymbol{\Delta} \in \mathbf{N}^p$. It is easy to verify that the multivariate geometric with independent geometric marginals given in (5.2.11) satisfies the above condition. Thus, the boundary class of MDMRL-VS and MIMRL-VS classes of distribution contains the multivariate geometric distribution with independent marginals.

Retaining the notations and results of previous section, the ageing classes based on the multivariate mean residual life vector are considered.

Definition 5.3.2. A p-dimensional discrete random vector $\mathbf{X}$ defined on $\mathbf{N}^p$ is said to have

(i) MDMRL-1 (MIMRL-1) if

$$m_i(\mathbf{x}+\mathbf{t}) \leq (\geq) m_i(\mathbf{x}),\ i=1,2,...,p$$

for all $\mathbf{x}$ and $\mathbf{t}$ in $\mathbf{N}^p$.

(ii) MDMRL-2(MIMRL-2) if

$$m_i(x_1,x_2,...,x_{i-1},x_i+t,x_{i+1},...,x_p) \leq (\geq) m_i(x_1,x_2,...,x_p)$$

for $i=1,2,...,p$ and t in $\mathbf{N}$.

(iii) MDMRL-3(MIMRL-3) if

$$m_i(x_1 + t, ..., x_n + t) \leq (\geq) m_i(x_1, ..., x_n)$$

for all $n \leq p$ and t in $\mathbf{N}$.
and

(iv) MDMRL-4(MIMRL-4) if

$$m_i(x + t, ..., x + t) \leq (\geq) m_i(x, ..., x)$$

for all $n \leq p$ and t in $\mathbf{N}$.

The interpretation of (i) is that the mean residual life of a p- component device, where the components are of different ages, decrease(increase) as the components age with different intensities. In (ii), the mean residual life decreases when a component is replaced by a younger one with the same life distribution. Case (iii) results when the time moves at the same rate for all components while initially the components are of different ages. Lastly, when the components are of the same age as well as time moves at the same rate for all of them, the mean residual life is decreasing(increasing) gives rise to version (iv).

We now discuss the properties of these classes. The following theorems characterize different types of multivariate geometric distributions.

Theorem 5.3.1. A random vector $\mathbf{X}$ is both MDMRL-1 and MIMRL-1 if and only if the distribution of $\mathbf{X}$ is multivariate geometric with independent geometric marginals in (5.2.11).

Proof. When the distribution is (5.2.11), $m_i(\mathbf{x}) = \dfrac{1}{1 - q_i}$ so that $m_i(\mathbf{x} + \mathbf{t}) = m_i(\mathbf{x})$ or $\mathbf{X}$ has MDMRL-1 and MIMRL-1. Conversely, when $\mathbf{X}$ has both MDMRL-1 and MIMRL-1 property, then $m_i(\mathbf{x}) = k_i$, a constant greater than unity for all $\mathbf{x}$ and $i = 1, 2, ..., p$. Taking $\dfrac{k_i - 1}{k_i} = q_i$, the stated distribution results on using (5.2.3). ■

Theorem 5.3.2. The only discrete distribution in $\mathbf{N}^p$ which belongs to both MDMRL-2 and MIMRL-2 class is the multivariate geometric law(Nair and Asha [95])

$$\mathbf{S}(\mathbf{x}) = \prod_{i=1}^{p} q_i^{x_i} \prod_{i<j}^{p} q_{ij}^{x_i x_j} ... q_{12...p}^{x_1 x_2 ... x_p} \tag{5.3.2}$$

$0 < q_1, q_2, ..., q_{12...p} < 1;\ 1 - \sum q_i + \sum_{i<j} q_{ij} + ... + (-1)^p q_{12...p} \geq 0.$

Proof. Assume that $\mathbf{X}$ is MDMRL-2 and MIMRL-2. Then

$$m_i(x_1, ..., x_{i-1}, x_i + t, x_{i+1}, ..., x_p) = m_i(\mathbf{x});\ i = 1, 2, ..., p$$

so that

$$m_i(\mathbf{x}) = g_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_p),$$

a function independent of x_i. Using this in (5.2.3)

$$\mathbf{S}(\mathbf{x}) = [g_1(x_2, ..., x_p)]^{x_1} [g_2(0, x_3, ..., x_p)]^{x_2} ... [g_p(0, 0, ..., 0)]^{x_p}.$$

Thus, $\log \mathbf{S}(\mathbf{x})$ is a linear function of x_1. Similarly from (5.2.10), $\log \mathbf{S}(\mathbf{x})$ is linear in x_2 and so on. Equating p expressions, we see that the only general solution that satisfies all the p equations is given by

$$\mathbf{S}(x) = \prod_{i=1}^{p} q_i^{x_i} \prod_{i<j=1}^{p} q_{ij}^{x_i x_j} ... q_{12...p}^{x_1 x_2 ... x_p}.$$

The conditions on the parameters are obtained from $\mathbf{S}(x_1, x_2, ..., x_r + 1) \leq \mathbf{S}(x_1, ..., x_r); r = 1, 2, ..., p.$ ■

Theorem 5.3.3. A discrete bivariate random vector $\mathbf{X}$ is MDMRL-2(MIMRL-2) if and only if $c_i(x_1 + 1, x_2 + 1, ..., x_p + 1) m_i(x_{(i)}, x_i + 1) < (>) 1;\ i = 1, 2, ..., p.$

The proof follows directly from (5.2.2).

Corollary 5.3.1. For a discrete p-variate random vector $\mathbf{X}$, the property

$$c_i(x_1 + 1, x_2 + 1, ..., x_p + 1) m_i(x_{(i)}, x_i + 1) = 1;\ i = 1, 2, ..., p$$

holds if and only if $\mathbf{X}$ has the survival function (5.3.2).
The proof follows from Theorem 5.3.2 and Theorem 5.3.3.

Theorem 5.3.4. The random vector $\mathbf{X}$ is MDMRL-3 and MIMRL-3 if and only if $\mathbf{X}$ is multivariate geometric(Nair and Asha [95]) with survival function

$$\mathbf{S}(\mathbf{x}) = q_{i_1}^{x_{i_1}} \left(\frac{q_{i_1 i_2}}{q_{i_1}}\right)^{x_{i_2}} ... \left(\frac{q_{i_1 i_2 ... i_p}}{q_{i_1 i_2 ... i_{p-1}}}\right)^{x_{i_p}}; \; x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_p}. \tag{5.3.3}$$

where $i_1, i_2, ..., i_p$ are the permutations of $(1, 2, ..., p)$.
The parameters satisfy

$$0 < q_{i_1 ... i_p} \leq ... \leq q_{i_1 i_2} \leq q_1, q_2, ..., q_p < 1,$$

$$q_{i_1 ... i_j} = q_{123...j} \text{ for } j = 1, 2, ..., p$$

and

$$1 - \sum_j q_j + \sum_{j<k} q_{jk} + ... + (-1)^{p-1} q_{123...p} \geq 0.$$

Proof. We prove the result in the bivariate case, which can be extended to the multivariate case. In the bivariate case, (5.3.3) can be written as

$$\mathbf{S}(x_1, x_2) = \begin{cases} q^{x_2} q_1^{x_1 - x_2}; \; x_1 \geq x_2 \\ q^{x_1} q_2^{x_2 - x_1}; \; x_2 \geq x_1. \end{cases} \tag{5.3.4}$$

We now get

$$(m_1(x_1, x_2), m_2(x_1, x_2))' = \begin{cases} \left(\frac{1}{1 - q_1}, \frac{1}{1 - q q_1^{-1}}\right)'; \; x_1 > x_2 \\ \left(\frac{1}{1 - q q_1^{-1}}, \frac{1}{1 - q_2}\right)'; \; x_1 < x_2 \\ \left(\frac{1}{1 - q_1}, \frac{1}{1 - q_2}\right)'; \; x_1 = x_2. \end{cases} \tag{5.3.5}$$

Thus, for $i = 1, 2$

$$m_i(x_1 + t, x_2 + t) = m_i(x_1, x_2) \tag{5.3.6}$$

for all x_1, x_2 and t taking values in $\mathbf{N}$. Thus, $\mathbf{X}$ is both MDMRL-3 and MIMRL-3.

Conversely, if (5.3.6) holds, we get from (5.2.2)

$$\begin{aligned} c_1(x_1+t+1, x_2+t+1) &= 1 - \frac{m_1(x_1+t, x_2+t)-1}{m_1(x_1+t+1, x_2+t)} \\ &= 1 - \frac{m_1(x_1, x_2)-1}{m_1(x_1+1, x_2)} \text{ by (5.3.6)} \\ &= c_1(x_1+1, x_2+1) \end{aligned}$$

which is a characterizing property of (5.3.4) (Nair and Asha [95]), which completes the proof. ■

Now, distribution of $\mathbf{X}$ is both MDMRL-4 and MIMRL-4, if and only if

$$A_i(x+t) = A_i(x),$$

where $A_i(x) = m_i(x, x, ..., x), i = 1, 2, ..., p$. The above equation is a univariate functional equation, from which a multivariate solution is difficult to emerge. Other interesting aspects of the above classes are mentioned below.

(i) If $\mathbf{X}$ is MDMRL-l(MIMRL-l), $l = 1, 2, 3, 4$, each non-empty subset $\mathbf{X}$ has the corresponding property.

(ii) If $\mathbf{X}$ is MDMRL-l(MIMRL-l) and $\mathbf{Y}$ also likewise, then $(\mathbf{X}, \mathbf{Y})$ is also MDMRL-l(MIMRL-l) provided $\mathbf{X}$ and $\mathbf{Y}$ are independent.

(iii) If $\mathbf{X}$ is MDMRL-l(MIMRL-l), then $a\mathbf{X}$ is MDMRL-l (MIMRL-l), $a > 0$.

(iv) $\mathbf{X}$ is MDMRL-1(MIMRL-1) if and only if $\mathbf{S}_{\mathbf{x}}(\mathbf{t})$ is decreasing in $\mathbf{x}$ for all $\mathbf{t}$. Taking $x_i = x$ and/or $t_i = t$; $i = 1, 2, ..., p$ we have a similar characterization of the other MDMRL classes.

For the further study, we need the definitions of multivariate IHR classes.

Definition 5.3.3. (Nair and Asha [95]) A random vector in the support of $\mathbf{N}^p$ or its subset is said to have

(i) MIHR-1(MDHR-1) if for all $\mathbf{x}$ and $\mathbf{t}$ in $\mathbf{N}^p$

$$c(\mathbf{x}+\mathbf{t}) \geq (\leq) c(\mathbf{x}).$$

(ii) MIHR-2(MDHR-2) if for all $\mathbf{x}$ in $\mathbf{N}^p$ and t in $\mathbf{N}$

$$c_i(x_1, x_2, ..., x_{i-1}, x_i + t, x_{i+1}, ..., x_p) \geq (\leq) c_i(\mathbf{x}); \ i = 1, 2, ..., p.$$

(iii) MIHR-3(MDHR-3) if for all $n \leq p$, $\mathbf{x}$ in $\mathbf{N}^n$ and t in $\mathbf{N}$

$$c_i(x_1 + t, ..., x_n + t) \geq (\leq) c_i(x_1, ..., x_n).$$

and

(iv) MIHR-4(MDHR-4) if for all $n \leq p$

$$c_i(x + t, x + t, ..., x + t) \geq (\leq) c_i(x, x, ..., x).$$

Theorem 5.3.5. $\mathbf{X}$ is MIFR-l(MDFR-l) $\Rightarrow$ $\mathbf{X}$ is MDMRL-l(MIMRL-l); $l = 1, 2, 3, 4$.

Proof. Since the proof is similar for $l = 1, 2, 3, 4$, we prove the result when $l = 1$. We prove the result only in the bivariate case which easily extends to the multivariate case. In the bivariate case, $(X_1, X_2)'$ is BIHR-1 if $c_i(x_1 + t_1, x_2 + t_2) \geq c_i(x_1, x_2)$ for all $(x_1, x_2)', (t_1, t_2)' \in \mathbf{N}^2$ and $i = 1, 2$.
When $i = 1$, we get

$$1 - \frac{\mathbf{S}(x_1 + t_1 + 1, x_2 + t_2)}{\mathbf{S}(x_1 + t_1, x_2 + t_2)} \geq 1 - \frac{\mathbf{S}(x_1 + 1, x_2)}{\mathbf{S}(x_1, x_2)}$$

or

$$\frac{\mathbf{S}(x_1 + t_1 + 1, x_2 + t_2)}{\mathbf{S}(x_1 + t_1, x_2 + t_2)} \leq \frac{\mathbf{S}(x_1 + 1, x_2)}{\mathbf{S}(x_1, x_2)}$$

for all $(x_1, x_2)' \in \mathbf{N^2}$. Now

$$\begin{aligned} m_1(x_1, x_2) &= \frac{1}{\mathbf{S}(x_1 + 1, x_2 + 1)} \sum_{r = x_1 + 1}^{\infty} \mathbf{S}(r, x_2 + 1) \\ &= 1 + \frac{\mathbf{S}(x_1 + 2, x_2 + 1)}{\mathbf{S}(x_1 + 1, x_2 + 1)} + \frac{\mathbf{S}(x_1 + 3, x_2 + 1)}{\mathbf{S}(x_1 + 2, x_2 + 1)} \frac{\mathbf{S}(x_1 + 2, x_2 + 1)}{\mathbf{S}(x_1 + 1, x_2 + 1)} + ... \end{aligned}$$

$$\geq 1 + \frac{\mathbf{S}(x_1+t_1+2, x_2+t_2+1)}{\mathbf{S}(x_1+t_1+1, x_2+t_2+1)}$$
$$+ \frac{\mathbf{S}(x_1+t_1+3, x_2+t_2+1)\mathbf{S}(x_1+t_1+2, x_2+t_2+1)}{\mathbf{S}(x_1+t_1+2, x_2+t_2+1)\mathbf{S}(x_1+t_1+1, x_2+t_2+1)} + ...$$
$$= m_1(x_1+t_1, x_2+t_2) \text{ for all } (x_1, x_2), (t_1, t_2) \in \mathbf{N^2}.$$

Similarly, when $i = 2$, $m_2(x_1, x_2) \geq m_2(x_1+t_1, x_2+t_2)$, which completes the proof. ■

5.4 Relationships between MIMRL and MDMRL classes

The chain of implications between the different ageing classes is given by

$$\begin{array}{ccccc} \text{MDMRL-1(MIMRL-1)} & \Rightarrow & \text{MDMRL-3(MIMRL-3)} & \Rightarrow & \text{MDMRL-4(MIMRL-4)} \\ \Downarrow & & & & \\ \text{MDMRL-2(MIMRL-2)} & & & & \end{array}$$

In order to prove that there exists no other implication between these classes, we consider the following examples.

Example 5.4.1. Let $\mathbf{X}$ be distributed as a bivariate geometric random vector with survival function in (5.2.1). Then,

$$m_1(x_1, x_2) = \left(1 - q_1\theta^{x_2+1}\right)^{-1} \text{ and } m_2(x_1, x_2) = \left(1 - q_2\theta^{x_1+1}\right)^{-1}.$$

It can be easily verified that $\mathbf{X}$ is both MIMRL-2 and MDMRL-2. But it is not MIMRL-4. Thus,

$$\text{MIMRL-2} \not\Rightarrow \text{MIMRL-4}$$
$$\text{MDMRL-2} \not\Rightarrow \text{MIMRL-4}$$

Example 5.4.2. Let $\mathbf{X}$ follow the bivariate Waring distribution(Nair and Asha [95]) with survival function

$$\mathbf{S}(x_1, x_2) = \frac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}};\ m, n > 0;\ x_i = 0, 1, 2, ...;\ i = 1, 2. \qquad (5.4.1)$$

Then

$$m_1(x_1, x_2) = m_2(x_1, x_2) = \frac{m + n + x_1 + x_2 + 1}{n - 1}.$$

It is easy to see that $\mathbf{X}$ is MIMRL-2. But $\mathbf{X}$ is neither MDMRL-2 nor MDMRL-4. Hence

MIMRL-2 $\nRightarrow$ MDMRL-2

MIMRL-2 $\nRightarrow$ MDMRL-4

Example 5.4.3. Let $\mathbf{X}$ has the bivariate version of the geometric distribution in (5.3.4), with survival function

$$\mathbf{S}(x_1, x_2) = \begin{cases} q^{x_2} q_1^{x_1 - x_2};\ x_1 \geq x_2 \\ q^{x_1} q_2^{x_2 - x_1};\ x_2 \geq x_1, \end{cases} \qquad (5.4.2)$$

$0 < q \leq q_i < 1;\ x_1 = 0, 1, 2, ...;\ 1 + q \geq q_1 + q_2;\ i = 1, 2.$
Then,

$$(m_1(x_1, x_2), m_2(x_1, x_2))' = \begin{cases} \left(\frac{1}{1 - q_1}, \frac{1}{1 - qq_1^{-1}}\right)';\ x_1 > x_2 \\ \left(\frac{1}{1 - qq_1^{-1}}, \frac{1}{1 - q_2}\right)';\ x_1 < x_2 \\ \left(\frac{1}{1 - q_1}, \frac{1}{1 - q_2}\right)';\ x_1 = x_2. \end{cases} \qquad (5.4.3)$$

It is MIMRL-3 and MDMRL-3. When $q > q_1 q_2$, $\mathbf{X}$ is not MIMRL-2. Similarly, when $q < q_1 q_2$, $\mathbf{X}$ is not MDMRL-2.
Thus,

MIMRL-3 $\nRightarrow$ MIMRL-2

MDMRL-3 $\nRightarrow$ MDMRL-2

Example 5.4.4. Let $\mathbf{X}$ be distributed with survival function

$$\mathbf{S}(x_1, x_2) = \frac{\binom{k+n-x_1-x_2}{n-x_1-x_2}}{\binom{k+n}{n}}; \; x_1 + x_2 \leq n; \; x_i = 0, 1, 2, ..., n; k > 0 \tag{5.4.4}$$

corresponding to the bivariate negative hyper-geometric distribution(Nair and Asha [95]). Then,

$$m_1(x_1, x_2) = m_2(x_1, x_2) = \frac{k+n-x_1-x_2-1}{k+1}.$$

It can be shown that $\mathbf{X}$ is MDMRL-1 and it does not belong to MIMRL-2 or MIMRL-4. Hence,

$$\text{MIMRL-1} \nRightarrow \text{MIMRL-2}$$

$$\text{MDMRL-1} \nRightarrow \text{MDMRL-4}$$

Example 5.4.5. Let $m_1(x_1, x_2) = \dfrac{x_1 + x_2 + c}{x_1 + d}$ and $m_2(x_1, x_2) = \dfrac{x_1 + x_2 + c}{x_2 + d}$; $c \geq d + 1$; $d > 0$. Now consider $m_1(x_1, x_2) - m_1(x_1 + 1, x_2) \geq 0$ for all $x_2 = 0, 1, 2, ...$ fixed and $x_1 = 0, 1, 2, ...$

$$\Rightarrow \frac{x_1 + x_2 + c}{x_1 + d} - \frac{x_1 + x_2 + c + 1}{x_1 + d + 1} \geq 0$$

$$\Rightarrow \frac{x_2 + c - d}{(x_1 + d)(x_1 + d + 1)} \geq 0$$

$$\Rightarrow x_2 + c - d \geq 0 \text{ for any fixed } x_2.$$

Thus, $m_1(x_1, x_2)$ is non-decreasing in x_1 for fixed x_2 when $c \geq d$. Similarly we can show that $m_2(x_1, x_2)$ is non-decreasing in x_2 for fixed x_1 when $c \geq d$. Hence, $m(\mathbf{x})$ is MDMRL-2 when $c \geq d$.
Now consider $m_1(x, x) - m_1(x + 1, x + 1) \geq 0$ for all $x = 0, 1, ...$

$$\Rightarrow \frac{2x + c}{x + d} - \frac{2x + c + 2}{x + d + 1} \geq 0$$

$$\Rightarrow \frac{c - 2d}{(x + d)(x + d + 1)} \geq 0 \text{ for all } x.$$

$$\Rightarrow c \geq 2d.$$

Similarly, we get $m_2(x,x) - m_2(x+1,x+1) \geq 0$ for all x implies $c \geq 2d$. Thus, $m(\mathbf{x})$ is MDMRL-4 for values of c and d such that $c \geq 2d$ and is MIMRL-4 for $c < 2d$.
If, in particular, we choose $c = 5$ and $d = 3$, $m(\mathbf{x})$ will be both MDMRL-2 and MIMRL-4. Thus,

$$\text{MDMRL-2} \not\Rightarrow \text{MDMRL-4}$$

Example 5.4.6. Consider $\mathbf{m}(\mathbf{x})$ in example 5.4.5. Then,

$$m_1(x_1,x_2) - m_1(x_1+1,x_2+1) = \frac{x_2 - x_1 + c - 2d}{(x_1+d)(x_1-d+1)}.$$

For this to be non-negative for all x_1 and x_2, $c - 2d - x_1 + x_2$ must be greater than or equal to 0 for any values of x_1 and x_2. Thus, $m(\mathbf{x})$ is neither MDMRL-3 nor MIMRL-3. In the previous example, we have seen that $m(\mathbf{x})$ is MDMRL-4 if $c \geq 2d$ and MIMRL-4 if $c < 2d$.
Hence

$$\text{MIMRL-4} \not\Rightarrow \text{MIMRL-3}$$
$$\text{MDMRL-4} \not\Rightarrow \text{MDMRL-3}$$

5.5 Conclusion

In the present chapter, we have extended the concept of mean residual life function into the multivariate discrete domain. Relationships with other multivariate reliability concepts have been discussed. We have studied different ageing classes based on multivariate mean residual life and derived their inter-relationships. Characterization results based on the monotonicity of MMRL function were obtained.

Chapter 6

Multivariate Variance Residual Life

6.1 Introduction

Various characteristics of residual life such as its mean, variance, coefficient of variation, higher moments and percentiles have been extensively studied in literature. Among these, the variance residual life(VRL) has attracted many researchers including Dallas [40], Karlin [76], Chen et al. [33], Gupta [53], Gupta et al. [58], Abouammoh et al. [4], Adatia et al. [5], Stein and Dattero [138], Gupta and Kirmani [55, 56], Stoyanov and Al-Sadi [139], Gupta [54] and Nair and Sudheesh [105], when lifetime is treated as a continuous random variable. These works emphasize the importance of variance residual life as (i) a reliability function useful in modelling lifetime data with special reference to inference procedures and characterizations (ii) a means to classify lifetime distributions through the monotonicity properties and (iii) through its relationship with the mean residual life in the same way as the mean to the variance; see Hall and Wellner [63]. In the discrete case, the topic has been well-studied by several authors that include Hitha and Nair [64], Roy [125], El-Arishy [43], Sudheesh and Nair [140], Khorashadizadeh et al. [80] and Al-Zahrani et al. [6]. The only study that appears to be made in higher dimensional discrete case is that of Roy [125] who characterized some bivariate discrete distributions by certain simple properties of the variance residual life. It appears that there is no systematic discussion on the concept of multivariate variance residual life (MVRL) in the discrete domain. The objective of the present chapter is to make a theoretical exposition of the properties of the multivariate discrete variance residual life. It includes properties of the variance residual life, characterization of life distributions and classes of life distributions based on the monotonic properties of the variance residual life. As a by-product, we also get some properties that do not seem to have been obtained in the univariate case, by specializing our results.

The chapter is organized as follows. In Section 6.2, we present the definition and properties of the p-dimensional variance residual life function. The MVRL funtion is expressed in-terms of MMRL function. Multivariate lifetime distributions are characterized by giving different functional forms for MVRL function. In Section 6.3, various classes of life distributions based on multivariate variance residual life are discussed. Characterization results are obtained, based on the monotonic nature of MVRL function. The major conclusions of the study are presented in Section 6.4.

