A STUDY ON LATTICE OF OPEN SETS.

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A Study on Lattice of Open Sets.

Ph.D. thesis in the field of Topology

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Certificate

This is to certify that the thesis entitled **"A Study on Lattice of Open Sets"** submitted to the Cochin University of Science and Technology by **Mrs.Vinitha.T** for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere. Also certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis and the work done is adequate and complete for the award of Ph.D Degree.

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Declaration

I, Vinitha.T., hereby declare that this thesis entitled '**A Study on Lattice of Open Sets**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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To

My Mother, Husband and Daughter
and in loving memory of
my Father

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“The power of God is with you at all times ; through the activities of mind, senses, breathing and emotions ; and is constantly doing all the work using you as a mere instrument.”-

(Bagavat Gita.)

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Vinitha.T

A STUDY ON LATTICE OF OPEN SETS.

Contents

Chapter 1

Introduction

"Mathematics, rightly viewed, possesses not only truth but supreme beauty"-

Bertrand Russel.

1.1 Introduction

Topology is an important tool in almost all areas of mathematics. Topology is divided in to different branches that includes point set topology, differential topology, algebraic topology, fuzzy topology, bi topological spaces etc.

Beginning of topology is due to Euler. It began in 1736 while working on the solution of Konigsberg bridge problem ; entitled 'The solution of a problem relating to geometry of position'. The title itself indicates that the work deals with another concept of geometry where the matter of distance is irrelevant. Before that no work was done without involving measurement.

The word Topology was first introduced by Listing. But his ideas were mainly due to Gauss who choose not to publish any of his work. He wrote a paper called 'Vorstudies Zue Topology' and later he published a paper dealing with study of Mobius Band and he also studied components of surfaces and connectivity. The first person who studied about connectivity was not Listing. Riemann done the same in 1851 and was studied eventually by Poincare. While studying connectivity Poincare introduced Fundamental Group of a variety and the idea of Homotopy was introduced in 1895 papers.

Generalization of idea of convergence was the second way in which topology was developed. This was done in 1817 by Bolzano while he associate convergence with sequence of numbers. In 1872 Cantor introduced concept of limit point, derived set etc. The fundamental concept of point set topology that is the open set was also introduced by Cantor. That leads to the concept of neighborhood which was introduced by Wierstrass while proving Bolzano Weirstrass theorem.

The concept of compactness was first introduced by Frechet in 1906. Later in 1909, the axiomatic approach to topology excluding concept of distance was developed by Riesz.

The third way in which topological concepts entered mathematics is through Functional Analysis. A further step in abstraction was done by Banach in 1932 when he considered inner product spaces.

Another major concept in which topology was studied is Lattice Theory. Charles.S.Pierce and Earnst Schroder introduced the concept of lattice at the end of nineteenth century as useful tool in investigating the axiomatic approach of Boolean Algebra. Richard Dedekind's study on ideals of algebraic numbers also led to same idea of lattice. The development of lattice theory began with the publication of two fundamental papers by Dedekind.

The notion of continuity properties in topological spaces was first studied by Hausdorff using notion of open sets. Till then topological space was known to possess a lattice structure with open sets although that lattice structure was not used there. American mathematician Marshall Stone started the application of lattice theory to topological spaces. The next work about the link between topology and lattice theory was through Stones Representation theorem. The first person who involved this idea was Henry Wallmann where he used lattice theoretic ideas to construct Wallmann Compactification of a T_1 topological space. The first text book which presented topology from lattice theoretic view point was written by Germann mathematician Nobeling.

Lattices are partially ordered sets in which least upper bound and greatest lower bound of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. A lattice is a set on which two operations are defined called meet and join denoted by $\wedge$ and $\vee$. Lattices behave better than posets lacking upper or lower bounds. General Topology and Lattice Theory are two related branches of mathematics influencing each other. Many mathematicians studied a lot of results [7], [19], [27] considering topology and lattice theory. Perhaps Birkhoff and Vaidyanathswamy are fore runners in this direction.

In 1954 Frink and V.K.Balachandran[6] introduced the concept of join irreducible and join prime elements in any lattice, while they were dealing with the solution for a problem of Birkhoff and Frink on the relation between completely prime and completely join irreducible ideals. Dually they defined meet irreducible and meet prime elements for any arbitrary lattice.

Another way of looking at prime elements in any lattice was done by Gierz et.al.[18] in 1980 in their work 'A compendium of continuous lattices'. They initiated the study by defining prime element in such a way that, Let L be any lattice an element $a \in L$ is said to be a prime element if whenever $b \wedge c \leq a$ for $b, c \in L$ then either $b \leq a$ or $c \leq a$. Johnstone[24] continued the study of join irreducible elements by considering the dual notion of it. He considered the lattice of open sets of any arbitrary topological space X and studied about the irreducible closed sets in the lattice of open sets of X ; thereby introduced sober spaces as spaces in which every irreducible closed subset is the closure of a unique point of X. Johnstone proved that any sober space is T_0. Also derived some results in frames and locale theory by extending the notion to it.

Later in 2004, Martin Maria Kovar[26] while solving an open problem by J.Lawson and M.Mislove established a necessary and sufficient condition for any element of a lattice to be prime in terms of filter. Thereby he partially solved the problem of J.Lawson and M.Mislove in which they ask for which directed complete posets, Scott topology has a basis of open filters and for which directed complete posets, the topology generated by Scott open filter is T_0.

Another notable work done using irreducible open sets was in the area of Frames and Locales. Let $(X, \Omega(X))$ be any topological space. In 2011 Jorge Picardo and Ales Pultr[40] analyzed certain situations in which complement of singleton sets become meet irreducible in the lattice $\Omega(X)$. This definition of meet irreducible element is actually motivated by the definition of prime element in any lattice by Gierz et.al[18]. Ales Pultr and Jorge Picardo extended the notion of prime to theory of filters and thereby they defined prime and completely prime filter, to prove an equivalent condition for a space to be sober. Any localic map sends meet irreducible ones to meet irreducible ones only and hence they deduce that corresponding to any meet irreducible open set there always exists a topological space consisting of all completely prime filters containing that irreducible open set which they called as spectrum of that locale denoted by $S_p(L)$ where L is the locale given. Also proved that $S_p(L)$ is always sober in nature.

Let (X, T) be any topological space then the collection of all open sets in T always forms a complete lattice and hence it is worthful to study prime/irreducible elements in any topological space. Motivated by definition of Gierz.et.al[18] we introduce prime open sets in any topological space and try to study some notions in topology using prime open sets.

The characterization of prime elements using filters itself implies existence of disjoint prime open sets corresponding to distinct points is not possible in this context. That lead us to consider some lower separation axioms for which basic foundation is generalised closed sets, semi-open sets etc. We wish to give a brief description and history of such important notions in topology. Generalised closed sets was first introduced by N.Levine[31] in 1970 as a generalization of closed sets. Spaces in which generalised closed sets and closed sets coincide are named as $T_{1/2}$ spaces by Levine. As the name itself indicates it lies between T_0 and T_1 axioms. $T_{1/2}$ axioms were characterized and more studies were done by William Dunham[16] in 1977. Dunham also provide structure theorems for minimal and maximal $T_{1/2}$ topologies on a given set. Levine introduced generalised closed sets in order to extend many of the important properties of closed sets to a larger family. It was shown by Levine that compactness, normality and completeness in uniform spaces are inherited by g-closed sets. Later Dunham and Levine continue the study of g-closed sets obtaining characterizations and other familiar theorems to this general context. Thus they derived the generalized Tietze extension theorem also.

Later in 1988 another characterisation of $T_{1/2}$ spaces was done by H.Maki et.al.[33] For that purpose they generalized the concept of v-set introduced by Maki in [4] and there by they initiated the study of generalised v-sets. Dunham[15] again in 1980 defined a new closure operator by using generalised closed sets, thus investigated a new topology and its properties. And the class of $T_{1/2}$ spaces is characterized by the new topology. He also proved that the new closure operator is a kuratowski closure operator and examined which properties are extended to new topology. Again in 1991 K.Balachandran et.al.[5] defined a class of mappings called generalised continuous mappings which contains the class of continuous mappings and Miguel Caldas Cueva[11] continued the study in 1993 by generalizing some theorems of N.Levine. Also in 1993, N.Palaniappan and K.Chandrashekara Rao[39] introduced regular generalised closed and open sets and studied its properties.

Another separation axiom called $T_{1/4}$ axiom was also introduced by Arenas, Dontchev and Ganster[3] in their study of generalised closed sets. Arenas et.al. introduced λ-closed sets and characterized T_0 spaces as those spaces where each singleton set is λ-closed and $T_{1/2}$ spaces as those spaces were each subset is λ-closed. The author also pointed out that class of $T_{1/4}$ spaces is strictly placed between the class of T_0 and the class of $T_{1/2}$ spaces.

Levine[30] call a topology τ on an arbitrary set X a D-topology whenever every non empty open set is dense in X. D-spaces are characterized as spaces in which every open set in X is connected. It was also proved that any topological space can be written as union of its maximal D-subsets. In [30] Levine also studied about star sub topologies which are topologies uniquely determined by D-subtopologies. In 1973, Pushpa Agashe and N.Levine[1] proved that every non D-topology has an immediate predecessor and consequently they concluded that every T_1 topology which is not a D-topology have immediate predecessor. D-spaces are also called as irreducible in the sense of Bourbaki and spaces in which every dense subset happens to be open is named as submaximal spaces by [9]. It was proved by Jullian Dontchev in [13] that every irreducible submaximal spaces are door spaces introduced by Kelley. Kelley[25] introduced door spaces as spaces in which every subset is either open or closed. Full identification of door spaces was completely done in [35] by S.D.Mccartan.

A subset A of a topological space is said to be semi-open if there exists an open set U such that $U \subseteq A \subseteq cl(U)$ where $cl(U)$ denotes the closure of U with respect to the topology given. This concept was also introduced by Levine[29]. Later a lot of work has been done in this area. In [21], T.R.Hamlett showed how different topologies on a set which determine the same class of semi-open sets can arise from functions and points out some implications of two topologies being related in this manner. Hamlett also introduced semi-continuous functions in [20] and basic questions involving semi-continuous functions into T_2 spaces were investigated. Later Dragan.S.Tankori, Noiri etc continued the study of semi-continuous functions. The notion of semi-closure was introduced by Grossely and Hildebrand in [10], G.Dimaio[12] studied about s-closed spaces utilizing semi-closure. The concept of semi-homeomorphism was introduced and studied by Biswas, Crossely and Hildebrand, J.P.Lee etc. Bhamini.M.P.Nayar and S.P.Arya[36] also continued the study by including the minimality structure in to it. Later notions of connectedness, compactness were also examined using semi-open sets by V.Pipitone, G.Russo, Thomas etc.

The concept of C-compactness was introduced by G.Viglino in [49]. Later it was characterized by [22] in terms of nets and filters. Let X be any arbitrary set and let $\Sigma(X)$ denote the collection of all topologies on X. Vaidyanathswamy proved that $\Sigma(X)$ is a complete lattice. Fröhlich in [17] proved that every topology is the infimum of all ultraspaces finer than the given topology. He also characterized ultraspaces in terms of filters. These two results helps us to study the lattice $\Sigma(X)$ and are also used to identify prime elements in the lattice of topologies.

In this thesis we have attempted to study problems related to irreducible open sets in the lattice of open sets of any arbitrary topological space mainly focused in the following contexts :

1. Studies on some generalised concepts of open and closed sets in topological spaces.
2. Various aspects of continuous transformations and separation properties.
3. Studies on some lattice theoretic view point.

1.2 Basic Concepts and Definitions

Definition 1.2.1. [8] A partially ordered set is a pair $(P, \leq)$ where P is any arbitrary set and $\leq$ is an order relation on P obeying Reflexivity, Anti-symmetry and Transitivity. The name "partially ordered set" is often abbreviated poset. In a partially ordered set P, the join and meet of a subset S are respectively the supremum (least upper bound) of S, denoted $\vee S$, and infimum (greatest lower bound) of S, denoted $\wedge S$. In general, the join and meet of a subset of a partially ordered set need not exist; when they do exist, they are elements of P.

Definition 1.2.2. [8] A partially ordered set in which all pairs have a join is a join-semilattice. Dually, a partially ordered set in which all pairs have a meet is a meet-semilattice. A partially ordered set that is both a join-semilattice and a meet-semilattice is a lattice. A lattice in which every subset, not just every pair, possesses a meet and a join is a complete lattice. The least element of a lattice is designated 0 and the greatest element is designated 1.

By "a covers b" in a lattice $(L, \leq)$ we mean $b < a$ and $b < c < a$ implies $b = c$ or $c = a$.

Definition 1.2.3. [8] An atom is an element which covers the least element. A lattice is atomic if every element other than 0 can be written as the join of atoms.

Definition 1.2.4. [8] An anti-atom/ dual atom is an element which is covered by 1. A lattice is anti atomic if every element other than 1 can be written as the meet of anti-atoms.

Definition 1.2.5. [8] An element 'a' is called the complement of 'b' in a lattice if $a \wedge b = 0$ and $a \vee b = 1$. A lattice is called complemented if every element has at least one complement.

Definition 1.2.6. [6] An element 'a' of a lattice L is called join irreducible if $a_1 \cup a_2 = a$ implies a_1 or $a_2 = a$, and join prime if $a_1 \cup a_2 \geq a$

implies a_1 or $a_2 \geq a$. Similarly a is called completely join irreducible if (for all existing joins $\bigcup_i a_i$) $\bigcup_i a_i = a$ implies some $a_i = a$, and completely join prime if $\bigcup_i a_i \geq a$ implies some $a_i \geq a$.

Definition 1.2.7. [18] An element $a \in L$ in any arbitrary lattice L is said to be prime if for any $b, c \in L$, whenever $b \wedge c \leq a$ then either $b \leq a$ or $c \leq a$.

Let (X, T) be any arbitrary topological space, then clearly the collection of open sets in T always forms a lattice under the operations of union and intersection. Johnstone defined irreducible closed sets in that lattice motivated by the above definitions.

Definition 1.2.8. [24] Let (X, T) be any topological space, a closed set F is said to be an irreducible closed set if it cannot be written as a union of two closed subsets F_1 and F_2 such that both are proper closed subsets of F. Also X is said to be sober, if every irreducible closed set in X is the closure of a unique point of X.

Definition 1.2.9. [26] A partially ordered set $(P, \leq)$ is said to be upward directed if for any 'a' and 'b' in P there must exist 'c' in P with $a \leq c$ and $b \leq c$ and the dual notion is called down directed and a partially ordered set is said to be directed if it is both upward and downward directed. Also a poset $(P, \leq)$ is directed complete, if every directed subset of P has a supremum.

Definition 1.2.10. [26] Let $(P, \leq)$ be any partially ordered set or briefly a poset. For any subset A of a poset $(P, \leq)$ define $\uparrow A = \{x/x \geq y; y \in A\}$ and $\downarrow A = \{x/x \leq y; y \in A\}$.

Definition 1.2.11. [26] Let $(X, \leq)$ be a directed complete poset. A subset $U \subseteq X$ is said to be Scott open, if $U = \uparrow U$ and whenever $D \subseteq X$ is a directed set with $supD \in U$, then $U \cap D \neq \phi$. Scott open sets of a directed complete poset always forms a topology called the Scott topology.

Kovar characterized those T_0 topological spaces X for which the Scott topology on the lattice of open sets of X has a basis of open filters and topology generated by Scott open filters is T_0 by using the notion of prime elements in a lattice.

Remark 1.2.1. [40] Let $(X, \Omega(X))$ be any topological space. In the lattice $\Omega(X)$, the elements of the form $X - \overline{\{x\}}$ have the following property : if $U \cap V \subseteq X - \overline{\{x\}}$ then either $U \subseteq X - \overline{\{x\}}$ or $V \subseteq X - \overline{\{x\}}$. Such elements are called as meet irreducible elements.

Definition 1.2.12. [40] A filter F in a lattice is prime if $a_1 \vee a_2 \in F$ implies that $a_i \in F$ for some 'i'. A filter is said to be completely prime filter if above condition holds for any joins.

Proposition 1.2.1. [40] A space X is sober if and only if the neighborhood filters are precisely the completely prime ones.

By extending the above definition of meet irreducible element to any lattice Ales Pultr and Jorge Picardo derived the following results, by calling the collection of all meet irreducible elements of any lattice L as the spectrum of L denoted by $Sp(L)$.

Theorem 1.2.1. *[40] Let L be any lattice, corresponding to any element of $Sp(L)$ there always exists a topological space consisting of all* *completely prime filters containing that meet irreducible element.*

Theorem 1.2.2. [*40*] *For any lattice L, $Sp(L)$ is always sober.*

Definition 1.2.13. [31] Let (X, T) be any topological space and $A \subseteq X$. A is generalised closed briefly g-closed iff $\overline{A} \subseteq O$ whenever $A \subseteq O$ and O is open.

Definition 1.2.14. [31] Spaces in which g-closed sets and closed sets coincide are said to satisfy the $T_{1/2}$ axiom.

Theorem 1.2.3. [*31*] *T_1 implies $T_{1/2}$ implies T_0 but none of the converse parts hold.*

Theorem 1.2.4. [*16*] *X is $T_{1/2}$ if and only if for each $x \in X$, either $\{x\}$ is open or $\{x\}$ is closed.*

Theorem 1.2.5. [*15*] *For a space (X, T),*
let $\mathcal{D} = \{A : A \subseteq X$ and A is g-closed$\}$. For any $E \subseteq X$, define $c^*(E) = \cap\{A : E \subseteq A \subseteq \mathcal{D}\}$.

c^* satisfies the following conditions :

1. $c^*(\phi) = \phi$
2. $E \subseteq c^*(E)$
3. $c^*(E_1 \cup E_2) = c^*(E_1) \cup c^*(E_2)$
4. $c^*(c^*(E)) = c^*(E)$

Thus c^* is a Kuratowski closure operator.

Theorem 1.2.6. *[15] Let T^* be the topology on X generated by c^* in the usual manner, then (X, T^*) is always $T_{1/2}$ for any topological space (X, T).*

Definition 1.2.15. [31] A mapping $f : X \to Y$ from a topological space X into a topological space Y is called g-continuous if inverse image of every closed set in Y is g-closed in X.

Definition 1.2.16. [33] A subset A of a topological space (X, T) is called λ-closed if $A = L \cap F$ where L is an intersection of open sets and F is a closed set.

Theorem 1.2.7. *[33] A space (X, T) is $T_{1/2}$ if and only if every subset is λ-closed.*

Definition 1.2.17. [30] A topology T for a set X is called a *D-topology* if every non-empty open set is dense in X and the corresponding topological space (X, T) is said to be a *D-space*.

Definition 1.2.18. [30] Every topology τ on a set X contains a uniquely defined D-subtopology τ^* called the star subtopology of τ, where $\tau^* = \{O^* : O^* \textit{ is dense in } X\}$.

Theorem 1.2.8. *[1] Every non D-topology has an immediate predecessor.*

Definition 1.2.19. [28] Let (X, T), (X, T^*) be two topological spaces and let $S.O.(X, T)$, $S.O.(X, T^*)$ be the collection of all semi-open sets on X with respect to T and T^*. If $S.O.(X, T) = S.O.(X, T^*)$ then the topologies T and T^* are said to be semi-correspondent.

Theorem 1.2.9. *[28] Semi-correspondent topologies on any arbi-*

trary topological space shares the same collection of nowhere dense sets.

Definition 1.2.20. [28] A function is said to be semi-continuous if inverses of open sets are semi-open and is said to be irresolute if inverses of semi-open sets are semi-open.

Definition 1.2.21. [47] A topological space (X, T) is said to be semi-connected if it is not the union of two non-empty disjoint semi-open sets and is said to be s-compact if every semi-open cover has a finite subcover.

If T_1 and T_2 are two topologies on the set X and every set in T_1 is also in T_2, then T_1 is said to be coarser than T_2 and T_2 finer than T_1. Under this order, the family $\Sigma(X)$ of all topologies on X forms a complete lattice.

Definition 1.2.22. [17] A topology T on X is an ultraspace if the only topology on X finer than T is the discrete topology.

Theorem 1.2.10. [17] *The ultraspaces on X are exactly the topologies of the form $\Im(x, U)$ where $x \in X$ and U is an ultrafilter on X ; such that $\Im(x, U)$ is the topology generated by subsets not containing 'x' and U.*

1.3 Summary of the Thesis

The thesis entitled **'A Study on Lattice of Open Sets'** is divided in to six chapters. The first chapter is an introductory one which contains basic definitions and results used in the formulation of the thesis. Also *chapter 1* contains a brief review of literature of basic concepts used in the thesis.

Chapter 2 introduces a new class of open sets called prime open sets in the lattice of open sets of any arbitrary topological space. Using Kovar's characterization of prime elements in any lattice we identified precisely the prime open sets in any hausdorff space. Consequently we presented a new concept of generalised p-closed sets as an analogous study of generalised closed sets, following Levine.

In *chapter 3* we consider the separation axioms involving prime open sets, generalised p-closed sets etc. and we introduce p-T_0, p-T_1, p-$T_{1/2}$ axioms and identified the spaces in which all the separation axioms coincides. Also introduced p-continuity, gp-continuity etc. and studied some of its properties. For p-$T_{1/2}$ spaces p-continuity and gp-continuity coincides.

Chapter 4 discusses about spaces in which any two p-open sets intersects which we named as non-prime isolated spaces. Characterization of non-prime isolated spaces is also done. We proved that any topological space can be written as union of its maximal non-prime isolated subsets. Applying concept of prime open sets we introduce p-irreducible, p-door and sub p-maximal spaces and proved that any non-prime isolated sub p-maximal spaces are p-$T_{1/2}$.

In *chapter 5* we introduce semi p-open sets and investigated its properties. Mean while we introduce nowhere p-dense sets and obtained that any semi p-open set can be written as the disjoint union of p-open and nowhere p-dense sets. A genre of mappings involving semi p-open sets are introduced and examined implications amongst each of the mappings. Also obtained that any p-homeomorphic image of a topological space of first category can be written as the union of nowhere p-dense sets in it.

The final chapter considers the concepts of compactness and C-compactness using p-open sets and discusses their characterizations using nets and filters. Also we introduces prime topological spaces and investigated its lattice properties. We also identified the prime elements in the lattice of all topologies on X.

The thesis ends by some concluding remarks and suggestions for further study.

Chapter 2

Generalised p-Closed Sets

2.1 Introduction

Consider any arbitrary set X with a topology τ. Clearly τ always forms a partially ordered set and it forms a lattice since τ is closed under finite intersections and unions. Thus extending the idea of prime element in a lattice to any arbitrary topological space is worthful. Motivated by the definition of prime element we introduced a new collection of open sets called *prime open sets* shortly *p-open sets* in any arbitrary topological space. It is quite natural to study various notions in topology using

Some results of this chapter are included in the following papers.

1. *Vinitha. T and T. P. Johnson* : On Generalised p-Closed Sets , International Journal of Pure and Applied Mathematics, Volume 117 No. 4 2017, 609-619.
2. *Vinitha. T and T. P. Johnson* : Results on Generalised p-closed Sets, Annals of Pure and Applied Mathematics , Vol. 16, No. 1, 2018, 91-103.

prime open sets. Kovar in [26] proved that a subset $A \subseteq L$ where L is a lattice is prime if and only if collection of all elements which does not belong to down set of A always forms a filter. By extending that result to topological space we prove that only prime open sets in any hausdorff topological space is the complement of singleton sets. In this chapter we introduce generalised p-closed sets shortly g-p.closed sets and study some of its properties. We established some equivalent conditions for a set to be g-p.closed. Also we studied when the equivalence of p-closed and g-p.closed happens. We proved that the collection of p-open sets and collection of p-closed sets of any arbitrary topological space coincides if and only if every subset of that topological space is a g-p.closed set. Also we introduced generalised p-open sets as complement of g-p.closed sets and studied some of its properties. Some properties of g-p.closed and g-p.open sets related to subspace topology is also considered.

We begin by stating some preliminary definitions and results useful in this chapter.

