# A Study on Some Information Measures using Quantile Functions

Thesis submitted to the
Cochin University of Science and Technology
for the Award of Degree of

**Doctor of Philosophy**

under the Faculty of Science

by

**Aswathy S Krishnan**

**DEPARTMENT OF STATISTICS**
**COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY**
**COCHIN-682 022**

June 2019

*To My Family*

**CERTIFICATE**

This is to certify that the thesis entitled **"A Study on Some Information Measures using Quantile Functions "** is a bonafide record of work done by Ms. Aswathy S Krishnan under our guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

| | | |
|---|---|---|
| Kochi 682 022 | **Dr. S. M. Sunoj** | **Dr. P. G. Sankaran** |
| June 2019 | Professor, | Professor, |
| | Department of Statistics, | Department of Statistics, |
| | Cochin University of | Cochin University of |
| | Science and Technology. | Science and Technology. |

## CERTIFICATE

Certified that all the relevant corrections and modifications suggested by the audience during pre-synopsis seminar and recommended by the Doctoral committee of the candidate have been incorporated in the thesis.

Kochi 682 022
June 2019

**Dr. S. M. Sunoj**
Professor,
Department of Statistics,
Cochin University of
Science and Technology.

**Dr. P. G. Sankaran**
Professor,
Department of Statistics,
Cochin University of
Science and Technology.

**DECLARATION**

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Kochi 682 022 **Aswathy S Krishnan**

June 2019

# *Acknowledgements*

*It is with all sincerity and high regards that I express my deep sense of gratitude to my supervising guide Dr. S. M. Sunoj, Professor, Department of Statistics, Cochin University of Science and Technology, for his meticulous guidance, consistent encouragement and valuable suggestions throughout my research period. This feat was possible only because of the unconditional support provided by him.*

*I also put in writing my obligation to my co-guide, Dr. P. G. Sankaran, Professor, Department of Statistics, CUSAT, Cochin, whose suggestions, inspiration and guidance throughout this work helped me in bringing out this thesis in the present form.*

*I wish to express my sincere gratitude to Dr. N. Unnikrishnan Nair, Former professor, Department of Statistics, CUSAT who willingly devoted so much time with great support, valuable advice, constructive criticism around my work which have provided good and smooth basis for my thesis work.*

*I profoundly thank the faculty members Prof. N. Balakrishna, Prof. K. C. James, Prof. Asha Gopalakrishnan, Dr. Rajesh G, Dr. Irshad M. R, Dr. K. G. Geetha, and Dr. Hitha of Department of Statistics, CUSAT for their valuable suggestions and help to complete this endeavour. I remember with deep gratefulness all my former teachers who gave me light in life through education. I extend my sincere thanks to all non-teaching staff of the Department of Statistics, CUSAT for their kind cooperation.*

*I owe my appreciation and thankfulness to Kerala State Council for Science, Technology and Environment (KSCSTE) for providing me financial support to carry out this work under KSCSTE research fellowship.*

*Words fail to express my gratitude to my best friends, who were giving me encouragement and stood behind me in ups and downs and gave inspiration and moral support for the completion of this task. I owe a lot to my friends and research scholars, Department of Statistics, CUSAT who helped, inspired and encouraged me whenever it needs. I am failing in words to express my feelings to my husband, parents, brother and mother in law for their love, care and support. I owe everything to them.*

*Besides this, several people have knowingly or unknowingly helped me in the successful completion of this project. I extend my sincere gratitude to all of them. Above all, I bow before the grace of the Almighty.*

***Aswathy S Krishnan.***

# Contents

# List of Tables

# List of Figures

# Chapter 1

# Introduction

Information theory is a branch of the mathematical theory of probability and mathematical statistics which studies the storage, quantification and communication of information. For the last few decades, there has been a tremendous growth in the volume of research in the field of information theory. The information theory plays an important role in modern communication theory, which formulates a communication system as a stochastic process. Entropy has been introduced to explain the concept of information completely by a single definition. The concept of entropy was originally developed by physicists in the context of equilibrium thermodynamics. Entropy is a key measure in information theory which quantifies the amount of uncertainty involved in the value of a random variable or the outcome of a random process. The statistical definition was developed by Ludwig Boltzmann in the 1870s by analysing the stochastic behaviour of the microscopic components of the system. Recently there has been a great deal of interest in the measurement of uncertainty associated with a probability distribution. Shannon (1948) introduced the concept of entropy which is widely used in the fields of communication theory, physics, information theory, economics, probability and statistics. Later, many researchers studied the generalized form of Shannon entropy like Renyi's entropy (see Renyi

(1961)), Tsallis entropy (Tsallis (1988)), etc.

Many generalizations of Shannon entropy (Shannon (1948)) are available in literature. The important properties of these generalized information measures are smoothness, large dynamic range with respect to certain conditions, etc. that make them more flexible in practice. In reliability and life testing, the data is often truncated for used items, so that the basic form of entropy measures become unsuitable. The role of information measures relating to the residual and past lifetime in reliability modelling has been extensively investigated during the last two decades, starting from the works of Muliere et al. (1993), Ebrahimi (1996) and Di Crescenzo & Longobardi (2002) with reference to Shannon's entropy.

Quantile functions are efficient and equivalent alternatives to distribution functions in modelling and analysis of statistical data (see Gilchrist (2000); Nair & Sankaran (2009)). A probability distribution can be specified either in terms of the distribution function or by the quantile functions (QF). Although both convey the same information about the distribution of the underlying random variable $X$, the concepts and methodologies based on the distribution function are employed in most forms of the lifetime data analysis. Quantile functions have several properties that are not shared by distribution functions. For example, the sum of two quantile functions is again a quantile function. In many cases, quantile function is more convenient as it is less influenced by extreme observations, and thus provides a straightforward analysis with a limited amount of information. There are explicit general distribution forms for the quantile function of order statistics. It is easy to generate random numbers from the quantile function. In reliability analysis, a single long term survivor can have a marked effect on mean life, especially in the cases where some heavy-tailed models are encountered that are commonly lifetime data. In these cases quantile-based estimates are generally more precise and robust against outliers. However, the use of quantile functions in the place of the distribution function $F(.)$ provides new models, alternative methodology, easier algebraic manipulations and methods of analysis in certain cases and some new results that are difficult to derive by using distribution function. There are certain distributions that do not have a tractable distribution function. In such cases, quantile functions are more convenient. For a detailed study on quantile function, its properties and its usefulness in identifying probability models we refer to Nair & Sankaran (2009), Sankaran & Nair (2009), Sankaran et al. (2010), Nair & Vineshkumar (2011), Nair et al. (2013), Sankaran & Kumar (2018), Sreelakshmi et al. (2018).

Even if abundant literature is available on characterizations of distributions using different statistical measures, little work has been found for modelling lifetime data using quantile versions of order statistics. Further, the study of information measures using quantile functions is of recent interest. Sunoj & Sankaran (2012) introduced a quantile version of Shannon entropy and studied its various properties. A quantile approach on residual Renyi's entropy and cumulative entropy measures for residual lifetime random variables are available in Nanda et al. (2014) and Sankaran & Sunoj (2017) respectively. In the meantime, the variant approach of employing the quantile version of various entropies was introduced with the objectives of providing alternative methodology, new results and different methods of stochastic comparisons. We may refer to Sunoj & Sankaran (2012), Yu & Wang (2013), Nanda et al. (2014) and Sunoj et al. (2017) for a review on this topic.

The objective of the present study is to develop information measures using quantile functions and provide certain characterisation theorems of distributions, bounds, ordering relations, ageing properties, non-parametric estimation, etc based on the proposed measure. The thesis is organized into seven chapters. After the introductory chapter, in Chapter 2 we give a brief review of the background materials required for the discussions in subsequent chapters starting from the definition and basic properties of quantile function along with the descriptive measures and $L-$ moments. Subsequently, we discuss basic reliability concepts such as hazard rate, mean residual life function, reversed hazard rate and reversed mean residual life function in both the distribution framework and quantile set up. We also provide a brief review of some popular quantile function models in literature such as lambda distributions by Ramberg & Schmeiser (1972), Freimer et al. (1988), the power-Pareto distribution Hankin & Lee (2006), Govindarajulu model (Govindarajulu (1977)), etc. Further, we give a brief review of entropy, extropy, order statistics and related measures. Order statistics play an important role in various fields of statistical theory and practice. Even if both convey the same information about the distribution, with different interpretations, the order statistics concepts and its methodologies based on distribution functions are traditionally employed in most of the statistical theory and practice. The quantile approach will be of more useful when no tractable distribution function exists but quantile function exists. Accordingly, we study some general distribution and reliability properties of order statistics using quantile functions. We also examine the quantile-based reliability ageing properties, moments and stochastic orders of order statistics.

In Chapter 3, we introduce a quantile-based Shannon entropy of order statistics and studied its properties. The quantile Shannon entropy of order statistics has several advantages. Firstly, unlike Shannon entropy of order statistics based on distribution function, the computation of its quantile version is quite simple especially in cases where the distribution functions are not tractable. Secondly, our approach gives an alternative methodology in the study of Shannon entropy of order statistics and its dynamic (residual) measure. Further, there are certain properties of quantile functions that are not shared by the distribution function approach. Thus a formulation of the definition and properties of entropy of order statistics in terms of quantile function is studied. Further, an explicit relationship between quantile entropy of order statistics and quantile density function can be obtained, thus uniquely determines the underlying distribution.

Rao et al. (2004) introduced an alternative measure of uncertainty that extends Shannon entropy to random variables with continuous distributions. Cumulative residual entropy can be easily computed from sample data and these computations asymptotically converge to the true values. Among different generalized entropy measures, an important one is the Tsallis entropy of order $\alpha$

(Tsallis (1988)). Tsallis entropy plays a major role in different fields viz. physics, chemistry, biology, economics and statistics. Unlike some other generalized information measures, Tsallis entropy is non-additive. For more details and recent works on Tsallis entropy we refer to Tsallis (1988), Mathai & Haubold (2007), Cartwright (2014), Kumar (2016), Baratpour & Khammar (2016), Rajesh & Sunoj (2016), Calì et al. (2017) and the references therein. Accordingly, in Chapter 4, we have introduced a quantile-based cumulative residual Tsallis entropy (CRTE) and quantile-based CRTE for order statistics. Unlike the cumulative residual Tsallis entropy measures in the distribution function approach due to Sati & Gupta (2015), Rajesh & Sunoj (2016) respectively, the corresponding quantile versions possess some unique properties. We obtain some characterizations for distributions based on the quantile versions of CRTE and derive certain bounds. We also study various properties of quantile-based CRTE for order statistics. Non-parametric estimation of CRTE based on quantile function is discussed. A simulation study is conducted to assess the performance of the estimator. Further, the proposed estimator is applied to Aarset data (Aarset (1987)).