6.2 Multivariate variance residual life

Let $\mathbf{X}$ be the p- dimensional random vector defined in Section 1.2. Following the same notations in Section 1.2 for reliability functions, the ith component of variance residual life vector $\boldsymbol{\sigma}^2(\mathbf{x})$ is defined as

$$\sigma_i^2(\mathbf{x}) = E\left[(X_i - x_i)^2|\mathbf{X} > \mathbf{x}\right] - m_i^2(\mathbf{x});\ x_i = -1, 0, 1, 2, ...;\ i = 1, 2, ..., p, \quad (6.2.1)$$

with $m_i(\mathbf{x})$ being the i th component of MMRL vector. The evaluation of (6.2.1) can be accomplished by the formula

$$\sigma_i^2(\mathbf{x}) = \frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} \sum_{u_i=t_i+1}^{\infty} \mathbf{S}(x_1+1, ..., x_{i-1}+1, u_i, x_{i+1}+1, ..., x_p+1) - m_i(\mathbf{x})(m_i(\mathbf{x}) - 1). \quad (6.2.2)$$

To prove this, we note that
$E[(X_i - x_i)^2|\mathbf{X} > \mathbf{x}]$

$$= \frac{1}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_1=x_1+1}^{\infty} \cdots \sum_{t_p=x_p+1}^{\infty} (t_i - x_i)^2 \mathbf{f}(\mathbf{x})$$

$$
\begin{aligned}
=&\frac{1}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\sum_{t_i=x_i+1}^{\infty}(t_i-x_i)^2\left[\mathbf{S}(x_1+1,...,x_{i-1}+1,t_i,x_{i+1}+1,...,x_p+1)\right.\\
&\qquad\left.-\mathbf{S}(x_1+1,...,t_i+1,...,x_p+1)\right]\\
=&\frac{1}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\sum_{t_i=x_i+1}^{\infty}\left[2(t_i-x_i)-1\right]\mathbf{S}(x_1+1,...,x_{i-1}+1,t_i,...,x_p+1)\\
=&\frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\sum_{t_i=x_i+1}^{\infty}(t_i-x_i)\mathbf{S}(x_1+1,...,t_i,...,x_p+1)+m_i(\mathbf{x})\\
=&\frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\sum_{t_i=x_i+1}^{\infty}\sum_{u_i=t_i+1}^{\infty}\mathbf{S}(x_1+1,...,x_{i-1}+1,u_i,x_{i+1}+1,...,x_p+1)\\
&\qquad+m_i(\mathbf{x}).
\end{aligned}
\tag{6.2.3}
$$

Substituting (6.2.3) into (6.2.1), we have (6.2.2).

Example 6.2.1. For the bivariate geometric distribution in (5.2.1), we have

$$m_i(x_1,x_2)=\left(1-q_i\theta^{x_{3-i}+1}\right)^{-1};\ i=1,2$$

and

$$\frac{2}{\mathbf{S}(x_1+1,x_2+1)}=\frac{2q_1\theta^{x_2+1}}{1-q_1\theta^{x_2+1}}.$$

Thus, from (6.2.2), when $p=2$ and $i=1$, we obtain

$$\sigma_1^2(x_1,x_2)=\frac{q_1\theta^{x_2+1}}{(1-q_1\theta^{x_2+1})^2}$$

and similarly for $i=2$,

$$\sigma_2^2(x_1,x_2)=\frac{q_2\theta^{x_1+1}}{(1-q_2\theta^{x_1+1})^2}.$$

Example 6.2.2. For the bivariate Waring distribution in (5.4.1), the bivariate mean residual life function is derived as

$$(m_1(x_1,x_2),m_2(x_1,x_2))'=\left(\frac{m+n+x_1+x_2+1}{n-1},\frac{m+n+x_1+x_2+1}{n-1}\right)'. \tag{6.2.4}$$

Some calculations are required to obtain the variance residual life function. Using (6.2.2), when $i=1$ and $p=2$, we write

$$\sigma_1^2(x_1,x_2) = \frac{2\,(m+n)_{x_1+x_2+2}}{(m)_{x_1+x_2+2}} \sum_{t=x_1+1}^{\infty}\sum_{u=t+1}^{\infty} \frac{(m)_{u+x_2+1}}{(m+n)_{u+x_2+1}} - m_1(x_1,x_2)\left[m_1(x_1,x_2)-1\right]. \quad (6.2.5)$$

To evaluate the first term, we write

$$\sum_{t=x_1+1}^{\infty}\sum_{u=t+1}^{\infty} \frac{(m)_{u+x_2+1}}{(m+n)_{u+x_2+1}} = \sum_{t=x_1+1}^{\infty} \frac{(m)_{t+x_2+2}}{(m+n)_{t+x_2+1}} \left(\frac{1}{m+n+t+x_2+} + \frac{m+t+x_2+2}{(m+n+t+x_2+1)(m+n+t+x_2+2)} + ...\right)$$

$$= \sum_{t=x_1+1}^{\infty} \frac{(m)_{t+x_2+2}}{(m+n)_{t+x_2+1}} \frac{m+t+x_2+2}{n-1}, \text{ by Waring expansion.}$$

$$= \frac{1}{n-1} \sum_{t=x_1+1}^{\infty} \frac{m\,(m+1)_{t+x_2+1}}{(m+n)_{t+x_2+1}}$$

$$= \frac{m\,(m+1)_{x_1+x_2+2}}{(n-1)(n-2)\,(m+n)_{x_1+x_2+1}}.$$

Hence the first term in (6.2.5) simplifies to

$$\frac{2\,(m+n)_{x_1+x_2+2}\, m\,(m+1)_{x_1+x_2+2}}{(m)_{x_1+x_2+2}\,(m+n)_{x_1+x_2+1}} = \frac{2(m+x_1+x_2+2)(m+n+x_1+x_2+1)}{(n-1)(n-2)} \quad (6.2.6)$$

and the second term to

$$m_1(x_1,x_2)\,(m_1(x_1,x_2)-1) = \frac{(m+x_1+x_2+2)(m+n+x_1+x_2+1)}{(n-1)^2}. \quad (6.2.7)$$

Using (6.2.6) and (6.2.7) in (6.2.5), we get

$$\sigma_1^2(x_1,x_2) = \frac{n(m+x_1+x_2+2)(m+n+x_1+x_2+1)}{(n-1)^2(n-2)}. \quad (6.2.8)$$

Similarly, we can evaluate $\sigma_2^2(x_1, x_2)$ to see that

$$\sigma_1^2(x_1, x_2) = \sigma_2^2(x_1, x_2).$$

6.2.1 Properties of variance residual life

1. If $i_1, i_2, ..., i_r; r = 1, 2, ..., p$ are permutations of the integers $(1, 2, ..., r)$, the variance residual life of the marginal distributions of $\mathbf{X}$ is obtained from (6.2.2) by setting $x_{i_{r+1}} ... = -1$ whenever $r < p$. In particular

$$\sigma_i^2(-\mathbf{e}) = \sigma_i^2,$$

the variance of the marginal distribution of X_i.

2. There exists a recurrence relation for $\sigma_i^2(\mathbf{x})$. Without loss of generality, we take $i = 1$ and state it as

$$\sigma_1^2(x_1+1, \mathbf{x}_{(1)}) = m_1(x_1+1, \mathbf{x}_{(1)}) \left[\frac{\sigma_1^2(\mathbf{x})}{m_1(\mathbf{x}) - 1} + m_1(\mathbf{x}) - m_1(x_1+1, \mathbf{x}_{(1)}) - 1 \right]. \tag{6.2.9}$$

Proof. When $i = 1$, (6.2.2) can be written as

$$\sigma_1^2(\mathbf{x}) = \frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_1=x_1+1}^{\infty} m_1(t_1, \mathbf{x}_{(1)}) \mathbf{S}(t_1+1, \mathbf{x}_{(1)}+\mathbf{e}_{p-1}) - m_1(\mathbf{x})(m_1(\mathbf{x})-1).$$

Thus,

$$\begin{aligned} &\left\{\sigma_1^2(\mathbf{x}) + m_1(\mathbf{x})[m_1(\mathbf{x}) - 1]\right\} \mathbf{S}(x_1+1, ..., x_p+1) \\ &\quad - \left\{\sigma_1^2(x_1+1, \mathbf{x}_{(1)}) + m_1(x_1+1, \mathbf{x}_{(1)}) \left[m_1(x_1+1, \mathbf{x}_{(1)}) - 1\right]\right\} \\ &\qquad \mathbf{S}(x_1+2, x_2+1, ..., x_p+1) \\ &\qquad = 2m_1(x_1+1, \mathbf{x}_{(1)}) \mathbf{S}(x_1+2, x_2+1, ..., x_p+1). \end{aligned} \tag{6.2.10}$$

Dividing (6.2.10) by $\mathbf{S}(x_1+1,...,x_p+1)$ and using (1.2.13), we obtain

$$\begin{aligned}&\left\{\sigma_1^2(\mathbf{x})+m_1(\mathbf{x})[m_1(\mathbf{x})-1]\right\}\\&-\left\{\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\left[m_1(x_1+1,\mathbf{x}_{(1)})-1\right]\right\}\frac{m_1(\mathbf{x})-1}{m_1(x_1+1,\mathbf{x}_{(1)})}\\&\qquad=2(m_1(\mathbf{x})-1),\end{aligned}$$

which leads to (6.2.9). ■

3. The variance residual life function can be expressed in-terms of mean residual life function as the following theorem shows.

Theorem 6.2.1. For $i=1$,

$$\sigma_1^2(\mathbf{x})=E\left[m_1(X_1,\mathbf{x}_{(1)})(m_1(X_1-1,\mathbf{x}_{(1)})-1)|\mathbf{X}>\mathbf{x}\right]. \tag{6.2.11}$$

Proof. From (6.2.10), we can write

$$\begin{aligned}&\left\{\sigma_1^2(\mathbf{x})+m_1(\mathbf{x})(m_1(\mathbf{x})-1)\right\}\mathbf{S}(\mathbf{x}+\mathbf{e})\\&=\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\mathbf{S}(x_1+2,x_2+1,...,x_p+1).\end{aligned}$$

Dividing by $\mathbf{S}(\mathbf{x}+\mathbf{e})$, we have

$$\begin{aligned}\sigma_1^2(\mathbf{x})=&\left[1-\frac{\mathbf{f}(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\right]\left[\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m^2(x_1+1,\mathbf{x}_{(1)})\right.\\&\left.+m_1(x_1+1,\mathbf{x}_{(1)})\right]-m_1(\mathbf{x})(m_1(\mathbf{x})-1).\end{aligned}$$

Now using (1.2.13),

$$\begin{aligned}\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-\sigma_1^2(\mathbf{x})&=m_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\\&-\left(\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\right)\\&\frac{1+m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})}{m_1(x_1+1,\mathbf{x}_{(1)})}-m_1(\mathbf{x})(m_1(\mathbf{x})-1)\\&=\frac{\mathbf{f}(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-\frac{\mathbf{f}(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{\mathbf{S}(\mathbf{x}+\mathbf{e})}\\&\qquad\left(m_1(x_1+1,\mathbf{x}_{(1)})(1-m_1(\mathbf{x}))\right).\end{aligned}$$

The last expression simplifies to

$$\sigma_1^2(\mathbf{x})\mathbf{S}(\mathbf{x}+\mathbf{e}) - \sigma_1^2(x_1+1,\mathbf{x}_{(1)})\mathbf{S}(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}) = \big[m_1(x_1+1,\mathbf{x}_{(1)}) \\ (m_1(\mathbf{x})-1)\big]\,\mathbf{f}(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}). \quad (6.2.12)$$

Adding the above identity for values of x_1,

$$\sigma_1^2(\mathbf{x})\mathbf{S}(\mathbf{x}+\mathbf{e}) \;=\; \sum_{t_1=x_1+1}^{\infty} m_1(t_1,\mathbf{x}_{(1)})(m_1(t_1-1,\mathbf{x}_{(1)})-1)\mathbf{f}(t_1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}),$$

which is same as (6.2.11). ■

Remark 6.2.1. In the univariate case, $(p=1)$

$$\sigma_1^2(x_1) = E\left[m_1(X_1)(m_1(X_1-1)-1)|X_1 > x_1\right],$$

a formula that does not seem to have appeared in literature. It can be used for obtaining quick estimates of $\sigma_1^2(x_1)$ based on the estimates of $m_1(x_1)$.

4. A problem of traditional interest in modelling situations is to characterize life distributions by properties of reliability functions that enable easy identification of the appropriate model. We give some characterization results below.

Theorem 6.2.2. A random vector $\mathbf{X}$ in $\mathbf{N}^p$ has a variance residual life of the form

$$\sigma_i^2(\mathbf{x}) = p_i(\mathbf{x}_{(i)});\ i = 1,2,...,p \quad (6.2.13)$$

for all $\mathbf{x}$ if and only if $\mathbf{X}$ follow the multivariate geometric distribution in (5.3.2).

Proof. By direct calculation,

$$\sigma_i^2(\mathbf{x}) = \frac{a(\mathbf{x}_{(i)})}{(1-a(\mathbf{x}_{(i)}))^2} \quad (6.2.14)$$

where

$$a(\mathbf{x}_{(i)}) = q_1 \prod_{j=2}^{p} q_{ij}^{x_j} \prod_{j,k=2;j<k}^{p} q_{ijk}^{x_j x_k} ... q_{12...p}^{x_2...x_p}.$$

This proves the 'if' part. Now assume that (5.3.2) holds. Using (6.2.12) with suffix 1 replaced by i, we get

$$p_i(\mathbf{x}_{(i)}) = m_i(x_i+1, \mathbf{x}_{(i)})(m_i(\mathbf{x})-1), \tag{6.2.15}$$

which cannot be true unless $m_i(\mathbf{x}) = a_i(\mathbf{x}_{(i)})$, a function independent of x_i. We first consider the univariate case. Taking $i=1$ and $p=1$, the mean residual life of X_1 is independent of x_1, say c_1. Then the survival function $S_1(.)$ of X_1 satisfies

$$S_1(x_1) = \prod_{t=0}^{x_1-1} \frac{m_1(x_1-1)-1}{m_1(x_1)} = \left(\frac{c_1-1}{c_1}\right)^{x_1} = q_1^{x_1};\ 0<q_1<1.$$

In general, $S_i(x_i) = q_i^{x_i}$. Similarly for $p=2$ and $i=1$, in the bivariate case

$$\begin{aligned}\mathbf{S}_2(x_1,x_2) &= \prod_{t=0}^{x_1-1} \frac{m_1(t-1,x_2)-1}{m_1(t,x_2)}\mathbf{S}_2(0,x_2) \qquad (6.2.16)\\ &= [b_1(x_2)]^{x_1}\, q_2^{x_2};\ b_1(x_2) = \frac{a_1(x_2)-1}{a_1(x_2)};\ 0<b_1(x_2)<1.\end{aligned}$$

Similarly working with $i=2$ and $p=2$, we obtain

$$\mathbf{S}_2(x_1,x_2) = [b_2(x_1)]^{x_2}\, q_1^{x_1};\ b_2(x_1) = \frac{a_2(x_1)-1}{a_2(x_1)};\ 0<b_2(x_1)<1. \tag{6.2.17}$$

From (6.2.16) and (6.2.17), we obtain

$$x_1\log b_1(x_2) + x_2\log q_2 = x_2\log b_2(x_1) + x_1\log q_1.$$

The left side of the above equation can be written as

$$x_1(\log b_1(x_2) - \log q_1) = x_2(\log b_2(x_1) - \log q_2),$$

which is linear in x_1 and hence right side must also be linear in x_1 and similarly for x_2. The only solution in this case is

$$b_1(x_2) = q_1 q_{12}^{x_2}$$

and

$$b_2(x_1) = q_2 q_{12}^{x_1}, \text{ for some } 0 < q_{12} < 1.$$

This gives

$$\mathbf{S}_2(x_1, x_2) = q_1^{x_1} q_2^{x_2} q_{12}^{x_1 x_2};\ 0 < q_1, q_2, q_{12} < 1.$$

Proceeding in this fashion, we arrive at (6.2.14) by mathematical induction and the proof is completed. ∎

Remark 6.2.2. The property $\left(\sigma_1^2(\mathbf{x}), ..., \sigma_p^2(\mathbf{x})\right)' = (c_1, c_2, ..., c_p)'$ where the c's are independent of $\mathbf{x}$ is satisfied if and only if the distribution of $\mathbf{X}$ is specified by (5.2.11).

Theorem 6.2.3. A bivariate random vector $(X_1, X_2)'$ in $\mathbf{N}^2$ has variance residual life of the form

$$\left(\sigma_1^2(x_1, x_2), \sigma_2^2(x_1, x_2)\right)' = \begin{cases} (c_1,\ c_2)';\ x_1 > x_2 \\ (c_3,\ c_4)';\ x_2 > x_1 \\ (c_1,\ c_4)';\ x_1 = x_2 \end{cases} \tag{6.2.18}$$

where $c_i;\ i = 1, 2, 3, 4$ are independent of x_1 and x_2 if and only if its survival function is

$$\mathbf{S}(x_1, x_2) = \begin{cases} q^{x_2} q_1^{x_1 - x_2};\ x_1 \geq x_2 \\ q^{x_1} q_2^{x_2 - x_1};\ x_2 \geq x_1;\ x_1, x_2 = 0, 1, 2, ... \\ \qquad\qquad 0 < q < q_1, q_2 < 1;\ 1 + q \geq q_1 + q_2. \end{cases} \tag{6.2.19}$$

Proof. We assume that the distribution of $(X_1, X_2)'$ is specified by (6.2.18). Then the mean residual life is calculated as

$$(m_1(x_1, x_2), m_2(x_1, x_2))' = \begin{cases} (k_1, k_2)';\ \text{if } x_1 > x_2 \\ (k_3, k_4)';\ \text{if } x_2 > x_1 \\ (k_1, k_4)';\ \text{if } x_1 = x_2 \end{cases} \tag{6.2.20}$$

where $k_1 = (1 - q_1)^{-1}$, $k_2 = \left(1 - \dfrac{q}{q_1}\right)^{-1}$, $k_3 = \left(1 - \dfrac{q}{q_2}\right)^{-1}$ and $k_4 = (1 - q_2)^{-1}$.

Also,

$$\left(\sigma_1^2(x_1,x_2),\sigma_2^2(x_1,x_2)\right)' = \begin{cases} \left(\dfrac{q_1}{(1-q_1)^2}, \dfrac{q}{q_1(1-\dfrac{q}{q_1})^2}\right)' ; \; x_1 > x_2 \\ \left(\dfrac{q}{q_2(1-\dfrac{q}{q_2})^2}, \dfrac{q_2}{(1-q_2)^2}\right)' ; \; x_2 > x_1 \\ \left(\dfrac{q_1}{(1-q_1)^2}, \dfrac{q_2}{(1-q_2)^2}\right)' ; \; x_1 = x_2 \end{cases} \tag{6.2.21}$$

showing that it is of the form stated in (6.2.19). Conversely, assuming (6.2.19), we see from (6.2.15) with $p = 2$ that $\boldsymbol{\sigma}^2(x_1, x_2) = (c_1, c_2)'$ for $x_1 > x_2$ gives

$$m_1(x_1 + 1, x_2)\,(m_1(x_1, x_2) - 1) = c_1$$

and similarly for $i = 2$,

$$m_2(x_1, x_2 + 1)(m_2(x_1, x_2) - 1) = c_2.$$

The solutions of these equations must be of the form

$$(m_1(x_1, x_2), m_2(x_1, x_2))' = (k_1, k_2)'$$

for some constants k_1 and k_2, both independent of x_1 and x_2. Similarly, we can work with the regions $x_2 > x_1$ and $x_1 = x_2$ to reach at (6.2.20). Substituting the values of $(m_1(.), m_2(.))'$ in formula (5.2.3), the bivariate geometric distribution of the form (6.2.19) is recovered. ■

Remark 6.2.3. The p-variate version of (6.2.19) can be stated as in (5.3.3). The method of proof used in Theorem 6.2.3 is applicable in this case also, but with lengthy expressions for the mean and variance residual lives according to the various regions of the sample space required by $x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_p}$. Note that the variance residual life is piece-wise constant.

The following theorem characterizes life distributions using the form of relationship between MVRL and MMRL.

Theorem 6.2.4. A random vector $\mathbf{X}$ taking values in $\mathbf{N}^p$ satisfies the property

$$\sigma_i^2(\mathbf{x}) = k_i m_i(\mathbf{x})\left[m_i(\mathbf{x}) - 1\right];\ i = 1, 2, ..., p \tag{6.2.22}$$

for all $\mathbf{x}$ if and only if the distribution of $\mathbf{X}$ is multivariate Waring with

$$\mathbf{S}(\mathbf{x}) = \frac{(A_0 + A_2x_2 + ... + A_px_p)_{x_1}}{(A_0 + A_1 + A_2x_2 + ... + A_px_p)_{x_1}} \cdots \frac{(B_0 + B_3x_3)_{x_2}\,(C_0)_{x_3}}{(B_0 + B_2 + B_3x_3)_{x_2}\,(C_0 + C_3)_{x_3}}, \tag{6.2.23}$$

$x_i = 0, 1, 2, ...;\ i = 1, 2, ..., p$ and negative hyper-geometric with

$$\mathbf{S}(\mathbf{x}) = \frac{\binom{\alpha_0 + \alpha_1 + \alpha_2x_2 + ... + \alpha_px_p - x_1}{\alpha_0 + \alpha_2x_2 + ... + \alpha_px_p - x_1}}{\binom{\alpha_0 + \alpha_1 + \alpha_2x_2 + ... + \alpha_px_p}{\alpha_0 + \alpha_2x_2 + ... + \alpha_px_p}} \cdots \frac{\binom{\beta_0 + \beta_1 + \beta_3x_{p-2} - x_{p-1}}{\beta_0 + \beta_3x_{p-2} - x_{p-1}}}{\binom{\beta_0 + \beta_1 + \beta_3x_{p-3}}{\beta_0 + \beta_3x_{p-3}}} \frac{\binom{\delta_0 + \delta_1 - x_p}{\delta_0 - x_p}}{\binom{\delta_0 + \delta_1}{\delta_0}}, \tag{6.2.24}$$

$$x_1 = 0, 1, 2, ..., \alpha_0; ...; x_p = 0, 1, 2, ..., \delta_0,$$

according as $k_i > 1$ and $0 < k_i < 1$.