Given any lattice L, Kovar in [26] proved an equivalent condition for any element $a \in L$ to be a prime element in terms of filter. The next definition and lemma is due to Kovar[26]

Definition 2.1.1. Let $(P, \leq)$ be any partially ordered set or briefly a poset. For any subset A of a poset $(P, \leq)$ define $\uparrow A = \{x/x \geq y; y \in A\}$ and $\downarrow A = \{x/x \leq y; y \in A\}$.

Lemma 2.1.1. Let $(X, \leq)$ be any poset. Then $L \subseteq X$ is prime if and only if $F = X - \downarrow L$ is a filter.

Definition 2.1.2. [8] Let $(L, \leq)$ be any lattice an element $a \in L$ is said to be an *atom* if there is no 'x' in L between 0 and a where 0 is the smallest element of the lattice.

Definition 2.1.3. [8] Let $(L, \leq)$ be any lattice an element $a \in L$ is said to be a *dual atom* if there is no 'x' in L between 1 and a where 1 is the largest element of the lattice.

Definition 2.1.4. [31] Let (X, T) be any topological space and $A \subseteq X$. A is generalised closed briefly g-closed iff $\overline{A} \subseteq O$ whenever $A \subseteq O$ and O is open.

2.2 p-Open Sets in The Lattice of Open Sets of Any Arbitrary Topological Space.

In this section we introduced the idea of prime open sets in the lattice of open sets of any arbitrary topological space and studied some of its properties. *Lemma : 2.1.1* proved an equivalent condition for an element in a lattice to be prime in terms of filters. Using that result we obtained a necessary and sufficient condition for an open set to be prime open and also identified precisely the p-open sets in a T_2 space.

Let (X, T) be any arbitrary topological space. The open sets in T forms a complete lattice with smallest element 0 and largest element 1 where $0 = \phi$ and $1 = X$. We define an open set in X to be prime as :

Definition 2.2.1. An open set G$\neq$ 1 in T is said to be a *prime open set* if $H \cap K \subseteq G \Rightarrow H \subseteq G$ or $K \subseteq G$; where H, K are open sets in T such that $H \cap K \neq \phi$. Clearly 0 and 1 are prime in T. Prime open sets are denoted by *p-open* sets. Complements of p-open sets are called *p-closed sets*.

Remark 2.2.1. p-open implies open but converse not true.

Example 2.2.1. Let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X , then $\{a, b\}, \{a, c\}$ are p-open sets in (X, T) but $\{a\}$ is not p-open .

Remark 2.2.2. The above example illustrates that intersection of two p-open sets need not be a p-open set.

In next result we analyzes a situation in which intersection of p-open sets is always a p-open set.

Proposition 2.2.1. Let (X, T) be any arbitrary topological space. Intersection of p-open sets is a p-open set if and only if it forms a chain.

Proof. Let $\{G_i : i \in I\}$ be a collection of p-open sets in T. Assume that $G_i \cap G_j$ is p-open for some i, j , to prove that $G_i \subseteq G_j$ or $G_j \subseteq G_i$. On contradiction assume that it does not forms a chain then $G_i \cap G_j = G_k$ for some 'k' ; $k \neq i, j$ and G_k is a p-open set. But since $G_k \subset G_i$ and $G_k \subset G_j$; G_k cannot be prime which is a contradiction . Hence either $G_i \subseteq G_j$ or $G_j \subseteq G_i$ and proof of sufficiency part is obvious.

□

While considering the closure property among intersection of p-open sets we obtained an equivalent condition for it but for union that is not the case :

Proposition 2.2.2. Let (X, T) be any arbitrary topological space. Union of p-open sets is a p-open set if it forms a chain.

Remark 2.2.3. Converse of *Proposition : 2.2.2* is not true, for example let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X , then all open sets are p-open but not a chain.

Since the collection of all open sets of any arbitrary topological space is a lattice , results in *Lemma : 2.1.1* can be extended to any arbitrary topological space (X, T) by defining upper set and lower set of any $G \subseteq X$ as : *upper set of* G denoted by $U(G)$ is defined as $U(G) = \{A \in T / A \supseteq G\}$ and *lower set of* G denoted by $L(G)$ is defined as $L(G) = \{A \in T / A \subseteq G\}$ and we obtain the result in *Lemma : 2.1.1* as :

Proposition 2.2.3. Let (X, T) be any arbitrary topological space and $G \in T$. G is p-open if and only if $T - L(G)$ is a filter ; where $T - L(G)$ is the collection of open sets which does not belongs to $L(G)$.

Using *Proposition : 2.2.3* we can say what are precisely the p-open sets in a hausdorff space.

Theorem 2.2.1. *Let (X, T) be a hausdorff space and $x \in X$ then the only p-open sets are $X - \{x\}$.*

Proof. Let $G \in T, G \neq X - \{x\}$ for any $x \in X$ and G be prime. There exists at least two elements $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $x_1, x_2 \notin G$. Since X is T_2 there exists disjoint open sets U_1, U_2 such that $x_1 \in U_1$ and $x_2 \in U_2$. Clearly $U_1 \in T - L(G)$ and $U_2 \in T - L(G)$ implies $U_1 \cap U_2 = \phi \in T - L(G)$ which is not possible since G is prime. Hence

$G = X - \{x\}$ always. □

Theorem 2.2.2. *Let (X, T) be a topological space and $G \in T$. Then G is not prime in T if and only if there exists two open super sets for G say G_1 and G_2 such that $G_1 \cap G_2 \subseteq G$*

Proof. Sufficiency part easily follows since existence of such open super sets trivially contradicts the definition of prime. To prove necessary part assume that G is not prime and let G_1 and G_2 be two open sets in T such that $G_1 \cap G_2 \subseteq G$. We have to prove that there exists two open super sets for G whose intersection is a subset of G. Given $G_1 \cap G_2 \subseteq G$ and G is not prime then both G_1 and G_2 are not subsets of G by definition of prime. Now consider G_1 and $G \cup G_2$ both of them are open sets and if G_1 is a super set of G, then the open sets G_1 and $G \cup G_2$ will be the required sets since
$(G \cup G_2) \cap G_1 = (G \cap G_1) \cup (G_2 \cap G_1) = G$.
Same is the case if G_2 is a super set of G. Now if both of them are not super sets of G but $G_1 \cap G_2 \subseteq G$ then consider $G \cup G_1$ and $G \cup G_2$ both are open super sets of G_1 and G_2 and then
$(G \cup G_1) \cap (G \cup G_2)$
$= G \cup (G \cap G_1) \cup (G \cap G_2) \cup (G_1 \cap G_2)$
$\subseteq G$
Hence there exists two open super sets of G always such that $G_1 \cap G_2 \subseteq G$ whenever G is not prime. □

2.3 p-closure, p-interior and p-open sets in relative topology

In order to proceed the study of notions in topology using p-open sets we have to define some basic notions like closure , interior etc using p-open sets. Throughout this section we try to study such notions and its properties by including p-open sets only.

Since the intersection of p-closed sets need not be p-closed there arise possibility for existence of more than one non-comparable p-closed super sets corresponding to any set. In such cases notion of minimal p-closed super sets have to be considered.

Definition 2.3.1. Let (X, T) be a topological space and let $A \subseteq X$, then the *p-closure* of A with respect to T is defined as the minimal p-closed super set of A in X and is denoted as p-$cl(A)$.

Example 2.3.1. Let $X = \{a, b, c, d\}$ and
$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ be a topology on X then p-cl$(\{c, d\}) = \{b, c, d\}$ and $\{a, c, d\}$.

Remark 2.3.1. Let (X, T) be a topological space for any $A \subseteq X$, $\overline{A} \subseteq$ p-$cl(A)$ where $\overline{A}$ is the closure of A with respect to T.

Proposition 2.3.1. Let (X, T) be a topological space, then for every p-open set $A \subseteq X$ there always exists a unique p-closed set containing A.

Proof. Let A be a p-open set and B^c be a p-closed super set of A.

On contradiction let C^c be another p-closed super set of A, then clearly B, C are p-open sets of X. If $B \cap C = \phi$, then A will not be prime since A can be written as intersection of open sets $A \cup B$ and $A \cup C$ which is not possible. If $B \cap C \neq \phi$, then B can be written as the intersection of open sets $B \cup C$ and $A \cup B$, that is B is not prime which is a contradiction. Hence there does not exists more than one p-closed super set of A. □

Remark 2.3.2. For a p-open set definition of p-closure becomes smallest p-closed set containing it.

Proposition 2.3.2. p-$cl(A) \cup p$-$cl(B) \subseteq p$-$cl(A \cup B)$; for any two subsets A and B of X.

Proof. Clearly

$$A \subseteq A \cup B; B \subseteq A \cup B \tag{2.1}$$

$$\Rightarrow p\text{-}cl(A) \subseteq p\text{-}cl(A \cup B); p\text{-}cl(B) \subseteq p\text{-}cl(A \cup B) \tag{2.2}$$

Thus *2.2* implies p-$cl(A) \cup p$-$cl(B) \subseteq p$-$cl(A \cup B)$. □

Remark 2.3.3. Let (X, T) be any arbitrary topological space and let $A, B \subseteq X$ then p-$cl(A \cup B) \nsubseteq p$-$cl(A) \cup p$-$cl(B)$; for example consider the discrete topological space (X, D) and let $A = \{x_1\}$, $B = \{x_2\}$ where $x_1, x_2 \in X$. Then p-$cl(A) = \{x_1\}$, p-$cl(B) = \{x_2\}$ and p-$cl(A \cup B) = X$. Hence p-$cl(A \cup B) \nsubseteq p$-$cl(A) \cup p$-$cl(B)$.

Proposition 2.3.3. Let (X, T) be a topological space, $A \subseteq X$ be a p-open set in X and $x \in X$. Then $x \in$p-$cl(A)$ if and only if every p-open set containing 'x' intersects A.

Proof. If $x \notin$p-$cl(A)$, then $U = X-$p-$cl(A)$ is a p-open set containing 'x' which does not intersects A. Thus if every p-open set containing 'x' intersects A then $x \in p\text{-}cl(A)$ and sufficiency part is proved. Next we have to prove that $x \in p\text{-}cl(A)$ implies every p-open set containing 'x' intersects A. On contradiction we assume that there exists a p-open set U containing 'x' which does not intersect A, which implies $X - U$ is a p-closed set containing A. Then either p-$cl(A) \subset X - U$ or $X - U$ itself happens to be p-$cl(A)$ by definition of p-closure and by Proposition : 2.3.1 which implies $x \notin$p-$cl(A)$ in both cases ; obtaining a contradiction. Hence necessary part is also proved. □

Remark 2.3.4. In general the above proposition is not true. Only the sufficiency part always holds, necessary part need not be true for non-prime open sets. For example Let $X = \{a, b, c, d\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ be a topology on X then p-cl($\{c, d\}$) = $\{b, c, d\}$ and $\{a, c, d\}$. Here 'a' belongs to p-$cl(\{c, d\})$ but it does not satisfies condition of the proposition.

Proposition : 2.3.3 gives a condition for a point to be in p-closure of any set and that lead us to the definition of p-limit point analogous to limit point of any set.

Definition 2.3.2. Let (X, T) be a topological space and $A \subseteq X$; an element $x \in X$ is called a *p-limit point/p-cluster point* of $A \subseteq X$ if every p-open set containing 'x' intersects A.

Remark 2.3.5. Every limit point is a p-limit point but converse not true. For example let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$.

Then (X, τ) is a topological space and if $A = \{b, d\}$ then 'a' is a p-limit point but not a limit point of A.

Proposition 2.3.4. Let A be a p-open subset of a topological space X, then A is p-closed if and only if A contains all its p-limit points.

Proof. Proof is trivial by definition of p-limit point and *Proposition : 2.3.3.* □

Analogous to the definition of p-closure of any arbitrary set, we can define the p-interior of any arbitrary set in a topological space using the concept of maximal p-open subsets and we proved that for any p-closed set there always exists a unique p-open set contained in it.

Definition 2.3.3. Let (X, T) be a topological space and let $A \subseteq X$, then the *p-interior* of A with respect to T is defined as the maximal p-open subset of A in X and is denoted as p-int(A).

Proposition 2.3.5. Let (X, T) be a topological space, then for every p-closed set there always exists a unique p-open set contained in A.

Proof. Proof is similar to that of *Proposition : 2.3.1* by taking dual. □

We conclude this section by studying the behavior of p-open sets in the relative topology.

Theorem 2.3.1. *Let (X, T) be a topological space and $Y \subseteq X$. If U is p-open in X then $U \cap Y$ is a p-open set in Y.*

Proof. Let X be any set with topology T, $Y \subseteq X$ and let T_Y be the relative topology on Y. Given U as a p-open set in X then by definition of p-open set

$$H \cap K \subseteq U \Rightarrow H \subseteq U \text{ or } K \subseteq U; \ H, K \in T, H \cap K \neq \phi \qquad (2.3)$$

We have to prove that $U \cap Y$ prime in Y ; that is to prove that $H' \cap K' \subseteq U \cap Y \Rightarrow H' \subseteq U \cap Y$ or $K' \subseteq U \cap Y$ where $H', K' \in T_Y$, $H' \cap K' \neq \phi$; but $H', K' \in T_Y$ implies $H' = H \cap Y$ for some $H \in T$ and $K' = K \cap Y$ for some $K \in T$. From (2.3) we have $(H \cap K) \cap Y \subseteq U \cap Y \Rightarrow H \cap Y \subseteq U \cap Y$ or $K \cap Y \subseteq U \cap Y$
that is $H' \cap K' \subseteq U \cap Y \Rightarrow H' \subseteq U \cap Y$ or $K' \subseteq U \cap Y$ where $H', K' \in T_Y$. Hence $U \cap Y$ is prime in T_Y. □

Proposition 2.3.6. Let (X, T) be a topological space and Y be a subspace of X. Then a set A is p-closed in Y if and only if it equals the intersection of a p-closed subset of X with Y.

Proof. For the necessary part assume that A is p-closed in Y, then $Y - A$ is p-open in Y that is $Y - A = U \cap Y$ where U is open in X. U must be prime because if U is not prime applying *Theorem : 2.2.2* we obtain that $U \cap Y = Y - A$ is not prime in Y but that is not possible, which implies $A = Y \cap (X - U)$. Hence A equals intersection of a p-closed set in X with Y. For sufficiency assume that A equals intersection of a p-closed set in X with Y ; that is $A = C \cap Y$ where C is p-closed in X which implies $Y - A = (X - C) \cap Y$. Hence $Y - A$ is p-open in Y by *Theorem : 2.3.1* so that A is p-closed in Y. □

Proposition 2.3.7. Let (X, T) be a topological space, Y be a subspace of X and A be any subset of Y. Let p-$cl(A)_X$ denote the p-closure of A in X. Then p-$cl(A)_Y$ in Y equals p-$cl(A)_X \cap Y$.

Proof. Let B denote the p-$cl(A)_Y$ in Y. p-$cl(A)_X$ is p-closed in X so p-$cl(A)_X \cap Y$ is p-closed in Y. Clearly $A \subset$p-$cl(A)_X \cap Y$ which implies

$$B \subseteq p\text{-}cl(A)_X \cap Y \tag{2.4}$$

On the other hand, B is p-closed in Y. Hence by above proposition $B = C \cap Y$ for some p-closed set C in X, then C is a p-closed set in X containing A and hence p-$cl(A)_X \subset C$ which implies

$$p\text{-}cl(A)_X \cap Y \subseteq C \cap Y = B \tag{2.5}$$

Hence (2.4) and (2.5) implies $B =$p-$cl(A)_X \cap Y$. □

2.4 Generalised p-closed sets

In *section 2.2* it was proved that only p-open sets in any T_2 space are the complement of singleton sets. Hence the existence of spaces with disjoint p-open sets corresponding to distinct points will not happen. Thus study of higher separation axioms using p-open sets is not possible which lead us to consider more weaker separation axioms like $T_{1/2}$ axiom for which the basic root is the generalised closed sets. In 1970 Levine [31] introduced the concept of generalised closed sets as a generalisation of closed sets. In this section we define generalised p-closed sets as a generalisation of p-closed sets. Any generalised p-closed set is generalised closed set and hence it is also a generalised concept of generalised closed sets.

Definition 2.4.1. Let (X, T) be a topological space and $A \subseteq X$ then A is said to be *generalised p-closed* shortly *g-p.closed* if p-$cl(A) \subseteq O$ whenever $A \subseteq O$ and O p-open in X.

Remark 2.4.1. Any p-closed set is generalised p-closed and converse need not be true. For example let $X = \{1, 2, 3, 4\}$ and $T = \{X, \phi, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ be a topology on X. Then $\{3, 4\}$ is generalised p-closed but not p-closed.

For any arbitrary topological space the existence of dual atoms are inevitable and dual atoms are always p-open. Dually the existence of p-closed atoms are also inevitable for any topological space. Using this result we proved an equivalent condition for a set to be g-p.closed in terms of p-closed sets and there by we proved that g-p.closed set is a generalisation of g-closed set. The next two theorems prove such results.

Theorem 2.4.1. *Let (X, T) be a topological space and $A \subseteq X$ then A is generalised p-closed if and only if p-$cl(A) - A$ contains no non-empty p-closed set.*

Proof. Assume that $A \subseteq X$ is g-p.closed to prove that p-$cl(A) - A$ contains no non-empty p-closed set. Let F be a non-empty p-closed set such that $F \subseteq p\text{-}cl(A) - A$ then $A \subseteq F^c$ which implies p-$cl(A) \subseteq F^c$ implies $F \subseteq X-$p-$cl(A)$ which is not possible since $F \subseteq p\text{-}cl(A)$ hence there does not exists such a non-empty p-closed subset . Conversely we assume that p-$cl(A) - A$ contains no non-empty p-closed set and let $A \subseteq O, O$ is p-open in X to prove that p-$cl(A) \subseteq O$. On contradiction let p-$cl(A) \cap O^c \neq \phi$ then p-$cl(A) \cap O^c$ is a closed subset of p-$cl(A) - A$. If p-$cl(A) - O$ is prime nothing more to prove , otherwise there exists closed subsets G, H such that $G \subseteq p\text{-}cl(A) \cap O^c$ and $H \subseteq p\text{-}cl(A) \cap O^c$ again if either G or H is prime proof is over otherwise continue until the atom element of T is reached and since all atoms are p-closed ; p-$cl(A) - A$ always contains a non-empty p-closed set which is not possible thus A is g-p.closed. □

Theorem 2.4.2. *Let (X, T) be a topological space and $A \subseteq X$ be such that A is generalised p-closed then A is g-closed.*

Proof. Given A as a g-p.closed set. Let A be not g-closed then $\overline{A} - A$ contains a non-empty closed set and $\overline{A} - A \subseteq p\text{-}cl(A) - A$ which implies p-$cl(A) - A$ contains non-empty closed set and proceeding as in the proof of above theorem p-$cl(A) - A$ contains a non-empty p-closed set which implies A is not g-p.closed. Hence g-p.closed implies g-closed. □

Remark 2.4.2. Converse of above theorem is not true for example ; let (X, D) be a discrete topological space with cardinality greater than two, consider $A = \{x, y\}$ $x, y \in X$ then A is g-closed but not g-p.closed.

Theorem 2.4.3. *Let (X, T) be a topological space, then the following conditions are equivalent for any subset $A \subseteq X$*

1. *A is g-p.closed.*
2. *For each x belongs to p-$cl(A)$, p-$cl(\{x\}) \cap A \neq \phi$.*
3. *$B \subseteq$p-$cl(A) - A$, $B \subseteq X$ and B p-closed implies $B = \phi$.*

Proof. We proceed through the following steps :
Step 1: Proof of (1) $\Rightarrow$ (2)

Suppose $x \in p\text{-}cl(A)$ and A is g-p.closed. To prove that p-$cl(\{x\}) \cap A \neq \phi$. On contradiction we assume that p-$cl(\{x\}) \cap A = \phi$ which implies $A \subseteq$ (p-$cl(\{x\}))^c$. Since A is g-p.closed and (p-$cl(\{x\}))^c$ is p-open we obtain p-$cl(A) \subseteq (p\text{-}cl(\{x\}))^c$ implies $x \notin p\text{-}cl(A)$ which is not possible. Hence p-$cl(\{x\}) \cap A \neq \phi$ always for $x \in p\text{-}cl(A)$.

Step 2: To prove (2) $\Rightarrow$ (3)

Assume that p-$cl(A) - A$ contains a non-empty p-closed set C and let $x \in C$; that is

$$x \in C \subseteq p\text{-}cl(A) - A \tag{2.6}$$

(2.6) implies $x \in p\text{-}cl(A) \Rightarrow p\text{-}cl(\{x\}) \cap A \neq \phi$.

Thus $\phi \neq p\text{-}cl(\{x\}) \cap A$

$\subseteq C \cap A$

$\subseteq (p\text{-}cl(A) - A) \cap A = \phi$. Hence we obtain a contradiction and therefore the only possibility is $C = \phi$. Thus p-$cl(A) - A$ contains no non-empty p-closed set.

Step 3: (3) $\Rightarrow$ (1) holds by *Theorem : 2.4.1*

Hence by steps 1, 2 and 3 ; equivalent conditions for a set to be g-p.closed are verified. □

In *Remark : 2.4.1* we already see that g-p.closed sets need not be p-closed. In next theorem we get a necessary and sufficient condition for the equivalence of p-closed and g-p.closed sets.

Theorem 2.4.4. *Let (X, T) be a topological space and $A \subseteq X$ be a g-p.closed set, then A is p-closed if and only if p-$cl(A) - A$ is p-closed.*

Proof. Given A as a g-p.closed set. Assume that A is p-closed then p-$cl(A) - A$ is empty set which is always p-closed. Conversely, p-$cl(A) - A$

is p-closed then it is a p-closed subset of itself and p-$cl(A) - A$ is empty. Hence A is p-closed. □

Remark 2.4.3. Union of two g-p.closed sets need not be g-p.closed, for example let (X, D) be a discrete topological space with cardinality greater than two and also let $A = \{x_1\}, B = \{x_2\}$ both A and B are g-p.closed but $A \cup B$ is not g-p.closed.

Remark 2.4.4. Intersection of two g-p.closed sets need not be g-p.closed, for example let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}\}$ be a topology on X. $A = \{a, b\}$ and $B = \{a, c\}$ are g-p.closed sets but $A \cap B$ is not g-p.closed.

Theorem 2.4.5. *Let (X, T) be a topological space. $A \subseteq Y \subseteq X$ and A is g-p.closed in X. Then A is g-p.closed relative to Y.*

Proof. Let O' be a p-open set in Y such that $A \subseteq O'$. We have to prove that p-$cl(A)_Y \subseteq O'$ where p-$cl(A)_Y$ is the p-closure of A with respect to relative topology on Y . O' p-open in Y implies that $O' = O \cap Y$ where O is p-open in X. Then $A \subseteq O'$ implies $A \subseteq Y \cap O$ implies $A \subseteq O$ which in turn implies p-$cl(A)_X \subseteq O$, since A is given to be g-p.closed in X. From which we obtain that p-$cl(A)_X \cap Y \subseteq O \cap Y$ which implies p-$cl(A)_Y \subseteq O \cap Y$ by *Proposition : 2.3.7.* Hence A is g-p.closed with respect to Y. □

Theorem 2.4.6. *Let (X, T) be a topological space. If A is g-p.closed and $A \subseteq B \subseteq$p-$cl(A)$ then B is g-p.closed.*

Proof. It is given that A is g-p.closed therefore p-$cl(A) - A$ has no non-empty p-closed sets in it. Clearly p-$cl(B) - B \subseteq$p-$cl(A) - A$ hence p-$cl(B) - B$ also contains no non empty p-closed set, that is B is g-p.closed. □

The next result analyses a condition for the equivalence of the collection of all p-open and the collection of all p-closed sets in terms of g-p.closed sets for any arbitrary topological space.