Measure of uncertainty in past lifetime plays an important role in different areas such as information theory, reliability theory, survival analysis, economics, business, forensic science and other related fields. However, in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. For instance, if at time $t$, a system which is observed only at certain preassigned inspection times, is found to be down; then the uncertainty of the system life relies on the past, *i.e.*, on which instant in $(0, t)$ it has failed. A wide variety of research is available on entropy measures and its applications in past lifetime.

For details, one can refer to Di Crescenzo & Longobardi (2002), Di Crescenzo & Longobardi (2009), Sachlas & Papaioannou (2014), Di Crescenzo & Toomaj (2015) and the references therein. Also, a study on the cumulative Tsallis entropy for past lifetime is available in Calì et al. (2017). In Chapter 5, we propose a cumulative Tsallis entropy in past lifetime based on quantile function. We obtain different characterizations based on the proposed measure and quantile-based reliability measures. We also study the quantile-based cumulative Tsallis entropy of order statistics in past lifetime. We discuss the non-parametric estimation of CRTE in past lifetime based on quantile function. A simulation study is conducted to examine the performance of the estimator. We illustrate the utility of the proposed model using a real life data set.

The entropy measure of a probability distribution has found plenty of useful applications in information sciences since its full-blown introduction in the extensive article of Shannon (1948). Due to its wide utility in thermodynamics as cited by Boltzmann and Gibbs, entropy has subsequently bloomed as a show piece in theories of communication, coding, probability and statistics. One of the statistical applications of extropy is to score the forecasting distributions using the total log scoring rule. In Chapter 6, we study the residual extropy function using distribution function and quantile function approaches. We also study the quantile-based extropy for order statistics and investigate extropy in past lifetime in both approaches. Some characterizations and ageing properties of these extropy functions are proposed. Different stochastic orders based the residual and past lifetime extropy function are also presented. Moreover, we introduce the quantile-based cumulative extropy and obtained some interesting results within the framework of quantile function. The non-parametric estimation of quantile-based extropy is also studied.

Finally, in Chapter 7 we summarize the major conclusions of the study and also indicate the directions of future study.

# Chapter 2

# Basic concepts and review of literature

The present chapter provides a brief review of some of the existing works on quantile functions, reliability concepts, entropy, extropy, stochastic orders and order statistics. We also discuss the quantile-based moments and stochastic orderings of order statistics.

## 2.1 Quantile functions

In this section, we present the definition and properties of quantile functions, some descriptive measures, $L-$ moments and reliability concepts, etc., in terms of quantile functions.

### 2.1.1 Definitions and properties

**Definition 2.1.** Let $X$ be a random variable with distribution function $F(x)$ which is continuous from right. Then quantile function $Q(u)$ of $X$ is defined as

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, \;\; 0 \leq u \leq 1. \tag{2.1}$$

For $-\infty < x < \infty$ and $0 < u < 1$, $F(x) \geq u$ if and only if $Q(u) \leq x$. Thus if there exists an $x$ such that $F(x) = u$, then $F(Q(u)) = u$ and $Q(u)$ is the smallest value of x satisfying $F(x) = u$. When $F(x)$ is continuous and strictly increasing,

$$Q(u) = inf\{x : F(x) = u\},$$

is the unique value of $x$ such that $F(x) = u$. In this case we can write the quantile function by solving $F(x) = u$ for $x$ in terms of $u$.

**Definition 2.2.** If $f(x)$ is the probability density function of $X$, then $f(Q(u))$ is called the density quantile function. The derivative of $Q(u)$, i.e., $q(u) = Q'(u)$ is known as the quantile density function of X. By differentiating $F(Q(u)) = u$, we have an identity

$$q(u)f(Q(u)) = 1. \tag{2.2}$$

Some important properties of quantile functions are given below:

1. For a general distribution function, from the definition of $Q(u)$ we have,

   - $Q(u)$ is non-decreasing on $(0, 1)$ with $Q(F(x)) \leq x$ for all $-\infty < x < \infty$ for which $0 < F(x) < 1$.
   - $F(Q(u)) \geq u$ for any, $0 < u < 1$.
   - $Q(u)$ is continuous from the left, ie, $Q(u^-) = Q(u)$ so that $Q(u)$ has limits from above.
   - Any jump points of $F(x)$ are flat points of $Q(u)$ and flat points of $F(x)$ are jump points of $Q(u)$.

2. Since for a uniform random variable $U$ over $(0, 1)$,

   $P\{Q(U) \leq x\} = P\{U \leq F(x)\} = F(x),$

   $Q(u)$ and $X$ are identically distributed.

3. The sum of two quantile functions is again a quantile function.

4. The product of two positive quantile functions is also a quantile function.

5. If $T(x)$ is non-decreasing function of $x$, then $T(Q(u))$ is a quantile function. On the other hand, if $T(x)$ is non-increasing, then $T(Q(1-u))$ is also a quantile function.

6. If $Q(u)$ is the quantile function of $X$ with continuous distribution function $F(x)$ and $T(u)$ is a non-decreasing function satisfying the boundary conditions $T(0) = 0$ and $T(1) = 1$, then $Q(T(u))$ is a quantile function of a random variable with the same support as $X$.

7. If $X$ has quantile function $Q(u)$, then $\frac{1}{X}$ has quantile function $\frac{1}{Q(1-u)}$

For further details on the properties of quantile function, we refer to Gilchrist (2000).

### 2.1.2 Descriptive Measures

In this section, we list out commonly used descriptive measures such as measures of location, scale, skewness and kurtosis based on quantile function. These quantile-based measures that reduce the shortcomings of the moment based ones can be thought of.

An important measure of location is the median, given by

$$M = Q\left(0.5\right).$$

To measure the dispersion, we use the inter-quantile range

$$IQR = Q_3 - Q_1,$$

where $Q_3 = Q\left(0.75\right)$ and $Q_1 = Q\left(0.25\right)$. Galton's measure of skewness is defined as

$$S = \frac{Q_1 + Q_3 - 2M}{Q_3 - Q_1}.$$

It can be seen that the Galton coefficient of skewness lies between $-1$ and 1, and the extreme positive skewness occurs when $Q_1 \to M$ and the extreme negative skewness is attained when $Q_3 \to M$. When a distribution is symmetric, $M = \frac{Q_1+Q_3}{2}$ and hence $S = 0$. Moors (1988) proposed a measure of kurtosis given by

$$T = \frac{Q\left(7/8\right) - Q\left(5/8\right) + Q\left(3/8\right) - Q\left(1/8\right)}{IQR}.$$

### 2.1.3 L-Moments

$L-$ moments are the competing alternatives to traditional moments. $L-$ moments are the expectations of linear combinations of order statistics. A unified theory on $L-$ moments was studied by Hosking (1990). The main advantages of $L-$ moments over the conventional moments are, the existence of first $L-$

moment ensures the existence of higher-order moments and they have generally lower sampling variances and are robust against outliers. Hosking (1989) Hosking (1992), Hosking (2006) and Hoskins & Wallis (1997) have made detailed studies on the properties of $L-$ moments, its application in summarizing and identifying probability distributions, estimation techniques based on $L-$ moments, characterizations of distributions by $L-$ moments and the comparison between the conventional moments and $L-$ moments in analysing measures of distributional shapes. The $r^{th}$ $L-$ moment is defined as

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} E\left(X_{r-k:r}\right), \quad r = 1, 2, ... \tag{2.3}$$

We have,

$$\begin{aligned} E(X_{r:n}) &= \int x f_r(x) dx \\ &= \frac{n!}{r!\,(n-r)!} \int_0^1 u^{r-1}(1-u)^{n-r} Q_r\left(u\right)\, du. \end{aligned} \tag{2.4}$$

Using (2.4), (2.3) becomes

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} \frac{r!}{k!\,(r-k-1)!} \int_0^1 u^{r-k-1}(1-u)^k Q\left(u\right)\, du. \tag{2.5}$$

Jones (2004) has given an alternative method of establishing the above relationship using the quantile function. In particular, the first four $L-$ moments are

$$L_1 = \int_0^1 Q(u)\, du = \mu, \tag{2.6}$$

$$L_2 = \int_0^1 (2u-1)\, Q(u)\, du, \tag{2.7}$$

$$L_3 = \int_0^1 \left(6u^2 - 6u + 1\right) Q(u)\, du, \tag{2.8}$$

$$\text{and} \quad L_4 = \int_0^1 \left(20u^3 - 30u^2 + 12u - 1\right) Q(u)\, du. \tag{2.9}$$

The equivalent formulae in terms of quantile density function are

$$L_1 = \int_0^1 (1-u)\, q(u)\, du, \tag{2.10}$$

$$L_2 = \int_0^1 \left(u - u^2\right) q(u)\, du, \tag{2.11}$$

$$L_3 = \int_0^1 \left(3u^2 - 2u^3 - u\right) q(u)\, du, \tag{2.12}$$

$$\text{and} \quad L_4 = \int_0^1 \left(u - 6u^2 + 10u^3 - 5u^4\right) q(u)\, du. \tag{2.13}$$

$L_1$ and $L_2$ represent the measures of location and spread respectively. Like the conventional moments, $L-$ moments can also be used to summarize the characteristics of probability distributions, to identify distributions and to fit models to data. It has been proved that the $L$-moments have several advantages over conventional moments. $L$-moments are capable of characterizing a wider range of distributions compared to the conventional moments. A distribution may be specified by its $L$-moments, even if some of its conventional moments do not exist (Hosking (1990)). Hosking (1992) illustrated that $L$-moments are preferable than conventional moments to provide summary measures of distributional shape. Yitzhaki (2003) compared the merits of mean difference and variance in the context of measuring variability. Nair & Vineshkumar (2010) has pointed out that the study of the measures of residual life based on $L-$ moments is worthy as the $L-$ moments are more advantageous than usual moments.

## 2.2 Reliability concepts

Reliability theory depicts the probability of a system completing its expected function during an interval of time. The term reliability is generally used to express a certain degree of assurance that a device or a system will operate successfully in a specified period of time. During the past decade, the development of reliability as a separate discipline has been rapid, mainly because of its applications in several branches such as, statistics, engineering, medicine, economics, demography, insurance, public policy, etc. Life distributions specified by their distribution functions and various concepts and characteristics derived from it are important topics of interest in reliability analysis.

### 2.2.1 Reliability function

Let $X$ be a non-negative continuous random variable defined on a probability space $(\Omega, A, P)$ with distribution function $F(x) = P(X \leq x)$. The random variable $X$ could represent the length of life of a device, measured in units of time. The function

$$\bar{F}(x) = P(X > x),\ x \geq 0,$$

is called a reliability function or survival function.