Proof. Since the proof of the theorem in the p- variate case is apparent from the tri-variate version, we consider the latter only, for brevity. Recall that

$$\sigma_i^2(\mathbf{x}) = \frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e})} \sum_{t_i = x_i+1}^{\infty} m_i(t_i, \mathbf{x}_{(i)})\mathbf{S}(t_i{+}1, \mathbf{x}_{(i)}{+}\mathbf{e}_{p-1}) - m_i(\mathbf{x})\left(m_i(\mathbf{x}) - 1\right), \tag{6.2.25}$$

$i = 1, 2, ..., p$. Taking $p = 3,\ i = 1$ and $\mathbf{x} = (x_1, x_2, x_3)'$, we can write the above identity when (6.2.22) holds as

$$(k{+}1)m_1(\mathbf{x})\left(m_1(\mathbf{x}) - 1\right)) = \frac{2}{\mathbf{S}(\mathbf{x}+\mathbf{e}_3)} \sum_{t_1 = x_1+1}^{\infty} m_1(x_1, x_2, x_3)\mathbf{S}(t_1{+}1, x_2{+}1, x_3{+}1).$$

Hence,

$$(k+1)m_1(\mathbf{x})\,(m_1(\mathbf{x})-1)\,\mathbf{S}(\mathbf{x}+\mathbf{e}_3)-(k+1)m_1(x_1+1,x_2,x_3)$$
$$(m_1(x_1+1,x_2,x_3)-1)\,\mathbf{S}(x_1+2,x_2+1,x_3+1)=2\mathbf{S}(x_1+2,x_2+1,x_3+1)$$
$$m_1(x_1+1,x_2,x_3).$$

Dividing by $\mathbf{S}(x_1+1,x_2+1,x_3+1)$ and invoking (6.2.11) with $p=3$, we get, after some simplifications, that

$$(k+1)\left[m_1(\mathbf{x})-m_1(x_1+1,x_2,x_3)\right]=2$$

or

$$m_1(\mathbf{x})-m_1(x_1+1,x_2,x_3)=\frac{k_1-1}{k_1+1}.$$

The solution of the above partial difference equation is

$$m_1(\mathbf{x})=\alpha_1x_1+p_1(x_2,x_3),\ \alpha=\frac{k_1-1}{k_1+1}.$$

Likewise for $i=2$ and 3, we further have from (6.2.25),

$$m_2(\mathbf{x})=\alpha_2x_2+p_2(x_1,x_3)$$

and

$$m_3(\mathbf{x})=\alpha_3x_3+p_3(x_1,x_2).$$

From (5.2.3), the survival function is written as

$$\mathbf{S}(\mathbf{x})=\prod_{r=0}^{x_1-1}\frac{\alpha_1(r-1)+p_1(x_2,x_3)-1}{\alpha_1r+p_1(x_2,x_3)}\prod_{r=0}^{x_2-1}\frac{\alpha_2(r-1)+p_2(0,x_3)-1}{\alpha_2r+p_2(0,x_3)}$$
$$\prod_{r=0}^{x_3-1}\frac{\alpha_3(r-1)+p_3(0,0)-1}{\alpha_3r+p_3(0,0)}$$

$$= \prod_{r=0}^{x_2-1} \frac{\alpha_2(r-1)+p_2(x_1,x_3)-1}{\alpha_2 r+p_2(x_1,x_3)} \prod_{r=0}^{x_3-1} \frac{\alpha_3(r-1)+p_3(x_1,0)-1}{\alpha_3 r+p_3(x_1,0)} \prod_{r=0}^{x_3-1} \frac{\alpha_1(r-1)+p_1(0,0)-1}{\alpha_1 r+p_1(0,0)}$$

$$= \prod_{r=0}^{x_3-1} \frac{\alpha_3(r-1)+p_3(x_1,x_2)-1}{\alpha_3 r+p_3(x_1,x_2)} \prod_{r=0}^{x_1-1} \frac{\alpha_1(r-1)+p_1(x_2,0)-1}{\alpha_1 r+p_1(x_2,0)} \prod_{r=0}^{x_2-1} \frac{\alpha_2(r-1)+p_2(0,0)-1}{\alpha_2 r+p_2(0,0)}. \tag{6.2.26}$$

When $k>1$ and $\alpha_i>0$, the terms under the product symbol can be written in-terms of the Pocchamer symbol

$$(t)_r = t(t+1)...(t+r-1).$$

Thus,

$$\mathbf{S}(\mathbf{x}) = \frac{\left(\frac{p_1(x_2,x_3)-1}{\alpha_1}-1\right)_{x_1}}{\left(\frac{p_1(x_2,x_3)}{\alpha_1}\right)_{x_1}} \frac{\left(\frac{p_2(0,x_3)-1}{\alpha_2}-1\right)_{x_2}}{\left(\frac{p_2(0,x_3)}{\alpha_2}\right)_{x_2}} \frac{\left(\frac{p_3(0,0)-1}{\alpha_3}-1\right)_{x_3}}{\left(\frac{p_3(0,0)}{\alpha_3}\right)_{x_3}}$$

and similarly the other two equivalent forms. A complete specification of $\mathbf{S}(\mathbf{x})$ requires the solution of the functions $p_1(x_2,x_3), p_2(x_1,x_3)$ and $p_3(x_1,x_2)$, for which, we consider

$$\frac{\mathbf{S}(x_1+1,x_2+1,x_3+1)}{\mathbf{S}(x_1,x_2,x_3)} = \frac{\mathbf{S}(x_1+1,x_2+1,x_3+1)}{\mathbf{S}(x_1,x_2+1,x_3+1)} \frac{\mathbf{S}(x_1,x_2+1,x_3+1)}{\mathbf{S}(x_1,x_2,x_3+1)} \frac{\mathbf{S}(x_1,x_2,x_3+1)}{\mathbf{S}(x_1,x_2,x_3)}$$

$$= \frac{\mathbf{S}(x_1+1,x_2+1,x_3+1)}{\mathbf{S}(x_1+1,x_2,x_3+1)} \frac{\mathbf{S}(x_1+1,x_2,x_3+1)}{\mathbf{S}(x_1+1,x_2,x_3)} \frac{\mathbf{S}(x_1+1,x_2,x_3)}{\mathbf{S}(x_1,x_2,x_3)}$$

$$= \frac{\mathbf{S}(x_1+1,x_2+1,x_3+1)}{\mathbf{S}(x_1+1,x_2+1,x_3)} \frac{\mathbf{S}(x_1+1,x_2+1,x_3)}{\mathbf{S}(x_1,x_2+1,x_3)} \frac{\mathbf{S}(x_1,x_2+1,x_3)}{\mathbf{S}(x_1,x_2,x_3)}.$$

Converting the right hand expressions in-terms of the mean residual life functions using (6.2.11) lead to three functional equations. One of these equations has the form

$$\frac{(m_1(x_1-1,x_2,x_3)-1)}{m_1(x_1,x_2,x_3)}\frac{(m_2(x_1-1,x_2-1,x_3)-1)}{m_2(x_1-1,x_2,x_3)}\frac{(m_3(x_1-1,x_2-1,x_3-1)-1)}{m_3(x_1-1,x_2-1,x_3)}=\frac{(m_1(x_1-1,x_2-1,x_3-1)-1)}{m_1(x_1,x_2-1,x_3-1)}$$

$$\frac{(m_2(x_1,x_2-1,x_3)-1)}{m_2(x_1,x_2,x_3)}\frac{(m_3(x_1,x_2-1,x_3-1)-1)}{m_3(x_1,x_2-1,x_3)}\frac{\frac{p_1(x_2,x_3)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2,x_3)}{\alpha_1}+x_1}$$

$$\frac{\frac{p_2(x_1-1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2}\frac{\frac{p_3(x_1,x_2)-1}{\alpha_3}+x_3-1}{\frac{p_3(x_1,x_2)}{\alpha_3}+x_3}$$

$$=\frac{\frac{p_1(x_2-1,x_3-1)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2-1,x_3-1)}{\alpha_1}+x_1}\frac{\frac{p_2(x_1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_2(x_1,x_3)}{\alpha_2}+x_2}$$

$$\frac{\frac{p_3(x_1,x_2-1)-1}{\alpha_3}+x_3-1}{\frac{p_3(x_1,x_2-1)}{\alpha_3}+x_3}$$

which can be rearranged into

$$\frac{\frac{p_1(x_2,x_3)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2,x_3)}{\alpha_1}+x_1}\frac{\frac{p_1(x_2-1,x_3-1)}{\alpha_1}+x_1}{\frac{p_1(x_2-1,x_3-1)-1}{\alpha_1}+x_1-1}$$

$$=\frac{\frac{p_1(x_1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_1(x_1,x_3)}{\alpha_2}+x_2}\frac{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2}{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2-1}$$

$$\frac{\frac{p_3(x_1,x_2-1)-1}{\alpha_3}+x_3-1}{\frac{p_3(x_1,x_2-1)}{\alpha_3}+x_3}\frac{\frac{p_3(x_1,x_2)}{\alpha_3}+x_3}{\frac{p_3(x_1,x_2)-1}{\alpha_3}+x_3-1}.$$

The terms on the left side are linear in x_1 and therefore the functions $p_1(x_1, x_3)$ and $p_2(x_1, x_3)$ must be linear in x_1. Similar arguments using two other equations of the same kind reveal that $p_1(x_1, x_3), p_2(x_1, x_3)$ and $p_3(x_1, x_2)$ should involve only linear terms in the respective variables. This enables to write the solution of the functional equations as

$$\begin{aligned} p_1(x_2, x_3) &= a_0 + a_2x_2 + a_3x_3 \\ p_2(x_1, x_3) &= b_0 + b_1x_1 + b_3x_3 \\ p_3(x_1, x_2) &= c_0 + c_1x_1 + c_2x_2. \end{aligned}$$

Substituting these in (6.2.26) and after renaming the constants, we get

$$\mathbf{S}(x_1, x_2, x_3) = \frac{(A_0 + A_2x_2 + A_3x_3)_{x_1}}{(A_0 + A_2x_2 + A_3x_3 + A_1)_{x_1}} \frac{(B_0 + B_3x_3)_{x_2}}{(B_0 + B_3x_3 + B_2)_{x_2}} \frac{(C_0)_{x_3}}{(C_0 + C_3)_{x_3}} \tag{6.2.27}$$

$$= \frac{(B_0 + B_3x_3 + B_1x_1)_{x_2}}{(B_0 + B_3x_3 + B_1x_1 + B_2)_{x_2}} \frac{(C_0 + C_1x_1)_{x_3}}{(C_0 + C_1x_1 + C_3)_{x_3}} \frac{(A_0)_{x_1}}{(A_0 + A_1)_{x_1}} \tag{6.2.28}$$

$$= \frac{(C_0 + C_1x_1 + C_2x_2)_{x_3}}{(C_0 + C_1x_1 + C_2x_2 + C_3)_{x_3}} \frac{(A_0 + A_2x_2)_{x_1}}{(A_0 + A_2x_2 + A_1)_{x_1}} \frac{(B_0)_{x_2}}{(B_0 + B_2)_{x_2}}, \tag{6.2.29}$$

as required. Now assuming the above distribution for $\mathbf{X}$, we have

$$\begin{aligned} m_1(\mathbf{x}) &= \frac{(A_0 + A_2(x_2+1) + A_3(x_3+1) + A_1)_{x_1+1}}{(A_0 + A_2(x_2+1) + A_3(x_3+1))_{x_1+1}} \\ &\qquad \sum_{t=x_1+1}^{\infty} \frac{(A_0 + A_2(x_2+1) + A_3(x_3+1))_t}{(A_0 + A_2(x_2+1) + A_3(x_3+1) + A_1)_t} \\ &= \frac{A_0 + A_2(x_2+1) + A_3(x_3+1) + A_1 + x_1}{(A_1 - 1)}, \end{aligned}$$

on using Waring expansion. Likewise

$$\begin{aligned} \sigma_1^2(\mathbf{x}) &= \frac{(A_0 + A_2(x_2+1) + A_3(x_3+1) + A_1 + x_1)}{(A_1-1)(A_1-2)} \\ &\qquad (A_0 + A_2(x_2+1) + A_3(x_3+1) + x_1 + 1) \end{aligned}$$

Thus, $\sigma_1^2(\mathbf{x}) = k_1 m_1(\mathbf{x})(m_1(\mathbf{x}) - 1),\ k_1 = \dfrac{A_1}{A_1 - 2} > 1.$

Using (6.2.28) and (6.2.29) in the same way, $k_2 = \dfrac{B_1}{B_1 - 2} > 1$ and $k_3 = \dfrac{C_1}{C_1 - 2} > 1$.

When $k_i < 1$, α is negative. The proof runs along the same lines as in the Waring case, except that in (6.2.26) the terms form a descending factorial expression resulting in a hyper-geometric function. The survival function takes the form

$$\mathbf{S}(x_1, x_2, x_3) = \frac{\binom{\alpha_0 + \alpha_1 + \alpha_2 x_2 + \alpha_3 x_3 - x_1}{\alpha_0 + \alpha_2 x_2 + \alpha_3 x_3 - x_1}}{\binom{\alpha_0 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_1}{\alpha_0 + \alpha_2 x_2 + \alpha_3 x_3}} \frac{\binom{\beta_0 + \beta_1 + \beta_3 x_3 - x_2}{\beta_0 + \beta_3 x_3 - x_2}}{\binom{\beta_0 + \beta_1 + \beta_3 x_3}{\beta_0 + \beta_3 x_3}} \frac{\binom{\delta_0 + \delta_1 - x_3}{\delta_0 - x_3}}{\binom{\delta_0 + \delta_1}{\delta_0}}$$

The mean and variance residual life functions are

$$m_1(\mathbf{x}) = \frac{\alpha_0 + \alpha_1 + \alpha_2(x_2 + 1) + \alpha_3(x_3 + 1) - x_1}{\alpha_1 + 1}$$

and

$$\sigma_1^2(\mathbf{x}) = \frac{(\alpha_0 + \alpha_1 + \alpha_2(x_2 + 1) + \alpha_3(x_3 + 1) - x_1)}{(\alpha_1 + 1)^2(\alpha_1 + 2)} (\alpha_0 + \alpha_2(x_2 + 1) + \alpha_3(x_3 + 1) - x_1 - 1)$$

and hence

$$\sigma_1^2(\mathbf{x}) = k_1 m_1(\mathbf{x})(m_1(\mathbf{x}) - 1);\ k_1 = \frac{\alpha_1}{\alpha_1 + 2} < 1.$$

This completes the proof. ■

Remark 6.2.4. When $k_i = 1$ the result corresponds to the multivariate geometric distribution (5.3.2).

Remark 6.2.5. It is evident from Theorem 6.2.4 that the multivariate Waring distribution and negative hyper-geometric distribution are characterized by a linear mean residual life function and a quadratic variance residual life function in $x_1, x_2, ..., x_p$.

Remark 6.2.6. The results in Theorem 6.2.4 are more general than that of Roy [125] in which he has taken $p = 2$ and $k_i = k;\ i = 1, 2$.

5. Let $\mathbf{X}$ and $\mathbf{Y}$ be two discrete random vectors defined on $\mathbf{N}^p$ with mean residual lives of the ith components as $m_{\mathbf{X}_i}(\mathbf{x})$ and $m_{\mathbf{Y}_i}(\mathbf{x})$. The corresponding variance residual lives are denoted by $\sigma^2_{X_i}(\mathbf{x})$ and $\sigma^2_{Y_i}(\mathbf{x})$. Then we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in multivariate mean residual life if $m_{\mathbf{X}_i}(\mathbf{x}) \leq m_{\mathbf{Y}_i}(\mathbf{x})$, for $i = 1, 2, ..., p$ and all $\mathbf{x}$ in $\mathbf{N}^p$ and is denoted by $\mathbf{X} \leq_{MMRL} \mathbf{Y}$. Similarly, we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in multivariate variance residual life if $\sigma^2_{X_i}(\mathbf{x}) \leq \sigma^2_{Y_i}(\mathbf{x})$, for $i = 1, 2, ..., p$ and all $\mathbf{x}$ in $\mathbf{N}^p$ and is denoted by $\mathbf{X} \leq_{MVRL} \mathbf{Y}$.
From Theorem 6.2.1, we see that
$$\mathbf{X} \leq_{MMRL} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{MVRL} \mathbf{Y}.$$

6. Corresponding to the vector $\mathbf{X}$, we can define a vector $\mathbf{Y} = (Y_1, Y_2, ..., Y_p)'$ in $\mathbf{N}^\mathbf{p}$ such that the distribution of $\mathbf{Y}$ is specified by the conditional probability mass functions
$$g_1(x_1|\mathbf{Y}_{(1)} > \mathbf{x}_{(1)}) = \frac{P[X_1 > x_1|\mathbf{X}_{(1)} > \mathbf{x}_{(1)}]}{E[X_1|\mathbf{X}_{(1)} > \mathbf{x}_{(1)}]}$$
$$g_2(x_2|\mathbf{Y}_{(2)} > \mathbf{x}_{(2)}) = \frac{P[X_2 > x_2|\mathbf{X}_{(2)} > \mathbf{x}_{(2)}]}{E[X_2|\mathbf{X}_{(2)} > \mathbf{x}_{(2)}]}$$
$$\cdots$$
$$g_p(x_p|\mathbf{Y}_{(p)} > \mathbf{x}_{(p)}) = \frac{P[X_p > x_p|\mathbf{X}_{(p)} > \mathbf{x}_{(p)}]}{E[X_p|\mathbf{X}_{(p)} > \mathbf{x}_{(p)}]}$$
where $\mathbf{x}_{(i)} = \mathbf{x} - \{x_i\}$. The above definitions are extensions to the multivariate case of the concept of continuous bivariate equilibrium distributions discussed in Gupta and Sankaran [60], Nair and Preeth [98] and Navarro and Sarabia [114]. Notice that the above conditional probability mass functions lead to a multivariate distribution if and only if
$$\frac{P[Y_i > x_i|\mathbf{Y}_{(i)} > \mathbf{x}_{(i)}]}{P[Y_j > x_j|\mathbf{Y}_{(j)} > \mathbf{x}_{(j)}]} = \frac{A_j(\mathbf{x}_{(j)})}{A_i(\mathbf{x}_{(i)})},$$
where $A_i(.)$ and $A_j(.)$ are survival functions. The distribution of $\mathbf{Y}$ is called the multivariate equilibrium distribution of the random vector $\mathbf{X}$. Denoting the mean

residual life function of $\mathbf{Y}$ as

$$\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x}), ..., r_p(\mathbf{x}))'$$

where

$$r_i(\mathbf{x}) = E[Y_i - x_i|\mathbf{Y} > \mathbf{x}];\ i = 1, 2, ..., p,$$

we see that

$$\begin{aligned} r_i(\mathbf{x}) &= \frac{1}{\mathbf{S_Y}(\mathbf{x}+\mathbf{e})} \sum_{t=x_1+1}^{\infty} \mathbf{S_Y}(t, x_2+1, ..., x_p+1) \\ &= \frac{\sum_{t=x_1+1}^{\infty} \sum_{u=t+1}^{\infty} \mathbf{S}(u, x_2+1, ..., x_p+1)}{\sum_{t=x_1+1}^{\infty} \mathbf{S}(t, x_2+1, ..., x_p+1)}. \end{aligned}$$

With the aid of (6.2.3) and (6.2.1),

$$\sigma_1^2(\mathbf{x}) + m_i(\mathbf{x})(m_i(\mathbf{x}) - 1) = 2r_i(\mathbf{x})(m_i(\mathbf{x}) - 1). \tag{6.2.30}$$

Writing

$$C_i^2(\mathbf{x}) = \frac{\sigma_i^2(\mathbf{x})}{m_i(\mathbf{x})(m_i(\mathbf{x}) - 1)},$$

$$r_i(\mathbf{x}) = \frac{1}{2}\left(1 + C_i^2(\mathbf{x})\right) m_i(\mathbf{x});\ i = 1, 2, ..., p.$$

It may be noticed that in the discrete case, $C_i^2(\mathbf{x}), i = 1, 2, ..., p$ enjoy properties analogous to the coefficient of variation of the residual life when $\mathbf{X}$ is continuous. For a discussion of the role of the coefficient of variation of residual life in reliability modelling, see Gupta and Kirmani [55] and Gupta [54].

6.3 Ageing classes based on variance residual life

Multivariate life distributions can be classified using the behaviour of their variance residual lives. In the multivariate case, there can be different ways of defining their monotonicity and as such we have an increasing(decreasing) multivariate variance residual life class MIVRL(MDVRL) corresponding to each mode of definition. Following Zahedi [146] and Nair and Asha [95], four different versions of classes are studied in this section.

A discrete random vector $\mathbf{X}$ defined on $\mathbf{N}^p$ is said to be

(i) MIVRL-1(MDVRL-1) if
$$\sigma_i^2(\mathbf{x}+\mathbf{t}) \geq (\leq)\sigma_i^2(\mathbf{x})$$
for all $\mathbf{x}$ and $\mathbf{t}$ in $\mathbf{N}^p$ and $i = 1, 2, ..., p$.

(ii) MIVRL-2(MDVRL-2) if
$$\sigma_i^2(x_1, x_2, ..., x_{i-1}, x_i + t, x_{i+1}, ...x_p) \geq (\leq)\sigma_i^2(\mathbf{x}),$$
for all $\mathbf{x} \in \mathbf{N}^p$ and $t \in \mathbf{N}$ and $i = 1, 2, ..., p$.

(iii) MIVRL-3(MDVRL-3) if
$$\sigma_i^2(x_1 + t, x_2 + t, ..., x_n + t) \geq (\leq)\sigma_i^2(\mathbf{x})$$
for all $n \leq p;\ i = 1, 2, ..., p$ and $t \in \mathbf{N}$.
and

(iv) MIVRL-4(MDVRL-4) if
$$\sigma_i^2(x + t, x + t, ..., x + t) \geq (\leq)\sigma_i^2(x, x, ..., x)$$
for all $x, t \in \mathbf{N}$ and $i = 1, 2, ..., p$.

The interpretation of (i) is that the variance residual life of a p- component device where the components are of different ages increase(decrease) with different intensities. In (ii) the variance residual life increases when a working component is replaced by a younger one, whereas in (iii), the components are initially of different ages and the variance residual life is reckoned after the same time for all of them. Lastly in (iv), the variances are compared after the same time when initially they are of the same age. From the definitions, it is easy to see that

$$\begin{array}{ccccc} \text{MDVRL-1(MIVRL-1)} & \Rightarrow & \text{MDVRL-3(MIVRL-3)} & \Rightarrow & \text{MDVRL-4(MIVRL-4)} \\ \Downarrow & & & & \\ \text{MDMRL-2(MIMRL-2)} & & & & \end{array}$$

Further, MDVRL-1 and MIVRL-1 are simultaneously satisfied when $\sigma_i^2(\mathbf{x}) = k_i$, a constant independent of $\mathbf{x}$. In this case, the distribution of $\mathbf{X}$ is multivariate geometric in 5.2.11. Likewise, $\mathbf{X}$ is both MIVRL-2 and MDVRL-2 if and only if $\sigma_i^2(\mathbf{x}) = p_i(\mathbf{x}_{(i)})$; $i = 1, 2, ..., p$ so that the corresponding distribution is as in (5.3.2). The multivariate geometric distribution in 5.3.3 satisfies the property of being both MDVRL-3 and MIVRL-3. Finally, distribution of $\mathbf{X}$ is both MIVRL-4 and MDVRL-4 is satisfied if and only if

$$A_i(x+t) = A_i(x),$$

where $A_i(x) = \sigma_i^2(x, x, ..., x), i = 1, 2, ..., p$. The above is a univariate functional equation, from which a multivariate solution is difficult to emerge. The Waring and negative hypergeometric laws are respectively MIVRL-k and MDVRL-k for $k = 1, 2, 3, 4$ so that all the classes are well defined.

Some properties of the MIVRL and MDVRL classes are given below.

Theorem 6.3.1. The random vector $\mathbf{X}$ is MIVRL-2(MDVRL-2) if and only if $\sigma_i^2(x_1, ..., x_{i-1}, x_i + 1, ..., x_p) \geq (\leq) m_i(x_1, x_2, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_p)(m_i(\mathbf{x}) - 1)$ for $i = 1, 2, ..., p$.

Proof. We have from (6.2.9)

$$\sigma_1^2(x_1+1, \mathbf{x_{(1)}}) - \sigma_1^2(\mathbf{x}) = \frac{m_1(x_1+1, \mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1}{m_1(x_1+1, \mathbf{x}_{(1)})} \left[\sigma_1^2(x_1+1, \mathbf{x}_{(1)}) - m_1(x_1+1, \mathbf{x}_{(1)})(m_1(\mathbf{x}-1))\right].$$

Since the above is an identity, $\mathbf{X}$ is MIVRL if and only if

$$\sigma_1^2(x_1+1, \mathbf{x}_{(1)}) \geq m_1(x_1+1, \mathbf{x}_{(1)})(m_1(\mathbf{x}) - 1).$$

The proof of $i = 2, 3, ..., p$ are similar. ∎

Theorem 6.3.2. $\mathbf{X}$ is MIMRL-2(MDMRL-2)$\Rightarrow$ $\mathbf{X}$ is MIVRL-2(MDVRL-2).

Proof. Using (6.2.2), we write

$$\sigma_1^2(x_1+1,\mathbf{x}_{(1)}) - m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1) = \frac{2}{\mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}$$
$$\sum_{t=x_1+2}^{\infty}\sum_{u=t+1}^{\infty}\mathbf{S}(u,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}) - m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)})-1)$$
$$- m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1)$$

$$= \frac{2}{\mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty} m_1(t,\mathbf{x}_{(1)})\mathbf{S}(t+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}) - m_1(x_1+1,\mathbf{x}_{(1)})$$
$$\big[m_1(x_1+1,\mathbf{x}_{(1)}) + m_1(\mathbf{x}) - 2\big]$$

$$= \frac{2}{\mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty}\big[m_1(t,\mathbf{x}_{(1)}) - m_1(x_1+1,\mathbf{x}_{(1)})\big]$$
$$+ \frac{2m_1(x_1+1,\mathbf{x}_{(1)})}{\mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\left[\sum_{t=x_1+2}^{\infty}\mathbf{S}(t,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}) - \mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})\right]$$
$$- m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)}) + m_1(\mathbf{x}) - 2)$$

$$= \frac{2}{\mathbf{S}(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty}\big[m_1(t,\mathbf{x}_{(1)}) - m_1(x_1+1,\mathbf{x}_{(1)})\big]\,\mathbf{S}(t+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})$$
$$+ m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x})).$$

When $\mathbf{X}$ is MDMRL-2, $m_1(t,\mathbf{x}_{(1)}) \leq m_1(x_1+1,\mathbf{x}_{(1)})$ for all $t \geq x_1+2$ and also $m_1(x_1+1,\mathbf{x}_{(1)}) \leq m_1(\mathbf{x})$. Hence the expression on the right is negative and hence by Theorem 6.4.1, $\mathbf{X}$ is MDVRL-2. The case of $i = 2, 3, ...$ are similar and so is the case of MIVRL-2. ■

The above result gives only a sufficient condition for $\mathbf{X}$ to be MIVRL-2, besides being the implication among the MMRL and MVRL classes. A stronger result is presented in the next theorem.

Theorem 6.3.3. Suppose that $\mathbf{S}(\mathbf{x})$ is strictly decreasing. Then $\mathbf{X}$ is MIVRL-2(MDVRL-2) if and only if the vector $\mathbf{Y}$ is MIMRL-2(MDMRL-2).