Theorem 2.4.7. *Let (X, T) be a topological space , $\mathcal{P}$ denote the collection of p-open sets in X and $\mathcal{F}$ the collection of p-closed sets in X then $\mathcal{P} = \mathcal{F}$ if and only if every subset of X is a g-p.closed set.*

Proof. First assume that $\mathcal{P} = \mathcal{F}$ to prove that every subset of X is g-p.closed. Let A be any subset of X such that $A \subseteq O$ where O is p-open in X which trivially implies p-$cl(A) \subseteq O$ since O is also p-closed by our assumption. Hence A is g-p.closed and since A is arbitrary every subset is g-p.closed. For sufficiency assume that every subset of X is g-p.closed. We have to prove that $\mathcal{P} = \mathcal{F}$. Let $O \in \mathcal{P}$ which implies O is a subset of p-open set O itself and since it is g-p.closed, p-$cl(O) \subseteq O$. Thus O is p-closed and $\mathcal{P} \subseteq \mathcal{F}$. Now let $O \in \mathcal{F}$ then $O^c \in \mathcal{P} \subseteq \mathcal{F}$. Hence $O \in \mathcal{P}$ and $\mathcal{P} \supseteq \mathcal{F}$. Therefore $\mathcal{P} = \mathcal{F}$. □

Theorem 2.4.8. *Let (X, T) be a topological space and let $A \subseteq X$, then A is g-p.closed if and only if $A = P - G$ where P is a p-closed subset of X and G is such that G contains no non-empty p-closed subset of X.*

Proof. Assume that A is g-p.closed to prove that $A = P - G$ where P is p-closed and G is such that G contains no non-empty p-closed subset of X. Now take $P = p\text{-}cl(A)$ and $G = p\text{-}cl(A) - A$ then P is a p-closed set and since A is g-p.closed ; G contains no non-empty p-closed set .

Thus P and G are the required sets. Now consider $P - G = p\text{-}cl(A) - (p\text{-}cl(A) - A)$=A ; that is A is of the required form and hence the necessary part is proved.
For sufficiency part let $A \subseteq X$ and A be of the form $A = P - G$ where P is p-closed and G contains no non empty p-closed set. We have to prove that A is g-p.closed. Let $A \subseteq O$ where 'O' is a p-open subset of X to prove that p-$cl(A) \subseteq O$. P and O^c are p-closed subsets of X hence $P \cap O^c$ is a closed subset of X and moreover $P \cap O^c$ is a closed subset of G, then two cases arise either $P \cap O^c$ is p-closed or it is only a closed set but not a p-closed set. If the second case occurs, since the existence of atoms which are p-closed is inevitable for a topological space ; $P \cap O^c$ contains at least atoms in T. Thus by definition of prime, in both cases $P \cap O^c$ contains a non-empty p-closed subset but as $P \cap O^c \subseteq G$ and so G contains a non-empty p-closed set if $P \cap O^c$ contains. Hence the only possibility is that $P \cap O^c = \phi$ which implies $P \subseteq O$. But $A \subseteq P \Rightarrow p\text{-}cl(A) \subseteq P \Rightarrow p\text{-}cl(A) \subseteq O$ and hence A is g-p.closed. □

Existence of g-p.closed sets itself implies the existence of its complement what we call it as g-p.open sets. In the forthcoming part of this section we study about g-p.open sets and its properties. Also derived some equivalent condition for a set to be g-p.open.

Definition 2.4.2. A set $A \subseteq X$ in a topological space (X, T) is said to be *generalised p-open* shortly *g-p.open* if A^c is g-p.closed.

Theorem 2.4.9. *Let (X, T) be a topological space and $A \subseteq X$ be a g-p.open set then it is g-open.*

Proof. Given that A is g-p.open $\Rightarrow A^c$ is g-p.closed , but then A^c is g-closed by *Theorem : 2.4.2* . Thus A is g-open. □

Remark 2.4.5. Converse of *Theorem : 2.4.9* is not true; Consider $X = \{x, y, z\}$ with discrete topology. $A = X - \{x, y\}$ is g-open but not g-p.open.

Theorem 2.4.10. *Let (X, T) be a topological space and let $A \subseteq X$. Then A is g-p.open if and only if $F \subseteq$ p-$int(A)$ whenever F is p-closed and $F \subseteq A$.*

Proof. Assume that A is g-p.open which implies A^c is g-p.closed
$\Rightarrow$ p-$cl(A^c) \subseteq O$ whenever $A^c \subseteq O$ and O is p-open.
$\Rightarrow O^c \subseteq [$p-$cl(A^c)]^c$ whenever $A^c \subseteq O$ and O is p-open.
$\Rightarrow O^c \subseteq$ p-$int(A)$ whenever $A^c \subseteq O$ and O is p-open. By taking $F = O^c$ as the p-closed set , the necessary part is proved.
Conversely we assume that F is p-closed and
$F \subseteq$ p-$int(A)$ whenever $F \subseteq A$
$\Rightarrow [$p-$int(A)]^c \subseteq F^c$ whenever $A^c \subseteq F^c$
$\Rightarrow A^c$ is g-p.closed. Thus A is g-p.open. □

Theorem 2.4.11. *Let (X, T) be a topological space and let $A \subseteq X$, then A is g-p.open if and only if $O = X$ whenever O is p-open and p-$int(A) \cup A^c \subseteq O$.*

Proof. Suppose A is g-p.open and p-$int(A) \cup A^c \subseteq O$ whenever O is p-open
$\Rightarrow O^c \subseteq [$p-$int(A) \cup A^c]^c$
$= ($p-$int(A))^c \cap A =$ p-$cl(A^c) - A^c$.
Hence p-$cl(A^c) - A^c$ contains a non-empty p-closed set but A^c is g-p.closed

and thus $O^c = \phi \Rightarrow O = X$.
For sufficiency part assume F as a p-closed set and $F \subseteq A$. It is enough to prove that $F \subseteq$ *p-int*(A) for showing A is g-p.open. Consider p-*int*$(A) \cup A^c \subseteq$ *p-int*$(A) \cup F^c$. Clearly p-*int*$(A) \cup F^c$ is open , then there arise two cases :

1. If p-*int*$(A) \cup F^c$ is prime then by assumption p-*int*$(A) \cup F^c = X$ and hence $F \subseteq$ *p-int*(A) which implies A is g-p.open.

2. If p-*int*$(A) \cup F^c$ is not prime then there exists two open sets G_1 and G_2 containing p-*int*$(A) \cup F^c$. Now if at least one of G_1 or G_2 is prime then by assumption the corresponding set becomes equal to X which is not possible by definition of prime. If both G_1 and G_2 are not prime then again there exists G_3 and G_4 containing the corresponding non-prime open set and again by the same reasoning as above that is not possible. Continuing this argument we reach the conclusion that whenever there exists open set containing p-*int*$(A) \cup F^c$ which is not prime, that will lead to a contradiction. Hence the only possibility is that p-*int*$(A) \cup F^c$ is prime and hence the result follows from case 1.

□

Theorem 2.4.12. *Let (X, T) be a topological space and $A \subseteq X$. If A is g-p.closed then p-cl$(A) - A$ is g-p.open.*

Proof. Assume that A is g-p.closed to prove that p-$cl(A) - A$ is g-p.open. That is to prove that $F \subseteq$ *p-cl*$(A) - A \Rightarrow F \subseteq$ *p-int*(*p-cl*$(A) - A$)

whenever F is p-closed. But $F \subseteq p\text{-}cl(A) - A$ implies $F = \phi$, since A is g-p.closed and F is p-closed . Hence result trivially follows. □

Proposition 2.4.1. Let (X, T) be a topological space and $A, B \subseteq X$. If p-$int(A) \subseteq B \subseteq A$ and A is g-p.open then B is g-p.open.

Proof. Given that p-$int(A) \subseteq B \subseteq A \Rightarrow A^c \subseteq B^c \subseteq (p\text{-}int(A))^c \Rightarrow A^c \subseteq B^c \subseteq p\text{-}cl(A^c)$
Since A is given to be g-p.open, A^c is g-p.closed which implies B^c is g-p.closed by Theorem : 2.4.6. Hence B is g-p.open. □

Chapter 3

Separation Axioms Involving p-Open Sets.

3.1 Introduction

Continuity and continuous transformations happening to sets is the basic key behind the development of the theory of topology. In this chapter

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Some results of this chapter are included in the following paper.

1. *Vinitha.T and T.P.Johnson* : On Generalised p-Closed Sets , International Journal of Pure and Applied Mathematics, Volume 117 No. 4 2017, 609-619.
2. *Vinitha.T and T.P.Johnson* : p-Compactness and C-p.compactness, Global Journal of Pure and Applied Mathematics, Volume 13, No.9 (2017), pp.5539-5550.
3. *Vinitha.T and T.P.Johnson* : Results on Generalised p-closed Sets, Annals of Pure and Applied Mathematics , Vol. 16, No. 1, 2018, 91-103.

we consider various mappings involving p-open, g-p.open and g-p.closed sets. Also try to identify the relation between such functions. Levine defined g-continuous functions as functions in which the inverse image of closed sets are g-closed. Analogously we define gp-continuous functions and established that concepts of p-continuity and gp-continuity are independent. Later in section 3 we proved that they coincides for p-$T_{1/2}$ spaces, the spaces for which g-p.closed sets and p-closed sets coincide. Also proved some necessary and sufficient conditions for a space to be gp-continuous. Meanwhile we introduced p-continuous functions, p-topological property etc and proved that p-$T_{1/2}$ is a p-topological property. Also any p-topological property happens to be a topological property.

In chapter 2 we see that for T_2 spaces what the p-open sets are and that implies consideration of higher separation axioms using p-open sets is not possible. However we can consider some weaker separation axioms using p-open sets and g-p.closed sets defined in chapter 2. The real motive of this chapter is to introduce such separation axioms and identify the spaces in which all this separation axioms coincides. We introduce p-T_0, p-$T_{1/2}$ and p-T_1 axioms and obtained some equivalent condition for this separation axioms. Also introduced prime symmetric spaces and observed that all the separation axioms p-T_0, p-$T_{1/2}$, p-T_1, T_1, $T_{1/2}$ and T_0 coincides for prime symmetric spaces. Also proved that corresponding to any arbitrary topological space there always exists a finer adjacent p-$T_{1/2}$ topological space.

Here are some definitions and results useful in this chapter.

Definition 3.1.1. [31] Let (X, T), (Y, T') be two topological spaces. Then $f : (X, T) \to (Y, T')$ is g-continuous if inverse image of closed sets are g-closed.

Definition 3.1.2. [31] A topological space (X, T) is said to be $T_{1/2}$ if every g-closed set is closed.

Definition 3.1.3. [31] A topological space (X, T) is said to be *symmetric* if for $x, y \in X$; $x \in \overline{\{y\}}$ implies that $y \in \overline{\{x\}}$

Theorem 3.1.1. *[31] If (X, T) is a symmetric space , then (X, T) is T_0 iff $T_{1/2}$ iff T_1.*

3.2 Mappings Involving p-open, g-p.open and g-p.closed Sets

The most inevitable part in the theory of topological spaces is the continuous mappings and various deformations happens to sets under this continuous mappings. Throughout this section we consider various mappings involving p-open, g-p.open and g-p.closed sets. We begin with the mappings involving p-open sets only.

Definition 3.2.1. Let $(X, T), (Y, T^{'})$ be two topological spaces and let $f : (X, T) \rightarrow (Y, T^{'})$ be a mapping between this two topological spaces. f is called *p-continuous* if the inverse image of p-open sets in $T^{'}$ are p-open in T. f is said to be p-open (p-closed) if p-open (p-closed) sets are mapped on to p-open (p-closed) sets only.

Even though all p-open sets are open, the concepts of continuity and p-continuity are independent of each other as the two succeeding remarks illustrates.

Remark 3.2.1. Let $(X,T),(Y,T^{'})$ be two topological spaces and let $f:(X,T)\rightarrow(Y,T^{'})$ be a continuous mapping between this two topological spaces. Then f need not be p-continuous ; for example let $X=Y=\{a,b,c\}$ and let $f:(X,D)\rightarrow(Y,T)$ be the identity mapping such that D is the discrete topology on X and $T=\{X,\phi,\{a\},\{a,b\}\}$ then f is continuous but not p-continuous.

Remark 3.2.2. Let $(X,T),(Y,T^{'})$ be two topological spaces and let $f:(X,T)\rightarrow(Y,T^{'})$ be a p-continuous mapping between this two topological spaces. Then f need not be continuous ; for example Let $X_1=R$ with co finite topology and $X_2=R$ with discrete topology. Identity function $f:X_1\rightarrow X_2$ is p-continuous but not continuous.

Now we are in a state to define p-homeomorphism and properties of sets preserved under p-homeomorphism are named as p-topological property. Clearly homeomorphism implies p-homeomorphism but converse is not true. Hence any p-topological property will become a topological property also.

Definition 3.2.2. Let $(X,T),(Y,T^{'})$ be two topological spaces and $f:(X,T)\rightarrow(Y,T^{'})$ be a mapping. f is said to be a *p-homeomorphism* if f is one-one, onto and both f, f^{-1} are p-continuous.

Theorem 3.2.1. *Homeomorphism implies p-homeomorphism .*

Proof. Let $(X,T),(Y,T^{'})$ be two topological spaces. Let $f:(X,T)\rightarrow(Y,T^{'})$ be a homeomorphism between this two topological spaces. Let G be a prime open set in T' to prove that $f^{-1}(G)$ is prime in T. Al-

ways $f^{-1}(G)$ is open in T since f is continuous. On contradiction assume that $f^{-1}(G)$ is not prime in T then there exists two open subsets $f^{-1}(H), f^{-1}(K)$ of X such that $f^{-1}(G) \subset f^{-1}(H)$, $f^{-1}(G) \subset f^{-1}(K)$ and $f^{-1}(H) \cap f^{-1}(K) \subseteq f^{-1}(G)$
$\Rightarrow f(f^{-1}(G)) \subset f(f^{-1}(H))$, $f(f^{-1}(G)) \subset f(f^{-1}(K))$ and $f(f^{-1}(H) \cap f^{-1}(K)) \subseteq f(f^{-1}(G))$
$\Rightarrow G \subset H$, $G \subset K$ and $H \cap K \subset G$, where H and K are open subsets of Y since f is an open continuous one one mapping . Hence G is not prime in T which is not possible and therefore inverse image of p-open sets are p-open that is f is p-continuous. Similarly we can prove that f^{-1} is also p-continuous. Hence f is a p-homeomorphism. □

Remark 3.2.3. Converse of above theorem is not true and example in *Remark : 3.2.2* illustrates it.

Definition 3.2.3. A property P is said to be a *p-topological property*, if whenever a space X has that property P then any space p-homeomorphic to that space also has the same property P. Hence any p-topological property is a topological property.

Let (X, T), (Y, T') be two topological spaces and $f : X \to Y$ be a p-continuous, p-closed function. Then we can prove that any g-p.closed set in X can be carried on to g-p.closed set in Y. But in order to prove that we have to first consider the following lemma :

Lemma 3.2.1. A function $f : X \to Y$ is p-continuous if and only if for every $A \subset X$; $f(p\text{-}cl(A)) \subseteq p\text{-}cl(f(A))$.

Proof. Assume that $f : X \to Y$ is p-continuous. Now consider $f(A) \subseteq p\text{-}cl(f(A))$

$\Rightarrow A \subseteq f^{-1}(f(A)) \subseteq f^{-1}[p\text{-}cl(f(A))]$.
Since f is p-continuous and p-$cl(f(A))$ is p-closed, $f^{-1}[p\text{-}cl(f(A))]$ is a p-closed set containing A
$\Rightarrow p\text{-}cl(A) \subseteq f^{-1}[p\text{-}cl(f(A))]$
$\Rightarrow f(p\text{-}cl(A)) \subseteq f(f^{-1}[p\text{-}cl(f(A))]) = p\text{-}cl(f(A))$. Hence $f(p\text{-}cl(A)) \subseteq p\text{-}cl(f(A))$.
Conversely assume that $f(p\text{-}cl(A)) \subseteq p\text{-}cl(f(A))$ to prove that f is p-continuous. Let B be a p-closed set in Y it is enough to prove that $f^{-1}(B)$ is p-closed in X. That is to prove that p-$cl(f^{-1}(B)) = f^{-1}(B)$.
Consider $f(p\text{-}cl(f^{-1}(B)) \subseteq p\text{-}cl(f(f^{-1}(B))) = p\text{-}cl(B) = B$
$\Rightarrow p\text{-}cl[f^{-1}(B)] \subseteq f^{-1}(B)$
$\Rightarrow f^{-1}(B)$ is p-closed and hence f is p-continuous. □

Theorem 3.2.2. *Let $(X,T),(Y,T')$ be two topological spaces. If A is a g-p.closed subset of X and $f : X \to Y$ be a p-continuous and p-closed function, then $f(A)$ is g-p.closed in Y.*

Proof. Let A be a g-p.closed subset and f be p-continuous and p-closed. Assume that $f(A) \subseteq O'$ where O' is p-open in Y which implies $A \subseteq f^{-1}(O')$. Since f is p-continuous and O' is p-open in Y, $f^{-1}(O')$ is p-open in X and again since A is g-p.closed, $p\text{-}cl(A) \subseteq f^{-1}(O')$ implies

$$f(p\text{-}cl(A)) \subseteq O' \tag{3.1}$$

but $f(p\text{-}cl(A))$ is p-closed and for any set $A \subseteq X$, $A \subseteq p\text{-}cl(A)$ which implies p-$cl(f(A)) \subseteq p\text{-}cl(f(p\text{-}cl(A))) = f(p\text{-}cl(A)) \subseteq O'$ by *Lemma : 3.2.1.* and (3.1)
$\Rightarrow f(A)$ is g-p.closed. □

Theorem 3.2.3. *Let (X,T), (Y,T') be any two topological spaces* *and $f : (X,T) \to (Y,T')$ be a p-continuous, p-closed mapping. If B is a g-p.closed subset of Y then $f^{-1}(B)$ is a g-p.closed subset of X.*

Proof. Given B is a g-p.closed subset of Y we have to prove that $f^{-1}(B)$ is g-p.closed in X , that is whenever $f^{-1}(B) \subseteq O$ where O is a p-open set in X we have to prove that p-$cl(f^{-1}(B)) \subseteq O$. For that it is enough to prove that p-$cl(f^{-1}(B)) \cap O^c = \phi$.
But $f(p\text{-}cl(f^{-1}(B)) \cap O^c) \subseteq p\text{-}cl(B) - B$. Since B is g-p.closed the only possibility is that $f(p\text{-}cl(f^{-1}(B)) \cap O^c) = \phi$ which implies p-$cl(f^{-1}(B)) \subseteq O$ whenever $f^{-1}(B) \subseteq O$. Hence $f^{-1}(B)$ is g-p.closed. □

Now we are going to consider functions involving g-p.closed and p-closed sets and we defined it as gp-continuous functions.

Definition 3.2.4. A map $f : X \to Y$ from a topological space X to another topological space Y is called *generalised p-continuous* shortly *gp-continuous* if inverse image of every p-closed set in Y is g-p.closed in X.

Remark 3.2.4. Let $f : X \to Y$ from a topological space X to another topological space Y be a p-continuous function then it is also gp-continuous. But converse need not be true. For example, Let $X = \{1, 2, 3, 4\}$. Also let $T = \{X, \phi, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}, T' = \{X, \phi, \{1, 2\}\}$ be two topologies on X. Then the identity mapping from (X,T) to (X,T') is gp-continuous but not p-continuous.

Example 3.2.1. Let R be the real line and let I be the identity mapping from the topological space R with cofinite topology to the topological space R with usual topology. Then I is gp-continuous but not g-continuous.

Example 3.2.2. Let $X = \{a, b, c, d\}$ and let I be the identity mapping from (X, D) to (X, T) where D is the discrete topology on X and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Then I is g-continuous but not gp-continuous.

Remark 3.2.5. The concepts of g-continuity and gp-continuity are independent of each other as the above two examples illustrates.

Theorem 3.2.4. *Let (X, T), (Y, T') be any two topological spaces and $f : (X, T) \to (Y, T')$ be a mapping between the two topological spaces. Then the following conditions are equivalent :*

1. *f is gp-continuous.*
2. *Inverse image of every p-open set in Y is g-p.open in X.*

Proof. Assume that $f : X \to Y$ is gp-continuous and let G be a p-open set in Y , then $Y - G$ is p-closed set in Y. Since f is gp-continuous, $f^{-1}(Y - G)$ is g-p.closed in X. Trivially $f^{-1}(Y - G) = X - f^{-1}(G)$. $Y - G$ is p-closed in Y which implies $f^{-1}(Y - G)$ is g-p.closed in X. Hence $X - f^{-1}(G)$ is g-p.closed in X and thus $f^{-1}(G)$ is g-p.open in X. Conversely we assume that inverse image of every p-open set in Y is g-p.open in X. To prove that f is gp-continuous. Let H be a p-closed set in Y, then $Y - H$ is p-open in Y which implies $f^{-1}(Y - H)$ is g-p.open in X. But $f^{-1}(Y - H) = X - f^{-1}(H)$; which implies $f^{-1}(H)$ is g-p.closed in X. Thus f is g-p.continuous. □

In Lemma : 3.2.1 we obtained an equivalent condition for p-continuity in terms of p-closure. Now we are going to examine whether the result will be valid for gp-continuity, for that purpose we defined g-p.$cl(A)$ for any subset $A \subseteq X$ in any arbitrary topological space X. We defined it as the intersection of g-p.closed super sets of A.

Definition 3.2.5. Let (X, T) be a topological space and let $A \subseteq X$ then *generalised p-closure* of A is defined as the intersection of all g-p.closed supersets of A and is denoted as g-$p.cl(A)$.

Remark 3.2.6. Since all p-closed sets are g-p.closed ; g-$p.cl(A) \subseteq p$-$cl(A)$ for any subset $A \subseteq X$.

Example 3.2.3. Let $X = \{a, b, c, d\}$ and
$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ be the topology on X. Consider $A = \{c, d\}$ then p-$cl(A) = \{d, b, c\}$ and $\{a, d, c\}$
and g-$p.cl(A) = A$ itself.

Theorem 3.2.5. *If $f : (X, T) \to (Y, T')$ is a gp-continuous function between the topological spaces (X, T) and (Y, T') then $f(g$-$p.cl(A)) \subseteq p$-$cl(f(A))$ for every subset $A \subseteq X$.*

Proof. Given f is gp-continuous. Let $A \subseteq X$ to prove that $f(g$-$p.cl(A)) \subseteq p$-$cl(f(A))$. Consider p-$cl(A)$ it is p-closed set in X and also p-$cl(f(A))$ is p-closed set in Y. Since f is gp-continuous $f^{-1}(p$-$cl(f(A)))$ is a gp-closed set in X. Clearly $A \subseteq f^{-1}(p$-$cl(f(A)))$ which implies g-$p.cl(A) \subseteq f^{-1}(p$-$cl(f(A)))$ which in turn implies $f(g$-$p.cl(A)) \subseteq p$-$cl(f(A))$. □

Remark 3.2.7. Converse of above proposition need not be true ; for example let $X = Y = \{1, 2, 3\}$ also let $T = \{X, \phi, \{1\}\}$, $T' = \{Y, \phi, \{1, 3\}\}$ be topologies on X and Y respectively. Define $f : (X, T) \to (Y, T')$ by $f(1) = 2, f(2) = 1$ and $f(3) = 3$. Condition of above theorem is satisfied here but the function is not gp-continuous.

Theorem 3.2.6. *Let (X,T), (Y,T') be any two topological spaces and $f:(X,T)\to(Y,T')$ be a mapping between the two topological spaces. Then the following conditions are equivalent :*

1. *Corresponding to each point $x \in X$ and each p-open set V containing $f(x)$ there exists a g-p.open set U containing 'x' such that $f(U) \subseteq V$*

2. *For every $A \subseteq X$; $f(g\text{-}p.cl(A)) \subseteq p\text{-}cl(f(A))$ holds.*

Proof. First we will prove (1) implies (2). Let $y \in f(g\text{-}p.cl(A))$. We have to prove that $y \in p\text{-}cl(f(A))$. Let V be a p-open set containing 'y' then there exists a point $x \in X$ and a g-p.open set U containing 'x' such that $f(x) = y$ and $f(U) \subseteq V$ by assumption.
$y \in f(g\text{-}p.cl(A))$
$\Rightarrow f^{-1}(y) \in g\text{-}p.cl(A)$
$\Rightarrow x \in g\text{-}p.cl(A)$.
Since U is a g-p.open set containing 'x' ; $U \cap A \neq \phi \Rightarrow f(U) \cap f(A) \neq \phi$ which in turn implies $V \cap f(A) \neq \phi$ since $f(U) \subseteq V$. Thus $V \cap f(A) \neq \phi$ for every p-open set containing 'y'. Hence $y \in p\text{-}cl(f(A))$ by *Proposition 2.3.3* and thus $f(g\text{-}p.cl(A)) \subseteq p\text{-}cl(f(A))$.