### 2.2.2 Hazard rate function

The failure rate (hazard rate) function of $X$ is defined as

$$h(x) = \lim_{\Delta t \to 0} \frac{P[x \leq X < x + \Delta x | X > x]}{\Delta x}. \tag{2.14}$$

For small $\Delta$, $\Delta h(x)$ is approximately the conditional probability that a unit will fail in the next small interval of time $\Delta$, given that the unit has survived age $x$. When $F(x)$ is absolutely continuous with probability density function $f(x)$, (2.14) reduces to

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \frac{d}{dx}\left[-\bar{F}(x)\right] \tag{2.15}$$

for all $t$ for which $\bar{F}(x) > 0$.

The hazard rate function is also referred to as the failure rate function, instantaneous death rate, the force of mortality, and intensity function in the fields respectively survival analysis, actuarial science, biosciences, demography and extreme value theory.

Integrating (2.15) over $(0, x)$ and using $F(0) = 0$, we can see that

$$\bar{F}(x) = \exp\left[-\int_0^x h(t)dt\right]. \tag{2.16}$$

From (2.16), it can be seen that $h(x)$ uniquely determines the distribution. The concept of hazard rate is widely used for characterizing lifetime distributions.

For example, constancy of hazard rate is a characteristic property of exponential distribution (Galambos & Kotz (1978)). A large volume of literature is available on characterizations and other properties of hazard rate function (see Barlow et al. (1963), Nanda & Shaked (2001), Noughabi et al. (2013), Hua et al. (2018), etc).

### 2.2.3 Mean residual life function

Given a component is of age $X$, the remaining life (residual life) after time $t$ is random. Define the residual random variable at age $t$ by $X_t = X - t|X > t$. The reliability function of $X_t$ is given by

$$\bar{F}_t(x) = \bar{F}(x|t) = \frac{\bar{F}(x+t)}{\bar{F}(t)} \quad x, t \geq 0, \tag{2.17}$$

which represents the conditional probability that a unit of age $t$ will survive for an additional $x$ unit of time. The mean residual life (MRL) function $m(t)$ for a random variable $X$ defined on the real line, with $E[X] < \infty$, is the expected value of the residual life function. i.e;

$$m(t) = E[X - t|X \geq t] \quad \forall\, t \geq 0. \tag{2.18}$$

The MRL function $m(t)$, represents the average lifetime remaining for a component, which has already survived up to time $t$. When $F(t)$ is absolutely continuous with respect to a Lebesgue measure, (2.18) becomes

$$\begin{aligned} m(t) &= \int_0^\infty \bar{F}(x|t)dx \\ &= \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx. \end{aligned} \tag{2.19}$$

If density function exists, then (2.19) can also be written as

$$m(t) = \frac{1}{\bar{F}(t)} \int_t^\infty (t-x)f(x)dx. \tag{2.20}$$

If $m(t)$ is differentiable then the following relationship exists between the hazard rate and the mean residual life function as

$$h(t) = \frac{1+m'(t)}{m(t)}. \tag{2.21}$$

The function $m(t)$ determines the distribution of $X$ uniquely by the relation

$$\bar{F}(t) = \frac{m(0)}{m(t)} \exp\left[-\int_0^t \frac{dt}{m(t)}\right]. \tag{2.22}$$

For more properties on $m(t)$, refer to Hall et al. (1981) , Mukherjee & Roy (1986), Gupta (2016) and references therein.

### 2.2.4 Reversed hazard rate

Let $X$ be a non-negative, absolutely continuous random variable and $[X|X \leq t]$ be a random variable which represents the variable pertaining to lifetimes of components which has failed before attaining an age $t$ were originated. The revered hazard rate of a random variable $X$ is defined by (see Barlow et al. (1963))

$$\lambda(x) = \lim_{\Delta \to 0} \frac{P[x - \Delta < X \leq x | X \leq x]}{\Delta}. \tag{2.23}$$

When probability density function (p.d.f) $f(x)$ of $X$ exists, (2.23) can be written as

$$\lambda(x) = \frac{f(x)}{F(x)}, \tag{2.24}$$

for all $X$ for which $F(x) > 0$. Thus $\lambda(x)dx$ can be used as an approximate probability of a failure in $(x - dx, x]$ given that the failure had occurred in $[0, x]$. $\lambda(x)$ uniquely determines the distribution through the relationship

$$F(x) = \exp\left[ -\int_{x}^{\infty} \lambda(t) dt \right]. \tag{2.25}$$

For more details on reversed hazard rate, one can refer to Gupta & Nanda (2001), Nanda & Shaked (2001), Nair & Asha (2004), Sunoj & Maya (2006), Sankaran et al. (2007) and Kundu & Ghosh (2017).

### 2.2.5 Reversed mean residual life

The random variable ${}_xX = [x - X | X \leq x]$ denotes the inactivity time or reversed residual life of $X$. It represents the time elapsed since the failure of a unit given that its lifetime is at most $x$ with the distribution function

$$ {}_xF(t) = \frac{F(x) - F(x-t)}{F(x)}. \tag{2.26} $$

The reversed mean residual life of $X$ is defined as

$$ \begin{aligned} r(x) &= E(x - X | X \leq x) \\ &= \frac{1}{F(x)} \int_0^x F(t) dt. \end{aligned} \tag{2.27} $$

For more details on reversed mean residual life functions, we refer to Kayid & Ahmad (2004), Ahmad & Kayid (2005), Gandotra et al. (2011) and Kundu & Ghosh (2017) and references therein.

## 2.3 Quantile-based reliability concepts

In reliability theory, several functions can be used to find the failure patterns in different mechanisms or systems as a function of age. This is accomplished by identifying the probability distribution of the lifetime random variable. In this section, we discuss the basic concepts of reliability using quantile functions. Nair & Sankaran (2009) has formulated the important reliability concepts using the quantile function approach which are discussed below.

### 2.3.1 Hazard quantile function

Setting $x = Q(u)$ in (2.15) and using (2.2), the hazard quantile function is defined as

$$\begin{aligned} H(u) &= h(Q(u)) \\ &= [(1-u)\,q(u)]^{-1}. \end{aligned} \tag{2.28}$$

In this definition, $H(u)$ is interpreted as the conditional probability of the failure of a unit in the next small interval of time given the survival of the unit at $100\,(1-u)\,\%$ point of the distribution. $H(u)$ uniquely determines the distribution through the relationship

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{2.29}$$

Sankaran & Nair (2009) studied the estimation of the hazard quantile function based on the right censored data. Midhu et al. (2014) introduced a class of distributions that has linear hazard quantile function and obtain various distributional properties and reliability characteristics of the class. For various properties of $H(u)$, one could refer to Nair et al. (2013).

### 2.3.2 Mean residual quantile function

The quantile version of mean residual life function obtained by letting $F(x) = u$ in (2.20). We have the mean residual quantile function as

$$\begin{aligned} M(u) &= m(Q(u)) \\ &= (1-u)^{-1} \int_u^1 [Q(p) - Q(u)]\, dp. \end{aligned} \tag{2.30}$$

In terms of the quantile density function, (2.30) can be written as

$$M(u) = (1-u)^{-1} \int_u^1 (1-p)\; q(p)\, dp. \tag{2.31}$$

Using (2.31) we have

$$M(u) = (1-u)^{-1} \int_u^1 (H(p))^{-1}\, dp. \tag{2.32}$$

Differentiating (2.32) we get

$$(H(u))^{-1} = M(u) - (1-u)\, M'(u)\,. \tag{2.33}$$

The distribution is uniquely determined by $M(u)$ through the identities

$$Q(u) = \mu - M(u) + \int_0^u (1-p)^{-1} M(p)\, dp. \tag{2.34}$$

Midhu et al. (2013) proposed a class of distributions that has the linear mean residual quantile function and studied various distributional properties and reliability characteristics of the class. Sankaran & Dileep Kumar (2018) introduced a new class of distributions with quadratic mean residual quantile function and studied distributional properties and various reliability characteristics of the proposed model.

### 2.3.3 Reversed hazard quantile function

Analogous to reversed hazard rate, reversed hazard quantile function is defined as

$$\Lambda(u) = \lambda(Q(u)) = (uq(u))^{-1}. \tag{2.35}$$

The quantile function is uniquely determined by $\Lambda(u)$ through the relationship

$$Q(u) = \int_0^u (pq(p))^{-1} dp. \tag{2.36}$$

For more properties, we may refer to Nair & Sankaran (2009).

### 2.3.4 Reversed mean residual quantile function

The reversed mean residual quantile function is defined by

$$\begin{aligned} R(u) &= r(Q(u)) \\ &= \frac{1}{u}\int_0^u (Q(u) - Q(p))dp \\ &= \int_0^u pq(p)dp. \end{aligned} \tag{2.37}$$

$R(u)$ determines the distribution uniquely through the relationship

$$Q(u) = R(u) + \int_0^u p^{-1} R(p) dp. \tag{2.38}$$

Nair & Sankaran (2009) derived that

$$(\Lambda(u))^{-1} = R(u) + uR'(u). \tag{2.39}$$

## 2.4 Ageing concepts

In the reliability context, life distributions are generally classified into different classes based on the monotonic behaviour of the failure rate and mean residual life function. There can be no ageing, positive ageing or negative ageing. Positive ageing means the residual lifetime of a unit decreases with the increase in the age of the unit. Negative ageing is the dual concept of positive ageing which has a beneficial effect on the life of the unit as the age increases and no ageing means that the age of a component has no effect on the distribution of residual lifetime of the unit (see Abouammoh et al. (2000), Deshpande et al. (1986)).

Most of the ageing concepts existing in the literature are described based on the measures defined in terms of the distribution function. For example, Barlow & Proschan (1975) established some closure properties of order statistics based on the reliability ageing classes such as increasing failure rate (IFR), increasing failure rate average (IFRA) or new better than used (NBU) classes. Takahasi (1988) and Nagaraja (1990) further studied some of these classes based on order statistics. Nair & Sankaran (2009) have identified some quantile functions as suitable models for lifetime data analysis. Nair & Vineshkumar (2011) modified the existing definitions based on distribution functions to study the ageing properties of distributions which do not have a tractable distribution function. However, a quantile-based study on these ageing classes is of recent interest due to Nair & Vineshkumar (2011). In this section, we discuss some of the important results in this area and translate the basic definitions to make them amenable for a quantile-based analysis.

### 2.4.1 Ageing Concepts based on hazard quantile function

The concept of increasing and decreasing failure rates for univariate distributions have been used as a useful tool in the study of the failure patterns of components / devices. A random variable $X$ is said to have an increasing (decreasing) failure rate ( IFR (DFR)) if its hazard rate $h(x)$ is increasing (decreasing).

In the quantile framework, a lifetime random variable $X$ is IHR (DHR) if its hazard quantile function $H(u)$ is increasing (decreasing) in $u$. This implies that $H'(u)$, the derivative of $H(u)$ satisfies

$$H'(u) \geq (\leq) 0, \;\; 0 < u < 1.$$

Thus, all distributions specified in terms of $F(x)$ that are IHR (DHR) preserve the same property when specified by $Q(u)$ as well. For further reference see Nair & Vineshkumar (2011).