Proof. Using (6.2.30), we can write for $i = 1$,

$$2\left(r_1(x_1+1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\right) = 2\left[\frac{\sigma_1^2(x_1+1,\mathbf{x}_{(1)}) + m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})}{2m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})} - 1\right]$$

$$= \frac{\sigma_1^2(x_1+1,\mathbf{x}_{(1)})}{m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})} - 1. \tag{6.3.1}$$

Hence,

$$\frac{\sigma_1^2(x_1+1,\mathbf{x}_{(1)})}{m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1)} = \frac{\left[m_1(x_1+1,\mathbf{x}_{(1)}) - 1\right]\left[1 + r_1(x_1+1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\right]}{m_1(\mathbf{x})-1}. \tag{6.3.2}$$

Also, from (6.3.1),

$$\frac{\sigma_1^2(\mathbf{x})}{m_1(\mathbf{x})(m_1(\mathbf{x})-1)} = 2\left[r_1(\mathbf{x}) - r_1(x_1-1,\mathbf{x}_{(1)}) + 1\right]. \tag{6.3.3}$$

The last two equations provide

$$\sigma_1^2(\mathbf{x}) - \sigma_1^2(x_1+1,\mathbf{x}_{(1)}) = \left(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\right)\left[m_1(\mathbf{x}) - m_1(x_1+1,\mathbf{x}_{(1)})\right] \left[1 + r_1(x_1+1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\right]$$

$$= \frac{\left[m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\right]}{1 + r_1(\mathbf{x}) - r_1(x_1-1,\mathbf{x}_{(1)})}\left[r_1(x_1-1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\right].$$

On using the identity,

$$m_1(\mathbf{x}) = r_1(\mathbf{x})\left[1 + r_1(\mathbf{x}) - r_1(x_1-1,\mathbf{x}_{(1)})\right],$$

further simplification yields

$$\sigma_1^2(\mathbf{x}) - \sigma_1^2(x_1+1,\mathbf{x}_{(1)}) = \left(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\right)(m_1(\mathbf{x}) - r_1(\mathbf{x})) \tag{6.3.4}$$

$$= m_1(\mathbf{x}) \left(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\right) \left(r_1(x_1-1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\right). \tag{6.3.5}$$

By (1.2.13), $m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1 > 0$ since $\mathbf{S}(\mathbf{x})$ is strictly decreasing. Moreover the sign of the left side of (6.3.5) is the same as that of $r_1(x_1-1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})$. This proves the assertion for $i = 1$. The same method applies to $i = 2, 3, ..., p$. ■

Remark 6.3.1. Equation (6.3.4) reveals that $\mathbf{X}$ is MIVRL-2(MDVRL-2) if and only if $m_i(\mathbf{x}) \geq (\leq) r_i(\mathbf{x})$. This is also equivalent to the statement

$$\mathbf{X} \text{ is MIVRL-2(MDVRL-2)} \iff \mathbf{X} \geq_{MMRL} (\leq_{MMRL})\mathbf{Y}.$$

Example 6.3.1. Let $\mathbf{X}$ follow the bivariate Waring distribution in (5.4.1). We see that the components of the variance residual life vector are equal and are increasing in x_1 and x_2. Hence the distribution is MIVRL-2. Now, we evaluate the mean residual life function $\mathbf{r}(\mathbf{x})$ of $\mathbf{Y}$ as

$$r_1(x_1,x_2) = \frac{\sum\limits_{t=x_1+1}^{\infty} \sum\limits_{u=t+1}^{\infty} \mathbf{S}(u, x_2+1)}{\sum\limits_{t=x_1+1}^{\infty} \mathbf{S}(t, x_2+1)} = \frac{\sum\limits_{t=x_1+1}^{\infty} \sum\limits_{u=t+1}^{\infty} \dfrac{(m)_{u+x_2+1}}{(m+n)_{u+x_2+1}}}{\sum\limits_{t=x_1+1}^{\infty} \dfrac{(m)_{t+x_2+1}}{(m+n)_{t+x_2+1}}}.$$

After expanding the terms and simplifying with the use of Waring expansion, we get

$$r_1(x_1,x_2) = \frac{m(n-1)\,(m+1)_{x_1+x_2+2}\,(m+n)_{x_1+x_2+1}}{(n-1)(n-2)(m+x_1+x_2+2)\,(m+n)_{x_1+x_2+1}\,(m)_{x_1+x_2+2}} = \frac{1}{n-2}.$$

Similarly, we can evaluate $r_2(x_1,x_2)$ to see that $r_1(x_1,x_2) = r_2(x_1,x_2) = \dfrac{1}{n-2}$ for $x_1, x_2 = 0, 1, 2, ...$

It is easy to verify that $m_i(x_1,x_2) \geq r_i(x_1,x_2);\ i = 1, 2$. Thus, $\mathbf{X} \geq_{MMRL} \mathbf{Y}$.

6.4 Conclusion

In this chapter, we have presented several properties of multivariate variance residual life in discrete time. Multivariate lifetime distributions are characterized by giving different functional forms for MVRL function. Also, the classification of life distributions based on the monotonicity of the concept were discussed. The results will be useful in modelling and analysis of multivariate discrete data, which is not much seen in reliability literature.

Chapter 7

Covariance Residual Life and Measures of Association

7.1 Introduction

In multivariate set-up, an important aspect to be considered in modelling lifetimes is the nature of dependence between the constituent variables. The model has to be chosen such that it reflects the quantum and nature of association exhibited by the data. Traditionally, covariance between the variables or some coefficients based on it are considered for this purpose. In the case of residual lives, the appropriate concept to be chosen for determining the association becomes the covariance of residual lives. Nair et al. [106] have discussed this role and properties of this notion and Navarro et al. [115] found its application in the context of equilibrium distributions, when the lifetimes are continuous. Sankaran et al. [129] proposed a dependence measure between a pair of continuous lifetime variables based on covariance residual life function. It was shown that zero covariance (correlation) residual life implies independence between the variables. Further, they proposed a non-parametric estimator for the dependence measure. Using that measure, they developed a test for independence among the variables. Even though, the definitions of hazard rate, mean residual life, etc. have been extended to the bivariate discrete case, a discussion of the covariance residual life is elusive in literature. Accordingly, the present chapter is an attempt to fill this gap. The importance of the work arises from the fact that in analysing bivariate discrete data, ascertaining the nature and extent of association makes the model building easier and meaningful. Also, one can develop time-dependent measures of association with the help of covariance residual life function. Finally, it is an essential component of the variance-covariance matrix of the residual lives, when one studies the properties of the bivariate mean residual life function from the data. From a practical view point also, the discussions on covariance residual life becomes meaningful. As an instance, the time of maximal association from the time of remission to relapse or of time of relapse and time of death, both measured in number of months, is a familiar case in medical studies. Also is the case of determining the genetic character of a disease by measuring the association between lifetimes of mono-zygotic twins (Hougaard [66]). The measures of association discussed in the sequel can be useful in this connection.

A sumary of the chapter is as follows. The product moment of residual life and covariance residual life are defined in the discrete domain in Section 7.2. Properties of these functions are discussed . It is shown that zero covariance residual life implies independence of the corresponding random variables. A measure of association for bivariate discrete data is proposed in Section 7.3. The new measure is compared with existing dependence concepts. In Section 7.4, the application of the theoretical results for real data is provided by way of illustration. The major conclusion of the study is given in Section 7.5.

7.2 Covariance residual life and its properties

Let $\mathbf{X}$ be a discrete random vector taking values in $\mathbf{N^p}$. The covariance residual life between X_i and X_j is defined as

$$\sigma_{ij}(x_i, x_j) = M_{ij}(x_i, x_j) - m_i(x_i, x_j)m_j(x_i, x_j); i, j = 1, 2, ..., p; i < j, x_i, x_j = -1, 0, 1, ..., \tag{7.2.1}$$

where

$$M_{ij}(x_i, x_j) = E\left[(X_i - x_i)(X_j - x_j)|X_i > x_i, X_j > x_j\right]$$

is the bivariate product moment of residual life (PMRL) and $m_i(x_i, x_j)$ denotes the i th component of the mean residual life of $(X_i, X_j)'$. Since working with any X_i and X_j is similar, without loss of generality, we take $i = 1, j = 2$ in the sequel. Dropping the suffixes in $M_{12}(.,.)$,

$$\begin{aligned} M(x_1,x_2) &= E\left[(X_1-x_1)(X_2-x_2)|X_1>x_1,X_2>x_2\right] \\ &= \frac{1}{\mathbf{S}(x_1+1,x_2+1)}\sum_{t_1=x_1+1}^{\infty}\sum_{t_2=x_2+1}^{\infty}(t_1-x_1)(t_2-x_2)\mathbf{f}(t_1,t_2) \\ &= \frac{1}{\mathbf{S}(x_1+1,x_2+1)}\sum_{t_1=x_1+1}^{\infty}\sum_{t_2=x_2+1}^{\infty}\mathbf{S}(t_1,t_2), \end{aligned} \tag{7.2.2}$$

where $\mathbf{S}(x_1,x_2)$ is the survival function and $\mathbf{f}(x_1,x_2)$ is the probability mass function, corresponding to $(X_1,X_2)'$.

Example 7.2.1. Let $\mathbf{X}$ be distributed as bivariate Waring in (5.4.1). Then

$$m_1(x_1,x_2) = \frac{m+n+x_1+x_2+1}{n-1} = m_2(x_1,x_2)$$

and

$$M(x_1,x_2) = \frac{(m+n+x_1+x_2+1)(m+n+x_1+x_2)}{(n-1)^2},$$

giving

$$\sigma_{12}(x_1,x_2) = -\frac{m+n+x_1+x_2+1}{(n-1)^2}.$$

More examples can be seen in the subsequent discussions. Some important properties of the covariance residual life are mentioned below.

1. $\sigma_{12}(-1,-1) = Cov(X_1,X_2)$, the usual covariance between X_1 and X_2.

2. There are some identities connecting $M(x_1,x_2)$ or $\sigma_{12}(x_1,x_2)$ and the bivariate reliability functions. Changing x_1 to x_1+1 in (7.2.2) and subtracting from (7.2.2), we get

$$\begin{aligned} &M(x_1,x_2)\mathbf{S}(x_1+1,x_2+1) - M(x_1+1,x_2)\mathbf{S}(x_1+2,x_2+1) \\ &\qquad = \sum_{t_2=x_2+1}^{\infty}\mathbf{S}(x_1+1,t_2) = \mathbf{S}(x_1+1,x_2+1)m_2(x_1,x_2). \end{aligned} \tag{7.2.3}$$

Using (1.2.13),

$$M(x_1, x_2)m_1(x_1 + 1, x_2) - [m_1(x_1, x_2) - 1]M(x_1 + 1, x_2)$$
$$= m_1(x_1 + 1, x_2)m_2(x_1, x_2), \quad (7.2.4)$$

a relationship connecting PMRL with the mean residual life. It also gives a recurrence relation connecting PMRL's as

$$M(x_1 + 1, x_2) = \frac{m_1(x_1 + 1, x_2)M(x_1, x_2)}{m_1(x_1, x_2) - 1} - \frac{m_2(x_1, x_2)m_1(x_1 + 1, x_2)}{m_1(x_1, x_2) - 1}.$$

Changing x_2 to $x_2 + 1$ in (7.2.2), we also have

$$m_2(x_1, x_2 + 1)M(x_1, x_2) - [m_2(x_1, x_2) - 1]M(x_1, x_2 + 1)$$
$$= m_1(x_1, x_2)m_2(x_1, x_2 + 1). \quad (7.2.5)$$

Relationships (7.2.4) and (7.2.5) can easily be converted into those involving $\sigma_{12}(.,.)$ by virtue of (7.2.1).

3. Equating the expressions for $M(x_1 + 1, x_2 + 1)$ in (7.2.3) and (7.2.4), we get a necessary condition for $M(x_1, x_2)$ to be a PMRL as

$$m_2(x_1, x_2) + \frac{m_1(x_1, x_2) - 1}{m_1(x_1 + 1, x_2)}M(x_1 + 1, x_2) = m_1(x_1, x_2) + \frac{m_2(x_1, x_2) - 1}{m_2(x_1, x_2 + 1)}M(x_1, x_2 + 1).$$

4. In general, neither $M(x_1, x_2)$ nor $\sigma_{12}(x_1, x_2)$ determine the distribution $\mathbf{S}(x_1, x_2)$ uniquely. This can be seen from the survival functions

$$\mathbf{S}(x_1, x_2) = p_1^{x_1} p_2^{x_2};\ 0 < p_1, p_2 < 1;\ x_1, x_2 = 0, 1, 2, ...$$

and

$$\mathbf{G}(x_1, x_2) = p_2^{x_1} p_1^{x_2};\ 0 < p_1, p_2 < 1;\ x_1, x_2 = 0, 1, 2, ...$$

for which

$$M(x_1, x_2) = (1 - p_1)^{-1}(1 - p_2)^{-1} \text{ and } \sigma_{12}(x_1, x_2) = 0,$$

but the probability mass functions are different when $p_1 = \frac{1}{4}$ and $p_2 = \frac{3}{4}$. However, with some additional information on the functional form of $(m_1(.,.), m_2(.,.))'$ we can determine $\mathbf{S}(x_1, x_2)$. To demonstrate this, we use (7.2.5) to write

$$\frac{\mathbf{S}(x_1+2, x_2+1)}{\mathbf{S}(x_1+1, x_2+1)} = \frac{M(x_1, x_2) - m_2(x_1, x_2)}{M(x_1+1, x_2)}.$$

Successive reduction gives

$$\mathbf{S}(x_1, x_2) = \prod_{r=0}^{x_1-1} \frac{M(r-1, x_2-1) - m_2(r-1, x_2-1)}{M(r, x_2-1)} \mathbf{S}(0, x_2) \qquad (7.2.6)$$

and similarly,

$$\mathbf{S}(x_1, x_2) = \prod_{r=0}^{x_2-1} \frac{M(x_1-1, r-1) - m_1(x_1-1, r-1)}{M(x_1-1, r)} \mathbf{S}(x_1, 0) \qquad (7.2.7)$$

Determining $\mathbf{S}(0, x_2)$ from (7.2.7) and $\mathbf{S}(x_1, 0)$ from (7.2.6) and substituting these expressions respectively in (7.2.6) and (7.2.7) we have the representations

$$\mathbf{S}(x_1, x_2) = \prod_{r=0}^{x_1-1} \frac{M(r-1, x_2-1) - m_2(r-1, x_2-1)}{M(r, x_2-1)} \prod_{r=0}^{x_2-1} \frac{M(-1, r-1) - m_1(-1, r-1)}{M(0, r)}$$

$$\mathbf{S}(x_1, x_2) = \prod_{r=0}^{x_2-1} \frac{M(x_1-1, r-1) - m_1(x_1-1, r-1)}{M(x_1-1, r-1)} \prod_{r=0}^{x_1-1} \frac{M(r-1, -1) - m_2(r-1, -1)}{M(r, 0)}$$

5. The following theorem characterizes the independence of random variables, which is an extension to the result of Sankaran et al. [129] in the continuous case.

Theorem 7.2.1. The covariance residual life is zero if and only if X_1 and X_2 are independent.

Proof. When X_1 and X_2 are independent, it is easy to see that $\sigma_{12}(x_1, x_2) = 0$. Conversely, $\sigma_{12}(x_1, x_2) = 0$ implies

$$M(x_1, x_2) = m_1(x_1, x_2)m_2(x_1, x_2).$$

Using this in (7.2.4),

$$[m_1(x_1, x_2) - 1]\,[m_2(x_1, x_2) - m_2(x_1 + 1, x_2)] = 0.$$

Since $m_1(x_1, x_2) > 1$,

$$m_2(x_1 + 1, x_2) = m_2(x_1, x_2) = m_2(-1, x_2). \tag{7.2.8}$$

Similarly considering (7.2.5),

$$m_1(x_1, x_2) = m_1(x_1, -1). \tag{7.2.9}$$

Equations (7.2.8) and (7.2.9) yield

$$M(x_1, x_2) = m_1(x_1, -1)m_2(-1, x_2)$$

so that X_1 and X_2 are independent. ■

6. Following Navarro and Sarabia [114] in the continuous case, the bivariate equilibrium distribution of the discrete random vector $(X_1, X_2)'$ is defined by the probability mass function

$$f_{(Y_1,Y_2)}(x_1, x_2) = \frac{\mathbf{S}(x_1 + 1, x_2 + 1)}{E[X_1X_2]};\; x_1, x_2 = 0, 1, 2, ... \tag{7.2.10}$$

provided $E[X_1X_2] < \infty$. We designate by $(Y_1, Y_2)'$, the random vector with probability mass function (7.2.10). Consequently, the survival function of $(Y_1, Y_2)'$ is

$$\mathbf{S}_{(Y_1,Y_2)}(x_1, x_2) = \frac{\sum_{t_1=x_1+1}^{\infty} \sum_{t_1=x_2+1}^{\infty} \mathbf{S}(t_1, t_2)}{E[X_1X_2]}. \tag{7.2.11}$$

Hence, the bivariate scalar hazard rate (1.2.3) of $(Y_1, Y_2)'$ is

$$a_{Y_1,Y_2}(x_1, x_2) = \frac{\mathbf{S}(x_1 + 1, x_2 + 1)}{\sum_{t=x_1+1}^{\infty} \sum_{t=x_2+1}^{\infty} \mathbf{S}(t_1, t_2)} = \frac{1}{M(x_1, x_2)}. \tag{7.2.12}$$

7. If $(\gamma_1(x_1, x_2), \gamma_2(x_1, x_2))'$ is the vector hazard rate of $(Y_1, Y_2)'$, from (1.2.5), we have

$$\begin{aligned}\gamma_1(x_1, x_2) &= 1 - \frac{\mathbf{S}_{Y_1,Y_2}(x_1 + 1, x_2)}{\mathbf{S}_{Y_1,Y_2}(x_1, x_2)} \\ &= 1 - \frac{\mathbf{S}(x_1 + 2, x_2 + 1)M(x_1 + 1, x_2)}{\mathbf{S}(x_1 + 1, x_2 + 1)M(x_1, x_2)} \\ &= 1 - \frac{M(x_1 + 1, x_2)(m_1(x_1, x_2) - 1)}{M(x_1, x_2)m_1(x_1 + 1, x_2)} \\ &= \frac{m_2(x_1, x_2)}{M(x_1, x_2)}, \text{ on using (7.2.4).}\end{aligned}$$

Similarly,

$$\gamma_2(x_1, x_2) = \frac{m_1(x_1, x_2)}{M(x_1, x_2)}.$$

Thus,

$$\sigma_{12}(x_1, x_2) = M(x_1, x_2) \left(1 - M(x_1, x_2)\gamma_1(x_1, x_2)\gamma_2(x_1, x_2)\right)$$

and

$$M^2(x_1, x_2) = \frac{m_1(x_1, x_2)m_2(x_1, x_2)}{\gamma_1(x_1, x_2)\gamma_2(x_1, x_2)}.$$

7.3 Measures of association

An important application of the notion of covariance of residual life is in defining a measure of association as

$$\alpha(x_1, x_2) = \frac{M(x_1, x_2)}{m_1(x_1, x_2)m_2(x_1, x_2)}. \tag{7.3.1}$$

We say that $(X_1, X_2)'$ is positively associated if $\alpha(x_1, x_2) > 1$, negatively associated if $\alpha(x_1, x_2) < 1$ and not associated if $\alpha(x_1, x_2) = 1$. In this section, we examine the justification of $\alpha(x_1, x_2)$ as a time-dependent measure of association, its relationship with other measures and various properties in the context of reliability analysis.

First it is observed that

$$\alpha(-1,-1) = \frac{Cov(x_1, x_2)}{(1+\mu_1)(1+\mu_2)} + 1;\ \mu_i = E[X_i],\ i = 1, 2,$$

so that
$\alpha(x_1, x_2) > 1$ for all $(x_1, x_2) \Rightarrow \alpha(-1,-1) > 1 \Rightarrow Cov(X_1, X_2) > 0.$

Among the six commonly used positive dependence measures, total positivity of order 2, stochastic increase (SI), right tail increase (RTI), association, positive quadrant dependence (PQD) and $Cov(X_1, X_2) \geq 0$, written in the order of stringency, positive covariance is the weakest. Thus, $\alpha(x_1, x_2)$ is a stronger condition than $Cov(x_1, x_2) \geq 0$ and provides a sufficient condition for the latter. From Theorem 7.2.1, $\alpha(x_1, x_2) = 1$ implies independence of X_1 and X_2.

There exist some relationships between $\alpha(x_1, x_2)$ and dependence concepts.

We say that a discrete random vector $(X_1, X_2)'$ is weakly positive quadrant dependent (WPQD) if

$$\sum_{t_1=x_1}^{\infty} \sum_{t_2=x_2+1}^{\infty} [P[X_1 > t_1, X_2 > t_2] - P[X_1 > t_1]P[X_2 > t_2]] \geq 0. \tag{7.3.2}$$

See Alzaid [8] for the definition and properties of WPQD in the continuous case. Equation (7.3.2) is equivalent to

$$\sum_{t_1=x_1+1}^{\infty} \sum_{t_2=x_2+1}^{\infty} \mathbf{S}(t_1, t_2) \geq \sum_{t_1=x_1+1}^{\infty} S_1(t_1) \sum_{t_2=x_2+1}^{\infty} S_2(t_2) \tag{7.3.3}$$

where $S_i(t_i)$ is the marginal survival function of $X_i, i = 1, 2$. Since $\mathbf{S}(x_1, x_2) \leq S_i(x_i); i = 1, 2$, (7.3.3) reduces to

$$\mathbf{S}(x_1+1, x_2+1)M(x_1, x_2) \geq \mathbf{S}(x_1+1, x_2+1)m_1(x_1, x_2)\mathbf{S}(x_1+1, x_2+1)m_2(x_1, x_2)$$

and

$$M(x_1, x_2) \geq \mathbf{S}(x_1+1, x_2+1)m_1(x_1, x_2)m_2(x_1, x_2).$$

Thus,

$$(X_1, X_2)' \text{ is WPQD} \Rightarrow \alpha(x_1, x_2) \geq \mathbf{S}(x_1, x_2)$$

and hence the WPQD class contains both positively and negatively associated distributions in the sense of $\alpha(x_1, x_2)$. Also,

$$\begin{aligned}(X_1, X_2)' \text{ is PQD } &\iff P[X_1 > x_1, X_2 > x_2] \geq P[X_1 > x_2]P[X_2 > x_2] \\ &\Rightarrow (X_1, X_2)' \text{ is WPQD.}\end{aligned}$$

The above results show that PQD does not imply positive dependence defined by $\alpha(x_1, x_2) > 1$ nor that $\alpha(x_1, x_2) > 1$ is weaker than PQD in the hierarchy of dependence concepts mentioned above.

The discrete analogue of the Clayton [35] measure of association is

$$\theta(x_1, x_2) = \frac{\mathbf{f}(x_1, x_2)\mathbf{S}(x_1, x_2)}{[\mathbf{S}(x_1, x_2) - \mathbf{S}(x_1 + 1, x_2)]\,[\mathbf{S}(x_1, x_2) - \mathbf{S}(x_1, x_2 + 1)]}. \tag{7.3.4}$$

When $\theta(x_1, x_2) > (<)1$, X_1 and X_2 are positively (negatively) associated and $\theta(x_1, x_2) = 1$ implies independence of X_1 and X_2.

An alternative interpretation of $\theta(x_1, x_2)$ is possible by expressing it in-terms of conditional hazard rates. Denote $e_i(x_1, x_2)$ and $e_i^*(x_1, x_2)$ as hazard rates of X_i given $X_j = x_j$ and X_i given $X_j \geq x_j$; $i, j = 1, 2$; $i \neq j$. Using the definition of conditional probability, (7.3.4) becomes

$$\begin{aligned}\theta(x_1, x_2) &= \frac{P[X_1 = x_1, X_2 = x_2]P[X_1 \geq x_1, X_2 \geq x_2]}{P[X_1 = x_1, X_2 \geq x_2]P[X_1 \geq x_1, X_2 = x_2]} \\ &= \frac{P[X_1 = x_1|X_2 = x_2]P[X_1 \geq x_1|X_2 \geq x_2]}{P[X_1 \geq x_1|X_2 = x_2]P[X_1 = x_1|X_2 \geq x_2]} \\ &= \frac{e_1(x_1, x_2)}{e_1^*(x_1, x_2)}. \end{aligned} \tag{7.3.5}$$

A similar expression is obtained as

$$\theta(x_1, x_2) = \frac{e_2(x_1, x_2)}{e_2^*(x_1, x_2)}. \tag{7.3.6}$$

Thus, $\theta(x_1, x_2) > (<)1$ if and only if $e_i(x_1, x_2) < (>)e_i^*(x_1, x_2)$; $i = 1, 2$. Further,

$\theta(x_1, x_2) > 1$ if and only if

$$\frac{P[X_1 > x_1 + 1, X_2 > x_2]}{P[X_1 > x_1, X_2 > x_2]} > \frac{P[X_1 > x_1 + 1, X_2 > x_2 + 1]}{P[X_1 > x_1, X_2 > x_2 + 1]}. \tag{7.3.7}$$

The above inequality shows that $(X_1, X_2)'$ is right convex set increasing (RCSI). Similarly, when $\theta(x_1, x_2) < 1$, the inequality in (7.3.7) gets reversed so that $(X_1, X_2)'$ is right convex set decreasing (RCSD).