Next to prove (2) $\Rightarrow$ (1). Assume that $\forall A \subseteq X$; $f(g\text{-}p.cl(A)) \subseteq p\text{-}cl(f(A))$. Also let $x \in X$ and V be a p-open set containing $f(x)$. Take $A = f^{-1}(V^c)$ then if $x \in A$, $f(x) \in f(A) = V^c$ which is not possible since V is a p-open set containing $f(x)$. Hence the only possibility is that $x \notin A$.
Now consider

g-$p.cl(A) \subseteq f^{-1}(f(g\text{-}p.cl(A)))$
$\subseteq f^{-1}(p\text{-}cl(f(A))$
$= f^{-1}(p\text{-}cl(V^c))$
$= f^{-1}(V^c) = A$
and then the only possibility is that g-$p.cl(A) = A$. Since $x \notin A$, $x \notin g$-$p.cl(A)$ which implies there exists a g-p.open set U containing 'x' such that $U \cap A = \phi$ which implies $U \subseteq A^c$ and hence $f(U) \subseteq f(A^c) \subseteq V$ that is $f(U) \subseteq V$. Hence (1) is proved. □

Remark 3.2.8. Composition of gp-continuous functions need not be gp-continuous. For example, Let $X = Y = Z = \{1, 2, 3\}$ also $T = \{X, \phi, \{1, 2\}\}$, $T' = \{Y, \phi, \{1\}, \{2, 3\}\}$ and $T'' = \{Z, \phi, \{1, 3\}\}$. Define $f : (X, T) \to (Y, T')$ by $f(1) = 3, f(2) = 2, f(3) = 3$ and g is the identity function from Y to Z. Clearly both $f$and g are gp-continuous but gof is not gp-continuous.

3.3 Separation axioms using p-open sets and g-p.closed sets

Norman Levine introduced the weaker separation axiom $T_{1/2}$ using generalised closed sets and he proved that for any symmetric space the separation axioms T_0, $T_{1/2}$ and T_1 coincides. As mentioned in section 2.4 , no higher separation axiom beyond T_1 can be considered using p-open sets. In this section we discuss p-T_0 , p-T_1 axioms etc using p-open sets and proved that T_1 and p-T_1 axioms always coincides. Also identified spaces in which all the separation axioms defined coincides.

Definition 3.3.1. Let (X, T) be any topological space. X is said to be p-T_0 if for every two distinct points $x, y \in X$ there exists p-open set U such that $x \in U, y \notin U$.

Remark 3.3.1. p-T_0 implies T_0 but converse is not true for example

1. Let X be any set with overlapping interval topology then X is T_0 but not p-T_0

2. Consider any set X with either or topology then only p-open set is $X - \{0\}$ and hence it is T_0 but not p-T_0.

Definition 3.3.2. Let (X, T) be any topological space. X is said to be p-T_1 if for every two distinct points $x, y \in X$ there exists p-open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$.

Most important fact noticed is that T_1 and p-T_1 axioms always coincides and for proving that we have to prove a necessary condition for a space to be T_1.

Proposition 3.3.1. Let (X, T) be any topological space. X is p-T_1 if and only if $\{x\}$ is p-closed.

Proof. Necessary part trivially follows since p-T_1 implies T_1. For sufficiency let x, y be two distinct points in X and assume that $\{x\}$ is p-closed which implies $X - \{x\}$ is a p-open set containing 'y' and not 'x'. Similarly by assuming $\{y\}$ to be p-closed we get a p-open set containing 'x' and not 'y'. Hence X is p-T_1. □

Theorem 3.3.1. *Let (X, T) be any topological space. X is T_1 if and only if it is p-T_1.*

Proof. Proof is trivial by using the last proposition. □

Definition 3.3.3. Let (X, T) be any topological space then X is p-$T_{1/2}$ if every g-p.closed set is p-closed.

Next result proves an equivalent condition for a space to be p-$T_{1/2}$ in terms of p-closed sets and as a corollary we have obtained that any p-$T_{1/2}$ space is a $T_{1/2}$ space and converse is not true.

Theorem 3.3.2. *A topological space (X, T) is p-$T_{1/2}$ if only if each singleton subset is either p-open or p-closed .*

Proof. Suppose X is p-$T_{1/2}$ and let $x \in X$. To prove that $\{x\}$ is p-open or p-closed. Assume that $\{x\}$ is not p-closed , then $X - \{x\}$ is not p-open and the only p-open set containing it is X which implies $X - \{x\}$ is g-p.closed and since X is p-$T_{1/2}$, $X - \{x\}$ is p-closed which implies $\{x\}$ is p-open. Hence $\{x\}$ is either p-open or p-closed. For converse part we assume that $\{x\}$ is either p-open or p-closed. Then there arise two cases. For case 1 assume $\{x\}$ is p-closed. Since $x \in p\text{-}cl(A)$ and A is g-p.closed ; p-$cl(\{x\}) \cap A \neq \phi$ by *Theorem 2.4.3* which implies $\{x\} \cap A \neq \phi$ which in turn implies $x \in A$. Thus p-$cl(A) \subseteq A$. Hence p-$cl(A) \subseteq A$ implies A is p-closed. Thus X is p-$T_{1/2}$ since A is an arbitrary g-p.closed set. As a second case we assume each singleton set to be p-open then (X, T) becomes a discrete space and thus p-$T_{1/2}$ trivially. □

Corollary 1. Any p-$T_{1/2}$ topological space is also $T_{1/2}$.

Proof. Since p-closed sets are always closed, the proof is trivial by last theorem. □

Remark 3.3.2. $T_{1/2}$ does not implies p-$T_{1/2}$ for example, Let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X then (X, T) is a $T_{1/2}$ space but not p-$T_{1/2}$.

Theorem 3.3.3. *If (X, T) is a p-$T_{1/2}$ topological space and $Y \subseteq X$, then (Y, T_Y) is also p-$T_{1/2}$.*

Proof. Let $y \in Y \subseteq X$. Consider $\{y\}$. Since X is p-$T_{1/2}$, it is p-open or p-closed in X. Then $\{y\}$ is p-open or p-closed in Y by *Theorem : 2.3.1* . □

A property is said to be an expansive property if whenever a topological space has that property then any topological space finer to it will also has the same property. And a property is said to be contractive if whenever a topological space has that property then any topological space coarser to it will also has the same property.

$T_{1/2}$ property is an expansive but not contractive property, whereas p-$T_{1/2}$ is neither expansive nor contractive.

Remark 3.3.3. p-$T_{1/2}$ is not an expansive property as the following example illustrates :
Let $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$,
$U = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be two topologies on X. Clearly $T \subset U$ and (X, T) is p-$T_{1/2}$ but (X, U) is not.

Remark 3.3.4. p-$T_{1/2}$ is not a contractive property for example let $X = \{a, b\}$ and $T = \{X, \phi, \{a\}\}$, $U = \{X, \phi, \{b\}\}$ be topologies on X. Then both T, U is p-$T_{1/2}$ but $T \cap U$ is not p-$T_{1/2}$.

Now we are moving to the most important part of the section. We try to find the implications between the separation axioms defined and identified the spaces in which all of them coincides. For that purpose we defined prime symmetric space analogous to symmetric space defined by N. Levine and later proved that all separation axioms coincide for a prime symmetric space.

Theorem 3.3.4. *Let (X, T) be a p-$T_{1/2}$ topological space then it is always p-T_0.*

Proof. Let (X, T) be a non p-T_0 topological space then there exists $x, y \in X$; $x \neq y$, such that any p-open set containing 'x' contains 'y' and vice versa. Consider $G = p\text{-}cl(\{x\}) \cap \{x\}^c$
Claim : G is g-p.closed but not p-closed.
First we will prove that $O \cap G \neq \phi$ for every p-open set O containing 'x' which in turn implies 'x' is a p-limit point of G and since $x \notin G$; G becomes a non p-closed set. In order to prove that $O \cap G \neq \phi$ for every p-open set O containing 'x' we will prove that $\{y\} \subseteq O \cap G$; that is to prove that $\{y\} \subseteq O \cap p\text{-}cl(\{x\}) \cap \{x\}^c$.

$$clearly; y \in \{x\}^c \tag{3.2}$$

Since X is non p-T_0

$$y \in O \tag{3.3}$$

again non p-T_0 implies 'y' is a p-limit point of $\{x\}$ and hence

$$y \in p\text{-}cl(\{x\}) \tag{3.4}$$

Thus (3.2), (3.3) and (3.4) implies $\{y\} \subseteq O \cap G$ and G becomes a non p-closed set. Again we claim that G is g-p.closed ; for that we assume that $G \subseteq O_1 \in T$ where O_1 is p-open in X to prove that p-$cl(G) \subseteq O_1$. But p-$cl(G)$ $=$ p-$cl(p$-$cl(\{x\}) \cap \{x\}^c)$ $\subseteq$ p-$cl(\{x\})$. Now it is enough to prove that p-$cl(\{x\}) \subseteq O_1$. But if $x \in O_1$, $G \cup \{x\} \subseteq O_1$ which implies p-$cl(\{x\}) \subseteq O_1$. Thus it is enough to prove that $x \in O_1$. On contradiction if $x \notin O_1$ then $(O_1)^c$ is a p-closed super set of $\{x\}$ which implies p-$cl(\{x\}) \subseteq (O_1)^c$, but $y \in p$-$cl(\{x\})$ by (3.4) which implies $y \in (O_1)^c$. Also $y \in G \subseteq O_1$ that is $y \in O_1$ and $y \in (O_1)^c$ which is not possible. Hence $x \in O_1$ and therefore there exists a g-p.closed but not p-closed set in X . Thus X is not p-$T_{1/2}$ if it is not p-T_0 and the theorem is proved. □

Remark 3.3.5. Let (X, T) be a p-T_0 topological space then it need not be p-$T_{1/2}$. For example Let $X = \{a, b, c\}$ and
$T = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ then (X, T) is p-T_0 but not p-$T_{1/2}$.

Theorem 3.3.5. *Let (X, T) be a p-T_1 topological space then it is always p-$T_{1/2}$.*

Proof. In order to prove that X is p-$T_{1/2}$ it is enough to prove that every g-p.closed set is p-closed. Let A be a non p-closed set then p-$cl(A) \neq A$ then there exists an $x \in X$ such that $\{x\} \subseteq p$-$cl(A) - A$ which implies p-$cl(A) - A$ contains a p-closed set since the given space is p-T_1 which implies A is not g-p.closed. Hence all g-p.closed sets are p-closed that is X is p-$T_{1/2}$. □

Remark 3.3.6. p-$T_{1/2}$ does not implies p-T_1. For example Let $X = \{1, 2\}$ and $T = \{X, \phi\, \{1\}\}$, then (X, T) is p-$T_{1/2}$ but not p-T_1.

Definition 3.3.4. A topological space (X, T) is said to be *prime symmetric* if $x \in p\text{-}cl(\{y\}) \Longrightarrow y \in p\text{-}cl(\{x\})$.

Using direct definition of prime symmetric space it is not easy to find examples and getting results. In order to solve that problem we obtained an equivalent condition for a prime symmetric space in terms of p-closed sets and hence we proved that all prime symmetric spaces are symmetric.

Theorem 3.3.6. *A topological space (X, T) is prime symmetric if and only if $\{x\}$ is g-p.closed for every $x \in X$.*

Proof. Assume (X, T) to be prime symmetric to prove that $\{x\}$ is g-p.closed. Let $\{x\} \subseteq O$ where O is p-open in X to prove that p-$cl(\{x\}) \subseteq O$. On contradiction $O^c \cap p\text{-}cl(\{x\}) \neq \phi$ and let $y \in O^c \cap p\text{-}cl(\{x\}) \subseteq p\text{-}cl(\{x\})$ but then $x \in p\text{-}cl(\{y\}) \subseteq O^c$, that is $x \in O^c$ which is not possible. Hence $\{x\}$ is g-p.closed.
For sufficiency assume that $\{x\}$ is g-p.closed for every $x \in X$ to prove that X is prime symmetric. On contradiction we assume that X is not prime symmetric, then there exists $y \neq x$ in X such that $x \in p\text{-}cl(\{y\})$ but $y \notin p\text{-}cl(\{x\})$ which implies $\{y\} \subseteq (p\text{-}cl(\{x\}))^c$ implies p-$cl(\{y\}) \subseteq (p\text{-}cl(\{x\}))^c$ since singleton sets are assumed to be g-p.closed ; but then $x \in p\text{-}cl(\{y\}) \subseteq (p\text{-}cl(\{x\}))^c$ which is not possible. Hence X is prime symmetric. □

Corollary 2. Let (X, T) be a prime symmetric topological space, then it is always a symmetric space.

Proof. Proof is trivial by last theorem since g-p.closed sets are always g-closed. □

Proposition 3.3.2. Let (X, T) be a p-T_1 topological space then it is a prime symmetric one.

Proof. Since the space is p-T_1, singleton sets are all p-closed and hence g-p.closed which in turn implies prime symmetry. □

Remark 3.3.7. Prime symmetry need not implies p-T_1 and indiscrete topology serves as an example.

Proposition 3.3.3. A topological space (X, T) is prime symmetric and p-T_0 if and only if (X, T) is p-T_1.

Proof. Sufficiency part is trivial by definition of p-T_0 and p-T_1. For necessary part assume that (X, T) is prime symmetric and p-T_0. Clearly then each singleton set is g-p.closed and hence g-closed. Also p-T_0 implies T_0. But if a space is T_0 and each singleton subset is g-closed then it must be T_1. Now by Theorem : 3.3.1, (X, T) is p-T_1 also and hence the equivalent condition is proved. □

Proposition 3.3.4. Let (X, T) be a prime symmetric topological space then (X, T) is p-$T_0 \Leftrightarrow$ *p*-$T_{1/2} \Leftrightarrow$ *p*-T_1.

Proof. Proof is trivial by *Theorem : 3.3.4*, *Theorem : 3.3.5* and *Proposition : 3.3.3.* . □

Now we are about to reach the solution of the problem. Above result indicates that p-$T_0 \Leftrightarrow$ *p*-$T_{1/2} \Leftrightarrow$ *p*-T_1 for prime symmetric spaces. Also we proved that prime symmetric spaces are always symmetric and it is known that $T_0 \Leftrightarrow T_{1/2} \Leftrightarrow T_1$. But by *Theorem : 3.3.1*, p-$T_1 \Leftrightarrow T_1$ always. Hence for prime symmetric spaces we have the following result.

Theorem 3.3.7. *Let (X, T) be a prime symmetric topological space then (X, T) is p-$T_0 \Leftrightarrow$ p-$T_{1/2} \Leftrightarrow T_1 \Leftrightarrow T_{1/2} \Leftrightarrow T_0$.*

Proof. Proof directly follows from *Theorem : 3.1.1*, *Theorem : 3.3.1*, *Corollary : 2* and *Proposition : 3.3.4*. □

In the rest of this section we discuss some more properties of p-$T_{1/2}$ spaces

Theorem 3.3.8. *Let (X, T) be a p-$T_{1/2}$ topological space and $f : X \to Y$ is p-continuous, p-closed and onto. Then Y is p-$T_{1/2}$.*

Proof. Let $B \subseteq Y$ be a g-p.closed set then by *Theorem : 3.2.3*, $f^{-1}(B)$ is g-p.closed and since X is p-$T_{1/2}$, $f^{-1}(B)$ is p-closed. Hence $B = f(f^{-1}(B))$ is p-closed in Y and thus Y is p-$T_{1/2}$. □

As a corollary we obtain that p-$T_{1/2}$ is a p-topological property and hence a topological property.

Corollary 3. p-homeomorphic image of p-$T_{1/2}$ space is p-$T_{1/2}$.

Proof. Proof is trivial by last theorem. □

In section 3.2 we obtained a counter example showing that gof is not gp-continuous even when g and f are gp-continuous. Now we established a condition when the composition of gp-continous functions become gp-continuous.

Theorem 3.3.9. *Let X, Y, Z be any three topological spaces, moreover Y be a p-$T_{1/2}$ space. Also let $f : X \to Y$ and $g : Y \to Z$ be gp-continuous. Then $gof : X \to Z$ is also gp-continuous.*

Proof. We have to prove that $gof : X \to Z$ is gp-continuous ; that is to prove that inverse image of p-closed set in Z is g-p.closed in X. Let H be a p-closed set in Z then $g^{-1}(H)$ is g-p.closed in Y and since Y is p-$T_{1/2}$, $g^{-1}(H)$ is p-closed in Y which implies $f^{-1}(g^{-1}(H))$ is g-p.closed in X provided H is p-closed in Z. Hence gof is gp-continuous. □

Remark 3.3.8. In *section 3.2*, *Remark : 3.2.4* indicates that concepts of p-continuity and gp-continuity are independent, but when the domain space is p-$T_{1/2}$ both the two concepts coincide.

Let (X_α, T_α) be an indexed family of topological spaces and $X = \prod X_\alpha$ be their product space. Let us examine some results concerning the product topological space and p-$T_{1/2}$ axiom.

Definition 3.3.5. Let $\{(X_i, T_i)/i \in I\}$ be a collection of topological spaces and let $(X = \prod X_i, T)$ be their product space. Then the p-open sets in T are sets of the form $\prod U_i$; where $U_i = X_i$ for infinitely many i's and other U_i's are all prime open in T_i.

Lemma 3.3.1. Projection functions are p-continuous.

Proof. Let $\{(X_i, T_i) : i = 1, 2..n\}$ be a collection of topological spaces and let (X, T) be their product space. We have to prove that the projection map $\pi_i : X \to X_i$ is p-continuous. Let $V_i \in T_i$ be a p-open set. Clearly $(\pi_i)^{-1}(V_i) = X_1 \times X_2 \times ...... \times X_{i-1} \times V_i \times X_{i+1} .... \times X_n \times ......$

which is clearly p-open by definition of p-open set in product topology. Hence each projection map is p-continuous. □

Theorem 3.3.10. *Let $\{(X_\alpha, T_\alpha) : \alpha \in I\}$ be a collection of topological spaces and let $X = \prod X_\alpha$ be their product topological space. If X is p-$T_{1/2}$ then X_α is p-$T_{1/2}$ for every $\alpha \in I$.*

Proof. X contains a subspace p-homeomorphic to X_α and by using above lemma and *Corollary 3* X_α is p-$T_{1/2}$. □

Theorem 3.3.11. *Let $\{(X_\alpha, T_\alpha) : \alpha \in I\}$ be a collection of topological spaces and let $X = \prod X_\alpha$ be their product topological space. Then X is p-$T_{1/2}$ if and only if X is p-T_1.*

Proof. Sufficiency part is trivial since p-$T_1 \Leftrightarrow T_1 \Rightarrow p$-$T_{1/2}$ by *Theorem : 3.3.1* and *Theorem : 3.3.5*. For necessary part, consider $\{x\}$, it is not open in product space and hence trivially not p-open in product topology and since X is p-$T_{1/2}$, $\{x\}$ is p-closed always for every 'x' which implies X is p-T_1. □

Corollary 4. Let $X = \prod X_\alpha$. Then X is p-$T_{1/2}$ if and only if X_α is p-T_1 for every $\alpha \in I$.

Proof. Proof is trivial by last result and by p-$T_1 \Leftrightarrow T_1$. □

We conclude this chapter by proving that corresponding to any topological space, there always exists a finer adjacent p-$T_{1/2}$ topological space.

Theorem 3.3.12. *Let (X_α, T_α) be a collection of p-$T_{1/2}$ topological spaces and if $\{T_\alpha / \alpha \in I\}$ forms a chain with inclusion as the order, then*

$(X, \cap\{T_\alpha/\alpha \in I\})$ *is* p-$T_{1/2}$.

Proof. We have to prove that $(X, \cap\{T_\alpha/\alpha \in I\})$ is p-$T_{1/2}$. Let $x \in X$ it is enough to prove that $\{x\}$ is either p-open or p-closed in $\cap\{T_\alpha/\alpha \in I\}$. Asssume $\{x\}$ not p-open in $\cap\{T_\alpha/\alpha \in I\}$. Then two cases arise :

1. $\{x\}$ is not open in $\cap\{T_\alpha/\alpha \in I\}$.
2. $\{x\}$ is open in $\cap\{T_\alpha/\alpha \in I\}$ but is not prime in $\cap\{T_\alpha/\alpha \in I\}$.

If case :1 occurs, then there exists $\beta \in I$ such that $\{x\} \notin T_\beta$. Since T_β is p-$T_{1/2}$, $X - \{x\} \in T_\beta$ and is always prime. Now if $T_\beta \subseteq T_\alpha$, then $X - \{x\} \in T_\alpha$ and is always prime in T_α. If $T_\beta \supseteq T_\alpha$ and $X - \{x\}$ is not p-open in T_α, then $\{x\} \in T_\alpha \subseteq T_\beta$ which implies $\{x\} \in T_\beta$ which is a contradiction. Hence in both cases , that is if $T_\beta \subseteq T_\alpha$ and $T_\beta \supseteq T_\alpha$; $X - \{x\}$ is p-open in T_α for all $\alpha \in I$. Thus $\{x\}$ becomes p-closed in $\cap\{T_\alpha/\alpha \in I\}$ and that implies $\cap\{T_\alpha/\alpha \in I\}$ is p-$T_{1/2}$.
If case : 2 occurs , that is if $\{x\}$ is open in $\cap\{T_\alpha/\alpha \in I\}$ but not is prime in $\cap\{T_\alpha/\alpha \in I\}$. Then there exists U, $V \in \cap\{T_\alpha/\alpha \in I\}$ such that $U \cap V \subseteq \{x\}$ and $\{x\} \subset U$, $\{x\} \subset V$ which implies $\{x\}$, U, V are open in T_α for every $\alpha \in I$ implies $\{x\}$ is not prime in T_α for every $\alpha \in I$. But since each T_α is p-$T_{1/2}$ by *Theorem : 3.3.2*, $X - \{x\}$ is p-open in T_α for every $\alpha \in I$ and thus $X - \{x\}$ is p-open in $\cap\{T_\alpha/\alpha \in I\}$. Hence $(X, \cap\{T_\alpha/\alpha \in I\})$ is p-$T_{1/2}$.
Therefore in both cases $\{x\}$ is either p-open or p-closed in $\cap\{T_\alpha/\alpha \in I\}$ for each $x \in X$ and that implies the result. □

Theorem 3.3.13. *Let (X, τ) be any topology on X, then there exists a topology U on X such that*

1. $\tau \subseteq U$.
2. (X, U) *is* p-$T_{1/2}$.
3. *If* (X, γ) *is* p-$T_{1/2}$ *where* (X, γ) *is such that* $\tau \subseteq \gamma \subseteq U$, *then* $\gamma = U$.

Proof. Let $\mathcal{G} = \{\tau_\alpha / \alpha \in I\}$ be the indexed family of p-$T_{1/2}$ topologies on X finer than τ. $\mathcal{G} \neq \phi$ since $\mathcal{G}$ contains atleast the discrete topology. Consider a chain of subsets of $\mathcal{G}$ say $\{\tau_\alpha / \alpha \in J\}$ then $\cap \{\tau_\alpha / \alpha \in J\}$ is p-$T_{1/2}$ and $\tau \subseteq \cap \{\tau_\alpha / \alpha \in J\}$. But then $\cap \{\tau_\alpha / \alpha \in J\}$ belongs to $\mathcal{G}$ and by applying dual statement of Zorn's lemma it contains a minimal element U such that $\tau \subseteq U$ and U is p-$T_{1/2}$ and by minimality, condition 3 is also satisfied. Hence the theorem is proved. □

Chapter 4

Non-prime Isolated, p-Irreducible, p-Door and Sub p-maximal Spaces

4.1 Introduction

In [30] Levine introduced the concept of D-spaces as spaces in which any two open sets intersects. Now it is worth studying about the spaces in which any two p-open sets intersects. In present work we try to introduce such spaces called non-prime isolated spaces and study some of its properties. We proved that being a non-prime isolated space is

Some results of this chapter are included in the following paper.
Vinitha.T and T.P.Johnson, Non-prime Isolated, p-Irreducible, p-Door and Sub p-maximal Spaces, Bulletin of Kerala Mathematical Association, Vol.14, Dec 2017, No.2.

a p-topological property and also a productive property. We obtained that any topological space can be written as the union of its maximal non-prime isolated subsets. Meanwhile we introduce p-irreducible, sub p-maximal and p-door spaces using the concept of p-open sets. We proved that p-irreducible and non-prime isolated spaces are equivalent. Also obtained the equivalent condition for a topological space to be sub p-maximal and proved that every non-prime isolated sub p-maximal spaces are p-$T_{1/2}$.