### 2.4.2 Ageing concepts based on mean resdiual quantile function

A random variable $X$ with mean residual life function $m(x)$ is said to be in the increasing (decreasing) mean residual life or IMRL (DMRL) class if $m(x)$ is increasing (decreasing) in $x > 0$. In terms of quantile function, the IMRL (DMRL) is defined as follows. A random variable $X$ with $E(X) < \infty$ is said to be IMRL (DMRL) if and only if

$$M(u_1) \geq (\leq) M(u_2); u_1 \geq (\leq) u_2.$$

### 2.4.3 Concepts based on survival function

The ageing properties in this class are obtained by comparing survival function at different points of time. New better (worse) than used (NBU (NWU)) is the most cited one in this category and new better (worse) than used in expectation (NBUE (NWUE)) and harmonic new better (worse) than used in expectation (HNBUE (HNWUE)) are the classes derived from NBU (NWU). We say that X is NBU (NWU) if and only if $\bar{F}(x+t) \leq (\geq) \bar{F}(x)\bar{F}(t)$, for all $x, t > 0$.

Within the framework of quantile functions, a random variable $X$ with $Q(0) = 0$ is said to be NBU (NWU) if and only if, $Q(u + v - uv) \leq (\geq) Q(u) + Q(v)$ for all $0 \leq u, v < 1$. A lifetime $X$ is new better (worse) than used in hazard rate if and only if $H(0) \leq (\geq) H(u)$ for $u \geq 0$ (see Nair et al. (2013)).

## 2.5 Stochastic orders

Stochastic orders are useful for a global comparison of two distributions in terms of certain characteristics. Stochastic orders have been used during the last few decades in many diverse areas of probability and statistics such as reliability theory, biology, economics, queuing theory, survival analysis, insurance, actuarial science, operations research, and management science. Stochastic orders are used to compare distributions in terms of their characteristics. Definitions of the stochastic orders given below, unless otherwise specified, can be seen in Shaked & Shanthikumar (2007). The purpose of stochastic orders is the comparison of distributions using a variety of information about the models.

### 2.5.1 Usual stochastic order

Let $X$ and $Y$ be two random variables with survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ and the corresponding quantile functions $Q_X(u)$ and $Q_Y(u)$, respectively. Then $X$ is said to be smaller than $Y$ in the usual stochastic order $X \leq_{st} Y$ if and only if

$$\bar{F}_X(x) \leq \bar{F}_Y(x), \forall x$$

or equivalently

$$Q_X(u) \leq Q_Y(u), \forall u \in (0, 1).$$

### 2.5.2 Hazard rate order

In hazard rate order, we compare two distributions by means of the relative magnitude of their hazard rates. The idea behind this comparison is that when the hazard rate becomes larger, the variable becomes stochastically smaller. $X$ is said to be smaller than $Y$ in hazard rate order $X \leq_{hr} Y$ if $h_X(x) \geq h_Y(x)$ for all $x$ or equivalently, $H_X(u) \geq H^*{}_Y(u)$, where $H_X(u) = h_X(Q_X(u))$, $H^*{}_Y(u) = h_Y(Q_X(u))$ and $h(.)$ denotes the hazard rate function.

### 2.5.3 Reversed hazard function order

Let $X$ and $Y$ be two non-negative random variables. Then $X$ is said to be smaller than $Y$ in reversed hazard quantile function order ($X \leq_{rhq} Y$), if and only if $\lambda_X(x) \leq \lambda_Y(x)$ for all $x$ or equivalently, $\Lambda_X(u) \leq \Lambda^*_Y(u)$ for all $u \in (0,1)$, where $\Lambda^*_Y(u) = \lambda_Y(Q_X(u))$.

### 2.5.4 Convex order

For two random variables $X$ and $Y$ if the condition

$$E(\phi(X)) \leq E(\phi(Y))$$

for all convex functions $\phi : R \to R$, provided the expectations exist then $X$ is said to be smaller than $Y$ in the convex order. It is denoted by $X \leq_c Y$. Within the framework of quantile function, $X \leq_c Y$ if and only if

$$\int_0^u Q_X(p)dp \geq \int_0^u Q_Y(p)dp.$$

### 2.5.5 Transform orders

(i) We say that $X$ is smaller than $Y$ in convex transform order, denoted by $X \leq_{cx} Y$, if $G^{-1}F(x)$ is convex in $x$ on the support of $F$ or $\frac{q_Y(u)}{q_X(u)}$ is increasing in $u$.

(ii) We say that $X$ is smaller than $Y$ in star order, written as $X \leq_* Y$, if and only if $\frac{G^{-1}F(x)}{x}$ is increasing for $x \geq 0$ or $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u$.

(iii) We say that $X$ is smaller than $Y$ in super-additive order, denoted by $X \leq_{su} Y$ if $Q_Y(F_X(x))$ is super-additive.

### 2.5.6 Dispersive ordering

Dispersive ordering can be used to compare the spread among probability distributions. We say that $X$ is smaller than $Y$ in dispersive order, denoted by $X \leq_{disp} Y$, if $Q_Y(F_X(x)) - Q_X(u)$ is increasing in $u \in (0,1)$.

## 2.6 Quantile function models

The main objective of quantile-based reliability analysis is to make use of quantile functions as models in lifetime data analysis. The lambda distributions were originally developed as a formula for transforming uniform random numbers to simulate new distributions with a rich variety of shapes. The lambda distributions are used, when the physical characteristics that govern the failure pattern in a specific problem are unknown to choose a particular distribution function. There are members of lambda families that can either exactly or approximately represent most of the continuous distributions by a good choice of the parameters. The lambda distribution originated with Tukey (1962).

### 2.6.1 Lambda family

A family of distributions by a quantile function introduced by Hastings Jr et al. (1947). During the past 60 years, considerable efforts were made to generalize this family of distributions and the refined model by Tukey (1962) and to study their new applications and inference procedures. The basic model from which all other generalizations originate is the Tukey lambda distribution with quantile function. Tukey's Lambda family of distributions is given by

$$Q(u) = \begin{cases} \frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda}, \lambda \neq 0 \\ \frac{\log(u)}{1-u}, \lambda = 0 \end{cases} \tag{2.40}$$

For $\lambda = 1$ and $\lambda = 2$, it is easy to verify that (2.40) becomes uniform over $(-1, 1)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$, respectively. The density functions are $U$ shaped for $1 < \lambda < 2$ and unimodal for $\lambda < 1$ or $\lambda > 2$. With (2.40) being symmetric and having the range for negative values of $X$, it has limited use as a lifetime model.

### 2.6.2 Generalized lambda distribution

The generalized lambda distribution is an extension of Tukey's lambda distribution [Hastings Jr et al. (1947) Tukey (1962)].

$$Q\left(u\right)=\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-\left(1-u\right)^{\lambda_4}\right),\quad 0\leq u\leq 1, \tag{2.41}$$

where $\lambda_1$ is a location parameter, $\lambda_2$ is a scale parameter, while $\lambda_3$ and $\lambda_4$ determine the shape. Ramberg & Schmeiser (1974) generalized the Tukey lambda distribution to a four parameter distribution specified by quantile function in (2.41). A major limitation of the generalized lambda family discussed above is that the distribution is valid only for certain regions in the parameter space. Freimer et al. (1988) introduced a modified generalized lambda distribution defined by

$$Q(u)=\lambda_1+\frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}}{\lambda_3}-\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right), \tag{2.42}$$

which is well defined for the values of the shape parameters $\lambda_3$ and $\lambda_4$ over the entire two-dimensional space.

### 2.6.3 van Staden & Loots model

A four-parameter distribution that belongs to lambda family proposed by van Staden & Loots (2009). They generated the model by considering the generalized

Pareto model with quantile function

$$Q_1(u) = \begin{cases} \frac{-1}{\lambda_4}\left((1-u)^{\lambda_4} - 1\right), & \lambda_4 \neq 0 \\ -\ln(1-u) & , \lambda_4 = 0 \end{cases} \tag{2.43}$$

and its reflection

$$Q_2(u) = \begin{cases} \frac{1}{\lambda_4}\left(u^{\lambda_4} - 1\right), & \lambda_4 \neq 0 \\ \log u & , \lambda_4 = 0 \end{cases} \tag{2.44}$$

A weighted sum of these two quantile functions with respective weights $\lambda_3$ and $1-\lambda_3$, $0 \leqslant \lambda_3 \leqslant 1$, along with the introduction of a location parameter $\lambda_1$ and a scale parameter $\lambda_2$, provide the new form. Thus, the quantile function of this model is

$$Q(u) = \lambda_1 + \lambda_2\left[(1-\lambda_3)\frac{u^{\lambda_4}-1}{\lambda_4} - \lambda_3\frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right], \ \lambda_2 > 0. \tag{2.45}$$

### 2.6.4 Five-parameter lambda family

Gilchrist (2000) proposed a five-parameter family of distributions with quantile function

$$Q(u) = \lambda_1 + \frac{\lambda_2}{2}\left[(1-\lambda_3)\left(\frac{(u^{\lambda 4}-1)}{\lambda_4}\right) - (1+\lambda_3)\left(\frac{(1-u)^{\lambda_5}-1}{\lambda_5}\right)\right], \tag{2.46}$$

as an extension to the Freimer et al. (1988) model in (2.42).

### 2.6.5 Power-Pareto distribution

The quantile function of the power distribution is of the form

$$Q_1(u) = \alpha u^{\lambda_1}, 0 \leq u \leq 1; \alpha, \lambda_1 > 0, \tag{2.47}$$

while that of the Pareto distribution is

$$Q_2(u) = \sigma(1-u)^{-\lambda_2}. \tag{2.48}$$

A new quantile function can then be formed by taking the product of these two as

$$Q(u) = \frac{Cu^{\lambda_1}}{(1-u)^{\lambda_2}}, 0 \leq u \leq 1, C, \lambda_1, \lambda_2 > 0. \tag{2.49}$$

The distribution of a random variable X with (2.49) as its quantile function is called the power-Pareto distribution. Gilchrist (2000) and Hankin & Lee (2006) have studied the properties of this distribution.

### 2.6.6 Govindarajulu distribution

Govindarajulu (1977) model is the earliest attempt to introduce a quantile function, not having an explicit form of the distribution function, for modelling data on failure times. Govindarajulu (1977) introduced the quantile function

$$Q(u) = \theta + \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right), \quad \theta,\ \sigma,\ \beta > 0, \quad 0 \leq u \leq 1, \tag{2.50}$$

and demonstrated its potential as a lifetime model by fitting it to the data on the failure times of a set of 25 refrigerators which were run to destruction under advanced stress conditions. However, other than proposing the model, Govindarajulu (1977) did not investigate the various characteristics of the distribution as a general model as well as its role in reliability analysis.