For the equilibrium vector $(Y_1, Y_2)'$, from (7.2.10) and (7.2.11),

$$\begin{aligned}\theta_{Y_1,Y_2}(x_1, x_2) &= \frac{\mathbf{S}(x_1 + 1, x_2 + 1) \sum_{t_1=x_1+1}^{\infty} \sum_{t_2=x_2+1}^{\infty} \mathbf{S}(t_1, t_2)}{\sum_{t_2=x_2+1}^{\infty} \mathbf{S}(x_1 + 1, t_2) \sum_{t_1=x_1+1}^{\infty} \mathbf{S}(t_1, x_2 + 1)} \\ &= \frac{M(x_1, x_2)}{m_1(x_1, x_2) m_2(x_1, x_2)} = \alpha_{X_1,X_2}(x_1, x_2). \end{aligned} \tag{7.3.8}$$

The relationship (7.3.8) enables us to write several properties of $\alpha(x_1, x_2)$ in-terms of the properties of the equilibrium random vector. Using (7.3.4) for $(Y_1, Y_2)'$ and (7.3.8) gives

$$\alpha_{X_1,X_2}(x_1, x_2) = \frac{a_{Y_1,Y_2}(x_1, x_2)}{\gamma_1(x_1, x_2)\gamma_2(x_1, x_2)}$$

where $a_{Y_1,Y_2}()$ is the scalar hazard rate of $(Y_1, Y_2)'$ and $\gamma_1(.)$ and $\gamma_2(.)$ are the components of vector hazard rate of $(Y_1, Y_2)'$ defined in Section 7.2.
Hence, X_1 and X_2 are positively(negatively) associated if and only if

$$a_{Y_1,Y_2}(x_1, x_2) > (<)\gamma_1(x_1, x_2)\gamma_2(x_1, x_2).$$

Making use of (1.2.6), an alternative condition is that

$$[1 - \gamma_2(x_1, x_2)] \left[\gamma_1(x_1, x_2 + 1) - \gamma_1(x_1, x_2)\right] > (<)0.$$

Thus, $\alpha(x_1, x_2) > (<)1$ if and only if either

$$\gamma_1(x_1, x_2 + 1) > (<)\gamma_1(x_1, x_2)$$

or

$$\gamma_2(x_1 + 1, x_2) > (<)\gamma_2(x_1, x_2).$$

Remark 7.3.1. The above result means that positive(negative) association using $\alpha(x_1, x_2)$ can be verified through only one component of the vector hazard rate of the equilibrium distribution by showing that $\gamma_i(x_1, x_2)$ is increasing(decreasing) in x_{3-i} for all $x_i,\ i = 1, 2$. Also, association can be inferred from the monotonicity of $\gamma_i(x_1, x_2)$.

We conclude this section by giving examples of some bivariate discrete distributions and their measure of association $\alpha(x_1, x_2)$.

Example 7.3.1. Consider the Waring distribution in (5.4.1). Then

$$\alpha(x_1, x_2) = \frac{m + n + x_1 + x_2}{m + n + x_1 + x_2 + 1}$$

which is less than unity for all x_1 and x_2. Thus, X_1 and X_2 have negative association that decreases with larger values of x_1 and x_2.

Example 7.3.2. The negative hyper-geometric law in (5.4.4) has

$$m_1(x_1, x_2) = \frac{k + n - x_1 - x_2 - 1}{k + 1} = m_2(x_1, x_2)$$

and

$$M(x_1, x_2) = \frac{(k + n - x_1 - x_2 - 1)(k + n - x_1 - x_2)}{(k + 1)^2}.$$

Thus,

$$\alpha(x_1, x_2) = \frac{k + n - x_1 - x_2}{k + n - x_1 - x_2 - 1} > 1,$$

so that X_1 and X_2 are positively associated.

Example 7.3.3. For the bivariate geometric distribution in (5.2.1), the mean residual life function has components

$$m_1(x_1, x_2) = \left(1 - q_1\theta^{x_2+1}\right)^{-1} \text{ and } m_2(x_1, x_2) = \left(1 - q_2\theta^{x_1+1}\right).$$

The product moment of residual life is calculated as

$$\begin{aligned} M(x_1, x_2) &= \left[q_1^{x_1+1} q_2^{x_2+1} \theta^{(x_1+1)(x_2+1)}\right]^{-1} \sum_{t_1=x_1+1}^{\infty} \sum_{t_1=x_2+1}^{\infty} q_1^{t_1} q_2^{t_2} \theta^{t_1 t_2} \\ &= \sum_{t_1=x_1+1}^{\infty} \frac{q_1^{t_1} \left(q_2\theta^{t_1}\right)^{x_2+1}}{1 - q_2\theta^{t_1}} \end{aligned}$$

$$= q_2^{x_2+1} \sum_{t=x_1+1}^{\infty} \left(q_1\theta^{x_2+1}\right)^{t_1} \left(1 - q_2\theta^{t_1}\right)^{-1}$$

$$= q_1^{x_1+1} q_2^{x_2+1} \theta^{(x_1+1)(x_2+1)} \sum_{r=0}^{\infty} \frac{\left(q_1\theta^{x_1+1}\right)^r}{1 - q_2\theta^{x_2+r+1}},$$

by expansion of $(1 - q_2\theta^{t_1})^{-1}$ and rearrangement of terms.
Thus,

$$M(x_1, x_2) \leq \frac{q_1^{x_1+1} q_2^{x_2+1} \theta^{(x_1+1)(x_2+1)}}{\left(1 - q_1\theta^{x_2+1}\right)\left(1 - q_2\theta^{x_1+1}\right)}$$

or

$$\alpha(x_1, x_2) \leq q_1^{x_1+1} q_2^{x_2+1} \theta^{(x_1+1)(x_2+1)} < 1.$$

Hence X_1 and X_2 are negatively associated.

7.4 Application to real data

Andrews and Herzberg [10] presented a multiple tumour recurrence data of patients with bladder cancer. The data were obtained in a randomized clinical trial conducted by the Veterans Administration Co-operative Urological Research Group (VACURG). All the patients had superficial bladder tumours when they entered into the trial. These tumours were removed transurethrally and patients were assigned randomly to one of three treatments, viz., placebo pills, pyridoxine pills or periodic instillation of a chemotherapeutic agent, thiotepa, into the bladder. At subsequent follow-up visits, tumours noticed were removed and the treatment was continued. The goal of the study was to determine the effect of treatment on the frequency of tumour recurrence. We now consider the dataset for the patients assigned to the thiotepa treatment group only, since our aim is to demonstrate the utility of our discrete measures of association rather than the complete analysis of the data. Let us denote the time (in months) to first and second recurrence of a tumour by X_1 and X_2 respectively. The data are given in Table 7.1.

We observe that the data contains 10 pairs of observations.When $(x_{1i}, x_{2i})'; i = 1, 2, ..., 10$ denote the observations, the empirical bivariate survival function can be obtained as

$$\hat{\mathbf{S}}(a, b) = \frac{\sum_{i=1}^{n} I[x_{i1} \geq a,\ x_{i2} \geq b]}{10}, \tag{7.4.1}$$

Table 7.1: Tumour data

Patient i	1	2	3	4	5	6	7	8	9	10
X_1	1	17	6	26	22	4	24	1	2	4
X_2	3	19	12	35	23	16	26	27	20	24

Table 7.2: Empirical survival function

x_1 \ x_2	0	3	12	16	19	20	24	26	27	35
0	1.0	1.0	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0.1
1	1.0	1.0	0.9	0.8	0.7	0.6	0.4	0.3	0.2	0.1
2	0.8	0.8	0.8	0.7	0.6	0.5	0.3	0.2	0.1	0.1
4	0.7	0.7	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.1
6	0.5	0.5	0.5	0.4	0.4	0.3	0.2	0.2	0.1	0.1
17	0.4	0.4	0.4	0.4	0.4	0.3	0.2	0.2	0.1	0.1
22	0.3	0.3	0.3	0.3	0.3	0.3	0.2	0.2	0.1	0.1
24	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.1
26	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

where $I[.,.]$ is the indicator function. The estimates of the survival function are presented in Table 7.2.

From the empirical survival function, the mean residual life functions $m_1(x_1, x_2)$ and $m_2(x_1, x_2)$ are estimated as

$$\hat{m}_1(x_1, x_2) = \frac{1}{\hat{\mathbf{S}}(x_1 + 1, x_2 + 1)} \sum_{t=x_1+1}^{\infty} \hat{\mathbf{S}}(t, x_2 + 1) \tag{7.4.2}$$

and

$$\hat{m}_2(x_1, x_2) = \frac{1}{\hat{\mathbf{S}}(x_1 + 1, x_2 + 1)} \sum_{t=x_2+1}^{\infty} \hat{\mathbf{S}}(x_1 + 1, t). \tag{7.4.3}$$

Table 7.3: Estimates of $\alpha(x_1, x_2)$

x_1 \ x_2	0	3	12	16	19	20	24	26
0	1.2149	1.1372	1.1829	1.1643	1.3004	1.1558	1.2389	1.7407
1	1.1593	1.1846	1.2482	1.2267	1.3719	1.1666	1.0288	1.0000
2	1.1594	1.1844	1.2099	1.1240	1.1544	1.1764	1.0301	1.0000
4	1.1449	1.1666	1.0709	1.1001	1.0444	1.0500	1.0329	1.0000
6	1.0425	1.0481	1.0797	1.1124	1.0493	1.0555	1.0364	1.0000
17	1.0408	1.0457	1.0714	1.0952	1.1269	1.1428	1.0865	1.0000
22	1.0491	1.0545	1.0810	1.1034	1.1304	1.1428	1.2307	1.0000
24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

We also estimate $M(x_1, x_2)$ as

$$\hat{M}(x_1, x_2) = \frac{1}{\hat{\mathbf{S}}(x_1 + 1, x_2 + 1)} \sum_{t_1 = x_1 + 1}^{\infty} \sum_{t_2 = x_2 + 1}^{\infty} \hat{\mathbf{S}}(t_1, t_2). \tag{7.4.4}$$

We then estimate $\alpha(x_1, x_2)$ as

$$\hat{\alpha}(x_1, x_2) = \frac{\hat{M}(x_1, x_2)}{\hat{m}_1(x_1, x_2)\hat{m}_2(x_1, x_2)}. \tag{7.4.5}$$

The estimates of $\alpha(x_1, x_2)$ are presented in Table 7.3. From the estimates, we see that X_1 and X_2 are positively correlated for small and moderate values of X_1 and X_2, since most of the $\hat{\alpha}(x_1, x_2)$ values are larger than one. But for large values of X_1 and X_2, $\hat{\alpha}(x_1, x_2)$ is unity, indicating that as time advances, the influence of treatment on tumour recurrence is gradually disappearing.

We also estimate $\theta(x_1, x_2)$ as

$$\hat{\theta}(x_1, x_2) = \frac{\hat{\mathbf{f}}(x_1, x_2)\hat{\mathbf{S}}(x_1, x_2)}{\left[\hat{\mathbf{S}}(x_1, x_2) - \hat{\mathbf{S}}(x_1 + 1, x_2)\right]\left[\hat{\mathbf{S}}(x_1, x_2) - \hat{\mathbf{S}}(x_1, x_2 + 1)\right]}, \tag{7.4.6}$$

where $\hat{\mathbf{f}}(x_1, x_2) = \hat{\mathbf{S}}(x_1 + 1, x_2) + \hat{\mathbf{S}}(x_1, x_2 + 1) - 2\hat{\mathbf{S}}(x_1, x_2) + \hat{\mathbf{S}}(x_1 + 1, x_2 + 1)$. The estimates are given in Table 7.4. From the table, it follows that X_1 and X_2 have positive

Table 7.4: Estimates of $\theta(x_1, x_2)$

X_1	1	17	6	26	22	4	24	1	2	4
X_2	3	19	12	35	23	16	26	27	20	24
$\hat{\theta}(x_1, x_2)$	5	4	5	1	3	3	2	2	5	3

dependence.

7.5 Conclusion

In the present chapter, we have studied the concepts of product moment of residual life and covariance residual life in the discrete case. A characterization result for independence of random variables has been derived, using the covariance residual life. A new measure of association was proposed and its properties were studied. The new measure was compared with existing dependence concepts. The utility of these measures was illustrated through a real dataset.

Chapter 8

Multivariate Reversed Hazard Rates

8.1 Introduction

The concept of reversed hazard rate function is employed extensively for modelling and analysis of lifetime data in recent times. Keilson and Sumita [77], who first defined the reversed hazard rate in continuous time, called it as the dual hazard rate. Apart from uniquely determining the underlying distribution, this function has been used in various contexts such as estimation of distribution function under left censoring(Lawless [87]), analysis of parallel systems (Marshall and Olkin [91]), definition of new stochastic orders (Shaked and Shanthikumar [135]) and to derive repair and maintenance strategies (Marshall and Olkin [91]). For more properties of the function in continuous and discrete set-up, we refer to Block et al. [22], Finkelstein [45], Gupta et al. [61] and Nair and Sankaran [99].

In the continuous case, the concept of reversed hazard rate has been extended to the multivariate case in several ways. Gürler [62] introduced a bivariate version of the reversed hazard rate as a three component vector in the continuous case. Roy [124] defined the bivariate reversed hazard rate as a two component vector and studied its properties. Further, Roy [124] introduced a class of bivariate distributions using the reversed hazard rate vector. Later, Bismi [20] introduced a scalar definition of the bivariate reversed hazard rate and

Results in this chapter have been accepted for publication in "Communications in Statistics-Theory and Methods". (See Sankaran et al. [131])

used it to characterize a family of bivariate Burr distributions. Sankaran and Gleeja [128] considered various definitions of the reversed hazard rate function in the bivariate case and developed dependence measures using the bivariate reversed hazard rate functions.

In the discrete domain, the only study appears to be made is that of Rejeesh [121], who defined a bivariate version of the reversed hazard rate function and provided characterization results based on it. The objective of the present study is to introduce multivariate reversed hazard rate function in the discrete case. We present four definitions of the multivariate reversed hazard rate in the discrete domain and study their properties.

The chapter is organized as follows. In Section 8.2 we define the scalar reversed hazard rate and study its properties. Section 8.3 is devoted to the study of vector reversed hazard rate and its properties. It is followed by a multivariate version of the alternative reversed hazard rate in Section 8.4. Section 8.5 presents the conditional reversed hazard rate and its characteristics. The chapter ends with a brief summary in Section 8.6.

8.2 Scalar reversed hazard rate

Let $\mathbf{X} = (X_1, X_2, ..., X_p)'$ be a random vector taking values in $\mathbf{N^p}$, where $\mathbf{N} = \{0, 1, 2, ...\}$ with distribution function $\mathbf{F}(\mathbf{x}) = P(\mathbf{X} \le \mathbf{x})$, where $\mathbf{x} = (x_1, x_2, ..., x_p)'$. The notation $(\mathbf{X} \le \mathbf{x})$ means $(X_1 \le x_1,\ X_2 \le x_2$,...,$X_p \le x_p)$.

Unlike the univariate case, the reversed hazard rate in the multivariate case can be defined in more than one way. We first define the scalar reversed hazard rate.

Definition 8.2.1. The scalar reversed hazard rate of $\mathbf{X}$ is defined by

$$\rho(\mathbf{x}) = P[\mathbf{X} = \mathbf{x} | \mathbf{X} \le \mathbf{x}] = \frac{\mathbf{f}(\mathbf{x})}{\mathbf{F}(\mathbf{x})}, \tag{8.2.1}$$

where $\mathbf{f}(\mathbf{x}) = P[\mathbf{X} = \mathbf{x}]$ is the probability mass function of $\mathbf{X}$.

Obviously $0 \le \rho(\mathbf{x}) \le 1$. It is interpreted as the conditional probability that a p-component device fails at time $\mathbf{x}$ given that it fails by that time.

Example 8.2.1. Let the bivariate random vector $(X_1, X_2)'$ be distributed as bivariate uniform with p.m.f.

$$\mathbf{f}(x_1, x_2) = \frac{1}{b_1 b_2}\left[1 + \theta\left(1 - \frac{2x_1 - 1}{b_1}\right)\left(1 - \frac{2x_2 - 1}{b_2}\right)\right]; x_i = 1, 2, ..., b_i;\ i = 1, 2. \tag{8.2.2}$$

Then the distribution function is

$$\mathbf{F}(x_1, x_2) = \frac{x_1 x_2}{b_1 b_2}\left[1 + \theta\left(1 - \frac{x_1}{b_1}\right)\left(1 - \frac{x_2}{b_2}\right)\right], x_i = 0, 1, 2, ..., b_i;\ i = 1, 2, \tag{8.2.3}$$

so that

$$\rho(x_1, x_2) = \frac{1 + \theta\left(1 - \frac{2x_1 - 1}{b_1}\right)\left(1 - \frac{2x_2 - 1}{b_2}\right)}{x_1 x_2\left[1 + \theta\left(1 - \frac{x_1}{b_1}\right)\left(1 - \frac{x_2}{b_2}\right)\right]}, x_i = 0, 1, 2, ..., b_i;\ i = 1, 2, \tag{8.2.4}$$

The following results of the multivariate scalar reversed hazard rate are useful in reliability modelling and analysis.

(i) The distribution of $\mathbf{X}$ is not uniquely determined by $\rho(\mathbf{x})$. This can be verified in the case of the bivariate distribution functions

$$\mathbf{F}(x_1, x_2) = \frac{x_1(x_1 + 1)x_2(x_2 + 1)}{b_1(b_1 + 1)b_2(b_2 + 1)}; x_i = 0, 1, 2, ..., b_i;\ i = 1, 2 \tag{8.2.5}$$

and

$$\mathbf{G}(x_1, x_2) = \frac{1}{2}\left[\frac{x_1 x_2(x_1 + 1)(x_2 + 1)}{b_1(b_1 + 1)b_2(b_2 + 1)} + \frac{x_1 x_2(x_1 + 1)(x_2 + 1)}{c_1(c_1 + 1)b_2(b_2 + 1)}\right]; c_1 \neq b_1 \tag{8.2.6}$$

Both (8.2.5) and (8.2.6) have the same $\rho(\mathbf{x})$ given by

$$\rho(\mathbf{x}) = \frac{4}{(x_1 + 1)(x_2 + 1)}. \tag{8.2.7}$$

(ii) At $\mathbf{0_p}$, $\rho(\mathbf{0_p}) = 1$, irrespective of the distribution and $0 \leq \rho(\mathbf{x}) \leq 1$.

(iii) If $X_1, X_2, ..., X_{p-1}$ and X_p are independent random variables, then

$$\rho(\mathbf{x}) = \prod_{i=1}^{p} \lambda_i(x_i),$$

where $\lambda_i(x_i)$ is the univariate reversed hazard rate of $X_i, i = 1, 2, ..., p$.

We now provide the conditions under which the scalar reversed hazard rate uniquely determines its underlying distribution.

Theorem 8.2.1. For a discrete random vector $\mathbf{X} = (X_1, X_2, ..., X_p)'$ with bounded support, i.e., $X_i \leq b_i\,; 0 < b_i < \infty\,; i = 1, 2, ..., p$, the distribution of $\mathbf{X}$ is uniquely determined by

$$\rho_r(\mathbf{x_r});\ r = 1, 2, ..., p \tag{8.2.8}$$

where $\rho_r(\mathbf{x_r})$ is the marginal scalar reversed hazard rate of $\mathbf{X_r} = (X_1, X_2, ..., X_r)'$.

Proof. Since $\rho_1(x_1)$ is the reversed hazard rate of X_1, it is known that

$$F_1(x_1) = \prod_{t=x_1+1}^{b_1} (1 - \rho_1(t)), \tag{8.2.9}$$

where $\mathbf{F}_r(\mathbf{x_r})$ denotes the distribution function of $\mathbf{X_r},\ r = 1, 2, ..., p$

For the bivariate random vector $(X_1, X_2)'$ with probability mass function $\mathbf{f}_2(x_1, x_2)$ and distribution function $\mathbf{F}_2(x_1, x_2)$, (8.2.1) gives

$$\rho_2(x_1, x_2)\mathbf{F}_2(x_1, x_2) = \mathbf{f}_2(x_1, x_2).$$

Thus,

$$P[X_1 = x_1, X_2 \leq x_2 - 1] = P[X_1 = x_1, X_2 \leq x_2] - \rho_2(x_1, x_2)\mathbf{F}_2(x_1, x_2). \tag{8.2.10}$$

Taking summation from 0 to x_1 with respect to X_1 in (8.2.10) yields

$$\mathbf{F}_2(x_1, x_2 - 1) = \mathbf{F}_2(x_1, x_2) - \sum_{t=0}^{x_1} \rho_2(t, x_2)\mathbf{F}_2(t, x_2). \tag{8.2.11}$$

Setting $x_2 = b_2$ in (8.2.11), we get

$$\mathbf{F}_2(x_1, b_2 - 1) = F_1(x_1) - \sum_{t=0}^{x_1} \rho_2(t, b_2) F_1(t). \tag{8.2.12}$$

We can find $\mathbf{F}_2(x_1, x_2)$ on using the recurrence relation (8.2.11) with (8.2.12) as the starting value, since $F_1(x_1)$ is known from (8.2.9). Similarly we obtain

$$\mathbf{F}_3(x_1, x_2, x_3 - 1) = \mathbf{F}_3(x_1, x_2, x_3) - \sum_{t_1=0}^{x_1} \sum_{t_2=0}^{x_2} \rho_3(t_1, t_2, x_3) \mathbf{F}_3(t_1, t_2, x_3). \tag{8.2.13}$$

Now work with $x_3 = b_3$ and determine $\mathbf{F}_3(\mathbf{x_3})$ from (8.2.13). Finally, we have to use recursively

$$\mathbf{F}_p(\mathbf{x_{p-1}}, x_p - 1) = \mathbf{F}_p(\mathbf{x_p}) - \sum_{t_1=0}^{x_1} \sum_{t_2=0}^{x_2} ... \sum_{t_{p-1}=0}^{x_{p-1}} \rho_p(t_1, ..., t_{p-1}, x_p) \mathbf{F}_p(t_1, ..., t_{p-1}, x_p) \tag{8.2.14}$$

in the above manner to reach $\mathbf{F}_p(\mathbf{x_p})$. ■

Remark 8.2.1. The above theorem may not work for random vectors with unbounded support, since in that case $\rho_r(\mathbf{x_r})$ may happen to be zero. See for example the bivariate geometric distribution

$$\mathbf{F}(x_1, x_2) = \left(1 - {q_1}^{x_1+1}\right)\left(1 - {q_2}^{x_2+1}\right);\ x_1, x_2 \in \mathbf{N};\ 0 < q_1, q_2 < 1. \tag{8.2.15}$$

Remark 8.2.2. The forms of the marginal distributions have to be known in computing $\mathbf{F}(\mathbf{x})$ since the form of $\rho_1(x_1)$ is assumed.

Remark 8.2.3. Since the recursive method does not apply in the continuous case, there is no similar result in the continuous case.

Using the above theorem, we propose a characterization for the multivariate reversed geometric distribution with independent marginals.

Theorem 8.2.2. A random vector $\mathbf{X},\ 0 \le X_i \le b_i;\ 0 < b_i < \infty;\ i = 1, 2, ..., p$ satisfies the property

$$\rho_r(\mathbf{x_r}) = \prod_{j=1}^{r} I(a_{ij}),\ r = 1, 2, ..., p \tag{8.2.16}$$

if and only if $\mathbf{X}$ is distributed as multivariate reversed geometric with independent marginals specified by

$$\mathbf{F}(\mathbf{x}) = \prod_{i=1}^{p}(1+c_i)^{x_i-b_i}; x_i = 0,1,2,...,b_i;\ b_i, c_i > 0, i = 1,2,...,p \tag{8.2.17}$$

where

$$I(a_{ij}) = \begin{cases} \dfrac{c_j}{1+c_j} & : \text{ if } x_j \neq 0 \\ 1 & : \text{ if } x_j = 0 \end{cases}$$

and c_j's are constants independent of $\mathbf{x}$.