The following are some definitions useful in this chapter :

Definition 4.1.1. [34] A topological space (X, T) is said to be *hyper connected* if any two open sets intersects.

Definition 4.1.2. [30] A topology T for a set X is called a *D-topology* if every non-empty open set is dense in X and the corresponding topological space (X, T) is said to be a *D-space*.

Definition 4.1.3. [9] An *irreducible* topological space is a topological space such that it cannot be written as union of two disjoint closed subsets of it.

Definition 4.1.4. [13] A topological space in which every subset is either open or closed is called a *door* space.

Definition 4.1.5. [9] A topological space (X, T) is said to be *sub maximal* if every subset U such that $cl(U) = X$ should be open.

4.2 Non-prime isolated and p-irreducible spaces

Throughout this section we studied about spaces in which any two non-empty p-open sets intersects which we called as non-prime isolated spaces. The main result obtained in this section is that any topological space can be written as union of its maximal non-prime isolated subsets. Also we introduce p-irreducible spaces and proved that non-prime isolatedness and p-irreducibility are equivalent always.

Definition 4.2.1. Let (X, T) be any arbitrary topological space then (X, T) is said to be a *prime isolated* space if there always exists two p-open sets U, V such that $U \cap V = \phi$. Otherwise if there does not exists two disjoint p-open sets in X then such spaces are called *non-prime isolated* spaces.

Let us consider some examples of prime isolated and non-prime isolated spaces.

Example 4.2.1. Consider $X = \{a, b, c\}$ and
let $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be a topology on X. Then the topological space (X, T) is a prime isolated one.

Example 4.2.2. Let $X = \{a, b, c, d\}$ and
$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ be a topology on X. Then the topological space (X, T) is a prime isolated one.

Example 4.2.3. Let X be any arbitrary set and let the topology on it be the cofinite topology, then it is a non-prime isolated space.

Example 4.2.4. Consider $X = \{a, b, c\}$
and let $T = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ be a topology on X. Then the topological space (X, T) is a non-prime isolated one.

Hyper connected spaces are spaces for which intersection of any two non-empty open sets is non-empty. Since all p-open sets are open, if open sets intersects then p-open sets also intersects trivially.

Remark 4.2.1. Hyper connected spaces are always non-prime isolated but converse is not true ; for example any discrete space is non-prime isolated but not hyper connected.

In a non-prime isolated space any two non-empty p-open sets intersects and then by definition of p-limit point defined in chapter 2, for any p-open set in any arbitrary topological space X, any point of X happens to be a p-limit point and the converse also holds. Using this we obtain the following necessary and sufficient condition for a space to be non-prime isolated in terms of p-closure.

Theorem 4.2.1. *Let (X, T) be any arbitrary topological space then the following conditions are equivalent :*

1. *(X, T) is non-prime isolated.*
2. *For every p-open set U in X ; p-cl$(U) = X$.*

Proof. Assume that (X, T) is non-prime isolated and let U be a p-open set in X we have to prove that p-$cl(U) = X$. For that it is enough to prove that every $x \in X$ is a p-limit point of U by *Proposition 2.3.3* and *Definition 2.3.2*. Consider V as a p-open set containing 'x'. Since

X is non-prime isolated $V \cap U \neq \phi$ for every p-open set V in T which implies 'x' is a p-limit point of U. Since 'x' is arbitrary ; any point of X is a p-limit point of U. Hence p-$cl(U) = X$.
Conversely assume that p-$cl(U) = X$ for every p-open set U in X to prove that $U \cap V \neq \phi$ for any two p-open sets U, V in X. U and V p-open $\Rightarrow$ p-$cl(U)$ =p-$cl(V) = X$; which implies any point of X happens to be a p-limit point of U and V ; particularly any point of U happens to be a p-limit point of V and vice versa which implies $U \cap V \neq \phi$ always and hence X is non-prime isolated. □

N.Levine defined a topological space X to be a D-space if for every non-empty open set U in X, $\overline{U} = X$. Analyzing *Theorem 4.2.1* and this definition we can deduce the relationship between D-space and non-prime isolated space as follows :

Remark 4.2.2. Any D-space is non-prime isolated and converse is not true ; for example any hausdorff space is non-prime isolated but cannot be a D-space.

Next result shows that non-prime isolatedness is preserved under p-continuous mappings and hence it is a p-topological property and therefore a topological property.

Theorem 4.2.2. *The p-continuous image of a non-prime isolated space is also non-prime isolated.*

Proof. Let (X, T), (Y, T') be two topological spaces and $f : (X, T) \rightarrow (Y, T')$ be a p-continuous function also assume that X is non-prime isolated. We have to prove that $f(X)$ is non-prime isolated. On contradiction we assume that $f(X)$ is prime isolated. Then there exists two disjoint p-open sets U, V in $f(X)$, clearly then $f^{-1}(U)$ and $f^{-1}(V)$ becomes two disjoint p-open non-empty sets in X which is not possible. Hence $f(X)$ is also non-prime isolated. □

Theorem 4.2.3. *p-homeomorphic image of non-prime isolated space is also non-prime isolated.*

Proof. Let (X,T), (Y,T') be two topological spaces and $f:(X,T) \to (Y,T')$ be a p-homeomorphism between them. Since f and f^{-1} are p-continuous the result is trivial by last theorem. □

A topological property P is said to be productive if whenever product space has property P then each co-ordinate space also has the property P and vice versa. The next result shows that non-prime isolatedness is productive in nature.

Theorem 4.2.4. *Let (X_i, T_i) be a collection of topological spaces and $\prod X_i$ be their product space, then $\prod X_i$ is non-prime isolated if and only if each X_i is non-prime isolated.*

Proof. Necessary part is trivial by *Theorem : 4.2.2* and since projection functions are p-continuous and on to. To show sufficiency part let 'O' be a p-open set in X, then we have to prove that p-$cl(O) = X$ which implies non-prime isolatedness by *Theorem : 4.2.1.* Let $x \in O$, then there exists $i \in I$ and U_i p-open in T_i such that

$$x \in \bigcap \left\{\pi_i^{-1}(U_i)/i = 1, 2.......n\right\} \subseteq O \tag{4.1}$$

Consider $X = \bigcap \left\{\pi_i^{-1}(X_i)/i = 1, 2.......n\right\}$
$= \bigcap \left\{\pi_i^{-1}(p\text{-}cl(U_i))/i = 1, 2.......n\right\}$
$\subseteq p\text{-}cl(\bigcap \left\{\pi_i^{-1}(U_i)/i = 1, 2.......n\right\})$
$\subseteq p\text{-}cl(O)$ ....by (4.1)
$= X$
Hence p-$cl(O) = X$ and since 'O' is an arbitrary p-open set we obtained the result that X is non-prime isolated. □

Subspace of D-space is always a D-space but for non-prime isolatedness that is not the case.

Remark 4.2.3. Subspace of non-prime isolated space need not be non-prime isolated ; for example Let $X = \{a, b, c\}$ and T be the discrete topology on X. Also let $Y = \{a, b\}$ and $T_Y = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$ be the subspace topology on Y. Clearly (X, T) is non-prime isolated but (Y, T_Y) is prime isolated.

Definition 4.2.2. Let (X, T) be a topological space and $A \subseteq X$. Then A is said to be non-prime isolated if the subspace topology on A with respect to T that is T_A is non-prime isolated.

Proposition 4.2.1. Let (X, T) be a topological space and let Y be a non-prime isolated space such that p-$cl(Y) = X$ and Y is p-open in X ; then X itself is a non-prime isolated space.

Proof. Let U and V be non-empty p-open sets in X to prove that $U \cap V \neq \phi$. Since p-$cl(Y) = X$; $G \cap Y \neq \phi$ for every p-open set G in T which implies $U \cap Y \neq \phi$ and $V \cap Y \neq \phi$ which again implies $(U \cap V) \cap Y \neq \phi$, then the only possibility is that $U \cap V \neq \phi$. Hence X is non-prime isolated. □

Theorem 4.2.5. *If Y is a non-prime isolated sub space of X such that Y is p-open in X, then p-cl(Y) is also non-prime isolated.*

Proof. Let $Y' = p\text{-}cl(Y)$. Since Y is non-prime isolated and Y' contains Y ; Y' satisfies the condition of proposition : 4.2.1. Hence p-$cl(Y)$ is always non-prime isolated whenever Y is non-prime isolated. □

Levine proved that any topological space can be written as the union of its maximal D-subsets. Now the problem is whether the result holds using non-prime isolated subsets. In order to solve the problem we have to apply Zorn's lemma and succeeding propositions provide a route to the solution of the problem.

Proposition 4.2.2. Let (X, T) be a topological space and $\{Y_\alpha/\alpha \in I\}$ be a chain of non-prime isolated subsets of (X, T). Also let $Y = \bigcup Y_\alpha :$ $\alpha \in I$. Then Y is also non-prime isolated.

Proof. We have to prove that $Y = \bigcup Y_\alpha : \alpha \in I$ is non-prime isolated given each Y_α is non-prime isolated. On contradiction we assume that Y is prime isolated ; then there exists two disjoint p-open sets U, V in Y and they will be of the form $U = U_\alpha \cap Y$ and $V = V_\alpha \cap Y$ where U_α and V_α are p-open sets in X.
Let $x \in U_\alpha \cap Y \Rightarrow x \in Y_\alpha$ for some $\alpha \in I$
and $y \in V_\alpha \cap Y \Rightarrow y \in Y_\beta$ for some $\beta \in I$
Since $\{Y_\alpha\}$ is a chain either $Y_\alpha \subseteq Y_\beta$ or $Y_\alpha \supseteq Y_\beta$; we assume that $Y_\alpha \subseteq Y_\beta$, then $\{x, y\} \subseteq Y_\beta$ which implies $Y_\beta \cap U_\alpha$ and $Y_\beta \cap V_\alpha$ are disjoint p-open sets in Y_β. Hence Y_β becomes prime isolated which is a contradiction which implies Y is non-prime isolated. □

Proposition 4.2.3. Let Y be a non-prime isolated subset of a topological space (X, T). Then there exists a maximal non-prime isolated subset Y^* such that $Y \subseteq Y^*$ and if Y happens to be p-open in X, then that maximal element is always p-closed.

Proof. Let $\{Y_\alpha / \alpha \in I\}$ be a chain of non-prime isolated subsets of X containing Y. Then by above proposition $\{\bigcup Y_\alpha : \alpha \in I\}$ is also non-prime isolated ; that is every chain of non-prime isolated subsets has an upper bound. Now we can apply Zorn's lemma and that implies there exists a maximal element Y^* such that $Y \subseteq Y^*$ and Y^* is non-prime isolated. Next assume that Y is p-open in X, to prove that Y^* is p-closed ; If Y^* is not p-closed then there exists non-prime isolated subset p-$cl(Y^*)$ such that $Y^* \subseteq p\text{-}cl(Y^*)$ by *Theorem : 4.2.5*. But by maximality of Y^* the only possibility is $Y^* = p\text{-}cl(Y^*)$; that is Y^* is p-closed. □

Theorem 4.2.6. *Let (X, T) be any topological space, then X is the union of its maximal non-prime isolated subsets.*

Proof. Let $x \in X$. Consider $\{x\}$, it is a non-prime isolated subset always. By *Proposition : 4.2.3* there exists a maximal non-prime isolated subset containing $\{x\}$ and this is true for any $x \in X$. Clearly $X = \bigcup\{\{x\} : x \in X\}$ which implies $X = \bigcup\{Y_x : x \in X\}$ where Y_x is the maximal non-prime isolated subset containing $\{x\}$ for each $x \in X$. Thus X is the union of maximal non-prime isolated subsets. □

We conclude this section by proving another equivalence for non-prime isolatedness.

Definition 4.2.3. A topological space is said to be *p-irreducible* if and only if $X \neq G_1 \cup G_2$ for any p-closed sets G_1 and G_2 in X.

Remark 4.2.4. Irreducible implies p-irreducible by definition itself but converse need not be true ; for example Discrete space with cardinality greater than three is p-irreducible but not irreducible.

Theorem 4.2.7. *A topological space is p-irreducible if and only if it is non-prime isolated.*

Proof. In order to prove the necessary part we assume that X is p-irreducible and on contradiction we assume that there exists disjoint p-open sets G_1 and G_2 in X but then $X = (X - G_1) \cup (X - G_2)$ which implies X can be written as union of two p-closed subsets of X, which is not possible. Hence there does not exists such disjoint p-open sets G_1, G_2 in X which implies X is non-prime isolated. Proof of sufficiency part is similar to the proof of necessary part by applying De-Morgan's law. □

4.3 p-door spaces and sub p-maximal spaces

In chapter 3 we introduced the weaker separation axiom p-$T_{1/2}$ and proved that any topological space X is p-$T_{1/2}$ if only if each singleton subset is either p-open or p-closed. Now the question arises whether there exists spaces for which each subset is either p-open or p-closed. Dontchev in [13] introduced door spaces as spaces in which any subset is either open or closed by restricting definition to prime case we reach the answer to our question and we named such spaces as p-door spaces. Trivially p-door spaces are generalization of p-$T_{1/2}$ spaces. We also discuss relation between p-$T_{1/2}$ spaces and non-prime isolated spaces introduced in section 2. For that purpose we defined sub p-maximal spaces and proved that any non-prime isolated sub p-maximal spaces are p-$T_{1/2}$.

Definition 4.3.1. A topological space (X, T) in which every subset is either p-open or p-closed is called a *p-door space*.

Remark 4.3.1. Any p-door topological space is also a door space but converse is not true ; for example Let (X, T) be a topological space such that $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$, then (X, T) is a door space but not a p-door space.

The property of being a door space is expansive whereas property of being a p-door space is not expansive.

Remark 4.3.2. The property of being a p-door space is not expansive; for example consider $X = \{a, b, c\}$ and $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ be the topology on X then (X, T) is a p-door space but the discrete topology on X which is always finer than T is not p-door.

Theorem 4.3.1. *If (X, T) is p-door space and if $Y \subseteq X$, then (Y, T_Y) is also a p-door space.*

Proof. We have to prove that every subset of Y is either p-open or p-closed in Y. Let A be a subset of Y. Since $Y \subseteq X$, A happens to be a subset of Y and since X is p-door A is either p-open or p-closed in X . But by *Theorem : 2.3.1*, $A \cap Y$ is either p-open or p-closed in Y ; which implies $A \cap Y = A$ is either p-open or p-closed in Y. Since $A \subseteq Y$ is arbitrary Y is a p-door space. □

p-doorness is a p-topological property and hence a topological property.

Proposition 4.3.1. p-open image of p-door space is a p-door space.

Proof. Let $f : (X,T) \to (Y,T')$ be a p-open mapping and X be a p-door space. To prove that $f(X)$ is p-door space. Let $A \subseteq f(X)$ then $f^{-1}(A)$ is a subset of X and since X is p-door $f^{-1}(A)$ is p-open or p-closed which implies $f(f^{-1}(A)) = A$ is p-open or p-closed . Hence $f(X)$ is a p-door space. □

Theorem 4.3.2. *p-homeomorphic image of p-door space is a p-door space.*

Proof. Proof is trivial using *Proposition : 4.3.1*. □

Next we introduce spaces such that if p-closure of any set happens to be full set then such a set is always p-open analogous to the concept of sub maximal spaces introduced by N.Bourbaki. Also procured a necessary and sufficient condition for a space to be sub p-maximal in terms of p-open and p-closed sets. Using that equivalent condition we can deduce that sub p-maximality is a restricted version of sub maximality.

Definition 4.3.2. A topological space (X,T) is said to be *sub p-maximal* if every subset U such that p-$cl(U) = X$ should be p-open.

Theorem 4.3.3. *A topological space (X,T) is sub p-maximal if and only if any subset $S \subseteq X$ can be written as the intersection of p-open and p-closed set in X.*

Proof. For necessary part we assume that X is sub p-maximal and let S be any subset of X
Consider $O = X - [p\text{-}cl(S) \cap S^c]$
$= [p\text{-}cl(S)]^c \cup S$
$\Rightarrow p\text{-}cl(O) = p\text{-}cl[[p\text{-}cl(S)]^c \cup S] \supseteq p\text{-}cl[(p\text{-}cl(S))^c] \cup p\text{-}cl(S)$ by *Proposition : 2.3.2*
But p-$cl[(p\text{-}cl(S))^c] \cup p\text{-}cl(S) = X$
$\Rightarrow X \subseteq p\text{-}cl(O)$
$\Rightarrow X = p\text{-}cl(O)$. But by our assumption X is sub p-maximal which implies O is p-open and by definition of O it is clear that $S = O \cap p\text{-}cl(S)$; that is S can be written as the intersection of a p-open and a p-closed set in X. Thus the necessary part is proved.

In order to prove the other part let S be any subset of X and assume that any subset can be written as the intersection of a p-open and a p-closed set in X ; that is $S = O \cap C$ where O is p-open and C is p-closed in X. We have to prove that X is sub p-maximal , for that let us choose S in such a way that p-$cl(S) = X$. Now it is enough to prove that S is p-open. Consider p-$cl(S) = p\text{-}cl(O \cap C) \subseteq p\text{-}cl(O) \cap p\text{-}cl(C) = p\text{-}cl(O) \cap C$; since C is p-closed which implies $X \subseteq p\text{-}cl(O) \cap C \Rightarrow p\text{-}cl(O) = X$ and $C = X$. Hence $S = O \cap C$ implies $S = O$ and which in turn implies S is p-open. Since S is arbitrary, X is sub p-maximal. □

Corollary 5. Let (X, T) be a sub p-maximal space then it is sub maximal also.

Proof. Proof is trivial by last theorem since p-open and p-closed sets are open and closed sets respectively. □

Remark 4.3.3. Converse of above corollary is not true in general. For example, Let X be any discrete space such that cardinality of X is greater than three, then any subset is open and hence X is sub maximal, but consider $\{x_1, x_2\}$ where $x_1, x_2 \in X$, clearly p-$cl(\{x_1, x_2\}) = X$ but $\{x_1, x_2\}$ is not p-open and hence X is not sub p-maximal.

Next result analyzes relation between p-door and sub p-maximal spaces. We use directly the definition of p-door to prove it and established that all p-door spaces are sub p-maximal, but only p-irreducible sub p-maximal spaces are p-door.

Theorem 4.3.4. *Every p-door space is sub p-maximal.*

Proof. Let (X, T) be a p-door topological space and let $A \subseteq X$ be such that p-$cl(A) = X$. In order to prove that X is sub p-maximal it is enough to prove that A is p-open . On contradiction assume that A is not p-open , then since X is p-door A should be p-closed ; but that implies $A = p\text{-}cl(A) = X$ which is not possible always. Hence our assumption is wrong ; that is A is p-open which proves that X is sub p-maximal. □

Theorem 4.3.5. *Every p-irreducible sub p-maximal spaces are p-door.*

Proof. Let (X, T) be a p-irreducible, sub p-maximal topological space and let $A \subseteq X$ we have to prove that A is either p-open or p-closed. If p-$cl(A) = X$, then A is p-open since X is sub p-maximal. Otherwise there exists at least one point $x \in X$ in such a way that 'x' is not a p-limit point of A which implies there exists at least one p-open set U such that $x \in U$ and $U \cap A = \phi$ which in turn implies there exist a p-open set U such that $U \subseteq X - A$. Given that X is p-irreducible which implies p-$cl(U) = X$ by *Theorem : 4.2.7* and *Theorem : 4.2.1* and hence p-$cl(X-A) = X$ which implies $X - A$ is p-open since X is sub p-maximal ; that is A is p-closed. Thus in both cases A is either p-open or p-closed and hence X is p-door. □

Corollary 6. Every non-prime isolated sub p-maximal topological space is p-door.

Proof. Proof is trivial by last theorem and *Theorem : 4.2.7.* □

Proposition 4.3.2. Let (X, T) be a p-door topological space , then it is always p-$T_{1/2}$.

Proof. Let A be any g-p.closed set to prove that A is p-closed. Since (X, T) is p-door, A is either p-open or p-closed. If A is p-closed nothing to prove. If A is p-open, applying definition of generalised p-closed set we obtain that A is a p-closed set. □

We conclude this chapter by establishing the result that every non-prime isolated sub p-maximal spaces are p-$T_{1/2}$.

Theorem 4.3.6. *Every non-prime isolated sub p-maximal spaces are p-$T_{1/2}$.*

Proof. Proof follows from last proposition and *Corollary : 6* . □

Since non-prime isolatedness implies and implied by p-irreducibility we have

Corollary 7. Every p-irreducible sub p-maximal spaces are p-$T_{1/2}$.

Chapter 5

Semi p-Open Sets, Semi p-Homeomorphisms and Nowhere p-Dense Sets.

5.1 Introduction

Norman Levine introduced semi-open sets as a generalization of open sets in any arbitrary topological space and later many authors [10], [20], [28] worked on it. Levine also defined semi continuous functions as functions in which inverse image of open sets are semi-open.

In this chapter we try to apply the concept of p-open sets to semi open sets and thereby we introduce the notion of semi p-open sets. Mean while we introduce nowhere p-dense sets and obtained that any semi p-open set can be written as the disjoint union of p-open and nowhere p-dense sets.

We studied semi p-continuous, semi-irresolute and semi p-open mappings using p-open and semi p-open sets. Examined the implications amongst each of the mappings and analyzed the behavior of semi p-open sets, p-open sets and nowhere p-dense sets under such mappings. Also obtained that any p-homeomorphic image of a topological space of first category can be written as the union of nowhere p-dense sets in it.

Definition 5.1.1. [29] A set A in a topological space X will be termed semi-open if there exists an open set O such that $O \subseteq A \subseteq \overline{O}$; where $\overline{O}$ is the closure of O in X.

Definition 5.1.2. [10] A function f is said to be semi-continuous if inverse image of open sets are semi-open.

In [10] Crossely and Hildebrand studied about some more genre of functions involving semi-open sets such as irresolute and pre semi-open functions which are not necessarily continuous functions.

Definition 5.1.3. [28] A function $f : X \to Y$ is said to be irresolute if for every semi-open set S of Y $f^{-1}(S)$ is semi-open in X.

Definition 5.1.4. [28] Let X and Y be topological spaces , a function $f : X \to Y$ is pre semi-open if every semi-open set in X is mapped to semi-open set in Y only.

5.2 Semi p-open sets and Nowhere p-dense sets

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Definition 5.2.1. Let (X, T) be a topological space and $A \subseteq X$. A is said to be *semi p-open* if there exists a p-open set 'O' such that $O \subseteq A \subseteq p\text{-}cl(O)$ and A is said to be *semi p-closed* if its complement is semi p-open.

Remark 5.2.1. Trivially p-open implies semi p-open but converse is not true ; for example Let $X = \{a, b, c, d\}$ and
$\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ be a topology on X. In (X, τ) , $\{a, c\}$ is not p-open but it is semi p-open.

For a hausdorff space only prime open sets are complement of singleton sets and that implies the following result :

Remark 5.2.2. For a hausdorff space, p-open sets and semi p-open sets coincides.

Neither semi-openness nor semi p-openness implies each other. But we established a necessary and sufficient condition for a set to be semi p-open and that equivalence indicates a situation where semi p-open implies semi-open.

Remark 5.2.3. For any arbitrary topological space, semi open sets are not always semi p-open. For example consider real line with usual topology, then $(0, 1]$ is semi-open but not semi p-open.