## 2.7 Information measures

### 2.7.1 Entropy

Suppose $X$ is a non-negative continuous random variable denoting the lifetime of an item with a probability density function (pdf) $f(x)$, cumulative distribution function (cdf) $F(x)$ and survival function (sf) $\bar{F}(x) = 1 - F(x)$. Then a classical measure of uncertainty for $X$ is the differential entropy, known as Shannon entropy ((see Shannon (1948), Cover & Thomas (2012))) defined as

$$\eta(F) = -E(\log f(X)) = -\int_0^\infty (\log f(x))\, f(x)dx. \tag{2.51}$$

As a statistical average, $\eta(F)$ measures the expected uncertainty subsumed in $f(x)$ about the predictability of an outcome of $X$. When the age of the component of $X$ is non-zero, say $t$, then $\eta(F)$ in (2.51) is not appropriate and Ebrahimi (1996) modified $\eta(F)$ to the residual random variable $X_t = X - t|X > t$, called the residual Shannon entropy of $X$ at time $t$, given by

$$\eta(F;t) = -E(\log f(X_t) = -\int_t^\infty \left(\log \frac{f(x)}{\bar{F}(t)}\right) \frac{f(x)}{\bar{F}(t)} dx. \tag{2.52}$$

Clearly, $\eta(F, t)$ measures the expected uncertainty contained in the conditional density of $X - t$ given $X > t$ about the predictability of the remaining lifetime of the component. For more properties of (2.52), we refer to Ebrahimi (1996), Ebrahimi & Pellerey (1995), Asadi & Ebrahimi (2000), Belzunce et al. (2004) and Sunoj et al. (2009).

The wide applicability of $\eta(F)$ and its modified/generalized versions and the usefulness of order statistics in many applied problems motivated many researchers to study the information theoretic aspects of order statistics. Wong & Chen (1990) showed that the difference between the average entropy of order statistics is symmetric about the median. Park (1995) obtained some recurrence relationships to the entropy of order statistics. Ebrahimi et al. (2004) obtained bounds for the entropy of order statistics and studied entropy ordering of order statistics. The residual Renyi entropy of order statistics and record values are available in Zarezadeh & Asadi (2010). Characterizations of distributions based on Renyi entropy and record values are studied by Baratpour et al. (2007), Baratpour et al. (2008) and Abbasnejad & Arghami (2010). Thapliyal & Taneja (2012) obtained certain bounds on a generalized two parameters entropy of order statistics.

### 2.7.2 Tsallis entropy

Many generalizations of Shannon entropy (Shannon (1948)) are available in literature. These generalized information measures have many important properties such as smoothness, large dynamic range with respect to certain conditions that make them applicable in practice. Among different generalized entropy measures, an important one is the Tsallis entropy of order $\alpha$ (Tsallis (1988)). Let $X$ be an absolutely continuous non-negative random variable, with probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$ respectively. Then Tsallis entropy is defined as

$$S_\alpha(X) = \tfrac{1}{\alpha-1}E\left(1-(f(X))^{\alpha-1}\right) = \frac{1}{\alpha-1}\left(1-\int_0^\infty (f(x))^\alpha dx\right),\ \alpha>0,\ \alpha\neq 1.$$

It plays an important role in different areas viz. physics, chemistry, biology, economics and statistics. Many researchers used it in many physical applications such as, Nakamichi et al. (2002) developed the statistical mechanics of large scale astrophysical system , Taruya & Sakagami (2003) investigated thermodynamic properties of stellar self-gravitating system, Arimitsu & Arimitsu (2002) described fully developed turbulence, Weili et al. (2009) developed a new method for medical image segmentation based on improved PCNN (Pulse Coupled Neural Network) and Tsallis entropy and Tong et al. (2002) studied signal processing. When $\alpha \to 1$, $S_\alpha(X) = H(X) = -E(\ln f(X)) = -\int_0^\infty (\ln f(x))f(x)dx$, the expected uncertainty contained in the pdf about the predictability of the random variable $X$. Unlike Shannon or other generalized entropy measures, Tsallis entropy is non-additive. For recent works on $S_\alpha(X)$, we refer to Cartwright (2014), Preda et al. (2015), Sati & Gupta (2015) and the references therein.

### 2.7.3 Extropy

The Shannon's measure of entropy in the discrete case $H_X = -\sum_{i=1}^{n} p_i \log p_i$, where $p_i = P[X = x_i]$ has complementary dual

$$J_X = -\sum_{i=1}^{n} (1 - p_i) \log(1 - p_i),$$

which is called the extropy of the random variable $X$. It originated in environmental investigations by Ayres & Martinas (1995) in the name of $M$ potential and the term entropy was coined in Martinas (1997). The two measures entropy and extropy address the measurement of uncertainty in contrasting styles. A detailed discussion on the motivation, importance and properties of the extropy in the context of thermodynamics and statistical mechanics is given in Martinas & Frankowicz (2000). They argue that both entropy and extropy share similar mathematical properties and that the latter has some conceptual superiority in certain situations.

## 2.8 Order statistics

Suppose that $X_1, X_2, ..., X_n$ are independently and identically distributed (i.i.d) observations from a population with cumulative distribution function (cdf) $F(.)$ and probability density function (pdf) $f(.)$. Then the order statistics of the sample is defined by the arrangement of $X_1, X_2, ..., X_n$, from the smallest to largest denoted as $X_{1:n} \leq X_{2:n} \leq, ..., \leq X_{n:n}$. Let $X_{i:n}$ be the $i^{th}$ order statistic and $F_{i:n}(.)$ and $f_{i:n}(.)$ be the corresponding cdf and pdf. If a system has $n$ independent components with lifetimes $X_1, X_2, ..., X_n$, then the system functions if and only if at least $k$ of the components works ( $k$-out-of-$n$ system). The lifetime of such a system is described by the $(n-k+1)^{th}$ order statistic. The parallel $(k=1)$ and series $(k=n)$ systems are particular cases of $k$-out-of-$n$ systems. These systems play an important role in reliability theory and life testing. For example, an aircraft with four engines will not crash if at least three of them are working and for the communications system with three transmitters will send signals if at least two of them are functioning. The marginal pdf of $X_{i:n}$, given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)}(F(x))^{i-1}(1-F(x))^{n-i}f(x), \tag{2.53}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \;\; a, b > 0$. A wide variety of research is now available in literature on different aspects of order statistics, the applications include Balakrishnan & Rao (1998b), Balakrishnan & Rao (1998a). Recently, quantile-based studies are of special interest among many researchers as it has some unique properties that are not shared by distribution functions.

In terms of quantile function, the pdf of $X_{i:n}$ based on quantile functions (Sunoj et al. (2017)) becomes,

$$f_{i:n}(Q_X(u)) = \frac{n!}{(i-1)!(n-i)!}u^{i-1}(1-u)^{n-i}(q_X(u))^{-1} = \frac{g_i(u)}{q_X(u)}, \tag{2.54}$$

where $g_i(u) = \dfrac{1}{B(i, n-i+1)}u^{i-1}(1-u)^{n-i}$ is the pdf of beta distribution with parameters $(i, n-i+1)$ and $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$; $a, b > 0$, the beta function. The corresponding survival function of $X_{i:n}$ reduces to

$$\bar{F}_{i:n}(Q_X(u)) = \frac{\bar{B}_u(i, n-i+1)}{\bar{B}(i, n-i+1)},$$

where $\bar{B}_u(i, n-i+1) = \int_u^1 u^{i-1}(1-u)^{n-i}du$ is the incomplete beta function.

The study of various reliability measures and its properties based quantile function is an important topic considered many researchers in the past (see Nair & Sankaran (2009), Nair et al. (2013)). Ageing concepts play an important role in distinguishing life distributions as an aid in model building and also in understanding the pattern in which a unit is ageing. Nair & Vineshkumar (2011) studied different classes of life distributions based on ageing concepts using quantile functions. Motivated with these, we study some basic distribution properties such as moments and ageing pattern of order statistics using quantile functions.

### 2.8.1 Moments of order statistics

Let $X_1, X_2, ..., X_n$ be a random sample from a population with pdf $f(.)$ and cdf $F(.)$, and let $X_{1:n} \leqslant X_{2:n}, ... \leqslant X_{n:n}$, be the order statistics obtained from the above sample. Let us denote the raw moment $E(X_{i:n}^k)$ by $\mu_{i:n}^{(k)}$. Its quantile notation

$$\begin{aligned}\mu_{i:n}^{(k)} &= E(X_{i:n}^{(k)}) \\ &= \int_{-\infty}^{\infty} x^k f_{i:n}(x)dx \\ &= \frac{n!}{(i-1)!(n-i)!}\int_0^1 (Q_X(u))^k u^{i-1}(1-u)^{n-i}du. \end{aligned} \tag{2.55}$$

From (2.55), the first two raw moments of $i^{th}$ order statistic based on quantile functions are respectively given by

$$\mu_{i:n}^{(1)} = \frac{n!}{(i-1)!(n-i)!}\int_0^1 Q(u)u^{i-1}(1-u)^{n-i}du, \tag{2.56}$$

and

$$\mu_{i:n}^{(2)} = \frac{n!}{(i-1)!(n-i)!}\int_0^1 (Q(u))^2 u^{i-1}(1-u)^{n-i}du.$$

Then the variance of $i^{th}$ order statistics become

$$\sigma_{i:n}^2 = \mu_{i:n}^{(2)} - (\mu_{i:n}^{(1)})^2,$$

and the covariance between $X_{i:n}$ and $X_{j:n}$,

$$Cov(X_{i:n}, X_{j:n}) = \mu_{i,j:n} - \mu_{i:n}\mu_{j:n}, 1 \leqslant i \leqslant n,$$

where,

$$\begin{aligned}\mu_{i,j:n} &= E(X_{i:n}X_{j:n}) \\ &= \frac{n!}{(i-1)!(n-j)!(j-i-1)!}\int_0^1\int_{Q(u)}^1 Q(u)Q(v)u^{i-1}(1-u)^{j-i-1}(1-v)^{n-j}dvdu,\end{aligned}$$

Using the distribution function approach, the recurrence relationship connecting moments of consecutive order statistics for continuous random variables are defined by Cole (1951), given by

$$i\mu^{(k)}_{i+1:n} + (n-i)\mu^{(k)}_{i:n} = n\mu^{(k)}_{i:n-1}, for\ i = 1,2,...,n-1 \text{ and } k \geq 1. \tag{2.57}$$

The identity (2.57) provides an easy way of finding the moments of order statistics using its preceding moments. An alternative proof based on mixtures of discrete and continuous random variables are available in Arnold (1977); (see Balakrishnan & Malik (1986)). In the following example, we illustrate the usefulness of the quantile-based approach in deriving the recurrence relationship (2.57), where no closed form distribution functions exist.