Proof. Assuming (8.2.17), we have

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \prod_{i=1}^{p} c_i(1+c_i)^{x_i-b_i-1} & : x_i \neq 0 \\ \prod_{r=1}^{p-1} c_{i_r}(1+c_{i_r})^{x_{i_r}-b_{i_r}-1} \prod_{j=1}^{p-r}(1+c_{i_j})^{-b_{i_j}} & : x_{i_j} = 0 \\ \prod_{i=1}^{p}(1+c_i)^{-b_i} & : \mathbf{x} = \mathbf{0}. \end{cases} \tag{8.2.18}$$

where $(i_1, i_2, ..., i_p)$ are permutations of the integers $(1,2,...,p)$. From (8.2.17) and (8.2.18), we have (8.2.16). Conversely, when (8.2.16) holds, we have $\rho_1(x_1) = \dfrac{c_1}{1+c_1}$, from which we get

$$F_1(x_1) = (1+c_1)^{x_1-b_1}.$$

Now using (8.2.11), we get

$$\begin{aligned} \mathbf{F}_2(x_1, b_2-1) =& (1+c_1)^{x_1-b_1} - \left\{ \frac{c_2}{1+c_2}(1+c_1)^{-b_1} + \sum_{t=1}^{x_1} \frac{c_1 c_2}{(1+c_1)(1+c_2)}(1+c_1)^{t-b_1} \right\} \\ =& (1+c_1)^{x_1-b_1}(1+c_2)^{-1} \end{aligned}$$

and recursively

$$\mathbf{F}_2(x_1, b_2-m) = (1+c_1)^{x_1-b_1}(1+c_2)^{-m}. \tag{8.2.19}$$

Thus,

$$\mathbf{F}_2(x_1, x_2) = (1+c_1)^{x_1-b_1}(1+c_2)^{x_2-b_2}$$

showing that (8.2.16) holds for $p = 2$. Assuming the result is true for $p = r$, positive integer,

$$\mathbf{F}_r(x_1, ..., x_{r-1}, x_r - 1) = \mathbf{F}_r(\mathbf{x_r}) - \frac{c_1}{1+c_1} \cdots \frac{c_r}{1+c_r} \sum \cdots \sum \mathbf{F}(\mathbf{x_{r-1}}, x_r). \quad (8.2.20)$$

Setting $x_r = b_r - 1$,

$$\mathbf{F}_t(x_1, ..., x_{r-1}, b_r - 1) = \mathbf{F}_{r-1}(\mathbf{x_{r-1}}) \left(\frac{c_r}{1+c_r} \right)$$

and hence

$$\mathbf{F}_r(x_1, .., x_{r-1}, b_r - m) = \mathbf{F}_{r-1}(\mathbf{x_{r-1}}) \left(\frac{c_r}{1+c_r} \right)^{-m}$$

giving

$$\mathbf{F}_r(\mathbf{x_r}) = (1+c_1)^{x_1 - b_1} ... (1+c_r)^{x_r - b_r}.$$

The result now follows by induction. ■

Remark 8.2.4. When the conditions of Theorem 8.2.2 are satisfied, the univariate reversed hazard rate of X_i is

$$\lambda_i(x_i) = \frac{c_i}{1+c_i};\ i = 1, 2, ..., p.$$

For a comparison of the scalar reversed hazard rate with scalar hazard rate in the multivariate case, see Nair and Sankaran [102].

Remark 8.2.5. When the variables in $\mathbf{X}$ are unbounded, there does not exist a multivariate distribution with constant scalar reversed hazard rate $\rho_r(\mathbf{x_r})$ as in the above theorem. This follows from the fact that there is no univariate distribution with support $\mathbf{N}$ and constant reversed hazard rate. Also, there is no similar result in the continuous case as there is no constant reversed hazard rate on the positive real line.

8.3 Vector reversed hazard rate

The multivariate reversed hazard rate can also be viewed as a vector quantity and is defined as

$$\boldsymbol{\delta}(\mathbf{x}) = (\delta_1(\mathbf{x}), \delta_2(\mathbf{x}), ..., \delta_p(\mathbf{x}))', \quad (8.3.1)$$

where

$$\delta_i(\mathbf{x}) = P[X_i = x_i | \mathbf{X} \leq \mathbf{x}], i = 1, 2, ..., p.$$

The vector (8.3.1) means that the i th component of the multivariate reversed hazard rate is the conditional probability that component i of a p- component device fails at age x_i when it is known that the device cannot survive beyond $\mathbf{x}$.

Since

$$P[X_i = x_i] = \mathbf{F}(\mathbf{x}) - \mathbf{F}(x_1, x_2, ..., x_{i-1}, x_i - 1, x_{i+1}, ..., x_p)$$

we write (8.3.1) as

$$\delta_i(\mathbf{x}) = 1 - \frac{\mathbf{F}(\mathbf{x_{i-1}}, x_i - 1, \mathbf{x_i^*})}{\mathbf{F}(\mathbf{x})} \tag{8.3.2}$$

where $\mathbf{x_i} = (x_1, x_2, ..., x_i)'$ and $\mathbf{x_i^*} = (x_{i+1}, ..., x_p)'$, $i = 1, 2, ..., p-1$. In the bivariate case

$$\boldsymbol{\delta}(x_1, x_2) = (\delta_1(x_1, x_2), \delta_2(x_1, x_2))'$$

with

$$\delta_1(x_1, x_2) = 1 - \frac{\mathbf{F}(x_1 - 1, x_2)}{\mathbf{F}(x_1, x_2)} \tag{8.3.3}$$

and

$$\delta_2(x_1, x_2) = 1 - \frac{\mathbf{F}(x_1, x_2 - 1)}{\mathbf{F}(x_1, x_2)} \tag{8.3.4}$$

We have from (8.3.2),

$$\begin{aligned} \mathbf{F}(\mathbf{x_{i-1}}, x_i - 1, \mathbf{x_i^*}) &= (1 - \delta_i(\mathbf{x}))\mathbf{F}(\mathbf{x}) \\ &= (1 - \delta_i(\mathbf{x}))\left[(1 - \delta_i(\mathbf{x_{i-1}}, x_i + 1, \mathbf{x_i^*}))\mathbf{F}(\mathbf{x_{i-1}}, x_i + 1, \mathbf{x}_i^*)\right] \\ &= (1 - \delta_i(\mathbf{x})) \cdots (1 - \delta_i(\mathbf{x_{i-1}}, \infty, \mathbf{x_i^*}))\mathbf{F}_{p-1}(\mathbf{x_{i-1}}, \mathbf{x_i^*}) \end{aligned} \tag{8.3.5}$$

The distribution function on the right hand side is $(p-1)$ dimensional. Successive reduction can be achieved in the same manner, reaching finally at a one-dimensional marginal distribution function. Thus, when the limit of $\delta_i(\mathbf{x})$ as each of the arguments tends to infinity is finite, one can arrive at the expression for $\mathbf{F}(\mathbf{x})$ in-terms of $\boldsymbol{\delta}(\mathbf{x})$. Thus,

$$\mathbf{F}(\mathbf{x}) = \prod_{t_1 = x_1 + 1}^{\infty} (1 - \delta_1(t_1, x_2, ..., x_p)) \prod_{t_2 = x_2 + 1}^{\infty} (1 - \delta_2(\infty, t_2, x_3, ..., x_p)) \cdots$$

$$\prod_{t_p=x_p+1}^{\infty} (1-\delta_p(\infty,\infty,...,t_p)). \quad (8.3.6)$$

In view of (8.3.6) we have the following theorem.

Theorem 8.3.1. The distribution of $\mathbf{X}$ is uniquely determined by $\boldsymbol{\delta}(\mathbf{x})$.

Remark 8.3.1. Since we can begin with any one of the $\delta_i(\mathbf{x})$ in the reduction process indicated above, there are $p!$ different forms in which $\mathbf{F}(\mathbf{x})$ can be written in-terms of $\boldsymbol{\delta}(\mathbf{x})$.

Remark 8.3.2. The bivariate form of (8.3.6), which is quite useful in theoretical work, is given by

$$\mathbf{F}(x_1,x_2) = \prod_{t_1=x_1+1}^{\infty}\prod_{t_2=x_2+1}^{\infty} (1-\delta_1(t_1,x_2))(1-\delta_2(\infty,t_2)) \quad (8.3.7)$$

$$= \prod_{t_1=x_1+1}^{\infty}\prod_{t_2=x_2+1}^{\infty} (1-\delta_2(x_1,t_2))(1-\delta_1(t_1,\infty)), x_1,x_2=0,1,2,... \quad (8.3.8)$$

Example 8.3.1. Consider the bivariate discrete uniform distribution of Example 8.2.1. Then,

$$\delta_1(x_1,x_2) = 1 - \frac{(x_1-1)x_2\left[1+\theta\left(1-\frac{x_1-1}{b_1}\right)\left(1-\frac{x_2}{b_2}\right)\right]}{x_1x_2\left[1+\theta\left(1-\frac{x_1}{b_1}\right)\left(1-\frac{x_2}{b_2}\right)\right]}$$

$$= \frac{1+\theta\left(1-\frac{x_2}{b_2}\right)\left(1-\frac{2x_1-1}{b_1}\right)}{x_1\left[1+\theta\left(1-\frac{x_1}{b_1}\right)\left(1-\frac{x_2}{b_2}\right)\right]}. \quad (8.3.9)$$

Similarly, $\delta_2(x_1,x_2)$ is obtained as,

$$\delta_2(x_1,x_2) = \frac{1+\theta\left(1-\frac{x_1}{b_1}\right)\left(1-\frac{2x_2-1}{b_2}\right)}{x_2\left[1+\theta\left(1-\frac{x_1}{b_1}\right)\left(1-\frac{x_2}{b_2}\right)\right]}. \quad (8.3.10)$$

When prescribing models with specified functional forms of $\delta_1(\mathbf{x})$ and $\delta_2(\mathbf{x})$, it may be noted that they cannot be chosen arbitrarily. There exists a consistency condition for the hypothesized bivariate reversed hazard rate. We have

$$\rho(x_1, x_2) = \frac{\mathbf{f}(x_1, x_2)}{\mathbf{F}(x_1, x_2)} = \frac{\mathbf{F}(x_1, x_2) - \mathbf{F}(x_1, x_2 - 1) - \mathbf{F}(x_1 - 1, x_2) + \mathbf{F}(x_1 - 1, x_2 - 1)}{\mathbf{F}(x_1, x_2)}. \tag{8.3.11}$$

Writing

$$\begin{aligned}\frac{\mathbf{F}(x_1 - 1, x_2 - 1)}{\mathbf{F}(x_1, x_2)} &= \frac{\mathbf{F}(x_1 - 1, x_2 - 1)}{\mathbf{F}(x_1, x_2 - 1)} \frac{\mathbf{F}(x_1, x_2 - 1)}{\mathbf{F}(x_1, x_2)} \\ &= (1 - \delta_1(x_1, x_2 - 1)) \, (1 - \delta_2(x_1, x_2)) \, .\end{aligned}$$

Substituting the above in (8.3.11) , we get

$$\rho(x_1, x_2) = \delta_1(x_1, x_2) - \delta_1(x_1, x_2 - 1) + \delta_1(x_1, x_2 - 1)\delta_2(x_1, x_2). \tag{8.3.12}$$

Similarly, from

$$\frac{\mathbf{F}(x_1 - 1, x_2 - 1)}{\mathbf{F}(x_1, x_2)} = \frac{\mathbf{F}(x_1 - 1, x_2 - 1)}{\mathbf{F}(x_1 - 1, x_2)} \frac{\mathbf{F}(x_1 - 1, x_2)}{\mathbf{F}(x_1, x_2)},$$

$$\rho(x_1, x_2) = \delta_2(x_1, x_2) - \delta_2(x_1 - 1, x_2) + \delta_1(x_1, x_2)\delta_2(x_1 - 1, x_2). \tag{8.3.13}$$

Note that (8.3.12) or (8.3.13) is an identity connecting scalar and vector reversed hazard rates. Thus, the consistency condition is

$$\begin{aligned}\delta_1(x_1, x_2) - \delta_1(x_1, x_2 - 1) + \delta_1(x_1, x_2 - 1)\delta_2(x_1, x_2) &= \delta_2(x_1, x_2) - \delta_2(x_1 - 1, x_2) \\ &\quad + \delta_1(x_1, x_2)\delta_2(x_1 - 1, x_2).\end{aligned}$$

Remark 8.3.3. Unlike scalar reversed hazard rate, the vector reversed hazard rate provides the univariate reversed hazard rates as particular cases. That is

$$\lambda_i(x_i) = \delta_i(\infty, \infty, ..., x_i, \infty, ..., \infty), \; i = 1, 2, ..., p. \tag{8.3.14}$$

Remark 8.3.4. Being conditional probabilities, $0 \leq \delta_i(\mathbf{x}) \leq 1$ and hence unlike the multivariate hazard functions in continuous time, the range of $\delta_i(\mathbf{x})$ is limited to the unit intervals, whereas in the continuous case there is no such limitation to the values of the hazard rate components.

Theorem 8.3.2. The multivariate vector reversed hazard rate is of the form

$$\boldsymbol{\delta}(\mathbf{x}) = \left(\frac{c_1}{1+c_1}, \frac{c_2}{1+c_2}, ..., \frac{c_p}{1+c_p}\right)', \ c_i > 0, i = 1, 2, ..., p \tag{8.3.15}$$

if and only if the distribution of $\mathbf{X}$ is multivariate reversed geometric in (8.2.17).

The proof is direct. A more general result in this connection is presented in the next theorem, which is a direct consequence of (8.3.2).

Theorem 8.3.3. The random variables $X_1, X_2, ..., X_p$ are independent if and only if

$$\boldsymbol{\delta}(\mathbf{x}) = (\lambda_1(x_1), \lambda_2(x_2), ..., \lambda_p(x_p))'.$$

A bivariate distribution of interest can be obtained if we assume that the reversed hazard rate is locally constant.

Theorem 8.3.4. The bivariate vector reversed hazard rate is of the form $\boldsymbol{\delta}(\mathbf{x}) = (\alpha_1(x_2), \alpha_2(x_1))'$, if and only if the distribution is specified by

$$\mathbf{F}(x_1, x_2) = (1-\alpha_1)^{b_1-x_1}(1-\alpha_2)^{b_2-x_2}k_1^{(b_1-x_1)(b_2-x_2)}, \ x_i = 0, 1, 2, ..., b_i \tag{8.3.16}$$

where $\alpha_1 = \alpha_1(b_1), \ \alpha_2 = \alpha_2(b_2)$ and $\left[1-(1-\alpha_1)k_1^{b_1}\right]\left[1-(1-\alpha_2)k_1^{b_2}\right] + k_1 \geq 0, \ 0 < k_1 \leq 1; \ 0 < \alpha_1, \alpha_2 < 1$.

Proof. Under the given conditions, (8.3.7) and (8.3.8) lead to

$$\mathbf{F}(x_1, x_2) = (1-\alpha_1(x_2))^{b_1-x_1}(1-\alpha_2)^{b_2-x_2} \tag{8.3.17}$$

$$= (1-\alpha_1)^{b_1-x_1}(1-\alpha_2(x_1))^{b_2-x_2}. \tag{8.3.18}$$

This implies

$$\left(\frac{1-\alpha_1(x_2)}{1-\alpha_1}\right)^{\frac{1}{b_2-x_2}} = \left(\frac{1-\alpha_2(x_1)}{1-\alpha_1}\right)^{\frac{1}{b_1-x_1}} \tag{8.3.19}$$

for all $x_i = 0, 1, 2, ..., b_i; i = 1, 2$. Equation (8.3.19) means that each of the expressions in (8.3.17) and (8.3.18) is a constant, say k_1. Hence

$$(1-\alpha_1(x_2)) = (1-\alpha_1)\,k_1^{b_2-x_2}.$$

Inserting the above value in (8.3.17), we have (8.3.16). Conversely assuming (8.3.16), we obtain

$$\delta_1(x_1, x_2) = 1 - \frac{\mathbf{F}(x_1 - 1, x_2)}{\mathbf{F}(x_1, x_2)} = 1 - (1 - \alpha_1)k_1^{b_2 - x_2}$$

and similarly

$$\delta_2(x_1, x_2) = 1 - (1 - \alpha_2)k_1^{b_1 - x_1}$$

Thus, $\boldsymbol{\delta}(x_1, x_2)$ is of the given form, which completes the proof. ■

A concept of relevance in this context is the multivariate reversed lack of memory property, which is an extension of (1.1.24). Since this is transparent from the bivariate case, we only discuss the bivariate property and its implications.

Definition 8.3.1. (Rejeesh [121]) A discrete bivariate random vector $\mathbf{X} = (X_1, X_2)'$ with support $0 \leq X_i \leq b_i; b_i < \infty;\ i = 1, 2$ is said to possess the bivariate reversed lack of memory property if

$$P[X_1 \leq x_1, X_2 \leq x_2 | X_1 \leq x_1 + t_1, X_2 \leq x_2 + t_2] \\ = P[X_1 \leq 0, X_2 \leq 0 | X_1 \leq t_1, X_2 \leq t_2] \quad (8.3.20)$$

for all $(x_1, x_2)'$ and $(t_1, t_2)'$ in the set $\mathcal{A} = \{(y_1, y_2)' | y_i = 0, 1, 2, ..., b_i; i = 1, 2\}$.

An equivalent condition for (8.3.20) is

$$\mathbf{F}(x_1 + t_1, x_2 + t_2)\mathbf{F}(0, 0) = \mathbf{F}(x_1, x_2)\mathbf{F}(t_1, t_2). \quad (8.3.21)$$

We see that for left censored data, (8.3.20) is the analogue of the lack of memory property (LMP). The physical interpretation of (8.3.20) is that if $\mathbf{x}+\mathbf{t} = (x_1+t_1, x_2+t_2)'$ represents the number of cycles of operations of two components of a device before they fail, then the right hand side represents the probability that a new equipment with two components fails before it completes the first cycle given that the components fail before it completes $\mathbf{t} = (t_1, t_2)'$ cycles. Thus, the expected time elapsed since failure is independent of the age of the components whenever the reversed lack of memory property is satisfied.

Theorem 8.3.5. (Rejeesh [121]) The random vector $\mathbf{X} = (X_1, X_2)'$ satisfies bivariate reversed lack of memory property if and only if $\mathbf{X}$ is distributed as the bivariate reversed geometric distribution with

$$\mathbf{F}(\mathbf{x}) = (1+c_1)^{x_1-b_1}(1+c_2)^{x_2-b_2};\ x_i = 0,1,2,...,b_i; b_i, c_i > 0;\ i = 1,2. \qquad (8.3.22)$$

The reversed lack of memory property 8.3.20 holds only for independent random variables, which is much restrictive. A weaker version is stated below which could be useful in certain modelling problems.

Definition 8.3.2. A discrete random vector $(X_1, X_2)'$ with support in the set $\mathcal{A}$ is said to have bivariate local reversed lack of memory property if it satisfies

$$P[X_i \le x_i | X_i \le x_i + t_i, X_j \le x_j] = P[X_i \le 0 | X_i \le t_i, X_j \le x_j] \qquad (8.3.23)$$

for all x_i, x_j, t_i and $x_i + t_i$ in the set $\mathcal{A}$. Equivalent conditions are given by

$$\mathbf{F}(x_1 + t_1, x_2)\mathbf{F}(0, x_2) = \mathbf{F}(x_1, x_2)\mathbf{F}(t_1, x_2) \qquad (8.3.24)$$

and

$$\mathbf{F}(x_1, x_2 + t_2)\mathbf{F}(x_1, 0) = \mathbf{F}(x_1, x_2)\mathbf{F}(x_1, t_2). \qquad (8.3.25)$$

The physical interpretation of (8.3.23) is similar to that of the univariate reversed lack of memory property, for a system with two components, based on the conditional distribution of X_i given $X_j \le x_j;\ i,j = 1,2;\ i \ne j$.

Theorem 8.3.6. The following statements are equivalent;

(i) $(X_1, X_2)'$ has local reversed lack of memory property,

(ii) $(X_1, X_2)'$ is distributed as (8.3.16),

(iii) The bivariate vector reversed hazard rate is of the form $(\alpha_1(x_2), \alpha_2(x_1))'$ where $\alpha_i(x_j)$ is independent of x_i; $i,j = 1,2;\ i \ne j$.

Proof. The equivalence of (ii) and (iii) follows directly from Theorem 8.3.4. For proving the equivalence of (i) and (iii), we put $t_1 = 1$ in (8.3.24). Thus, we obtain

$$\frac{\mathbf{F}(x_1+1, x_2)}{\mathbf{F}(x_1, x_2)} = \frac{\mathbf{F}(1, x_2)}{\mathbf{F}(0, x_2)},$$

which implies that

$$\delta_1(x_1+1, x_2) = \delta_1(1, x_2),$$

for all x_1 and x_2 in the support of $(X_1, X_2)'$. Hence $\delta_1(x_1, x_2)$ must be independent of x_1. Similarly by putting $t_2 = 1$ in (8.3.25), we can show that $\delta_2(x_1, x_2)$ is independent of x_2. Thus, we see that (i) implies (ii) . Conversely, when (iii) holds, using (8.3.7) and (8.3.8), the joint distribution can be written as

$$\mathbf{F}(x_1, x_2) = (1-\alpha_1(x_2))^{(b_1-x_1)}(1-\alpha_2(b_1))^{(b_2-x_2)} \tag{8.3.26}$$

$$= (1-\alpha_2(x_1))^{(b_2-x_2)}(1-\alpha_1(b_2))^{(b_1-x_1)} \tag{8.3.27}$$

Thus, $\mathbf{F}(x_1, x_2)$ will satisfy both (8.3.24) and (8.3.25). Hence (i) and (iii) are equivalent. ■

Remark 8.3.5. For a comparison of the vector reversed hazard rate with vector hazard rate, see Nair and Asha [95].

8.4 Alternative reversed hazard rate

Reasons, similar for introducing the alternative reversed hazard rate mentioned in Chapter 1, in the univariate case, also hold in the multivariate case. The univariate alternative reversed hazard rate (1.1.22) allows extension to the multivariate case. In the bivariate case, the alternative reversed hazard rate (Rejeesh [121]) is defined as,

$$\boldsymbol{\delta}^*(\mathbf{x}) = (\delta_1^*(\mathbf{x}), \delta_2^*(\mathbf{x}))'$$

where

$$\delta_1^*(x_1, x_2) = \log\frac{\mathbf{F}(x_1, x_2)}{\mathbf{F}(x_1-1, x_2)} \tag{8.4.1}$$

and

$$\delta_2^*(x_1, x_2) = \log\frac{\mathbf{F}(x_1, x_2)}{\mathbf{F}(x_1, x_2-1)}. \tag{8.4.2}$$

There is no physical interpretation to $\delta_1^*(.)$ and $\delta_2^*(.)$. However, like the univariate reversed hazard rate, the alternative reversed hazard rate possesses additivity for parallel systems. Moreover, using this definition we can determine the properties of vector reversed hazard rate also, as the components of the alternative reversed hazard rate are related to $(\delta_1(\mathbf{x}), \delta_2(\mathbf{x}))'$ through

$$\delta_1(\mathbf{x}) = 1 - \exp\{-\delta_1^*(\mathbf{x})\} \text{ and } \delta_2(\mathbf{x}) = 1 - \exp\{-\delta_2^*(\mathbf{x})\}. \tag{8.4.3}$$

Theorem 8.4.1. The alternative reversed hazard rate function is of the form

$$\boldsymbol{\delta}^*(\mathbf{x}) = (\log(1 + c_1), \log(1 + c_2), ..., \log(1 + c_p))', \; c_i > 0, i = 1, 2, ..., p \tag{8.4.4}$$

if and only if the distribution of $\mathbf{X}$ is multivariate reversed geometric in (8.2.17).

The proof follows from (8.4.3) and Theorem 8.3.2.

There has been attempts from early days in multivariate distribution theory to define bivariate distributions that can be expressed in-terms of the marginals,(see Morgenstern [92], Cambanis [28], etc). We derive a new family of bivariate discrete distributions using the properties of $(\delta_1^*(\mathbf{x}), \delta_2^*(\mathbf{x}))'$.