Theorem 5.2.1. *Let (X, T) be a topological space and $A \subseteq X$. A is semi p-open iff $A \subseteq p\text{-}cl(p\text{-}int(A))$.*

Proof. For necessary part we assume A as a semi p-open set which implies there exists a p-open set G such that

$$G \subseteq A \subseteq p\text{-}cl(G) \tag{5.1}$$

Now $G \subseteq A$ and G is p-open, which implies $G \subseteq p\text{-}int(A)$ which again implies

$$p\text{-}cl(G) \subseteq p\text{-}cl(p\text{-}int(A)) \tag{5.2}$$

Now (5.1) and (5.2) implies $G \subseteq A \subseteq p\text{-}cl(G) \subseteq p\text{-}cl(p\text{-}int(A))$. Particularly $A \subseteq p\text{-}cl(p\text{-}int(A))$

Conversely assume that $A \subseteq p\text{-}cl(p\text{-}int(A))$. Take p-$int(A) = G$, then G is a p-open set such that $G \subseteq A \subseteq p\text{-}cl(G)$. That is A is semi p-open. □

Corollary 8. Let (X, T) be a topological space and let $A \subseteq X$ be a semi p-open set in X then A is semi-open if p-$cl(p\text{-}int(A)) \subseteq cl(int(A))$.

Remark 5.2.4. Generally semi p-open sets are not always semi-open. Let $X = \{1, 2, 3, 4\}$ and
$\tau = \{X, \phi, \{1\}, \{2, 3\}, \{1, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}$. In the topological space (X, τ) , $\{1, 2, 4\}$ is semi p-open but not semi-open.

Remark 5.2.5. Union of semi p-open sets need not be semi p-open. For example let $X = \{a, b, c\}$ and the topology on it be $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Here $\{a\}$ and $\{b\}$ are semi p-open but $\{a, b\}$ is not semi p-open.

Remark 5.2.6. Intersection of two semi p-open sets need not be semi p-open. Consider any arbitrary set with cardinality greater than three and with discrete topology, clearly $X - \{x_1\}, X - \{x_2\}$ are semi p-open but their intersection is not semi p-open.

Proposition 5.2.1. Let (X, T) be a topological space and let A be a semi p-open set in (X, T) . Also let $A \subseteq B \subseteq p\text{-}cl(A)$, then B is also semi p-open.

Proof. Given A as a semi p-open set then by definition of semi p-open set there exist a p-open set 'O' such that

$$O \subseteq A \subseteq p\text{-}cl(O) \tag{5.3}$$

Now $O \subseteq A$ and $A \subseteq B$ which implies

$$O \subseteq B \tag{5.4}$$

From (5.3) we obtain $A \subseteq p\text{-}cl(O)$
$\Rightarrow p\text{-}cl(A) \subseteq p\text{-}cl(O)$.
But we have $B \subseteq p\text{-}cl(A)$ by assumption. Hence $B \subseteq p\text{-}cl(A) \subseteq p\text{-}cl(O)$ and (5.4) implies $O \subseteq B \subseteq p\text{-}cl(O)$. Thus B is also semi p-open. □

Trivially any p-open set is also a semi p-open set and Proposition : 5.2.1 implies that if B is any subset of an arbitrary topological space X such that B lies between a semi p-open set and its p-closure, then B is also semi p-open. Comparing this two results we locate collection of sets satisfying the two results.

Theorem 5.2.2. *Let (X, T) be any topological space and $\mathcal{G} = \{G_\alpha\}$ be a collection of sets in X such that*

1. *Collection of p-open sets in T belongs to $\mathcal{G}$.*

2. *$G_\alpha \in \mathcal{G}$ and $G_\alpha \subseteq H \subseteq p\text{-}cl(G_\alpha)$ implies $H \in \mathcal{G}$.*

Then the collection of all semi p-open sets in X belongs to $\mathcal{G}$ and it is the smallest collection of sets in X satisfying 1 and 2.

If a set is semi p-open in the original space then it is semi p-open in the corresponding subspace but not conversely.

Theorem 5.2.3. *Let (X,T) be a topological space with a subspace (Y,T_Y) where $Y \subseteq X$. If $A \subseteq Y$ is semi p-open in (X,T), then A is semi p-open in (Y,T_Y).*

Proof. Given A is semi p-open in (X,T) then by definition of semi p-open set there exists a p-open set 'O' such that

$$O \subseteq A \subseteq p\text{-}cl_X(O) \tag{5.5}$$

where p-$cl_X(O)$ is the p-closure of 'O' with respect to (X,T).
We have $O \subset A \subset Y$ which implies $O \subset Y$.
Now (5.5) implies $O \cap Y \subset A \cap Y \subset p\text{-}cl_X(O) \cap Y$
$\Rightarrow O \cap Y \subset A \cap Y \subseteq p\text{-}cl_Y(O)$ by *Proposition : 2.3.7.*
Since $O \subset Y$, $O \cap Y = O$. Hence we obtain $O \subseteq A \subseteq p\text{-}cl_Y(O)$. Thus A is semi p-open in (Y,T_Y). □

Remark 5.2.7. Converse of above result need not be true. For example consider the discrete topological space $X = \{x_1, x_2, x_3\}$ and let $Y = \{x_1, x_2\}$. Then $Y - \{x_2\}$ is semi p-open in Y but not semi p-open in X.

In the forth coming part of this section we establish that any semi p-open set can be written as disjoint union of two sets. With that motive in mind we define nowhere p-dense sets. Also we introduced $D_p(A)$ as the set of all p-limit points of A where A is any subset of an arbitrary topological space, to examine some results related to semi p-open sets and corresponding set of p-limit points.

Definition 5.2.2. Let (X, T) be a topological space and let $A \subset X$ then we define $D_p(A)$ as the set of all p-limit points of A with respect to T.

Definition of p-limit point and *Remark : 2.3.5* implies the following result :

Remark 5.2.8. Clearly $D(A) \subseteq D_p(A)$ but not conversely where $D(A)$ denotes the set of all limit points of A.

Theorem 5.2.4. *Let (X, T) be a topological space and let $A \subseteq X$ be such that A is p-open, then p-cl$(A) = A \cup D_p(A)$.*

Proof. To prove that $A \cup D_p(A) \subseteq p\text{-}cl(A)$. Let $x \in A \cup D_p(A)$, then $x \in A$ or $x \in D_p(A)$.
If $x \in A$, then trivially $x \in p\text{-}cl(A)$. Now if $x \in D_p(A)$, then by definition 'x' is a p-limit point of A. By *Proposition : 2.3.3* and *Definition : 2.3.2*; $x \in p\text{-}cl(A)$. Thus in either cases $x \in p\text{-}cl(A)$; that is $A \cup D_p(A) \subseteq p\text{-}cl(A)$.
Conversely assume that $x \in p\text{-}cl(A)$. To prove that p-$cl(A) \subseteq A \cup D_p(A)$. If $x \in A$ the result trivially follows. If $x \notin A$ then since $x \in p\text{-}cl(A)$ every p-open set 'U' containing 'x' intersects A which implies 'x' is a p-limit point of A. Hence $x \in A \cup D_p(A)$. Thus p-$cl(A) \subseteq A \cup D_p(A)$ always and hence p-$cl(A) = A \cup D_p(A)$ □

Remark 5.2.9. Above theorem does not works in general. That is for any arbitrary set $A \subseteq X$, only p-$cl(A) \supseteq A \cup D_p(A)$ and the other part need not holds.

Next we initiate the study of nowhere p-dense sets and proved that it is a restricted variety of nowhere dense sets.

Definition 5.2.3. A subset A of a topological space (X, T) is said to be nowhere p-dense if p-$int(\overline{A}) = \phi$, where $\overline{A}$ is the closure of A with respect to T.

Remark 5.2.10. Nowhere p-dense does not implies nowhere dense. Consider (R, U) and let A be the set of all rationals between 0 and 1 then A is nowhere p-dense but not nowhere dense.

Remark 5.2.11. Trivially if $int(\overline{A}) = \phi$ then p-$int(\overline{A}) = \phi$. Hence nowhere dense implies nowhere p-dense.

Proposition 5.2.2. Let (X, T) be a topological space and $A \subseteq X$ then A is nowhere p-dense if and only if every non-empty p-open set in X contains a non-empty open set which is disjoint from A.

Proof. For necessity assume that A is nowhere p-dense, that is p-$int(\overline{A}) = \phi$. We have to prove that every non-empty p-open set in X contains a non-empty open set which is disjoint from A. Let U be the given non-empty p-open set. Clearly U is not a subset of $\overline{A}$, if $U \subseteq \overline{A}$ then p-$int(\overline{A}) \neq \phi$ which is not possible. Hence $U \cap (X - \overline{A})$ is a non-empty open set disjoint from $\overline{A}$ and thus disjoint from A and such that $U \cap (X - \overline{A}) \subset U$. Thus U contains a non-empty open set disjoint from A and since U is arbitrary this can be done for any arbitrary p-open set.

Thus proved the necessary part.
Conversely assume the sufficiency part in order to prove that p-$int(\overline{A}) = \phi$. On contradiction let p-$int(\overline{A}) \neq \phi$ then there exists a p-open set G such that $G \subset \overline{A}$. Thus any point of G happens to be a limit point of A ; that is all open sets containing points of G must intersects A which implies there does not exists an open set in G disjoint from A contradicting our assumption. Hence p-$int(\overline{A}) = \phi$. □

Proposition 5.2.3. Let (X, T) be a topological space and let O be p-open in X; then p-$cl(O) \cap O^c$ is nowhere p-dense in X.

Proof. By *Theorem : 5.2.4* we have p-$cl(O) = O \cup D_p(O)$ which implies p-$cl(O) \cap O^c \subseteq D_p(O)$. That is p-$cl(O) \cap O^c$ contains all p-limit points of O. We have to prove that p-$cl(O) \cap O^c$ is nowhere p-dense. Let U be a non-empty p-open set in X, it is enough to prove that U contains an open set disjoint from p-$cl(O) \cap O^c$. If $U \subseteq O$; then $U \cap$ (p-$cl(O) \cap O^c) = \phi$ that is U itself is a p-open set disjoint from p-$cl(O) \cap O^c$. Hence p-$cl(O) \cap O^c$ is nowhere p-dense. If $O \cap U = \phi$ then U contains no points of O which implies it does not contains any p-limit points of O which implies $U \cap$(p-$cl(O) \cap O^c) = \phi$. Now if both of the above cases fails that is if $O \cap U \neq \phi$ and U is not a subset of O then $U \cap O$ is a non-empty open subset of U as well as O. Now let $G =$ (p-$cl(O) \cap O^c) \cap (U \cap O)$, then clearly $G = \phi$. Thus in this case also U contains an open set disjoint from p-$cl(O) \cap O^c$ which implies p-$cl(O) \cap O^c$ is nowhere p-dense. □

In the next result we prove that any semi p-open set can be written as disjoint union of a p-open set and a nowhere p-dense set.

Theorem 5.2.5. *Let A be a semi p-open set in a topological space*

(X,T). Then A will be of the form $A = O \cup B$ where O is a p-open set in X such that $O \cap B = \phi$ and B is nowhere p-dense.

Proof. Given A as a semi p-open set which implies there exists a p-open set O such that $O \subseteq A \subseteq p\text{-}cl(O)$. Clearly any arbitrary set A can be written as $A = O \cup (A \cap O^c)$. Now let $B = A \cap O^c$, since $A \subseteq p\text{-}cl(O)$ we have $B \subseteq p\text{-}cl(O) \cap O^c$. By *Proposition : 5.2.3*, p-$cl(O) \cap O^c$ is nowhere p-dense and that implies B is also nowhere p-dense. Thus $A = O \cup B$ and it satisfies all conditions of the theorem. □

Merely by using definition of p-interior we obtained that the collection of p-open sets and collection of p-interior of all semi p-open sets in any topological space coincides and this result holds for any arbitrary topological space.

Theorem 5.2.6. *Let (X,T) be a topological space and let $\mathcal{P}$ denote the collection of p-open sets in T. If $\mathcal{G}$ denote the collection of p-interior of all semi p-open sets in X then $\mathcal{G} = \mathcal{P}$.*

Proof. Let $P \in \mathcal{P}$, then p-$int(P) = P$ itself which implies $P \in \mathcal{G}$. Hence $\mathcal{P} \subset \mathcal{G}$. Now let $G \in \mathcal{G} \Rightarrow G = p\text{-}int(G_1)$ for some G_1 semi p-open in X which is a maximal p-open subset of G_1 which implies $G \in \mathcal{P}$. Hence $\mathcal{G} \subset \mathcal{P}$ and thus $\mathcal{G} = \mathcal{P}$. □

Lemma 5.2.1. Let A be a semi p-open set in a topological space (X,T). Then there exists a p-open set O such that $(A \cap O^c) \subseteq D_p(O)$.

Proof. Given A as a semi p-open set then by definition of semi p-open set there exists a p-open set O such that $O \subseteq A \subseteq p\text{-}cl(O)$. Clearly

$A \cap O^c \subseteq p\text{-}cl(O) \cap O^c \subseteq D_p(O)$. Hence for any semi p-open set A there exists a p-open set O such that $A \cap O^c \subseteq D_p(O)$. □

Theorem 5.2.7. *Let (X, T) be a topological space with A as a semi p-open set. Then there always exists a p-open set O such that $D_p(A \cap O^c) \subseteq D_p(O)$.*

Proof. Let $y \in D_p(A \cap O^c)$ then any p-open set containing 'y' must contain points of $A \cap O^c$ but by above lemma $A \cap O^c \subseteq D_p(O)$ which implies any p-open set containing points of $A \cap O^c$ must contain points of O. Hence any p-open set containing 'y' must contain points of 'O' that is $y \in D_p(O)$. Thus $D_p(A \cap O^c) \subseteq D_p(O)$. □

5.3 Mappings Involving semi p-open sets and p-open sets

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In this section we consider variety of mappings involving p-open, semi p-open etc and try to find out the implications among them.

We begin this section by proving that being semi p-open is preserved under p-continuous, p-open mappings.

Theorem 5.3.1. *Let $f : (X, T) \rightarrow (Y, T')$ be a p-continuous, p-open mapping between the topological spaces (X, T) and (Y, T'). If A is semi p-open in (X, T), then $f(A)$ is semi p-open in (Y, T').*

Proof. Given A is semi p-open in (X, T) , then by *Theorem : 5.2.5*, $A = O \cup B$ where O is p-open and B is nowhere p-dense. Now from the proof of *Theorem : 5.2.5*, B is such that $B \subseteq p\text{-}cl(O) \cap O^c$.
Clearly $O \subseteq A$
$\Rightarrow f(O) \subseteq f(A)$
$= f(O) \cup f(B)$
But $B \subseteq p\text{-}cl(O) \cap O^c \subseteq p\text{-}cl(O)$
which implies $O \subseteq A \subseteq f(O) \cup f(p\text{-}cl(O))$
$\subseteq f(O) \cup p\text{-}cl(f(O))$ by *Lemma : 3.2.1*
$\subseteq p\text{-}cl(f(O))$.
Thus $f(O) \subseteq f(A) \subseteq p\text{-}cl(f(O))$. Since $f(O)$ is p-open in Y, $f(A)$ is semi p-open in Y. $\square$

Next we consider functions in which inverse image of each p-open set is semi p-open, which we call as semi p-continuous functions. We noticed that semi p-continuity does not implies p-continuity and even continuity. Also it is independent of semi-continuity. But trivially p-continuity implies semi p-continuity since all p-open sets are semi p-open also.

Definition 5.3.1. Let $f : (X, T) \to (Y, T')$ be a mapping between two topological spaces (X, T) and (Y, T') , then f is said to be *semi p-continuous* if inverse image of each p-open set in Y is semi p-open in X.

Example 5.3.1. Let $X = Y = \{a, b, c, d\}$.
Also let $T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$
and $T' = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$ be two topologies on X . Define $f : (X, T) \to (Y, T')$ by $f(a) = f(c) = c$, $f(b) = b, f(d) = d$. Then f is semi p-continuous.

Remark 5.3.1. The above example indicates that semi p-continuity does not implies p-continuity and it does not implies even continuity.

Remark 5.3.2. Trivially p-continuity implies semi p-continuity.

Remark 5.3.3. Semi-continuity neither implies nor implied by semi p-continuity. For example Let $X = Y = \{a, b, c\}$, T be the discrete topology and $T' = \{X, \phi, \{c\}\}$. Now define $f : (X, T) \to (Y, T')$ as the identity mapping . Then f is semi continuous but not semi p-continuous. Now consider the function $g : (R, U) \to (R, D)$ where R is the real line with Discrete topology D and usual topology U ; g is the identity mapping. Clearly g is semi p-continuous but not semi-continuous.

From *Remark : 5.2.2* it is clear that for all T_2 spaces p-open and semi p-open sets coincides and hence the following remark follows :

Remark 5.3.4. Let X be a T_2 space and f is a function such that $f : X \to Y$ is semi p-continuous then it is p-continuous.

Theorem 5.3.2. *Let (X, T), (Y, T') be two topological spaces and $f : (X, T) \to (Y, T')$ be a mapping such that f is a single valued function. If f is semi p-continuous then for any $f(x) \in G'$, G' p-open in Y there exists G semi p-open in X such that $x \in G$ and $f(G) \subset G'$.*

Proof. Let $f(x) \in G'$. Clearly $f^{-1}(G')$ is semi p-open in X and contains 'x'. Now let $G = f^{-1}(G')$ then $x \in G$ and $f(G) \subset G'$. □

Definition 5.3.2. Let (X, T), (Y, T') be two topological spaces ; then $f : (X, T) \to (Y, T')$ is said to be *semi-irresolute* if and only if inverse image of semi p-open set in Y is semi p-open in X.

Example 5.3.2. Let $X = Y = \{a, b, c, d\}$ and τ, τ' be two topologies on X such that $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\tau' = \{X, \phi, \{c\}\}$. Define a function $f : (X, \tau) \to (Y, \tau')$ as $f(a) = f(b) = c$, $f(c) = d$ and $f(d) = a$. Then f is semi- irresolute but not p-continuous.

Remark 5.3.5. Both p-continuity and semi p-continuity does not implies semi -irresoluteness. For example Let $X = Y = \{a, b, c, d\}$ and let $T = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$, $T' = \{X, \phi, \{a\}\}$ be two topologies on X. Consider $f : (X, T) \to (Y, T')$ as $f(a) = c, f(b) = b, f(c) = a, f(d) = d$. Then f is p-continuous and semi p-continuous but not semi -irresolute.

Every p-continuous, p-open functions are semi-irresolute. In order to prove this we have to use the following lemma. Also we define semi p-open function and proved that every p-continuous, p-open functions are semi-irresolute and semi p-open.

Lemma 5.3.1. If $f : X \to Y$ is p-continuous and p-open , then $f^{-1}(p\text{-}cl(A)) = p\text{-}cl(f^{-1}(A))$.

Proof. Since f is p-open , f^{-1} is p-continuous and hence $f^{-1}(p\text{-}cl(A)) \subseteq p\text{-}cl(f^{-1}(A))$. For the other part we have $A \subseteq p\text{-}cl(A)$ which implies

$$f^{-1}(A) \subseteq f^{-1}(p\text{-}cl(A)) \tag{5.6}$$

Since f is p-continuous and p-$cl(A)$ is p-closed always, $f^{-1}(p\text{-}cl(A))$ is p-closed and thus (5.6) implies p-$cl(f^{-1}(A)) \subseteq f^{-1}(p\text{-}cl(A))$. Thus $f^{-1}(p\text{-}cl(A)) = p\text{-}cl(f^{-1}(A))$. □

Theorem 5.3.3. *Let $f : X \to Y$ be p-continuous and p-open , then*

f is semi irresolute.

Proof. To prove that every semi p-open set in Y is mapped on to semi p-open set in X. Let G be a semi p-open set in Y then by definition of semi p-open set there exists a set O such that O is p-open and $O \subseteq G \subseteq p$-$cl(O)$

$$\Rightarrow f^{-1}(O) \subseteq f^{-1}(G) \subseteq f^{-1}(p\text{-}cl(O)) = p\text{-}cl(f^{-1}(O)) \tag{5.7}$$

Since f is p-continuous, $f^{-1}(O)$ is p-open in X and then (5.7) implies $f^{-1}(G)$ is semi p-open in X. □

Theorem 5.3.4. *Let (X, T), (Y, T') be two topological spaces then $f : (X, T) \to (Y, T')$ is a semi - irresolute function if and only if for every semi p-closed subset G of T', $f^{-1}(G)$ is semi p-closed in T.*

Proof. Proof is trivial by taking complements. □

Definition 5.3.3. Let (X, T), (Y, T') be two topological spaces, then a function $f : X \to Y$ is *semi p-open* if for every semi p-open set A in X; $f(A)$ is semi p-open in Y.

Example 5.3.3. Let f be the identity function from (R, D) to (R, U) where R is the real line , D is the discrete topology and U is the usual topology. Then f is semi p-open but not pre semi -open.

Theorem 5.3.5. *Composition of semi-irresolute functions are semi -irresolute.*

Theorem 5.3.6. *If $f : X \to Y$ is p-continuous and p-open then f is semi-irresolute and semi p-open.*

Proof. If f is given to be p-continuous and p-open, then f should be semi-irresolute by *Theorem : 5.3.3*. Also the proof of semi p-openness analogously follows from the proof of *Theorem : 5.3.3* and *Lemma : 5.3.1.* □

5.4 Semi p-homeomorphism and Nowhere p-dense Sets

Now we are in a situation to define semi p-homeomorphism. In this section we notice that nowhere p-dense sets are preserved under p-homeomorphisms and classified spaces for which nowhere p-dense sets remain the same.

Definition 5.4.1. A function $f : X \to Y$ is said to be a *semi p-homeomorphism* if f is one-one, onto, semi p-open and semi-irresolute.

Remark 5.4.1. Homeomorphism implies p-homeomorphism implies semi p-homeomorhism and none of the converse implications holds. For example , let $X = \{a, b, c, d\}$ and let $T = \{X, \phi, \{a, b\}, \{a\}, \{a, b, c\}, \{a, b, d\}\}$, $T' = \{X, \phi. \{a\}\}$ be two topologies on X. Consider the function $f : (X, T) \to (X, T')$ defined by $f(a) = b, f(b) = c, f(c) = d, f(d) = a$; then f is a semi p-homeomorphism but not a p-homeomorphism. The non occurrence of other implication follows from Chapter 3.

In *chapter 3, Lemma : 3.2.1* we proved an equivalent condition for p-continuity in terms of p-closure. It is worthful to check whether such an equivalent condition will be obtained for semi-irresolute function introduced in last section. For that purpose we defined semi p-closure. Also later we defined semi p-interior and analogously such an equivalent condition is true using semi p-interior also.

Definition 5.4.2. Let (X, T) be a topological space and let $A \subseteq X$ then semi p-closure of A denoted by *semi*$_p$*-cl(A)* is defined as the minimal semi p-closed super set of A.

Example 5.4.1. Let (X, T) be a topological space with $X = \{a, b, c\}$,
$T = \{X, \phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. Let $A = \{c, d\}$, then semi$_p$-$cl(\{c, d\}) = \{c, d\} = \overline{\{c, d\}}$ and p-$cl(\{c, d\}) = X$. Now let $B = \{b\}$, semi$_p$-$cl(\{b\}) = \{b\}$ and p-$cl(\{b\}) = \{b, c, d\} = \overline{\{b\}}$.

Proposition 5.4.1. Let (X, T) be a topological space and $A \subseteq X$, then semi$_p$-$cl(A) \subseteq$ *p*-$cl(A)$.

Proof. Since p-closed implies semi p-closed, the result is trivial. □

Lemma 5.4.1. A function $f : X \to Y$ is semi-irresolute if and only if f(*semi*$_p$-$cl(A)) \subseteq$ semi$_p$-$cl(f(A))$.

Proof. Let $A \subseteq X$ and consider semi$_p$-$cl(f(A))$ which is semi p-closed in Y. For necessity assume that f is semi-irresolute and hence f^{-1}(semi$_p$-$cl(f(A)))$ is semi p-closed in X.
But $f(A) \subseteq$semi$_p$-$cl(f(A))$
$f^{-1}(f(A)) \subseteq f^{-1}$(semi$_p$-$cl(f(A)))$. That is f^{-1}(semi$_p$-$cl(f(A)))$ is a semi p-closed super set of A and by definition of semi p-closure
semi$_p$-$cl(A) \subseteq f^{-1}$(semi$_p$-$cl(f(A)))$.