**Example 2.1.** Let $X$ be a random variable with Davies distribution (see Hankin & Lee (2006)) distribution in which no closed form expression for the distribution function exists. However, its quantile function is given by $Q(u) = c\frac{u^{\lambda_1}}{(1-u)^{\lambda_2}}, c, \lambda_1, \lambda_2 \geqslant 0$. Then $\mu^{(1)}_{1:n} = nc\frac{\Gamma(a+1)\Gamma(n-b)}{\Gamma(a-b+n+1)}$, using the relation (2.57) we obtain the first order moment of second order statistic as

$$\begin{aligned}\mu_{2:n}^{(1)} &= n\mu_{1:n-1}^{(1)} - (n-1)\mu_{1:n}^{(1)} \\ &= \left(\frac{Cn(n-1)\Gamma(1+\lambda_1)\Gamma(n-\lambda_2-1)}{\Gamma(n+\lambda_1-\lambda_2)}\right)\left(\frac{\lambda_1+1}{n+\lambda_1-\lambda_2}\right) \\ &= \left(\frac{Cn(n-1)\Gamma(2+\lambda_1)\Gamma(n-\lambda_2-1)}{\Gamma(n+\lambda_1-\lambda_2+1)}\right).\end{aligned}$$

### 2.8.2 Bounds based on moments of order statistics

Hartley & David (1954), Gumbel (1954), Moriguti (1951), Moriguti (1953) and Moriguti (1954) derived universal bounds for moments of order statistics. David (1986) derived some bounds for order statistics arising from $X_i^{\prime s}$. Balakrishnan (1990) provides simple improvements over the Hartley-David- Gumbel bounds. Consider an arbitrary population with mean 0 and variance 1. That is,

$$\int_0^1 Q(u)du = 0 \text{ and } \int_0^1 (Q(u))^2 du = 1. \tag{2.58}$$

Then the expected value of the $i^{th}$ order statistic $X_{i:n}$ as

$$\mu_{i:n} = \int_0^1 Q(u)\left(\frac{1}{B(i,n-i+1)}u^{i-1}(1-u)^{n-i}\right)du. \tag{2.59}$$

From calculus of variations (see Davis (1962)) we can find the extremal giving stationary values of (2.59) subject to (2.58) by first obtaining the unconditional extremal for

$$\mu_{i:n} = \int_0^1 Q(u)\left(\frac{1}{B(i,n-i+1)}u^{i-1}(1-u)^{n-i} - \lambda\right)du,$$

and then determining the constant $\lambda$ so as to satisfy (2.58). Using Schwarz's inequality, we obtain

$$\begin{aligned} \mu_{i:n} &\leqslant \left( \int_0^1 \frac{u^{2(i-1)}(1-u)^{2(n-i)}}{B^2(i,n-i+1)} - 2\lambda \frac{1}{B(i,n-i+1)} u^{i-1}(1-u)^{n-i} + \lambda^2 \right)^{\frac{1}{2}}, \\ &\leqslant \left( \frac{B(2i-1,2n-2i+1)}{(B(i,n-i+1))^2} - 2\lambda + \lambda^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{2.60}$$

Let $y = \left( \frac{B(2i-1,2n-2i+1)}{(B(i,n-i+1))^2} - 2\lambda + \lambda^2 \right)^{\frac{1}{2}}$, $\frac{dy}{d\lambda} = 0 \Rightarrow \lambda = 1$, and

$$\frac{d^2y}{d\lambda}|_{\lambda=1} = \frac{1}{\sqrt{\frac{B(2i-1,2n-2i+1)-(B(i,n-i+1))^2}{(B(i,n-i+1))^2}}} > 0.$$

Then the right hand side of (2.60) is minimum when $\lambda = 1$. Therefore,

$$|\mu_{i:n}| \leqslant \frac{\left(B(2i-1,2n-2i+1) - (B(i,n-i+1))^2\right)^{\frac{1}{2}}}{B(i,n-i+1)}. \tag{2.61}$$

This implies that $Q(u) = c(g_i(u) - 1), 0 < u < 1$. The constant of proportionality, $c$ is determined from $\int_0^1 Q(u)du = 0$ and $\int_0^1 (Q(u))^2 du = 1$.

$$\int_0^1 c^2(g_i^2(u) - 2g_i(u) + 1)du = 1, \Rightarrow c^2 \left( \frac{B(2i-1,2n-2i+1)}{(B(i,n-i+1))^2} - 1 \right) = 1.$$

Thus

$$c = \frac{B(i,n-i+1)}{\sqrt{B(2i-1,2n-2i+1) - (B(i,n-i+1))^2}}.$$

Similarly, by considering the arbitrary population with mean 0 and variance 1, the moment of largest order statistic,

$$\mu_{n:n} = \int_0^1 Q(u)(nu^{n-1} - \lambda)du. \tag{2.62}$$

Using Schwarz's inequality, (2.62) turns to

$$\begin{aligned}\mu_{n:n} &\leqslant \left(\int_0^1 (Q(u))^2 du\right)^{\frac{1}{2}} \left(\int_0^1 (nu^{n-1} - \lambda)^2 du\right)^{\frac{1}{2}}, \\ &\leqslant \left(\int_0^1 (n^2u^{2n-2} - 2n\lambda u^{n-1} + \lambda^2) du\right)^{\frac{1}{2}}, \\ &\leqslant \left(\frac{n^2}{2n-1} - 2\lambda + \lambda^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.63}$$

Let $z = \left(\frac{n^2}{2n-1} - 2\lambda + \lambda^2\right)^{\frac{1}{2}}$, $\frac{dy}{d\lambda} = 0 \Rightarrow \lambda = 1$, and $\frac{d^2y}{d\lambda}|_{\lambda=1} = \frac{1}{\sqrt{\frac{n^2}{2n-1} - 1}} > 0.$
Then the right hand side of (2.63) is minimum when $\lambda = 1$. Thus we obtain,

$$\mu_{n:n} \leqslant \frac{n-1}{(2n-1)^{\frac{1}{2}}}. \tag{2.64}$$

The bound is attained when $Q(u) = c(nu^{n-1} - 1), 0 < u < 1$. Also, the constant of proportionality $c$ can be determined using the initial conditions $c = \frac{(2n-1)^{\frac{1}{2}}}{n-1}$.

**Example 2.2.** Suppose $X$ follows Tukey-lambda distribution with quantile function

$$Q(u) = \frac{(u^\lambda - (1-u)^\lambda)}{\lambda}, 0 \leqslant u \leqslant 1,$$

for all non-zero lambda values. Then the moment of $i^{th}$order statistic is given by

$$\begin{aligned}\mu_{i:n} &= \int_0^1 \frac{\left(u^\lambda - (1-u)^\lambda\right)\left(\frac{u^{i-1}(1-u)^{n-i}}{B(i,-i+n+1)} - \lambda\right)}{\lambda} du, \\ &= \frac{\Gamma(i+\lambda)\Gamma(-i+n+1) - \Gamma(i)\Gamma(-i+n+\lambda+1)}{\lambda B(i,-i+n+1)\Gamma(n+\lambda+1)}.\end{aligned}$$

Now, the moment of first order statistic is obtained as

$$\begin{aligned}\mu_{1:n} &= \int_0^1 \frac{\left(u^\lambda - (1-u)^\lambda\right)\left(\frac{(1-u)^{n-1}}{B(1,n)} - \lambda\right)}{\lambda} du, \\ &= \frac{\Gamma(\lambda)\Gamma(n+1)}{\Gamma(n+\lambda+1)} - \frac{n}{\lambda(\lambda+n)}.\end{aligned}$$

The moment of $n^{th}$ order statistic is obtained as

$$\begin{aligned}\mu_{n:n} &= \int_0^1 \frac{\left(u^\lambda - (1-u)^\lambda\right)\left(\frac{u^{n-1}}{B(n,1)} - \lambda\right)}{\lambda} du, \\ &= \frac{n}{\lambda^2 + \lambda n} - \frac{\Gamma(\lambda)\Gamma(n+1)}{\Gamma(n+\lambda+1)}.\end{aligned}$$

Table 2.1 gives the sharpness of the universal bounds for the moment of the largest order statistic $\mu_{n:n}$ and the largest moment of Tukey-lambda distribution. We can infer from the table that as the sample size increases the bounds are less sharp.

### 2.8.3 Ageing properties of order statistics

In the present section we propose a quantile-based analysis on the ageing concepts based on order statistics. Like the distribution function approach, we now

TABLE 2.1: Universal bounds for $\mu_{n:n}$ and exact values for the largest moment of Tukey-lambda distribution

| Sample size $n$ | $\mu_{n:n}$ for Tukey lambda distribution | Universal bound for largest moment |
|---|---|---|
| 2 | 0.3333 | 0.5774 |
| 3 | 0.5000 | 0.8944 |
| 4 | 0.6000 | 1.1334 |
| 5 | 0.6667 | 1.3333 |
| 6 | 0.7143 | 1.5076 |
| 7 | 0.7500 | 1.6641 |
| 8 | 0.7778 | 1.8074 |
| 9 | 0.8000 | 1.9403 |
| 10 | 0.8181 | 2.0647 |

prove that the quantile-based ageing classes of order statistics preserve the same ageing classes.

## Increasing Hazard Rate (IHR)

An extensive review of various properties of IHR classes in the distribution function approach is available in Lai & Xie (2006) and a quantile-based study of the same is available in Nair & Vineshkumar (2011). Barlow & Proschan (1996) showed that if $F(.)$ is IHR then $F_{i:n}(.)$ is also IHR. Now the following theorem establishes the IHR property of $X_{i:n}$ based on quantile function.

**Theorem 2.3.** *If $X$ is IHR then the $i^{th}$ order statistic $X_{i:n}$ is IHR.*

*Proof.* By virtue of the definition of IHR, $X_{i:n}$ is said to be IHR iff the hazard quantile function of the $i^{th}$ order statistic, $H_{X_{i:n}}(u) = h_{i:n}(Q(u)) = \dfrac{f_{i:n}(Q(u))}{\bar{F}_{i:n}(Q(u))}$ is increasing in $u > 0$. When $H(u) = ((1-u)q(u))^{-1}$ is increasing in $u$ , then

$$\begin{aligned} H_{X_{i:n}}(u) &= \frac{u^{i-1}(1-u)^{n-i}(q(u))^{-1}}{\bar{B}_u(i,n-i+1)} \\ &= \frac{u^{i-1}(1-u)^{n-i+1}}{\bar{B}_u(i,n-i+1)}H(u), \end{aligned} \tag{2.65}$$

is also increasing in $u$ as $\dfrac{u^{i-1}(1-u)^{n-i+1}}{\bar{B}_u(i,n-i+1)} \geqslant 0$. Thus $X_{i:n}$ is IHR. □

For measuring the quantile-based IHR properties of simple systems with i.i.d components viz. series and parallel, $H_{X_{1:n}}(u)$ and $H_{X_{n:n}}(u)$ are of important. Accordingly, in the following example, we consider $H_{X_{1:n}}(u)$ to derive different properties, results of $H_{X_{n:n}}(u)$ can be obtained in a similar way. Also, for a certain population, the quantile function $Q(u)$ does not possess a closed form while the quantile density function $q(u)$ has a closed form. In the following example, we examine $H_{X_{1:n}}(u)$ for a family of distributions that can be represented only through $q(u)$.