Theorem 8.4.2. The alternative reversed hazard function is of the form

$$(\delta_1^*(\mathbf{x}), \delta_2^*(\mathbf{x}))' = (\lambda_1^*(x_1)c_1(x_2), \lambda_2^*(x_2)c_2(x_1))'$$

where $c_i(x_j)$ is a non-negative function of x_j only, $i = 1, 2;\ i \neq j$ and $\lambda_i^*(x_i)$ is the univariate alternative reversed hazard rate of $X_i,\ i = 1, 2$ if and only if

$$\mathbf{F}(x_1, x_2) = F_1(x_1)\,[F_2(x_2)]^{1+k\log F_1(x_1)};\ 0 \leq k \leq 1;\ x_1, x_2 = 0, 1, 2, ... \tag{8.4.5}$$

or

$$\mathbf{F}(x_1, x_2) = F_2(x_2)\,[F_1(x_1)]^{1+k\log F_2(x_2)};\ 0 \leq k \leq 1;\ x_1, x_2 = 0, 1, 2, ... \tag{8.4.6}$$

Proof. The alternative reversed hazard rate of (8.4.5) is obtained by employing (8.4.1) as

$$\delta_1^*(x_1, x_2) = \lambda_1^*(x_1)[1 + k\log F_2(x_2)] \tag{8.4.7}$$

and

$$\delta_2^*(x_1, x_2) = \lambda_2^*(x_2)[1 + k \log F_1(x_1)], \tag{8.4.8}$$

which are of the required form in the theorem. On the other hand, when $\delta_1^*(\mathbf{x})$ and $\delta_2^*(\mathbf{x})$ are of the stated form, we use (8.4.3), (8.3.7) and (8.3.8) to arrive at the functional equation

$$[F_1(x_1)]^{c_1(x_2)-1} = [F_2(x_2)]^{c_2(x_1)-1}$$

or

$$[F_1(x_1)]^{\frac{1}{c_2(x_1)-1}} = [F_2(x_2)]^{\frac{1}{c_1(x_2)-1}}. \tag{8.4.9}$$

The solution of (8.4.9) is

$$[F_i(x_i)]^{\frac{1}{c_{3-i}(x_i)-1}} = c, \text{ a constant}, i = 1, 2$$

or

$$c_{3-i}(x_i) = 1 + k \log F_i(x_i), \; k = (\log c)^{-1}.$$

This leads to

$$\begin{aligned}\mathbf{F}(x_1, x_2) &= [F_1(x_1)]^{c_1(x_2)} F_2(x_2) \\ &= F_2(x_2) \, [F_1(x_1)]^{1+k \log F_2(x_2)},\end{aligned}$$

which is same as (8.4.5). ■

Remark 8.4.1. Equation (8.4.5) defines a family of bivariate discrete distributions with marginals $F_1(x_1)$ and $F_2(x_2)$. Thus, substituting appropriate marginal distributions, we can realize the bivariate model. Alternatively, if one knows the forms of the marginal alternative reversed hazard rates $\lambda_1^*(x_1)$ and $\lambda_2^*(x_2)$, $\mathbf{F}(x_1, x_2)$ is determined. In modelling problems, it is not difficult to capture the marginal distributions or the marginal reversed hazard rate which makes the identification of the bivariate model easier.

Example 8.4.1. Let X_i, $i = 1, 2$ be discrete random variables with distribution functions

$$F_i(x_i) = (1 + c_i)^{x_i - b_1}; x_i = 0, 1, 2, ..., b_i; \; i = 1, 2. \tag{8.4.10}$$

Then using (8.4.5) we can obtain the bivariate distribution function as

$$\mathbf{F}(x_1, x_2) = (1 + c_1)^{x_1 - b_1} \left[(1 + c_2)^{x_2 - b_2}\right]^{1 + k(x_1 - b_1)\log(1 + c_1)}; x_i = 0, 1, 2, ..., b_i; i = 1, 2, \tag{8.4.11}$$

which is a new family of bivariate reversed time distribution.

8.5 Conditional reversed hazard rate

A fourth definition of multivariate reversed hazard rate is based on the conditional distributions. In the bivariate case, the conditional reversed hazard rate is defined as the vector

$$\xi(\mathbf{x}) = (\xi_1(\mathbf{x}), \xi_2(\mathbf{x}))', \tag{8.5.1}$$

where

$$\xi_1(\mathbf{x}) = \frac{P(X_1 = x_1 | X_2 = x_2)}{P(X_1 \leq x_1 | X_2 = x_2)} \text{ and } \xi_2(\mathbf{x}) = \frac{P(X_2 = x_2 | X_1 = x_1)}{P(X_2 \leq x_2 | X_1 = x_1)}. \tag{8.5.2}$$

The interpretation is similar to the univariate reversed hazard rate with the change that it is conditioned on the event $X_2 = x_2(X_1 = x_1)$ in the case of $\xi_1(\mathbf{x})$ $(\xi_2(\mathbf{x}))$. Thus, $\xi_1(\mathbf{x})$ is the probability that a two-component device with failure times X_1 and X_2, the first component fails at time x_1 when the lifetime of the second component is x_2. $\xi_2(\mathbf{x})$ can be similarly interpreted.

Example 8.5.1. Consider the bivariate discrete uniform distribution given in Example 8.2.1.
The conditional hazard rate vector is

$$(\xi_1(\mathbf{x}), \xi_2(\mathbf{x}))' = \frac{\left[1 + \theta\left(1 - \frac{2x_1 - 1}{b_1}\right)\left(1 - \frac{2x_2 - 1}{b_2}\right)\right]}{1 + \theta\left(1 - \frac{x_1}{b_1}\right)\left(1 - \frac{x_2}{b_2}\right)} \left(\frac{1}{x_1}, \frac{1}{x_2}\right)', \tag{8.5.3}$$

$x_i = 0, 1, 2, ..., b_i;\ i = 1, 2.$

Working as in the univariate case, we have the conditional distributions

$$P[X_1 \leq x_1 | X_2 = x_2] = \prod_{t=x_1+1}^{\infty} (1 - \xi_1(t, x_2)) \tag{8.5.4}$$

and

$$P[X_2 \leq x_2 | X_1 = x_1] = \prod_{t=x_2+1}^{\infty} (1 - \xi_2(x_1, t)). \tag{8.5.5}$$

The last two equations determine the joint distribution function of $(X_1, X_2)'$ provided that

$$\frac{P[X_1 \leq x_1 | X_2 = x_2]}{P[X_2 \leq x_2 | X_1 = x_1]} = \frac{A_1(x_1)}{A_2(x_2)}, \tag{8.5.6}$$

where $A_1(.)$ and $A_2(.)$ are distribution functions. Notice that when X_1 and X_2 are independent, $\xi_i(x_1, x_2) = \lambda_i(x_i), i = 1, 2$.

We now define the conditional reversed lack of memory property as follows

Definition 8.5.1. A discrete bivariate random vector $\mathbf{X} = (X_1, X_2)'$ with support $0 \leq X_i \leq b_i; b_i < \infty;\ i = 1, 2$ is said to possess the conditional reversed lack of memory property if

$$P[X_i \leq x_i | X_i \leq x_i + t_i, X_j = x_j] = P[X_i \leq 0 | X_i \leq t_i, X_j = x_j], \tag{8.5.7}$$

for all $(x_i, x_j'),\ i, j = 1, 2;\ i \neq j$, in the set $\mathcal{A}$ and t_i in $\{0, 1, 2, ..., b_i\}$, $i = 1, 2$, such that $x_i + t_i \in \mathcal{A}$. (8.5.7) can be equivalently expressed as

$$P[X_i \leq x_i + t_i | X_j = x_j] P[X_i \leq 0 | X_j = x_j] =$$
$$P[X_i \leq x_i | X_j = x_j] P[X_i \leq t_i | X_j = x_j]; i, j = 1, 2; i \neq j. \tag{8.5.8}$$

Theorem 8.5.1. The following statements are equivalent;

(i) $(X_1, X_2)'$ have conditional reversed lack of memory property,

(ii) $(X_1, X_2)'$ have the joint probability mass function

$$\mathbf{f}(x_1, x_2) = k(1 + c_1)^{x_1 - b_1} (1 + c_2)^{x_2 - b_2} k_1^{(x_1 - b_1)(x_2 - b_2)}; x_i = 0, 1, 2, ..., b_i, \tag{8.5.9}$$

$i = 1, 2;\ 0 < k_1 \leq 1.$

(iii) The conditional reversed hazard rate is of the form $\xi(x_1, x_2) = (\xi_1(x_2), \xi_2(x_1))'$.

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (i). We first establish that (i) implies (ii). From (8.5.8), for $t_i = 1$ we get

$$P[X_i \leq x_i | X_j = x_j] = \frac{P[X_i \leq 0 | X_j = x_j]}{P[X_i \leq 1 | X_j = x_j]} P[X_i \leq x_i + 1 | X_j = x_j]; i, j = 1, 2; i \neq j. \tag{8.5.10}$$

Solving the recurrence relation (8.5.10) in x_i, we get

$$P[X_i \leq x_i | X_j = x_j] = p_j(x_j)^{b_i - x_i};\ i, j = 1, 2;\ i \neq j, \tag{8.5.11}$$

where $p_j(x_j)$ is a function of x_j only. From (8.5.11), the conditional probability mass functions are obtained as

$$P[X_i = x_i | X_j = x_j] = q_j(x_j) p_j(x_j)^{b_i - x_i};\ i, j = 1, 2;\ i \neq j, \tag{8.5.12}$$

with $q_j(x_j) = 1 - p_j(x_j);\ j = 1, 2$. Hence the joint probability mass function can be expressed in two equivalent expressions as

$$\mathbf{f}(x_1, x_2) = \begin{cases} q_2(x_2) p_2(x_2)^{b_1 - x_1} g_2(x_2) \\ q_1(x_1) p_1(x_1)^{b_2 - x_2} g_1(x_1) \end{cases}, \tag{8.5.13}$$

where $g_1(x_1)$ and $g_2(x_2)$ are the probability mass functions of X_1 and X_2 respectively.

Now replacing x_2 with x_2+1 in (8.5.13) and dividing the resulting equations by (8.5.13) we get

$$\mathbf{f}(x_1, x_2) = \begin{cases} \dfrac{q_2(x_2+1) p_2(x_2+1)^{b_1 - x_1} g_2(x_2+1)}{q_2(x_2) p_2(x_2)^{b_1 - x_1} g_2(x_2)} \\ \dfrac{1}{p_1(x_1)} \end{cases}. \tag{8.5.14}$$

From (8.5.14), we see that

$$\frac{p_1(x_1)}{p_1(x_1+1)} = \frac{p_2(x_2)}{p_2(x_2+1)} = k_1, \text{ say.} \tag{8.5.15}$$

We evaluate the functional forms of $p_1(.)$ and $p_2(.)$ from (8.5.15), as

$$p_i(x_i) = k_1^{b_i - x_i} p_i(b_i);\ i = 1, 2. \tag{8.5.16}$$

Substituting the functional forms of $p_1(.)$ and $p_2(.)$ in (8.5.13) and after simplifying, we have

$$\frac{\left(1 - k_1^{b_2 - x_2} p_2(b_2)\right) g_2(x_2)}{p_1(b_1)^{b_2 - x_2}} = \frac{\left(1 - k_1^{b_1 - x_1} p_1(b_1)\right) g_1(x_1)}{p_2(b_2)^{b_1 - x_1}}, \tag{8.5.17}$$

for all $(x_1, x_2)'$ in the support of $(X_1, X_2)'$. Hence $g_i(.); i = 1, 2$ must be of the form

$$g_i(x_i) = a_i \frac{p_j(b_j)^{b_i - x_i}}{\left(1 - k_1^{b_i - x_i} p_i(b_i)\right)};\ i, j = 1, 2;\ i \neq j, \tag{8.5.18}$$

where a_1 and a_2 are normalizing constants. Substituting (8.5.18) in (8.5.13), we obtain the joint distribution as

$$\mathbf{f}(x_1, x_2) = k p_2(b_2)^{b_1 - x_1} p_1(b_1)^{b_2 - x_2} k_1^{(b_1 - x_1)(b_2 - x_2)}, \tag{8.5.19}$$

where k is a constant. Putting $p_i(b_i) = (1 + c_i)^{-1}; i = 1, 2$ we have the expression in (8.5.9).

To prove that (ii) implies (iii), consider

$$\begin{aligned} P[X_i \leq x_i, X_j = x_j] &= k(1 + c_j)^{x_j - b_j} \sum_{t=0}^{x_i} (1 + c_i)^{t - b_i} k_1^{(t - b_i)(x_j - b_j)} \\ &= k(1 + c_j)^{(x_j - b_j)} \left((1 + c_i) k_1^{(x_j - b_j)}\right)^{x_i - b_i} \left(\frac{1 - \left((1 + c_i) k_1^{(x_j - b_j)}\right)^{-(x_i + 1)}}{1 - \left((1 + c_i) k_1^{(x_j - b_j)}\right)^{-1}} \right). \end{aligned} \tag{8.5.20}$$

Since $\sum_{t=0}^{b_i} g_i(t) = 1$, we have $(1 + c_i)^{-(b_i + 1)};\ i = 1, 2$. Thus, (8.5.20) reduces to

$$P[X_i \leq x_i, X_j = x_j] = \frac{k(1 + c_j)^{(x_j - b_j)} \left((1 + c_i) k_1^{(x_j - b_j)}\right)^{x_i - b_i}}{1 - \left((1 + c_i) k_1^{(x_j - b_j)}\right)^{-1}}; i, j = 1, 2; i \neq j, \tag{8.5.21}$$

From (8.5.19) and (8.5.21), we get

$$\xi_i(x_1,x_2) = \frac{\mathbf{f}(x_1,x_2)}{P[X_i \le x_i, X_j = x_j]} = 1 - \left((1+c_i)k_1^{x_j-b_j}\right)^{-1};\ i,j=1,2;\ i \ne j.$$

Now we prove (iii) $\Rightarrow$ (i). Suppose that the conditional reversed hazard rate takes the form $\xi(x_1,x_2) = (\xi_1(x_2), \xi_2(x_1))$. Then the corresponding conditional distributions can be determined from (8.5.4) and (8.5.5) as

$$P[X_i \le x_i | X_j = x_j] = (1-\xi_i(x_j))^{b_i - x_i};\ i,j=1,2;\ i \ne j. \tag{8.5.22}$$

From (8.5.22), we obtain

$$\begin{aligned} P[X_i \le x_i + t_i | X_j = x_j] P[X_i \le 0 | X_j = x_j] &= (1-\xi(x_j))^{b_i - x_i - t_i}(1-\xi(x_j))^{b_i} \\ &= (1-\xi(x_j))^{b_i - x_i}(1-\xi(x_j))^{b_i - t_i} \\ &= P[X_i \le x_i | X_j = x_j] P[X_i \le t_i | X_j = x_j], \end{aligned}$$

$i,j = 1,2; i \ne j$, which implies (i). ■

Remark 8.5.1. The concept of conditional reversed hazard rate does not appear to have been discussed in the continuous case.

8.6 Conclusion

In the present chapter, we have introduced four versions of the multivariate reversed hazard rates and studied their properties. We have determined the criterion under which the scalar reversed hazard rate uniquely determines the underlying distribution. It has been shown that unlike scalar reversed hazard rate, the vector reversed hazard rate uniquely determines the distribution. The reversed lack of memory property useful in maintenance problems has been extended to the multivariate case. The multivariate reversed geometric distribution has been characterized using these multivariate reversed ageing concepts. Section 8.4 provided a general class of distributions, which can be used in reliability analysis by choosing appropriate marginal distributions.

Chapter 9

Schur-Constant Models

9.1 Introduction

Let $(X,Y)'$ be a vector of non-negative random variables with absolutely continuous survival function $\bar{F}(x,y)$. Then the vector $(X,Y)'$ have a Schur-constant distribution if $\bar{F}(x,y)$ can be written as

$$\bar{F}(x,y) = P[X \geq x, Y \geq y] = G(x+y);\ x,y > 0, \tag{9.1.1}$$

where $G(.)$ is a convex survival function. Barlow and Mendel [14] characterized (9.1.1) in-terms of the bivariate no-ageing property

$$P[X > x+t|X > x, Y > y] = P[Y > y+t|X > x, Y > y] \tag{9.1.2}$$

which means that the residual lifetimes of younger and older components with the same survival history have the same distribution. In Bayesian reliability theory, (9.1.1) gives that regardless of the ages of the components, one would bet the same amount on the next increment in life of either component. This concept can also be explained in-terms of the majorization order. The properties of (9.1.1) have been studied by various researchers including Barlow and Mendel [15], Caramellino and Spizzichino [30, 31], Bassan and Spizzichino [18], Nelsen [116], Chi et al. [34] and Nair and Sankaran [100, 101] .

Recently, Castañer et al. [32] have studied properties of discrete Schur-constant models. They have calculated mean, variance and correlation coefficient of the Schur-constant family. Under independence of random variables, it has been shown that the components have geometric distribution. Multivariate Schur-constant models play the same role in Bayesian reliability as multivariate geometric distribution in the classical discrete reliability analysis.

The aim of the present chapter is to investigate various properties of discrete Schur-constant models. Specifically, we study ageing phenomenon of bivariate Schur-constant models using univariate ageing concepts. A criteria employed for choosing a particular bivariate(multivariate) model for a given data set is the dependence relationship among the variables. The scalar measures of association, time-dependent measures and dependence concepts are commonly used in such contexts. The scalar measures like correlation coefficient and Kendall's tau have been studied in discrete case by Castañer et al. [32]. In the present work, we study time-dependent measures of discrete Schur-constant models and it is shown that such measures can be related to univariate ageing concepts. We also discuss implications between time-dependent association measures and dependence concepts for discrete Schur-constant models.

The rest of the chapter is organized as follows. In Section 9.2, we discuss basic properties of discrete Schur-constant models. Section 9.3 presents reliability characteristics of these models. The ageing phenomenon of the models is studied. In Section 9.4, we study the dependence structure of the models. We establish the relationships between the association measures and ageing classes corresponding to discrete Schur-constant models. The chapter ends with a brief conclusion in Section 9.5.

9.2 Basic properties

Let Z be a lifetime random variable taking values in $\mathbf{N} = \{0, 1, 2, ...\}$ with convex survival function $S(x) = P[Z \geq x]$ and probability mass function $f(x) = P[Z = x]$. Let $\mathbf{X} = (X_1, X_2)'$ be a bivariate random vector taking values in the support $\mathbf{N}^2$. The joint survival function of $\mathbf{X}$ is denoted by

$$\mathbf{S}(x_1, x_2) = P[X_1 \geq x_1, X_2 \geq x_2] \tag{9.2.1}$$

and the joint probability mass function is denoted by $\mathbf{p}(x_1, x_2)$. Let the scalar hazard rate of $\mathbf{X}$ be denoted as $a(x_1, x_2)$ and the vector hazard rate as $(c_1(x_1, x_2), c_2(x_1, x_2))'$. The vector MRL function is denoted by $\mathbf{m}(\mathbf{x}) = (m_1(x_1, x_2), m_2(x_1, x_2))'$.

Definition 9.2.1. The random vector $\mathbf{X}$ is said to have a Schur-constant joint survival function $\mathbf{S}(\mathbf{x})$ if for all $(x_1, x_2)' \in \mathbf{N^2}$,

$$\mathbf{S}(\mathbf{x}) = P[X_1 \geq x_1, X_2 \geq x_2] = S(x_1 + x_2). \tag{9.2.2}$$

The convexity of $S(.)$ implies that

$$S(x+2) - 2S(x+1) + S(x) \geq 0. \tag{9.2.3}$$

The probability mass function of bivariate discrete Schur-constant model (BSM) is

$$\mathbf{p}(x_1, x_2) = f(x_1 + x_2) - f(x_1 + x_2 + 1). \tag{9.2.4}$$

Since $\mathbf{p}(.)$ must be non-negative, $f(.)$ has to be a decreasing function. Thus, in order to construct BSM, one should have a decreasing probability mass function. Some bivariate Schur-constant models that are useful in reliability analysis are presented in Table 9.1. Various properties of BSM are listed below.

(a) The distribution of the total lifetime $T = X_1 + X_2$ has the survival function

$$S_T(x) = S(x) - x\Delta S(x). \tag{9.2.5}$$

(b) The conditional distribution of $X_1 | X_1 + X_2$ is uniform over $[0, x_2]$.

(c) The joint distribution of $M = max(X_1, X_2)$ and $V = min(X_1, X_2)$ has survival function

$$\mathbf{S}_{M,V}(m, v) = 2S(m+v) - S(2m).$$

Thus, the probability that at least one of the components of the device with lifetime represented by $(X_1, X_2)'$ survives time x is $2S(x) - S(2x)$ and both survives beyond x is $S(2x)$. Hence the bivariate reliability concepts of the Schur-constant models that represent the two systems can be evaluated in-terms of those of the components' lives alone.

Table 9.1: Bivariate Schur-constant distributions

Baseline Distribution	Survival function $S(x)$	Schur-constant model $\mathbf{S}(x_1, x_2)$
Geometric	q^x	$q^{x_1+x_2}$
Waring[107]	$\frac{(m)_x}{(m+n)_x}$; $m, n > 0$	$\frac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}}$
Power[91]	$1 - \left(\frac{x}{\alpha}\right)^{\beta}$; $\beta < 0$	$1 - \left(\frac{x_1+x_2}{\alpha}\right)\beta$; $0 < \beta < 1$
Logarithmic[74]	$\frac{-\theta^{x+1}}{(x+1)\log(1-\theta)}$; $0 < \theta < 1$	$\frac{-\theta^{x_1+x_2}}{(1+x_1+x_2)\log(1-\theta)}$
Pareto[74]	$\frac{1}{(1+x)^{\theta+1}\xi(1+\theta)}$; $\theta > 0$	$\frac{1}{(1+x_1+x_2)^{\theta+1}\xi(1+\theta)}$
Weibull I[113]	q^{x^β}; $\beta > 0$	$q^{(x_1+x_2)^\beta}$; $0 < \beta \leq 1$
Gompertz[122]	$q^{-\theta(1-e^x)}$; $\theta > 0$	$q^{-\theta(1-e^{x_1+x_2})}$; $\theta > 1$
Half-logistic[13]	$\frac{2}{1+e^{\frac{x}{\sigma}}}$; $\sigma > 0$	$\frac{2}{1+e^{\frac{x_1+x_2}{\sigma}}}$
Lindley[50]	$\frac{e^{-\theta x}(1+\theta+\theta x)}{1+\theta}$; $\theta > 0$	$\frac{e^{-\theta(x_1+x_2)}(1+\theta+\theta(x_1+x_2))}{1+\theta}$
Geometric-Weibull[85]	$\exp\left[-\lambda x - \{\lambda(x-a)^+\}^\beta\right]$	$\exp\left[-\lambda(x_1+x_2) - \{\lambda(x_1+x_2-a)^+\}^\beta\right]$
	$(x-a)^+ = \max(0, x-a)$; $\lambda, \beta, a > 0$	
Generalized exponential[85]	$1 - \left(1 - e^{-\lambda x}\right)^\alpha$; $\alpha, \lambda > 0$	$1 - \left(1 - e^{-\lambda(x_1+x_2)}\right)^\alpha$

(d) We can express the conditional means and variances in-terms of reliability functions as given in the following theorem.

Theorem 9.2.1.
$$E[X_i|X_j = x_j] = \frac{1 - h(x_j)}{h(x_j)}$$

and

$$V[X_i|X_j = x_j] = \frac{1 - h(x_j)}{h^2(x_j)}(2h(x_j)m(x_j) - 1);\ i, j = 1, 2;\ i \neq j,$$

where $h(x_j)$ and $m(x_j)$, respectively, are the hazard rate and the MRL function corresponding to the marginal random variable Z.

Proof. We prove the result for $i = 1$.

$$\begin{aligned} E[X_1|X_2 = x_2] &= \sum_0^\infty x_1 \frac{(f(x_1 + x_2) - f(x_1 + x_2 + 1))}{f(x_2)} \\ &= \frac{S(x_2 + 1)}{f(x_2)} = \frac{1 - h(x_2)}{h(x_2)} \\ E[X_1^2|X_2 = x_2] &= \sum_0^\infty x_1^2 \frac{(f(x_1 + x_2) - f(x_1 + x_2 + 1))}{f(x_2)} \\ &= \frac{S(x_2 + 1)}{f(x_2)} + 2\frac{S(x_2 + 2)m(x_2 + 1)}{f(x_2)} \end{aligned}$$

By using the relationship between $m(x_2)$ and $h(x_2)$ given in (1.1.9), we obtain

$$\begin{aligned} E[X_1^2|X_2 = x_2] &= \frac{1 - h(x_2)}{h(x_2)} + 2\frac{(1 - h(x_2))(1 - h(x_2 + 1))m(x_2 + 1)}{h(x_2)} \\ &= \frac{1 - h(x_2)}{h(x_2)}(2m(x_2) - 1). \end{aligned}$$

Thus,

$$\begin{aligned} V[X_1|X_2 = x_2] &= \frac{1 - h(x_2)}{h(x_2)}(2m(x_2) - 1) - \left(\frac{1 - h(x_2)}{h(x_2)}\right)^2 \\ &= \frac{1 - h(x_2)}{h^2(x_2)}(2m(x_2)h(x_2) - 1). \end{aligned}$$

■

Remark 9.2.1. Theorem 9.2.1 helps us to write the regression equations in-terms of the hazard rate of the marginal distribution.