Now taking f on both sides $f(\text{semi}_p\text{-}cl(A)) \subseteq f(f^{-1}(\text{semi}_p\text{-}cl(f(A)))) \subseteq \text{semi}_p\text{-}cl(f(A))$. Hence $f(\text{semi}_p\text{-}cl(A)) \subseteq \text{semi}_p\text{-}cl(f(A))$, thus the required condition is proved and hence the necessary part is proved.
Conversely let G be a semi p-closed set in Y to prove that $f^{-1}(G)$ is semi p-closed in X. Consider $f^{-1}(G)$ and applying our assumption on $f^{-1}(G)$ we have
$f(\text{semi}_p\text{-}cl(f^{-1}(G))) \subseteq \text{semi}_p\text{-}cl(f(f^{-1}(G)))$
$\subseteq \text{semi}_p\text{-}cl(G) = G$.
$\Rightarrow \text{semi}_p\text{-}cl(f^{-1}(G)) \subseteq f^{-1}(G)$ and then only possibility is $f^{-1}(G) = \text{semi}_p\text{-}cl(f^{-1}(G)$. Thus $f^{-1}(G)$ is semi p-closed and hence f is semi-irresolute. □

Lemma 5.4.2. A function $f : X \to Y$ is semi-irresolute if and only if for every $H \subseteq Y$; $\text{semi}_p\text{-}cl(f^{-1}(H)) \subseteq f^{-1}(\text{semi}_p\text{-}cl(H))$.

Proof. Necessarily we assume that f is semi-irresolute and consider $\text{semi}_p\text{-}cl(H)$ for $H \subseteq Y$. Since f is semi-irresolute, $f^{-1}(\text{semi}_p\text{-}cl(H))$ is semi p-closed in X.
But $H \subseteq \text{semi}_p\text{-}cl(H)$
$\Rightarrow f^{-1}(H) \subseteq f^{-1}(\text{semi}_p\text{-}cl(H))$
$\Rightarrow \text{semi}_p\text{-}cl(f^{-1}(H)) \subseteq f^{-1}(\text{semi}_p\text{-}cl(H))$.
Conversely let H be a semi p-closed set in Y to prove that $f^{-1}(H)$ is semi p-closed in X. Clearly $f^{-1}(H) \subseteq \text{semi}_p\text{-}cl(f^{-1}(H)) \subseteq f^{-1}(\text{semi}_p\text{-}cl(H)) = f^{-1}(H)$. Hence $f^{-1}(H)$ is semi p-closed in X and thus f is semi-irresolute. □

If the function happens to be a semi p-homeomorphism then using the preceding two lemmas we obtain :

Theorem 5.4.1. *If $f : X \to Y$ is a semi p-homeomorphism then $semi_p$-$cl(f^{-1}(B)) = f^{-1}(semi_p$-$cl(B))$ for every $B \subseteq Y$.*

Theorem 5.4.2. *If $f : X \to Y$ is a semi p-homeomorphism then $semi_p$-$cl(f(B)) = f(semi_p$-$cl(B))$ for every $B \subseteq Y$.*

Definition 5.4.3. Semi p-interior of $A \subseteq X$ in a topological space X is defined as maximal semi p-open subset of A and is denoted as *semi $_p$-int (A)*.

Theorem 5.4.3. *If $f : X \to Y$ is a semi p-homeomorphism then*

1. *$semi_p$-int$(f^{-1}(B)) = f^{-1}(semi_p$-int$(B))$.*

2. *$semi_p$-int$(f(B)) = f(semi_p$-int$(B))$*

Proof. Proof is trivial and analogous to proof of *Lemma 5.4.1* and *Lemma 5.4.2.* □

Preceding theorems give four equivalent condition for a function to be semi p-homeomorphic in terms of semi p-interior and semi p-closure. Using that conditions we prove that nowhere dense sets are preserved under p-homeomorphisms which in turn helps us to prove the main objective of this chapter. That is any p-homeomorphic image of a topological space of first category can be written as the union of nowhere p-dense sets in it.

Theorem 5.4.4. *Let (X, T) be a topological space and $A \subseteq X$. Then A is nowhere p-dense if and only if p-int$(semi_p$-$cl(A)) = \phi$.*

Proof. If p-int(semi$_p$-cl(A)) $= \phi$ then by *Proposition : 5.4.1*, p-int(p-cl(A)) $= \phi$ which in turn implies A is nowhere p-dense. Converse is trivial by definition of nowhere p-dense. □

Proposition 5.4.2. If $f : X \to Y$ is a p-homeomorphism and $A \subseteq X$ is nowhere p-dense in X then $f(A)$ is nowhere p-dense in Y.

Proof. Given A to be no where p-dense in X ; that is p-int($\overline{A}$) $= \phi$ by definition of nowhere p-dense and also last theorem implies

$$p\text{-}int(semi_p\text{-}cl(A)) = \phi. \tag{5.8}$$

We have to prove that $f(A)$ is nowhere p-dense that is to prove that p-int(semi$_p$-$cl(f(A))) = \phi$ again by applying last theorem. Given f is p-homeomorphic which implies f is semi p-homeomorphic also. Hence p-int(semi$_p$-$cl(f(A))) = f($p-int(semi$_p$-$cl(A))) = f(\phi) = \phi$ by *Theorem : 5.4.2*. Now (5.8) implies p-int(semi$_p$-$cl(f(A))) = \phi$. Hence $f(A)$ is nowhere p-dense in Y. □

Theorem 5.4.5. *Let (X, T) be a topological space of first category and $f : (X, T) \to (Y, T')$ be a p-homeomorphism from (X, T) to another topological space (Y, T'). Then (Y, T') can be written as union of no where p-dense sets in it.*

Proof. Given X is of first category ; that is

$$X = \bigcup_{i=1}^{\infty} G_i \tag{5.9}$$

where each G_i is nowhere dense in X. By *Remark : 5.2.10* each G_i is nowhere p-dense in X and then *Proposition : 5.4.2* implies each $f(G_i)$ is nowhere p-dense and (5.9) implies
$Y = f(X) = f(\bigcup_{i=1}^{\infty} G_i) = \bigcup_{i=1}^{\infty} f(G_i)$.
That is (Y, T') can be written as union of nowhere p-dense sets in it. □

Approaching the end of this chapter we noticed that any two topological spaces with same collection of p-open sets always shares same collection of nowhere p-dense sets.

Definition 5.4.4. Let X be any arbitrary set and τ, τ' be topologies on X, then τ and τ' are said to be *p-correspondent* topologies on X if (X, τ) and (X, τ') has the same collection of p-open sets.

Example 5.4.2. Any two hausdorff topologies on X is p-correspondent.

Theorem 5.4.6. *Any two p-correspondent topologies on any arbitrary set X determines precisely the same nowhere p-dense subsets.*

Proof. Proof is trivial by definition of p-correspondent topologies and by *Theorem : 5.4.5.* □

Chapter 6

p-Compactness, C-p.compactness and Some Lattice Theoretic Properties

6.1 Introduction

The chapter aims at studying the concepts of compactness and C-compactness using p-open sets. If a topological space can be covered by a collection of open sets , then surely such a cover can be constructed using closures of that open sets also. This idea leads to the development of a new

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Some results of this chapter are included in the following paper.
Vinitha.T and T.P.Johnson. : p-Compactness and C-p.compactness, Global Journal of Pure and Applied Mathematics, Vol.13, No.9 (2017).

Some results of this chapter were presented in.
National Seminar on Topology and Its Applications, 7-9 February 2018, St.Berchmanns College, Changanassery.

concept in topology called C-compactness which was first introduced by G.Viglino in 1969. Later in [22] Harry.L.Herrington and Paul. E. Long characterized C-compactness using nets and filters.

Analogously it is meaningful to consider covers using p-open sets only and thereby we introduce new types of compactness called p-Compactness and C-p.compactness. We characterized
p-Compactness and C-p.compactness using nets and filters. For that purpose we introduce the ideas of p-limit points and p-closure limit points for nets and filters, regular p-open sets etc. Also proved that p-Compactness is a p-topological property and thereby a topological property. Some product theorems on p-Compactness is also considered. Trivially p-Compactness and compactness are not equivalent concepts. Using characterizations of p-Compactness we identified spaces for which concepts of p-Compactness and compactness coincides.

In this chapter we also try to study about spaces in which all open sets happens to be prime. We define such spaces as *prime topological spaces* and we established a necessary and sufficient condition for a space to be prime topological space. Let $P_r(X)$ denote the collection of all prime topologies on an arbitrary set X, we proved that $P_r(X)$ forms a meet complemented atomic semi lattice.

We conclude the thesis by extending the idea of prime element to the lattice of all topologies on any arbitrary set X and we identified that ultraspaces are precisely the prime elements in $\Sigma(X)$, where $\Sigma(X)$ denote the lattice of all topologies on an arbitrary set X.

6.2 p-Compactness In Topological Spaces

For any arbitrary topological space compactness and discreteness occurs simultaneously only when the underlying set is finite. Introduction of p-Compactness gives a variety of infinite p-Compact discrete topological spaces ; in fact in $\Sigma(X)$ the largest element is always p-Compact where $\Sigma(X)$ denotes the collection of all topologies on X.

Definition 6.2.1. Let (X, T) be a topological space and let $A \subset X$ then the collection $\{Pi : i \in I\}$ of prime open sets in T is said to be a *p-open cover* of A if $A \subseteq \bigcup_i P_i$.

Definition 6.2.2. Let (X, T) be a topological space, (X, T) is said to be *p-Compact* if every p-open cover has a finite sub cover.

Example 6.2.1. Any sober space is p-Compact.

Theorem 6.2.1. *Any hausdorff space is p-Compact.*

Proof. Proof is trivial by Theorem : 2.2.1. □

Theorem 6.2.2. *Every compact space is p-Compact.*

Proof. Since any p-open cover is an open cover, existence of finite sub cover for p-open cover obviously follows from the compactness of the space. □

Remark 6.2.1. The converse of above theorem is not true ; for example Let X be any infinite set with discrete topology. Then X is p-Compact but not compact.

In *Chapter 2* we studied about primeness in relative topology, using that results we obtained that p-Compactness is an absolute property and hence we also prove that p-closed subset of a p-Compact space is also p-Compact.

Proposition 6.2.1. Let (X, T) be a topological space and $A \subset X$. Then A is p-Compact with respect to T if and only if it is p-Compact with respect to T_A ; where T_A is the relative topology on A with respect to T.

Proof. For necessity we assume that A is p-Compact with respect to T to prove that A is p-Compact with respect to T_A. Let $\{G_i : i \in I\}$ be a p-open cover of A where each $G_i \in T_A$. Now by applying *Proposition : 2.3.6* each G_i is of the form $H_i \cap A$ such that H_i is p-open in X but by our assumption H_i has a finite sub collection which covers A ; that is
$A \subseteq \bigcup_i G_i \subseteq \bigcup_i (H_i \cap A) \subseteq \bigcup_i H_i$
$\Rightarrow A \subseteq H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n}$
$\Rightarrow A \subseteq (H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n}) \cap A$
$= G_{i_1} \cup G_{i_2} \cup ...... \cup G_{i_n}$
That is A can be covered by finitely many members of G_i. Thus A is p-Compact with respect to T_A. Now in order to prove the sufficiency part let H_i be a p-open cover of A by p-open sets in T
Then $A \subseteq \bigcup_i H_i$
$\Rightarrow A \subseteq A \cap (\bigcup_i H_i) = \bigcup_i G_i$
where each $G_i = A \cap H_i$ and is p-open in T_A again by applying *Proposition : 2.3.6* which implies
$A \subseteq G_{i_1} \cup G_{i_2} ..... \cup G_{i_n}$
$\subseteq A \cap (H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n})$
$\subseteq H_{i_1} \cup H_{i_2} \cup ...... \cup H_{i_n}$. Hence A is p-Compact with respect to T. Thus

proving the sufficiency part. □

Theorem 6.2.3. *Every p-closed subset of a p-Compact space is p-Compact.*

Proof. Let Y be a p-closed subset of a p-Compact space X we have to prove that Y is p-Compact. Let G be a p-open covering of Y by sets p-open in X then $H = G \cup \{X - Y\}$ is a p-open covering of X and since X is p-Compact H has a finite sub cover in particular G has a finite sub cover which covers Y. Hence Y is p-Compact. □

Theorem 6.2.4. *The p-continuous image of a p-Compact space is p-Compact.*

Proof. Let $f : X \to Y$ be p-continuous and let X be p-Compact. We have to prove that $f(X)$ is p-Compact, that is to prove that any p-open covering of $f(X)$ by sets p-open in Y has a finite sub cover. Let $\{G_i : i \in I\}$ be a p-open covering of $f(X)$, where each G_i is p-open in Y. Since f is given to be p-continuous each $f^{-1}(G_i)$ is p-open in X.
But we have $f(X) \subseteq \bigcup\limits_i G_i$
$\Rightarrow X \subseteq \bigcup\limits_i f^{-1}(G_i)$.
Hence $\{f^{-1}(G_i) : i \in I\}$ forms a p-open cover of X and since X is p-Compact
$X \subseteq \bigcup\limits_{i=1}^{i=n} f^{-1}(G_i)$
$\Rightarrow f(X) \subseteq \bigcup\limits_{i=1}^{i=n} G_i$.
That is finitely many of $\{G_i : i \in I\}$ say $G_1, G_2 .... G_n$ cover $f(X)$. Then $f(X)$ is p-Compact and therefore p-continuous image of a p-Compact

space is p-Compact. □

Theorem 6.2.5. *p-Compactness is a p-topological property.*

Proof. Let $(X,T),(Y,T^{'})$ be two topological spaces and $f:(X,T)\rightarrow (Y,T^{'})$ be a p-homeomorphism between them. Let X be p-Compact we have to prove that Y is also p-Compact. Since p-continuous image of a p-Compact space is p-Compact and f is onto ; we obtain $f(X)=Y$ as a p-Compact space. Hence p-Compactness is a p-topological property. □

6.3 Characterizations of p-Compactness

Proposition 6.3.1. Let (X,T) be a topological space and let $\{F_i\}$ be a collection of p-closed subsets of X, then X is p-Compact if and only if whenever $\bigcap\limits_i Fi=\phi$, $\{F_i\}$ must contains a finite subclass $\{F_{i_1},............F_{i_m}\}$ with $\bigcap\limits_{j=1}^{j=m} F_{i_j}=\phi$.

Proof. For necessary part assume X to be p-Compact. Given $\bigcap\limits_i Fi=\phi$ implies $X=\bigcup\limits_i (F_i)^c$, so $\{(F_i)^c\}$ is a p-open cover of X, and since X is p-Compact there exists a finite sub cover $(F_{i_1})^c,............(F_{i_m})^c$ such that $X=\bigcup\limits_{j=1}^{j=m} F_{i_j}^c$ implies $\phi=\bigcap\limits_{j=1}^{j=m} F_{i_j}$ and thus required condition is proved.

Now to prove the sufficiency part we assume that whenever $\bigcap\limits_i F_i=\phi$ there exists a finite subclass $\{F_{i_1},............F_{i_m}\}$ with $\bigcap\limits_{j=1}^{j=m} F_{i_j}=\phi$. Let $\{O_i\}$

be a p-open cover of X, that is $X = \cup_i O_i$ which implies $\phi = \cap_i (O_i)^c$. Since each O_i is p-open, $\{O_i^c\}$ is a class of p-closed sets and has an empty intersection. Hence there exists $(O_{i_1})^c, (O_{i_2})^c, ...(O_{i_m})^c$ such that $\bigcap_{j=1}^{j=m} O_{i_j}^c = \phi$. Thus $X = \bigcup_{j=1}^{j=m} O_{i_j}$. Accordingly X is p-Compact. □

Theorem 6.3.1. *A topological space X is p-Compact if and only if every class $\{F_i\}$ of p-closed subsets of X which satisfies the finite intersection property has itself a non-empty intersection.*

Proof. Proof is trivial by last proposition. □

Definition 6.3.1. Let (X, T) be a topological space and let $S : D \to X$ be a net. A point $x \in X$ is said to be a *p-cluster point/p-limit point* if for every p-open set U containing 'x' and $m \in D$, there exists an $n \in D$ such that $n \geq m$ and $S(n) \cap U \neq \phi$. Also S is said to *p-converges* to a point $x \in X$ if for every p-open set U containing 'x', there exists an $m \in D$ such that for all $n \geq m, n \in D$; $S(n) \cap U \neq \phi$.

Definition 6.3.2. Let (X, T) be a topological space and F be a filter on X. A point $x \in X$ is said to be a *p-cluster point /p-limit point* of F if every p-open set containing 'x' intersects every member of F. Also F is said to *p-converges* to a point $x \in X$ if every p-open set containing 'x' is a member of F.

Remark 6.3.1. Every limit point is a p-limit point but converse need not be true ; for example let N the set of all natural numbers be the directed set and let (N, D) be the given discrete topological space. Also let $I : N \to N$ be the identity mapping, that defines a net on N. Clearly any point of N is a p-limit point but not a limit point of the net

I.

Theorem 6.3.2. *Let (X, T) be a topological space then the following statements are equivalent*

1. *X is p-Compact.*

2. *Every net/filter in X has a p-cluster point.*

Proof. Case :1 To prove that (1) $\Longrightarrow$ (2)

Let $S : D \rightarrow X$ be a net in X, we have to prove that S has a p-cluster point. On contradiction we assume that S has no p-cluster point. Then by definition of p-cluster point, for each $x \in X$ there exist a p-open set U_x such that $x \in U_x$ and $m_x \in D$ such that for every $n \in D$, $n \geq m_x \Rightarrow S(n) \in X - U_x$. For each $x \in X$ there exists such U_x, hence $X = \bigcup\limits_{x \in X} U_x$. Since X is p-Compact there exists $x_1, x_2, ...x_k$ such that

$$X = \bigcup_{i=1}^{i=k} U_{x_i} \tag{6.1}$$

For each $x_i \in X$, there exists $m_{x_i} \in D$ such that $S(n) \in X - U_{x_i}$. Since D is a directed set there exists $n \in D$ such that $n \geq m_{x_i}$ for $i = 1, 2, ...k$. But then $S(n) \in \bigcap \{X - U_{x_i}/i = 1, 2, ...k\}$ which is equal to null set by (6.1) not possible by definition of net. Hence our assumption is wrong and S has at least one p-cluster point in X. Then since S is arbitrary, (1) $\Rightarrow$ (2).

Case : 2 To prove that (2) $\Rightarrow$ (1)

We assume that every net in X has a p-cluster point in X. To prove that X is p-Compact. Consider $\mathcal{F}$ as a family of p-closed sets in X having finite intersection property. We define $\mathcal{D}$ as the family of all finite intersection of members of $\mathcal{F}$. Now we make $\mathcal{D}$ a directed set by defining for $D_1, D_2 \in \mathcal{D}$; $D_1 \geq D_2 \Rightarrow D_1 \subset D_2$. Now define a net $S : D \to X$ by $S(D) =$ any point in D. By our assumption S has a p-cluster point say $x \in X$. In order to prove p-Compactness it is enough to prove $\mathcal{F}$ itself has non-empty intersection. For that we will prove $x \in \bigcap\limits_{F \in \mathcal{F}} F$. On contradiction we assume that $x \notin \bigcap\limits_{F \in \mathcal{F}} F$ which implies there exists $F \in \mathcal{F}$ such that $x \notin F$. Then $X - F$ is a p-open set containing 'x' . Since 'x' is a p-cluster point, by definition there exists $F_1 \in \mathcal{D}$ such that $F_1 \geq F$ and $S(F_1) \in X - F$. Now $F_1 \geq F$ implies $F_1 \subseteq F$ and $S(F_1) \in X - F_1$ which is a contradiction to our definition of net. Hence $x \in \bigcap\limits_{F \in \mathcal{F}} F$ and X is p-Compact. □

Theorem 6.3.3. *A topological space is p-Compact iff every ultra filter/ultra net in it is p-convergent.*

Proof. Let (X, T) be a topological space and assume that every ultra filter in it is p-convergent. To show that X is p-Compact it is enough to prove that every filter on X has a p-cluster point. Suppose $\mathcal{F}$ is a filter on X then there exists an ultra filter $\mathcal{F}^*$ containing $\mathcal{F}$. By our assumption $\mathcal{F}^*$ p-converges to a point say 'x' on X which implies 'x' is a p-cluster point of $\mathcal{F}^*$. So every p-open set containing 'x' meets every member of $\mathcal{F}^*$ and in particular every member of $\mathcal{F}$ since $\mathcal{F} \subset \mathcal{F}^*$. Hence 'x' is a p-cluster point of $\mathcal{F}$. Thus every filter on X has a p-cluster point which implies p-Compactness of X. Necessary part is trivial by *Theorem : 6.3.2.* □

Analyzing *Theorem 6.3.1*, *Theorem 6.3.2* and *Theorem 6.3.3* we characterize p-Compactness of any arbitrary topological space according to the following result.

Theorem 6.3.4. *Let (X, T) be any topological space. The following statements are equivalent*

1. *X is p-Compact.*
2. *Each family of p-closed subsets of X with the finite intersection property has non-empty intersection.*
3. *Every net/filter in X has a p-cluster point.*
4. *Every ultra filter in X p-converges.*

Theorem 6.3.5. *Let (X, T) be a topological space. Then p-Compactness of X implies compactness of X if and only if there exists no net S in X such that S has a p-cluster point but has no cluster points.*

Proof. Necessary Condition

Assume that X is p-Compact and p-Compactness implies compactness. Compactness of X implies that every net in X has a cluster point which in turn implies that there exists no net S in X such that S has a p-cluster point but has no cluster points in X.

Sufficiency Condition

Assume that there exists no net S in X such that S has a p-cluster point but has no cluster points and X is p-Compact. To prove that X is compact. p-Compactness of X implies every net in X has a p-cluster point and by our assumption above every net has a cluster point hence X is compact. □

Towards the end of this section we discuss some product theorems related to p-Compactness.

Theorem 6.3.6. *Finite product of p-Compact spaces is p-Compact.*

Proof. We shall prove that the product of two p-Compact spaces is p-Compact and the theorem follows by induction for finite product and we proceed through the following steps :

Step : 1

Let (X,T) and (Y,T') be two topological spaces given with Y p-Compact. Take $a_0 \in X$ and consider $a_0 \times Y$. For each $y \in Y$, let $G_y \times H_y$ be a p-open cover of $a_0 \times Y$. Clearly $y \in H_y$ and $a_0 \in G_y$ for every 'y'. Then the p-open sets $\{H_y\}_{y\in Y}$ forms a p-open cover of Y. As Y is p-Compact there exists finite sub cover $H_{y_1}, H_{y_2}, ..., H_{y_n}$ such that $Y = \bigcup\limits_{i=1}^{i=n} H_{y_i}$. Now take $G = \bigcap\limits_{i=1}^{i=n} G_{y_i}$, if $\bigcap\limits_{i=1}^{i=n} G_{y_i}$ is prime, otherwise choose G as a p-open set such that $\bigcap\limits_{i=1}^{i=n} G_{y_i} \subseteq G \subseteq G_{y_i}$, for each $i = 1, 2, ...n$. Now if both the above conditions does not holds choose G as $G = G_{y_i}$ for some $i = 1, 2, ...n$. Then in each case G is a p-open set containing a_0. Next we will prove that $G \times Y$ can be covered by $\{G_{y_i} \times H_{y_i}\} / i = 1, 2........n$. But since for each $x \times y \in G \times Y$, $x \in G_{y_i}$ for some $i = 1, 2......n$. Thus $G \times Y$ can be covered by $\{G_{y_i} \times H_{y_i}\} / i = 1, 2........n$.