**Example 2.3.** When $X$ is distributed with quantile density function given by, $q(u) = Ku^{\delta}(1-u)^{-(A+\delta)}$, where $K, \delta$ and $A$ are real constants. This quantile function contains several well-known distributions, which include the exponential ($\delta = 0, A = 1$), Pareto ($\delta = 0; A < 1$), rescaled beta ($\delta = 0; A > 1$), and Govindarajulu distribution ($\delta = \beta - 1, A = -\beta$).

Figure 2.1 indicates that exponential distribution is both IHR and DHR (since $H_{X_{1:n}}(u)$ is constant), Pareto distribution is IHR($H_{X_{1:n}}(u)$ is increasing), rescaled beta is DHR ($H_{X_{1:n}}(u)$ is decreasing) and Govindarajulu distribution is IHR when $\beta \leqslant 1$ and bathtub-shaped for $\beta > 1$.

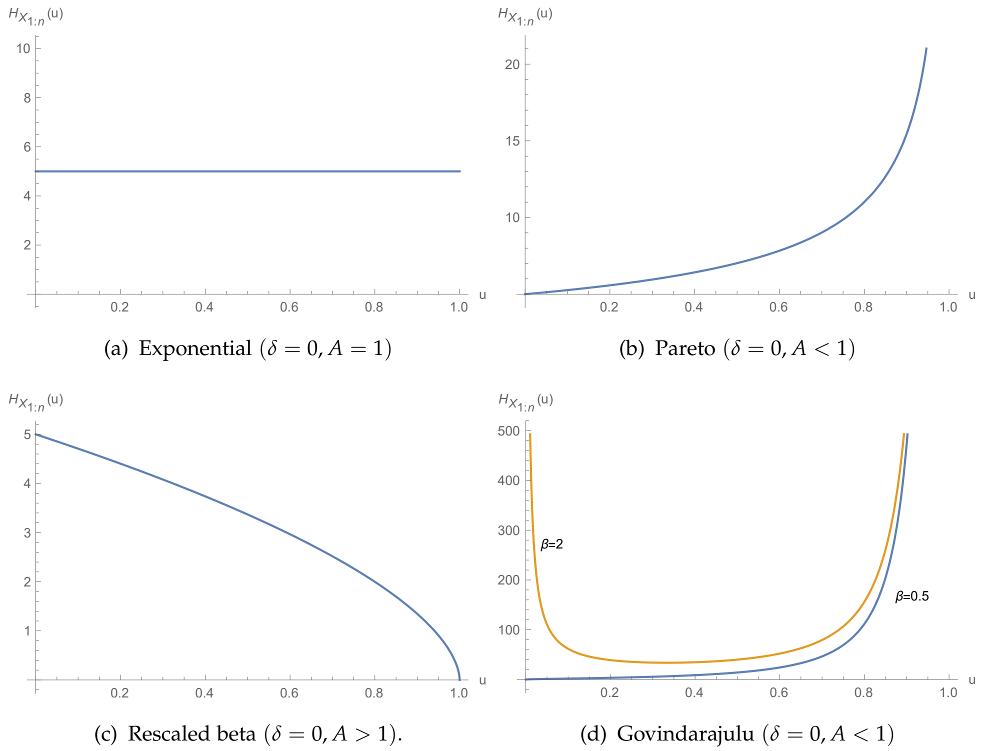



FIGURE 2.1: Plots of Hazard quantile function of first order statistic, $H_{X_{1:n}}(u)$ against $u$.

## Increasing Hazard Rate Average (IHRA)

A lifetime random variable $X_{i:n}$is IHRA iff $-\dfrac{\log \bar{F}_{i:n}(x)}{x}$ is increasing in $x$. The equivalent definition using quantile function is given in the following example.

**Definition 2.4.** We say that $X_{i:n}$ is IHRA iff $\frac{-1}{Q(u)} \log \frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}$ is increasing in $u$. For the smallest order statistic, $X_{1:n}$ is IHRA iff $-\frac{\log(1-u)^n}{Q(u)}$ is increasing in $u$, and for the largest order statistic, $X_{n:n}$ is IHRA iff $-\frac{\log(1-u^n)}{Q(u)}$ is increasing in $u$.

The following theorem gives the equivalent conditions for IHRA of first order statistic (series systems).

**Theorem 2.5.** *The following conditions are equivalent* *(i)* $X_{1:n}$ *is IHRA*

*(ii)* $\frac{\int_0^u H_{X_{i:n}}(p)q(p)dp}{\int_0^u q(p)dp}$

*(iii)* $H_{X_{i:n}}(u) \geqslant -\frac{n\log(1-u)}{Q(u)}$

*Proof.* Assume that $X_{1:n}$ is IHRA.

$$\Leftrightarrow -\frac{\log(1-u)^n}{Q(u)} \text{ is increasing in } u$$

$$\Leftrightarrow \frac{n}{Q(u)}\int_0^u \frac{1}{1-p}dp \text{ is increasing in } u \quad \Leftrightarrow n\frac{\int_0^u H_X(p)q(p)dp}{\int_0^u q(p)dp} \text{ is increasing in } u,$$

$$\Leftrightarrow \frac{\int_0^u H_{X_{1:n}}(p)q(p)dp}{\int_0^u q(p)dp} \text{ is increasing in } u.$$

To prove $(ii) \Leftrightarrow (iii)$, assume (ii) holds. $\frac{\int_0^u H_{X_{1:n}}(p)q(p)dp}{\int_0^u q(p)dp}$ is increasing in $u$,

$$\Leftrightarrow Q(u)H(u)q(u) - q(u)\int_0^u H(p)q(p)dp \geqslant 0,$$

$$\frac{Q(u)}{1-u} - q(u)(-\log(1-u)) \geqslant 0 \text{ (from (2.65)} \quad H_{X_{1:n}}(u) = nH_X(u)),$$

$$\Leftrightarrow \frac{1}{(1-u)q(u)} \geqslant \frac{-\log(1-u)}{Q(u)},$$

$$H(u) \geqslant \frac{-\log(1-u)}{Q(u)}.$$

□

**Theorem 2.6.** *If $X_{1:n}$ is IHR then $X_{1:n}$ is IHRA but the converse need not be true.*

*Proof.* The first part of the theorem is straightforward. If $X_{1:n}$ is IHR then

$$H_{X_{1:n}}(u_2) \geqslant H_{X_{1:n}}(u_1),\ for\ u_2 > u_1,$$

equivalently

$$\frac{\int_0^u H_{X1:n}(p_2)q(p_2)dp_2}{\int_0^u q(p_2)dp_2} \geqslant \frac{\int_0^u H_{X_{1:n}}(p_1)q(p_1)dp_1}{\int_0^u q(p_1)dp_1},$$

the condition for IHRA. The converse part is proved using a counter example that IHRA does not implies IHR. Consider power-Pareto distribution with quantile function

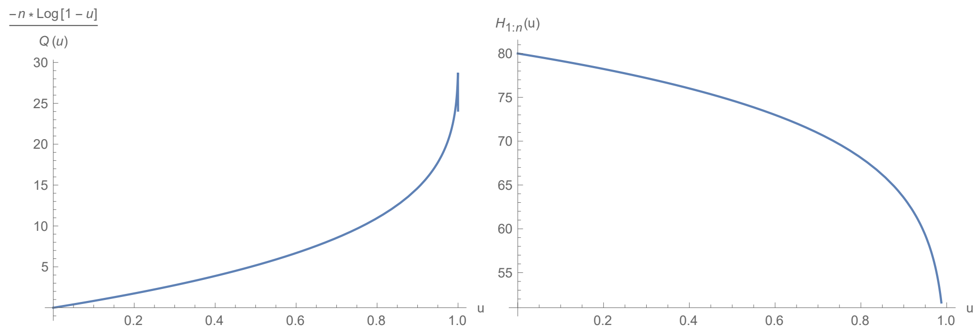
$$Q(u) = \frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, \lambda_1, \lambda_2 > 0.$$

Then, from the Figure 2.2, it is clear that the condition for IHRA of $X_{1:n}$ is

$$\frac{-1}{Q(u)}\log(1-u)^n = -\frac{\log(1-u)\left(n(1-u)^{\lambda_2}\right)}{cu^{\lambda_1}}$$

is increasing while

$$H_{X_{1:n}}(u) = \frac{n(1-u)^{\lambda_2}}{c\lambda_2}\lambda_1 = 0, 0 < \lambda_2 < 0.1, c > 0,$$



(a) IHRA graph with $\lambda_1 = 0, 0 < \lambda_2 < 0.1, c > 0$ (b) Hazard quantile function of first order statistic $H_{X_{1:n}}(u)$ versus $u$

FIGURE 2.2: Plots of IHRA and Hazard quantile function

is not increasing.

□

## Decreasing (Increasing) mean residual life class (DMRL (IMRL))

**Definition 2.7.** A random variable $X_{i:n}$ is said to have decreasing (increasing) mean residual life if $M_{i:n}(u)$ is decreasing (increasing) in $u$, $0 < u < 1$ or $M_{i:n}(u_1) \leqslant M_{i:n}(u_2)$ for $0 \leqslant u_2 \leqslant u_1 < 1$, where $M_{X_{i:n}}(u) = \dfrac{1}{\bar{B}_u(i, n-i+1)}$ $\left(\int_u^1 \bar{B}_p(i, n-i+1) q(p) dp\right)$, is the quantile-based mean residual life function for the random variable $X_{i:n}$.

**Example 2.4.** Suppose $X$ follows the power-Pareto distribution. Its quantile function is given by

$$Q(u) = \frac{c u^{\lambda_1}}{(1-u)^{\lambda_2}}, 0 \leqslant u \leqslant 1, c, \lambda_1, \lambda_2 \geqslant 0.$$

Power-Pareto distribution is obtained by taking the product of quantile functions of power and Pareto distributions (see Gilchrist (2000)). Then

$$\begin{aligned}
M_{X_{1:n}}(u) &= \frac{1}{(1-u)^n}\int_u^1 (1-p)^n q(p)dp \\
&= \frac{1}{(1-u)^n}\int_u^1 (1-p)^n \left(\frac{cp^{\lambda_1}}{(1-p)^{\lambda_2+1}}\right)(\lambda_1(1-p)+\lambda_2 p)dp \\
&= \frac{c}{(1-u)^n}\left(\lambda_1 \bar{B}_u(\lambda_1, n-\lambda_2+1) + \lambda_2 \bar{B}_u(\lambda_1+1, n-\lambda_2\right).
\end{aligned}$$

Here $M'_{X_{1:n}}(u) < (>)0$, if $\lambda_2 \leqslant (>)1$ so that $F(.)$ is DMRL (IMRL) according as $\lambda_2 \leqslant (>)1$.