9.3 Reliability properties

In this section, we relate the bivariate reliability concepts to the univariate reliability functions of the marginal distribution. In the case of BSM, the scalar hazard rate of $(X_1, X_2)'$ defined in (1.2.3) can be written as

$$\begin{aligned} a(x_1, x_2) &= \frac{\mathbf{p}(x_1, x_2)}{\mathbf{S}(x_1, x_2)} = \frac{f(x_1 + x_2) - f(x_1 + x_2 + 1)}{S(x_1 + x_2)} \\ &= h(x_1 + x_2) - h(x_1 + x_2 + 1)(1 - h(x_1 + x_2)) \end{aligned} \tag{9.3.1}$$

where $h(x)$ is the hazard rate of Z.

Definition 9.3.1. (Nair and Sankaran [102]) The random vector $(X_1, X_2)'$ possesses bivariate increasing(decreasing) hazard rate property (BIHR/BDHR) if $h_1(x_1)$ is increasing (decreasing) in x_1 and $a(x_1, x_2)$ is increasing(decreasing) in x_1 and x_2, where $h_1(x_1)$ is the marginal hazard rate of X_1.

Now we have the following proposition connecting ageing properties of $(X_1, X_2)'$ and Z.

Proposition 9.3.1. Z is IHR(DHR) $\Longleftrightarrow$ $(X_1, X_2)'$ is BIHR(BDHR).

The proof follows from (9.3.1).
For discrete Schur-constant models, the vector hazard rate components satisfy

$$c_1(x_1, x_2) = c_2(x_1, x_2) = h(x_1 + x_2). \tag{9.3.2}$$

The following proposition is immediate.

Proposition 9.3.2. Z is IHR(DHR) $\Longleftrightarrow$ $(X_1, X_2)'$ is BIHR-2(BDHR-2)

Definition 9.3.2. Suppose that $(X_1, X_2)'$ and $(Y_1, Y_2)'$ are two random vectors defined in $\mathbf{N^2}$. The vector hazard rate of $(X_1, X_2)'$ is $(c_1(x_1, x_2), c_2(x_1, x_2))'$ and $(Y_1, Y_2)'$ has the vector hazard rate $(d_1(x_1,x_2), d_2(x_1,x_2))'$. Then $(X_1,X_2)'$ is less than $(Y_1,Y_2)'$ in bivariate hazard rate $((X_1,X_2) \leq_{BHR} (Y_1,Y_2))$ if $c_i(x_1,x_2) \geq d_i(x_1,x_2);\ i = 1,2\ \ x_1,x_2 = 0,1,2,...$

Now we discuss the relation between the univariate ordering $\leq_{hr}$ and the bivariate ordering $\leq_{BHR}$, in the case of BSM.

Proposition 9.3.3. $Z \leq_{hr} Y \iff (X_1,X_2)' \leq_{BHR} (Y_1,Y_2)'$ where Y and $(Y_1,Y_2)'$ are related in the same manner as Z and $(X_1,X_2)'$.

Proof.

$$\begin{aligned} Z \leq_{hr} Y &\iff h_Z(x) \geq h_Y(x) \iff h_Z(x_1+x_2) \geq h_Y(x_1+x_2) \\ &\iff c_i(x_1,x_2) \geq d_i(x_1,x_2);\ i = 1,2 \\ &\iff (X_1,X_2)' \leq_{BHR} (Y_1,Y_2)' \end{aligned}$$

■

For the BSM, we obtain relationships for the MRL functions as

$$m_1(x_1,x_2) = \frac{1}{S(x_1+x_2+2)} \sum_{t=x_1+1}^{\infty} S(t+x_2+1) = m(x_1+x_2+1) \tag{9.3.3}$$

and

$$m_2(x_1,x_2) = m(x_1+x_2+1). \tag{9.3.4}$$

Like the hazard rate, the components of MRL have identical values.

Proposition 9.3.4. Z is IMRL(DMRL) $\iff$ $(X_1,X_2)'$ is MIMRL-1(MDMRL-1).

The proof follows from the identity $m_i(x_1,x_2) = m(x_1+x_2+1);\ i = 1,2.$
The following proposition based on the bivariate MRL ordering is direct.

Proposition 9.3.5. $Z \leq_{hr} Y \Rightarrow Z \leq_{mrl} Y \iff (X_1,X_2)' \leq_{BMRL} (Y_1,Y_2)'$.

9.4 Dependence concepts

We examine the relationship between time-dependent measures of association and ageing properties of bivariate discrete Schur-constant models. Time-dependent measures are of importance in survival analysis, where identification of the age at which association is maximum is of special interest. One of the popular time-dependent measures is Clayton [35] measure. In the discrete set-up, the measure is defined by (7.3.4). Dividing by $\mathbf{S}^2(x_1, x_2)$, (7.3.4) becomes

$$\theta(x_1, x_2) = \frac{a(x_1, x_2)}{c_1(x_1, x_2)c_2(x_1, x_2)} \tag{9.4.1}$$

which expresses $\theta(x_1, x_2)$ in-terms of the hazard rates. Further, by the relationship between $c_i(x_1, x_2)$ and $a(x_1, x_2)$ given in (1.2.6), we obtain

$$\theta(x_1, x_2) = \frac{c_1(x_1, x_2+1)c_2(x_1, x_2) + [c_1(x_1, x_2) - c_1(x_1, x_2+1)]}{c_1(x_1, x_2)c_2(x_1, x_2)}. \tag{9.4.2}$$

For BSM, (9.4.2) becomes

$$\theta(x_1, x_2) = \frac{h(x_1+x_2+1)h(x_1+x_2) + h(x_1+x_2) - h(x_1+x_2+1)}{h^2(x_1+x_2)}. \tag{9.4.3}$$

Hence, $\theta(x_1, x_2) > 1$ is equivalent to

$$(1 - h(x_1+x_2))\left(h(x_1+x_2) - h(x_1+x_2+1)\right) > 0.$$

Thus, X_1 and X_2 are positively associated if $h(x_1+x_2) > h(x_1+x_2+1)$ which means that Z must be strictly DHR. Similarly, $\theta(x_1, x_2) < 1 \iff Z$ is strictly IHR.

When X_1 and X_2 are independent $(\theta(x_1, x_2) = 1)$, $h(x)$ is constant(Z is geometric) and conversely.

Example 9.4.1. Let $(X_1, X_2)'$ have bivariate Waring distribution in (5.4.1). The distribution of Z has survival function

$$S(x) = \frac{(m)_{(x)}}{(m+n)_{(x)}}, x = 0, 1, 2, ... \tag{9.4.4}$$

and hazard rate $h(x) = \dfrac{n}{m+n+x}$, which is DHR.

Then

$$\theta(x_1, x_2) = \frac{(n+1)(m+n+x_1+x_2)}{m(m+n+x_1+x_2+1)} > 1.$$

Accordingly, X_1 and X_2 are positively associated.

Example 9.4.2. Let $(X_1, X_2)'$ have the bivariate version of Weibull-I distribution, with survival function

$$\mathbf{S}(x_1, x_2) = q(x_1 + x_2)^{\beta};\ x_1, x_2 = 0, 1, 2, ...;\ \beta > 0. \tag{9.4.5}$$

The distribution of Z has survival function

$$S(x) = q^{x^{\beta}};\ x = 0, 1, 2, ...; \beta > 0. \tag{9.4.6}$$

The distribution is IHR, when $\beta > 1$. From (9.4.2), we get

$$\theta(x_1, x_2) = \frac{q^{(x_1+x_2)^{\beta}}\left(q^{(x_1+x_2)^{\beta}} - 2q^{(x_1+x_2+1)^{\beta}} + q^{(x_1+x_2+2)^{\beta}}\right)}{\left(q^{(x_1+x_2)^{\beta}} - q^{(x_1+x_2+1)^{\beta}}\right)^{2}}. \tag{9.4.7}$$

To verify that $\theta(x_1, x_2) < 1$, assume that the inequality holds. This is possible when

$$q^{(x_1+x_2)^{\beta}}\left(q^{(x_1+x_2)^{\beta}} - 2q^{(x_1+x_2+1)^{\beta}} + q^{(x_1+x_2+2)^{\beta}}\right) < \left(q^{(x_1+x_2)^{\beta}} - q^{(x_1+x_2+1)^{\beta}}\right)^{2}$$

$$\Longleftrightarrow \left(q^{(x_1+x_2)^{\beta} - q^{(}x_1+x_2+2)^{\beta}}\right)^{2} + q^{(x_1+x_2)^{\beta}+(x_1+x_2+2)^{\beta}} - q^{2(x_1+x_2+1)^{\beta}}$$
$$< \left(q^{(x_1+x_2)^{\beta} - q^{(}x_1+x_2+2)^{\beta}}\right)^{2}$$

$$\Longleftrightarrow 1 - \frac{q^{(x_1+x_2+1)^{\beta}}}{q^{(x_1+x_2)^{\beta}}} < 1 - \frac{q^{(x_1+x_2+2)^{\beta}}}{q^{(x_1+x_2+1)^{\beta}}}$$
$$\Longleftrightarrow h(x_1 + x_2) < h(x_1 + x_2 + 1). \tag{9.4.8}$$

The inequality in (9.4.8) holds when $\beta > 1$ since Z is IHR in this case. Hence $\theta(x_1, x_2) < 1$ and accordingly, X_1 and X_2 are negatively associated, when $\beta > 1$.

Bjerve and Doksum [21] defined two measures of association to study the dependence among two random variables. The first one measures the strength of association between X_1 and X_2 as a function of X_1 called correlation curve. In the discrete set-up, this measure $\rho(x_1)$ is defined as

$$\rho(x_1) = \frac{\sigma_1 \Delta E[X_2|X_1 = x_1]}{(\sigma_1 \Delta E^2[X_2|X_1 = x_1] + \sigma^2(x_1))^{\frac{1}{2}}}, \tag{9.4.9}$$

where $\sigma^2(x_1) = V[X_2|X_1 = x_1]$ and $\sigma_1^2 = V[X_1]$.

From Theorem 9.2.1, we get

$$\rho(x_1) = \frac{\sigma_1 \left(h(x_1) - h(x_1+1)\right)}{h(x_1)h(x_1+1)\left[\sigma_1^2 \frac{h(x_1) - h(x_1+1)}{h(x_1)h(x_1+1)} + 2m(x_1)h(x_1) - 1\right]^{\frac{1}{2}}} \tag{9.4.10}$$

Recalling that $\rho(x_1) > (<)0$ implies positive(negative) association, (9.4.10) provides the following results.

Proposition 9.4.1. a) When Z is DHR(IHR) in the strict sense, $\rho(x_1) > (<)0$ and conversely.

b) X_1 and X_2 are independent $(\rho(x_1) = 0) \iff X_1(X_2)$ is geometric $\iff Z$ is geometric.

c) $\theta(x_1, x_2) > (<)1 \iff Z$ is strictly DHR(IHR) $\iff \rho(x_1) > (<)0$.

A second measure of the association, suggested by Bjerve and Doksum [21] is the conditional correlation curve defined by

$$\xi(x_1) = \frac{\sigma_1 \Delta E[X_2|X_1 = x_1]}{\sigma^2(x_1)}. \tag{9.4.11}$$

For discrete Schur-constant models,

$$\begin{aligned}\xi(x_1) &= \frac{\sigma_1 \left(h(x_1) - h(x_1+1)\right.}{h(x_1)h(x_1+1)\left[\frac{1-h(x_1)}{h^2(x_1)}\left(2m(x_1)h(x_1) - 1\right)\right]^{\frac{1}{2}}} \\ &= \frac{\sigma_1 \left(h(x_1) - h(x_1+1)\right)}{h(x_1+1)\left[\left(1-h(x_1)\right)\left(2m(x_1)h(x_1) - 1\right)\right]^{\frac{1}{2}}}.\end{aligned} \tag{9.4.12}$$

Thus, $(X_1, X_2)'$ is positively(negatively) associated when $\xi(x_1) > (<)0$. Thus, we have the following result from (9.4.12).

Proposition 9.4.2. $\theta(x_1, x_2) > (<)1 \iff Z$ is strictly DHR(IHR) $\iff \xi(x_1) > (<)0$.

Example 9.4.3. Let $(X_1, X_2)'$ follow the Waring distribution in (5.4.1). We have already shown that $\theta(x_1, x_2) > 1$. The mean residual life of Z is given by

$$m(x) = \frac{m+n+x}{n-1};\ m > n > 1. \tag{9.4.13}$$

From (9.4.10), we get

$$\rho(x) = \frac{\sigma_1}{n\sqrt{\dfrac{\sigma_1^2}{n} + \dfrac{n+1}{n-1}}}, \tag{9.4.14}$$

which is greater than zero when $n > 1$. Using (9.4.12), we evaluate

$$\xi(x) = \frac{\sigma_1}{\sqrt{\dfrac{(n+1)(m+x)}{(n-1)(m+n+x)}}(m+n+x)}, \tag{9.4.15}$$

which is greater than zero when $n > 1$.

Instead of considering regression function, Anderson et al. [9] employed the ratio of MRL function $m_i(x_1, x_2);\ i = 1, 2$ in suggesting a measure of association. In the discrete case, the measure is defined by

$$\phi(x_1, x_2) = \frac{m_1(x_1, x_2)}{m_1(x_1, -1)} \tag{9.4.16}$$

Values of $\phi(x_1, x_2)$ very different from unity indicate strong association between X_1 and X_2. If X_1 and X_2 are positively associated, $\phi(x_1, x_2)$ should increase for increasing values of x_2. Also, $\phi(x_1, x_2) = 1$ if and only if X_1 and X_2 are independent and geometrically distributed. For Schur-constant models,

$$\phi(x_1, x_2) = \frac{m(x_1 + x_2 + 1)}{m(x_1)}. \tag{9.4.17}$$

A similar relation is obtained using $m_2(x_1, x_2)$. Hence, we obtain the following result.

Proposition 9.4.3. $\phi(x_1, x_2) > (<)1 \iff m(x_1 + x_2 + 1) > (<)m(x_1) \iff Z$ is strictly IMRL(DMRL).

Another measure proposed in Anderson et al. [9] is based on the ratio of survival functions. Its discrete analogue is given by

$$\psi(x_1, x_2) = \frac{P[X_1 \geq x_1 | X_2 \geq x_2]}{P[X_1 \geq x_1]}. \tag{9.4.18}$$

When X_1 and X_2 are independent, $\psi(x_1, x_2) = 1$ and large values of $\psi(x_1, x_2)$ indicate positive association. Assuming Schur-constancy for $(X_1, X_2)'$, we obtain

$$\psi(x_1, x_2) = \frac{S(x_1 + x_2)}{S(x_1)S(x_2)} \tag{9.4.19}$$

so that

$$\begin{aligned} \frac{\psi(x_1 + 1, x_2)}{\psi(x_1, x_2)} &= \frac{S(x_1 + x_2 + 1)S(x_2)}{S(x_1 + x_2)S(x_2 + 1)} \\ &= \frac{1 - h(x_1 + x_2)}{1 - h(x_2)} \end{aligned} \tag{9.4.20}$$

From (9.4.20), we obtain the following result.

Proposition 9.4.4. $\psi(x_1, x_2) > 1 \iff Z$ is strictly DHR

From the above discussions, the following result is immediate.

Theorem 9.4.1. a) Z is strictly DHR(IHR) $\iff \theta(x_1, x_2) > (<)1 \iff \rho(x_1) > (<)0 \iff \xi(x_1) > (<)0 \iff (X_1, X_2)'$ is RCSI(RCSD) $\iff \psi(x_1, x_2) > (<)1 \Rightarrow \phi(x_1, x_2) > (<)1$

b) $\rho(x_1) \geq 0 \iff X_1$ is stochastically increasing(SI) in X_2.

c) $\psi(x_1, x_2) > 1 \iff (X_1, X_2)'$ is PQD.

Example 9.4.4. For the bivariate Waring distribution in (5.4.1), we compute $\phi(x_1, x_2)$, using (9.4.17), as

$$\phi(x_1, x_2) = \frac{m + n + x_1 + x_2 + 1}{m + n + x_1} > 1, \tag{9.4.21}$$

agreeing with the IMRL property of Z. Now, using (9.4.19), we write

$$\frac{\psi(x_1, x_2 + 1)}{\psi(x_1, x_2)} = \frac{(m + x_1 + x_2)(m + n + x_2)}{(m + x_2)(m + n + x_1 + x_2)}, \tag{9.4.22}$$

which is greater than one since $\frac{m+x}{m+n+x}$ is increasing in x. Thus, $\psi(x_1,x_2)$ is increasing in x_2. Now,

$$\frac{\frac{P[X_1>x_1+1,X_2>x_2]}{P[X_1>x_1,X_2>x_2]}}{\frac{P[X_1>x_1+1,X_2>x_2+1]}{P[X_1>x_1,X_2>x_2+1]}}=\frac{(m+x_1+x_2+1)\,(m+n+x_1+x_2)}{(m+x_1+x_2)\,(m+n+x_1+x_2+1)}>1, \quad (9.4.23)$$

implying that $(X_1,X_2)'$ is RCSI. Since RCSI implies PQD, $(X_1,X_2)'$ is PQD. To check whether X_1 is stochastically increasing(SI) in X_2, consider

$$\begin{aligned} P[X_1>x_1|X_2=x_2] &= \frac{S(x_1,x_2)-S(x_1,x_2+1)}{S(x_2)-S(x_2+1)} \\ &= -\frac{\Gamma\,(m+x_1+x_2)\,\Gamma\,(m+n+x_2+1)}{\Gamma\,(m+x_2)\,\Gamma\,(m+n+x_1+x_2+1)}. \end{aligned} \quad (9.4.24)$$

To find the monotonicity of (9.4.24), consider the ratio

$$\frac{P[X_1>x_1|X_2=x_2+1]}{P[X_1>x_1|X_2=x_2]}=\frac{(m+x_1+x_2)\,(m+n+x_2+1)}{(m+x_2)\,(m+n+x_1+x_2+1)}, \quad (9.4.25)$$

which is greater than unity implying that $P[x_1>x_1|X_2=x_2]$ is increasing in x_2.

Remark 9.4.1. Among bivariate distributions, there is no direct relationship between SI and RCSI. Further, PQD neither implies RCSI nor SI.

Remark 9.4.2. From Theorem 9.4.1, it is obvious that negative(positive) ageing is equivalent to the positive(negative) association in the case of BSM.

9.5 Conclusion

In this chapter, we have discussed basic properties of discrete Schur-constant models. Reliability characteristics of the models were studied. The dependence structure of the models has been discussed. We have established the relationships between the association measures and ageing classes corresponding to the models.

Chapter 10

Conclusions and Future Study

10.1 Conclusions

There are many real life situations, in which failure times are measured in discrete time. For example, a piece of equipment operates in cycles and the observation is the number of cycles completed before failure, so that the lifetime is clearly discrete. The lack of accuracy of the measuring devices may also generate discrete lives. There are occasions to prefer counts over clock time even when the latter is available. There are conceptual and mathematical problems in developing discrete reliability theory. These situations encourage researchers to study reliability concepts in discrete time. Motivated by these facts, in the present work, we have studied the modelling and analysis of lifetime data in discrete time.

In Chapter 2, we have studied ageing classes for discrete life distributions using two different versions of the hazard rate. The relationships among these ageing classes were derived. It may be noted that properties of various ageing classes based on the hazard function, in the continuous set-up, are not directly transformed into discrete set-up. Various ageing criterion discussed in this chapter play a fundamental role in the development of reliability theory and practice. An attempt is made to establish some properties of the class of distributions with BT or UBT hazard rates which could be useful in practice. Being general results, they can be readily applied in finding bounds for the reliability.

In Chapter 3, we have presented some theorems that help in detecting the shape of the hazard rate function when lifetime is treated as discrete. All the results will work out when the probability mass function alone is known. Following this, we have discussed various methods of construction of discrete bathtub distributions. We have provided examples in which the models were applied to real data and we have studied the properties of discretized quadratic hazard model in detail. These supplement the existing list of BT models in literature.

The role of relative ageing concepts is either to compare the ageing patterns of two devices at a fixed time or to investigate whether the same device is ageing more positively (negatively) at different points of time. In Chapter 4, we have presented some concepts and results that lead to a quantitative assessment of which of two devices is ageing faster. Also, the impact of spent life of a device on its residual life can also be numerically evaluated. It was proved that the relative ageing concepts are related to the well-known ageing classes such as IHR, NBU, etc.

Chapters 5, 6 and 7 were devoted to the study of residual life functions in the discrete multivariate domain. We have discussed the multivariate mean residual life function, variance residual life function and covariance residual life function. Characterizations based on these concepts were derived. Ageing classes based on these functions were proposed and their inter-relationships were studied. The results in these chapters are of great use for modelling and analysis of multivariate discrete lifetime data .

In Chapter 8, we have introduced four versions of the multivariate reversed hazard rates and studied their properties. We have determined the criterion under which the scalar reversed hazard rate uniquely determines the underlying distribution. Unlike scalar reversed hazard rate, the vector reversed hazard rate uniquely determines the distribution. The reversed lack of memory property useful in maintenance problems has been extended to the multivariate set-up. The multivariate reversed geometric distribution has been characterized using these multivariate reversed ageing concepts. Section 8.4 provided a general class of distributions, which can be used in reliability analysis by choosing appropriate marginal distributions.

Finally in Chapter 9, we have discussed basic properties of discrete Schur-constant models. Reliability characteristics of the models were studied. The dependence structure of the models has been discussed. We have established the relationship between the association measures and ageing classes corresponding to these models.

10.2 Future study

In Chapter 2, we have derived some new properties of discrete ageing classes based on hazard rate function. The application and properties of IHR(2)/DHR(2) class are yet to be studied. There is a scope for studying the properties of ageing classes based on other reliability functions such as mean residual life, variance residual life, etc. New ageing classes can be proposed on the basis of reliability functions such as odds function and residual odds function, in the discrete domain.

The results in Chapter 3 mainly deals with BT and UBT hazard rate distributions. While studying the BT and UBT distributions, models with only one change point have been considered in the present study. However, we can extend our study to the two-change point case by suitably modifying our results. There are other non-monotone hazard rate functions that take forms like periodic, roller coaster, etc. Results in the present chapter need to be modified in order to accommodate these type of hazard rate functions. Regarding the construction methods discussed, only a few methods have been included in the discussion. New methods using total time on test transforms, additive hazard rate models, etc. can be developed in the discrete domain. The work in this direction will be carried out later.

We have discussed stochastic ordering by ageing concepts in Chapter 4, in which, we have studied $\leq_{IHR}$ and $\leq_{IHRA}$ orderings, which are based on hazard rate function. In a similar way, other stochastic orderings based on survival function, mean residual life function, etc. can be studied. Testing procedures for differentiating positive and negative ageing, based on the values of specific and relative ageing factors, can be developed. New discrete lifetime models can be developed using the ageing intensity function. The relative ageing concepts in the univariate domain can be extended into higher dimensions for comparing the efficiencies of multicomponent systems.

We have studied multivariate residual life functions in Chapters 5, 6 and 7. In the case of multivariate mean, variance, and covariance residual lives, it is required to propose estimation and testing procedures for dealing with real life datasets. Non-parametric estimators of covariance residual life could be used for testing independence of random variables. Stochastic orderings can be developed for comparing multivariate discrete life distributions in-terms of its residual lives. Necessary and sufficient conditions for a vector to be multivariate mean residual life and variance residual life should be derived for proposing new models based on these ageing concepts.

We have studied discrete multivariate reversed hazard rates in Chapter 8. The residual life functions such as mean residual life, variance residual life and covariance residual life can be introduced in the reversed time scenario also. Such a study will be complementary to the study of reversed hazard rates. We have obtained a new family of multivariate discrete distribution in Chapter 8. The reliability properties of the family are yet to be studied.

The reliability properties of discrete Schur-constant models were studied in Chapter 9. These results are analogues of the results in the continuous case, given in Nair and Sankaran [100]. There is a scope for extending the study to Schur-constant equilibrium distributions, as in the continuous case, given in Nair and Sankaran [101].

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2. Sankaran, P. G., Nidhi, P. R., and Nair, N. U. Multivariate discrete reversed hazard rates. *Communications in Statistics - Theory and Methods*, 2016. (to appear)

3. Nair, N. U., Sankaran, P. G., and Nidhi, P. R. Determination of hazard rate shape for discrete lives. *International Journal of Reliability, Quality and Safety Engineering*, 23 (04), 1650015, 2016.

4. Nair, N. U., Sankaran, P. G., and Nidhi, P. R. Discrete distributions with bathtub-shaped hazard rates. *South African Statistical Journal*, 51, 1-22, 2016.

5. Nair, N. U., Sankaran, P. G., and Nidhi, P. R. Quantification of relative ageing in discrete time. *Metron*, 74, 339-355, 2016.

6. Nair, N. U., Sankaran, P. G., and Nidhi, P. R. Some properties of discrete bathtub-shaped distributions. *Communications in Statistics - Theory and Methods*, 2016. (to appear)

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