Step : 2

Now consider two p-Compact topological spaces (X,T) and (Y,T') and let $\mathcal{G}$ be a p-open covering of $X \times Y$. Let $a_0 \in X$ and then by step : 1, $a_0 \times Y$ is p-Compact and can be covered by $\{G_i \times Y/i = 1, 2......m\}$. Thus for every $x \in X$ there exists a p-open set G_x such that $G_x \times Y$ can be covered by finitely many elements of $\mathcal{G}$. Clearly $\{G_x\}$ is a p-open covering of X and by p-Compactness there exists $\{G_{x_1}, G_{x_2}.................G_{x_n}\}$ covering X. Clearly then each of $G_{x_1} \times Y, G_{x_2} \times Y.......$ can be covered by finitely many elements of $\mathcal{G}$. Since $G_{x_1} \times Y, G_{x_2} \times Y.......$ covers $X \times Y$, it can be covered by finitely many elements of $\mathcal{G}$. Hence $X \times Y$ is p-Compact and any finite case follows by induction. □

Lemma 6.3.1. Let X be the topological product of an indexed family of spaces $\{(X_i, T_i)/i \in I\}$. Let $\mathcal{F}$ be a filter on $X$and let $x \in X$. If for each $i \in I$, the image filter of $\mathcal{F}$ under π_i p-converges to $\pi_i(x)$ in X_i then $\mathcal{F}$ p-converges to 'x'.

Proof. Given for each $i \in I$, the image filter of $\mathcal{F}$ under π_i p-converges to $\pi_i(x)$ in X_i, we have to show that $\mathcal{F}$ p-converges to 'x' in X. Let V be a p-open set in X containing 'x'. Then $V = \prod V_i$ where each V_i is a p-open set in X_i and $V_i = X_i$ for infinitely many 'i ' s except for $i = i_1, i_2...............i_n$. Given $\pi_{i_k}(\mathcal{F})$ p-converges to $\pi_{i_k}(x)$ for $i = 1, 2, ...n$ which implies $V_{i_k} \in \pi_{i_k}(\mathcal{F})$ and hence there exists $F_k \in \mathcal{F}$ such that

$V_{i_k} \supset \pi_{i_k}(F_k)$ for $k = 1, 2, ...n$

$\Rightarrow \pi_{i_k}^{-1}(V_{i_k}) \supset F_k$ for $k = 1, 2, ...n$

$\Rightarrow \pi_{i_k}^{-1}(V_{i_k}) \in \mathcal{F}$

$\Rightarrow V \in \mathcal{F}$ since $\mathcal{F}$ is a filter. Hence $\mathcal{F}$ p-converges to 'x'. □

Theorem 6.3.7. *Let (X_i, T_i) be a collection of non-empty topolog-*

ical spaces and let X be its topological product. Then X is p-Compact if and only if each X_i is p-Compact.

Proof. Necessary part is trivial by *Theorem : 6.2.4* and *Lemma : 3.3.1.* Conversely suppose each X_i is p-Compact to show that X is p-Compact. It is enough to prove that every ultrafilter in it is p-convergent by *Theorem : 6.3.4.* So let $\mathcal{F}$ be an ultrafilter on X and for $i \in I$ let F_i be its image filter under π_i. Then F_i is a filter on X_i and infact it is an ultrafilter on X_i. Since X_i is p-Compact F_i p-converges to x_i which implies $\mathcal{F}$ p-converges to 'x'. Thus X is p-Compact. □

6.4 C-p.compactness and Its Characterizations

The notion of covering a topological space by closures promoted the study of C-compactness in topological spaces and G.Viglino defined C-compactness as :

Definition 6.4.1. [49] A space X is said to be C-compact if for each closed $A \subset X$ and each open cover $\{U_\alpha / \alpha \in I\}$ of A, there exists a finite sub collection $\{U_{\alpha_i} / i = 1, 2..n\}$ such that $A \subseteq \bigcup\limits_{i=1}^{i=n} cl(U_{\alpha_i})$.

Similarly if p-open sets cover a particular set then definitely the p-closures of corresponding sets also forms a cover. Such a conception bring out the definition of C-p.compactness and in this section we introduced C-p.compactness. Also obtained characterization of C-p.compactness using nets and filters.

Definition 6.4.2. Let (X, T) be a topological space. A set $U \subseteq X$ is said to be *regular prime open(regular p-open)* if p-int(p-$cl(U)$) $= U$ and U is called regular p-closed set if it is the complement of a regular p-open set.

Example 6.4.1. Consider $X = \{a, b, c, d\}$,
$T = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ then (X, T) is a topological space and $\{a\}$ is a regular p-open set .

Definition 6.4.3. [41] Let (X, T) be a topological space. A set $U \subseteq X$ is said to be regular open if $\text{int}(cl(U)) = U$.

Remark 6.4.1. In a discrete space all subsets are regular open but all of them are not regular p-open hence regular open sets need not be regular p-open.

Theorem 6.4.1. *p-interior of a set $A \subset X$ is regular p-open, if it is p-closed.*

Proof. In order to prove that p-int(A) is regular p-open it is enough to prove that p-int(A)=p-int(p-cl(p-$int(A)$)). Since A is p-closed p-cl(p-$int(A)$) is A itself. Hence the result. □

Corollary 9. For any set $U \subseteq X$, p-int(p-$cl(U)$) is a regular p-open set always.

Proof. Proof is trivial by last theorem since p-$cl(U)$ is always p-closed. □

Proposition 6.4.1. Let (X, T) be a topological space, then for every p-open set there always exists a unique regular p-open set containing

A.

Proof. Proof is trivial by *Proposition : 2.3.1* and *Proposition : 2.3.5* □

Definition 6.4.4. A space X is said to be *C-p.compact* if for each p-closed $A \subset X$ and each p-open cover $\{U_\alpha / \alpha \in I\}$ of A, there exists a finite sub collection $\{U_{\alpha_i} / i = 1, 2..n\}$ such that $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{\alpha_i})$.

Remark 6.4.2. Compactness implies p-Compactness implies C-p.compactness.

Remark 6.4.3. C-p.compactness need not implies C-compactness ; in order to prove this we consider an example due to S.Sakai [42]. Let $X = \{(a, b) : n, m \in N\}$ such that $a = 1/n, b = 1/m$ or $a = 1/n, b = 0$ or $a = 0, b = 0$ where N stands for the set of all positive integers. Also let $\{N_i / i \in N\}$ be the partition of N to infinitely many disjoint classes. Define subsets of X as follows:
$H_{ik} = \{\{(1/i, 0)\} \cup \{(1/i, 1/m)/m \geq k\}\} \cup \{(1/n, 1/m)/n \geq k, m \in N_i\}$,
$L_k = \{(0, 0)\} \cup \{(1/n, 1/m)/n > k, m \notin N_i, 1 \leq i \leq k\}$.
Let T be the topology on X generated by $\{\{(1/n, 1/m)\} / n, m \in N\} \cup \{H_{ik} / i, k \in N\} \cup \{L_k / k \in N\}$ .Then (X, T) is a C-compact hausdorff space. Now let $Y = \{y_0, y_1, y_2....\}$ be a one point compactification of a countable discrete space $\{y_1, y_2....\}$. Consider $X \times Y$ S.Sakai in [42] proved that this is a non C-compact space, but this space is hausdorff since both X and Y are hausdorff and hence it happens to be a p-Compact and thus C-p.compact space.

Theorem 6.4.2. *A space (X, T) is C-p.compact if and only if for each p-closed $A \subseteq X$ and regular p-open cover (Cover in which all ele-*

ments are regular p-open) $\{U_\alpha/\alpha \in I\}$ *there exists a finite sub collection* $\{U_{\alpha_i} : i = 1, 2...n\}$ *such that* $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{\alpha_i})$.

Proof. Necessary Part:

If X is C-p.compact all p-open covers satisfies the given condition we have to prove that each regular p-open cover $\{U_\alpha/\alpha \in I\}$ satisfies the given condition but since each U_α is regular p-open ; p-int(p-$cl(U_\alpha)) = U_\alpha$ for every $\alpha \in I$ that is each U_α is p-open and since each p-open cover satisfies the given condition $\{U_\alpha/\alpha \in I\}$ also satisfies the given condition. Hence necessary part is trivial.

Sufficiency Part:

Assume that for each p-closed $A \subseteq X$ and regular p-open cover $\{V_\alpha/\alpha \in I\}$ there exists a finite sub collection $\{V_{\alpha_i} : i = 1, 2...n\}$ such that $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(V_{\alpha_i})$. Let $\{U_\alpha/\alpha \in I\}$ be any p-open cover of A. Then $\{p\text{-}int(p\text{-}cl(U_\alpha))/\alpha \in I\}$ is a regular p-open cover of A. So there exists a finite sub collection $\{p\text{-}int(p\text{-}cl(U_{\alpha_i})) : i = 1, 2....n\}$ such that
$A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(p\text{-}int(p\text{-}cl(U_{\alpha_i})))$.
Clearly p-$cl(p\text{-}int(p\text{-}cl(U_{\alpha_i}))) = p\text{-}cl(U_{\alpha_i})$
hence $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{\alpha_i})$; that is A is C-p.compact. □

Definition 6.4.5. Let (X, T) be a topological space and let $X \neq \phi$, **F** a filter on X. Then **F** is said to be *p-closure convergent* to a point $a \in X$ if for every p-open set V in X containing 'a' ; p-$cl(V) \in$ **F** and $a \in X$ is said to be a *p-closure limit point* of **F** if every p-open set V containing 'a' is such that p-$cl(V) \cap A \neq \phi$ for every $A \in$ **F**.

Remark 6.4.4. p-convergence implies p-closure convergence but converse is not true. Similarly for p-closure limit point.

Example 6.4.2. Let $(R, T = \{R, \phi, Q, Q'\})$ be a topological space and a filter be defined on it as $F = U(x)/x \in Q'$ that is principal filter generated by 'x'.Then any rational point is a p-closure convergent point but not a p-convergent point.

Definition 6.4.6. Let (X, T) be a topological space and let $S : D \to X$ be a net on X , then S *p-closure converges* to a point $x \in X$ if for every p-open set U containing 'x' there exists $m \in D$ such that for every $n \geq m$; $S(n) \cap p\text{-}cl(U) \neq \phi$ and 'x' is said to be a *p-closure limit point* of S if for every p-open set U containing 'x' and $m \in D$; there exists $n \in D$ such that $n \geq m$ and $S(n) \cap p\text{-}cl(U) \neq \phi$.

Now we characterize C-p.compactness as follows :

Theorem 6.4.3. *Let (X, T) be a topological space then the following conditions are equivalent:*

1. *X is C-p.compact.*
2. *Corresponding to each regular p-open cover (Cover in which all elements are regular p-open) $\{U_\alpha/\alpha \in I\}$ of an arbitrary p-closed set A, there exists a finite sub collection $\{U_{\alpha_i} : i = 1, 2...n\}$ such that $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{\alpha_i})$ and this is true for each p-closed set.*
3. *Corresponding to each family of non-empty regular p-closed sets $\{F_\alpha/\alpha \in I\}$ such that $(\bigcap_\alpha F_\alpha) \cap A = \phi$ where A is any arbitrary p-closed set, there exists a finite sub family $\{F_{\alpha_i}/i = 1, 2......n\}$ such*

that $(\bigcap_{i=1}^{i=n} p\text{-}int(F_{\alpha_i})) \cap A = \phi$.

4. *For each p-closed* $A \subseteq X$ *and each filter with base* $F = \{A_\alpha/\alpha \in I\}$ *in* A*, there exists an* $a \in A$ *such that the filter has 'a' as a p-closure cluster point.*

Proof.

- (1) $\Leftrightarrow$ (2) already proved.

- To prove (1) $\Rightarrow$ (3).

  Assume that X is C-p.compact. Given A as a p-closed set and collection of non-empty regular p-closed sets $\{F_i/i \in I\}$ such that $(\bigcap_i F_i) \cap A = \phi$. To prove that there exists a finite sub family such that $(\bigcap_{i=1}^{i=n} p\text{-}int(F_i)) \cap A = \phi$.
  Consider $\mathcal{U} = \{U_i/U_i = X - F_i/i \in I\}$ then each U_i is a regular p-open set. Also $(\bigcap_i F_i) \cap A = \phi$
  $\Rightarrow A \subseteq \bigcup_i (X - F_i)$
  $\Rightarrow A \subseteq \bigcup_i U_i$.
  That is A can be covered by a collection of regular p-open sets but since X is C-p.compact and A is p-closed
  $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_i)$
  Now consider $\bigcap_{i=1}^{i=n} p\text{-}int(F_i) = X - \bigcup_{i=1}^{i=n} p\text{-}cl(U_i) \subseteq X - A$ and hence $(\bigcap_{i=1}^{i=n} p\text{-}int(F_i)) \cap A = \phi$.

- To prove (3) $\Rightarrow$ (2)

Let $\{U_i/i \in I\}$ be a regular p-open cover of A, to prove that there exists a finite sub collection $\{U_i : i = 1, 2...........n\}$ such that $A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_i)$. Given $A \subseteq \bigcup_i U_i$ which implies $A \cap (\bigcap_i (X - U_i)) = \phi$. Since $X - U_i$ is regular p-closed now we can apply condition (3) and that implies there exists a finite sub collection $\{U_i : i = 1, 2...........n\}$ such that

$(\bigcap_{i=1}^{i=n} p\text{-}int(X - U_i)) \cap A = \phi$ which implies

$A \subseteq X - (\bigcap_{i=1}^{i=n} p\text{-}int(X - U_i))$

$= \bigcup_{i=1}^{i=n} (X - p\text{-}int(X - U_i))$

$= \bigcup_{i=1}^{i=n} p\text{-}cl(U_i)$ Hence condition (2) is satisfied.

- To prove condition (1) $\Rightarrow$ (4)

  Suppose there exists a filter **F** on A with filter base $\mathbf{F} = \{A_i/i \in I\}$ in A to prove that **F** has a p-closure limit point. On contradiction assume that **F** has no p-closure limit point. Then for each $a_j \in A$ there exists a p-open set U_{a_j} and some $A_{i(a_j)} \in \mathbf{F}$ such that $A_{i(a_j)} \cap p\text{-}cl(U_{a_j}) = \phi$. Such a p-open set U_{a_j} exists for each $a_j \in A$ and hence $\{U_{a_j}/a_j \in A\}$ forms a p-open cover of A and C-p.compactness of X implies

$$A \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{a_j}). \tag{6.2}$$

  Now corresponding to each $U_{a_j}; j = 1, 2......n$ there exists $A_{i(a_j)} \in \mathbf{F}$ which implies

  $\bigcap_{i=1}^{i=n} A_{i(a_j)}$ belongs to the filter

which implies there exists $A_{i(a_0)}$ belongs to the filter base and is such that $A_{i_{(a_0)}} \subseteq \bigcap_{i=1}^{i=n} A_{i(a_j)}$.
But clearly $A_{i_{(a_0)}} \subseteq A$ and hence (6.2) implies
$A_{i_{(a_0)}} \subseteq \bigcup_{i=1}^{i=n} p\text{-}cl(U_{a_j})$ which implies
$A_{i(a_j)} \cap p\text{-}cl(U_{a_j}) \neq \phi$ for every j which is a contradiction. Hence the result.

- To prove (4) $\Rightarrow$ (3)

In order to prove condition (3) it is enough to prove its contra positive statement. Suppose there exists a p-closed set $A \subseteq X$ and a collection of regular p-closed sets $\{F_i/i \in I\}$ such that each finite sub collection $\{F_i/i = 1, 2.......n\}$ has the property that $(\bigcap_{i=1}^{i=n}(p\text{-}int(F_i))) \cap A \neq \phi$ but $(\bigcap_i F_i) \cap A = \phi$.
Then the sets $\{(p\text{-}int(F_i)) \cap A/i \in I\}$ together with all finite intersections of the form $(\bigcap_{i=1}^{i=n}(p\text{-}int(F_i))) \cap A$ will form a filter base for a filter **F** on A ; then by our assumption **F** has a p-closure limit point say $a \in A$. Then for any p-open set $U(a)$ containing 'a' and each $p\text{-}int(F_i)$; p-$cl(U(a)) \cap (p\text{-}int(F_i) \cap A) \neq \phi$. But $F_i \cap A \neq \phi$ for every $i \in I$ and $(\bigcap_i F_i) \cap A = \phi$ together implies there exists $i_0 \in I$ such that $a \notin F_{i_0}$. Therefore $a \notin p\text{-}int(F_{i_0})$ implies $a \in X - F_{i_0} \subset p\text{-}cl(X - F_{i_0}) \subset X - p\text{-}int(F_{i_0})$ which implies $p\text{-}cl(X - F_{i_0}) \cap p\text{-}int(F_{i_0}) = \phi$ which implies 'a' is not a p-closure limit point of F which is a contradiction. Hence $(\bigcap_i F_i) \cap A \neq \phi$. Hence proving the contra positive statement of (3).

□

Remark 6.4.5. Let X be a topological space and 'x' in X is said to be a p-closure cluster point of the filter F on X if and only if $'x'$ is a p-closure cluster point for the net associated with the filter and conversely. Hence the above equivalent condition for C-p.compactness holds for any net on any p-closed subset of the corresponding topological space.

6.5 Prime Topological Spaces and Its Lattice Structure

Definition 6.5.1. A topological space (X, T) is said to be a *prime topological space* shortly *p-topological space* if every open set in T is a p-open set.

Example 6.5.1. Consider the real line R with $T = \{R, \phi, \{(-\infty, a)/a \in R\}\}$. Then (R, T) is a prime topological space.

Remark 6.5.1. A topological space in which the open sets forms a chain will always be a prime topological space. Converse of this remark is not true , for example Consider real line R with $T = \{R, \phi, Q, Q'\}$ where Q and Q' are the rational and irrational numbers respectively is a p-topological space but collection of all open sets in it will not form a chain.

Theorem 6.5.1. *A topological space (X, T) is a prime topological space if and only if either T is a chain or $|B| \leq 2$ where B is the sub base for T and elements of B are disjoint.*

Proof. Sufficiency part is trivial and for proving necessary part we assume that (X, T) is a prime topological space and let the collection of open sets in it be not a chain. Then we consider two cases : either $|B| \leq 2$ or $|B| > 2$ where B is a sub base for T.

If the first case occurs and if B is not of the required form , then $T = \{X, \phi, A, B, A \cap B, A \cup B\}$ then clearly $A \cap B$ is not prime in T and that is not possible since by our assumption T is a prime topology on X. If the second case occurs then cardinality of B is at least three and for mean time we assume that $|B| = 3$. Let $B = \{A, B, C\}$ then again there arise two cases :

1. $A \cap B \neq \phi$, $B \cap C = \phi$ then
   $T = \{X, \phi, A \cap B, A, B, C, A \cup B, A \cup C, B \cup C\}$. Clearly here $A \cap B$ is not prime obtaining a contradiction. Similarly we can prove for cases when $B \cap C \neq \phi$, $A \cap C \neq \phi$. If each of them intersects mutually then trivially there always exists non prime open sets.

2. A, B, C are mutually disjoint , then
   $T = \{X, \phi, A, B, C, A \cup B, A \cup C, B \cup C\}$. Then A, B, C are clearly not prime.

If cardinality of B is greater than three , then also B is of the required form and the same proof above holds. Hence either B is a chain or $|B| \leq 2$ and elements of B must be disjoint. □

Definition 6.5.2. [2] A topological space is said to be principal if arbitrary intersection of open sets are open.

Remark 6.5.2. Analyzing *Theorem : 6.5.1*, we can deduce that all prime topological spaces are principal. Converse of this remark need not be true since all finite topological spaces are principal but all are clearly not prime topologies.

Proposition 6.5.1. Let X be any arbitrary set and let T, T' be two prime topologies on X, then $T \cap T'$ is also a prime topology on X.

Proof. Let $G \in (T \cap T')$ and assume that G is not prime then there exists H, $K \in T \cap T'$ such that $H \cap K \subseteq G$ but H not a subset of G and K not a subset of G. Given H, K, G belongs to $T \cap T'$ which means H, K, G belongs to both T and T' implies G not prime in T and T' which is not possible. Hence $T \cap T'$ is a prime topology on X. □

Remark 6.5.3. Join of two prime topologies need not form a prime topology. For example Let $X = R$, real line with $T = \{(-\infty, a)/a \in R\}$ and $T' = \{(b, \infty)/b \in R\}$. Here both T and T' are prime topologies on X, but $T \vee T'$ is the discrete topology on R which is not a prime topology.

Let $\mathcal{P}_r(X)$ denote the collection of all prime topologies on any arbitrary set X, then $\mathcal{P}_r(X)$ forms a meet semi lattice on X. The minimal element in $\mathcal{P}_r(X)$ is the indiscrete topology on X and atoms of $\mathcal{P}_r(X)$ is same as atoms of the lattice of all topologies on X.

Theorem 6.5.2. *$\mathcal{P}_r(X)$ is meet complemented.*

Proof. Let $T \in \mathcal{P}_r(X)$ we have to prove that there exists $T' \in \mathcal{P}_r(X)$ such that $T \cap T' = \underline{O}$, the indiscrete topology. If we take any prime topology T then either it is a chain or not . In either case there exists at least one singleton set $\{x_0\}$, $x_0 \in X$ such that $\{x_0\} \notin T$. Then clearly $T \cap \{X, \phi, \{x_0\}\} = \underline{O}$. If there does not exists $x_0 \in X$ such that $\{x_0\} \notin T$ then T becomes discrete topology which is not a prime topology. Hence there always exists such a singleton set and the atom involving that element serves as the meet complement of prime topology chosen. □

Remark 6.5.4. Analyzing *Theorem : 6.5.1* we can see that every element of $\mathcal{P}_r(X)$ can be written as join of atoms. Hence $\mathcal{P}_r(X)$ is atomic.

Theorem 6.5.3. *For any arbitrary set X , $\mathcal{P}_r(X)$ is meet complemented atomic semi lattice.*

Proof. Proof is trivial by *Proposition :6.5.1, Theorem : 6.5.2* and *Remark : 6.5.4.* □

Theorem 6.5.4. *Arbitrary product of prime topological spaces is a prime topological space.*

Proof. Let $\{(X_i, T_i) : i \in I\}$ be a collection of p-topological spaces and let $(X = \prod X_i, T)$ be their product topological space. Then the open sets in T are sets of the form $\prod U_i$ where $U_i = X_i$ for infinitely many i's and other U_i's are open in T_i. But since U_i open in T_i , U_i is p-open in T_i for every 'i' and hence each open set in X becomes a p-open set. Thus X is a prime topological space. □

6.6 Prime Elements in the Lattice of Topologies

Let X be any arbitrary set and let T_1 and T_2 be any two topologies on X. If every set in T_1 is also in T_2, then T_1 is said to be coarser than T_2 and T_2 finer than T_1. If $\Sigma(X)$ denote the collection of all topologies on X, then Vaidyanathswamy[48] proved that $\Sigma(X)$ forms a complete lattice under the above order of coarser and finer topologies.

Definition 6.6.1. Let $\Sigma(X)$ denote the collection of all topologies on an arbitrary set X. Let $T \in \Sigma(X)$, then T is said to be prime in $\Sigma(X)$ if $T_1 \wedge T_2 \leq T$ implies $T_1 \leq T$ or $T_2 \leq T$ where T_1,$T_2 \in \Sigma(X)$ with $T_1 \wedge T_2 \neq \underline{O}$, the indiscrete topology.

Definition 6.6.2. [44] A topology $T \in \Sigma(X)$ is called an ultraspace if the only topology on X finer than T is the discrete topology on X.

Frohlich[17] characterised ultraspaces in terms of filters. He defined a family $\delta(x, F)$ for each filter F on X as $\delta(x, F) = \mathcal{P}[X - \{x\}] \cup F$ where $\mathcal{P}[X - \{x\}]$ is the collection of all subsets of X which does not contain 'x'. And then he observed that ultraspaces on X are exactly topologies of the form $\delta(x, U)$ where $x \in X$ and U is an ultrafilter on X.

Theorem 6.6.1. [*17*] *Every topology T on any arbitrary set X is the infimum of all ultraspaces on X which are finer than T.*

Theorem 6.6.2. *Let X be any arbitrary set, then the prime elements in $\Sigma(X)$ are precisely the ultraspaces in $\Sigma(X)$.*

Proof. Proof follows directly from *Theorem : 6.6.1* and *Definition : 6.6.1*. □

Concluding remarks and suggestions for further study

We have introduced p-open sets in the lattice of open sets of any arbitrary topological space and studied the concepts of generalised closed sets, semi open sets etc using p-open sets. Also analyzed some weaker separation axioms like p-$T_{1/2}$, p-T_0, p-T_1. However some more weaker separation axioms like $T_{1/4}$, $T_{3/4}$ and properties like connectedness are not yet analyzed. Application of this concept in other branches of mathematics is yet to be investigated.

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