## New better than used (NBU) class

Barlow & Proschan (1975) introduced NBU for the $i^{th}$ order statistic. $X_{i:n}$ is said to be NBU if

$$\bar{F}_{i:n}(x+t) \leqslant \bar{F}_{i:n}(x)\bar{F}_{i:n}(t). \tag{2.66}$$

We obtain now the quantile-based NBU of order statistics.

**Theorem 2.8.** *The first order statistic of a lifetime random variable, $X_{1:n}$ with quantile function $Q(.)$ is NBU iff $Q(u+v-uv)-Q(v) \leqslant Q(u)$ holds.*

*Proof.* From (2.66), we can write

$$\bar{B}_{F(x+t)}(i, n-i+1) \leqslant \frac{\bar{B}_{F(x)}(i, n-i+1)\bar{B}_{F(t)}(i, n-i+1)}{B(i, n-i+1)} \tag{2.67}$$

In terms of quantile function (2.67) becomes (see Nair et al. (2013)),

$$\bar{B}_{Q^{-1}(Q(u)+Q(v))}(i, n-i+1) \leqslant \frac{\bar{B}_u(i, n-i+1)\bar{B}_v(i, n-i+1)}{B(i, n-i+1)} \tag{2.68}$$

For the first order statistic (2.68) becomes

$$\bar{B}_{Q^{-1}(Q(u)+Q(v))}(1, n) \leqslant \frac{\bar{B}_u(1, n)\bar{B}_v(1, n)}{B(1, n)},$$

equivalently,

$$\int_{Q^{-1}(Q(u)+Q(v))}^{1} (1-p)^{n-1} dp \leqslant \frac{(1-u)^n(1-v)^n}{n}$$

That is

$$Q(u+v-uv) - Q(v) \leqslant Q(u).$$

Hence the proof. □

**Example 2.5.** Suppose $X$ follows the power-Pareto distribution with quantile function $Q(u) = \frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, c, \lambda_1, \lambda_2 > 0$, which is both NBU and NWU, since for $\lambda_2 = 0$ it is power distribution, which is NBU and for $\lambda_1 = 0$, it is Pareto, which is NWU.

**Theorem 2.9.** *If $X_{1:n}$ is IHRA then $X_{1:n}$ is NBU also.*

*Proof.* We need to show that $IHRA \implies NBU$. Using the condition of IHRA (see Nair et al. (2013)) we can write

$$\begin{aligned}
\frac{-n\log(1-u)}{Q_X(u)} &\leqslant \frac{-n\log(1-(u+v-uv))}{Q_X(u+v-uv)}, 0<u<v<1,\\
\implies \frac{Q_X(u+v-uv)}{Q_X(u)} &\leqslant \frac{-n\log(1-(u+v-uv))}{-n\log(1-u)},\\
\implies \frac{Q_X(u+v-uv)}{Q_X(u)} &\geqslant \frac{\log((1-u)(1-v))}{\log(1-u)},\\
&\geqslant 1+\frac{\log(1-v)}{\log(1-u)},
\end{aligned}$$

equivalently,

$$\begin{aligned}
\frac{Q_X(u+v-uv)}{Q_X(u)}-1 &\leqslant \frac{Q_X(v)}{Q_X(u)}\left(\frac{\frac{-\log(1-v)^n}{Q_X(v)}}{\frac{-\log(1-u)^n}{Q_X(u)}}\right), 0<u<v<1\\
&\leqslant \frac{Q_X(v)}{Q_X(u)}, (\textit{ since } X_{1:n} \textit{ is } IHRA)\\
\implies Q_X(u+v-uv)-Q_X(u) &\leqslant Q_X(v).
\end{aligned}$$

Hence the proof. □

### 2.8.4 Stochastic orders of order statistics

In the usual stochastic ordering, $X$ is said to be stochastically smaller than $Y$ if $\bar{F}_X(t) \leqslant \bar{F}_Y(t)$. From the definition of order statistics, it is easy to show that $X_{i:n} \leqslant_{st} X_{j:n}$, for any $i<j$. Shaked et al. (1995) done a comprehensive study on stochastic orders. Boland et al. (1994) obtained hazard rate ordering from usual stochastic order. The hazard rate ordering plays an important role in reliability and survival analysis. The likelihood ordering is a powerful one.

In this section, we examine various quantile-based ordering relationships viz. stochastic order, hazard rate order, dispersive order, mean residual order, convex order and star-shaped order based on quantile-based order statistics. Barlow & Proschan (1981) represented the distribution function of the $i^{th}$ order statistic $X_{i:n}$ as $F_{X_{(i)}}(t) = B^n_{(i)}(F_X(t))$, where $B^n_{(i)}(p) = \frac{n!}{(i-1)!(n-i)!}\int_0^p u^{i-1}(1-u)^{n-i}du$, for $0 \leqslant p \leqslant 1$.

**Definition 2.10.** Let $X$ and $Y$ be two random variables with quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. We say that $X_{i:n}$ is smaller than $Y_{i:n}$ in the usual stochastic order if $Q_{X_{i:n}}(u) \leqslant Q_{Y_{i:n}}(u)$.

**Example 2.6.** Suppose that $X$ follows beta distribution with quantile function $Q_X(u) = 1-(1-u)^{\frac{1}{c}}, c > 0$ and $Y$ follows Pareto II distribution with quantile function $Q_Y(u) = (1-u)^{\frac{-1}{c}} - 1, c > 0$. Using the relation $Q_{X_{i:n}}(u) = \int_0^u \left((1-u)H_{X_{i:n}}(u)\right)^{-1} du$, we get $Q_{X_{1:n}}(u) = \frac{Q_X(u)}{n}$,

$$\begin{aligned} Q_{X_{1:n}}(u) - Q_{Y_{1:n}}(u) &= \frac{1}{n}\left(1-(1-u)^{\frac{1}{c}} - \left((1-u)^{\frac{-1}{c}} - 1\right)\right) \\ &= \frac{1}{n}\left(1-(1-u)^{\frac{1}{c}} - \left(\frac{1-(1-u)^{\frac{1}{c}}}{(1-u)^{\frac{1}{c}}}\right)\right) \\ &= -(1-u)^{\frac{-1}{c}}\left(1-(1-u)^{\frac{1}{c}}\right)^2 \leqslant 0, \end{aligned}$$

for all $u$. Thus $X_{1:n} \leqslant_{st} Y_{1:n}$.

**Definition 2.11.** Let $X$ and $Y$ be two random variables with hazard quantile functions $H_X(u)$ and $H_Y(u)$ respectively. We say that $X_{i:n}$ is smaller than $Y_{i:n}$ in the hazard rate order if $H_{Y_{i:n}}(u) \leqslant H_{X_{i:n}}(u)$, where $H_{X_{i:n}}(u) = h_{X_{i:n}}(Q_X(u))$ and $H_{Y_{i:n}}(u) = h_{Y_{i:n}}(Q_X(u))$.

**Theorem 2.12.** *If $X \leqslant_{hr} Y$ then $X_{i:n} \leqslant_{hr} Y_{i:n}$.*

*Proof.* We have $H_X(u) \geqslant H_Y(u)$. Using the relation (2.65) it is clear that $H_{X_{i:n}}(u) \geqslant H_{Y_{i:n}}(u)$. Thus the result. □

**Definition 2.13.** Let $X_{i:n}$ and $X_{j:n}$ be $i^{th}$ and $j^{th}$ order statistics with density quantile functions $f_{i:n}\left(Q_X(u)\right)$ and $f_{j:n}\left(Q_X(u)\right)$ respectively. We say that $X_{i:n}$ is smaller than $X_{j:n}$ in the likelihood order if $\frac{f_{j:n}(Q_X(u))}{f_{i:n}(Q_X(u))}$ is increasing in $u$.

**Theorem 2.14.** *For the $i^{th}$ and $j^{th}$ order statistic, $X_{i:n} \leqslant_{lr} X_{j:n}$, for $i < j$.*

*Proof.* Using (2.54), we get

$$\begin{aligned}\frac{f_{j:n}(Q_X(u))}{f_{i:n}(Q_X(u))} &= \frac{g_j(u)}{g_i(u)} \\ &= \frac{B(i, n-i+1)}{B(j, n-j+1)} \frac{u^{j-1}(1-u)^{n-j}}{u^{i-1}(1-u)^{n-i}} \\ &= c\left(\frac{u}{(1-u)}\right)^{j-i},\end{aligned}$$

is increasing in $u$ for $j - i > 0$. Thus $X_{i:n} \leqslant_{lr} X_{j:n}$. □

**Definition 2.15.** We say that $X_{i:n}$ is smaller than $Y_{i:n}$ in increasing convex order if, $X_{i:n} \leqslant_{icx} Y_{i:n}$ if $\int_t^\infty \bar{F}_{i:n}(x)dx \leqslant \int_t^\infty \bar{G}_{i:n}(x)dx$, for all $t$. In quantile notation $X_{i:n} \leqslant_{icx} Y_{i:n}$ if $\int_u^1 \bar{B}_p(i, n-i+1)q_X(p)dp \leqslant \int_u^1 \bar{B}_p(i, n-i+1)q_Y(p)dp$.

**Definition 2.16.** We say that $X$ is smaller than $Y$ in star order, $X \leqslant_* Y$ iff $F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x))$ is star-shaped in $x$.

By the definition of the star-shaped functions, $\frac{1}{x}F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x))$ should be increasing in $x \geqslant 0$. We use the relationship from Barlow & Proschan (1975) that $F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x)) = (B^n_{(i)}(F_Y))^{-1}(B^n_{(i)}(F_X(x))) = F^{-1}_Y(F_X(x))$. Then, $\frac{1}{x}F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x))$ is increasing in $x$, which is the same as $\frac{1}{x}F^{-1}_Y(F_X(x))$ increasing in $x$. This is equivalent to $\frac{Q_Y(u)}{Q_X(u)}$ is increasing in $u$.

Since $X_{i:n} \leqslant_c Y_{i:n}$ implies that $\frac{q_Y(u)}{q_X(u)}$ is increasing in $u$, it follows that $X_{i:n} \leqslant_c Y_{i:n} \implies X_{i:n} \leqslant_* Y_{i:n}$.

**Definition 2.17.** We say that $X_{i:n}$ is smaller than $Y_{i:n}$ in dispersive order, $X_{i:n} \leqslant_{disp} Y_{i:n}$ if $F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x)) - x$ is increasing in $x$.

Since $F^{-1}_{Y_{i:n}}(F_{X_{i:n}}(x)) = F^{-1}_{Y}(F_X(x))$, (see Barlow & Proschan (1975)) we say that $X_{i:n} \leqslant_{disp} Y_{i:n}$ if $Q_Y(u) - Q_X(u)$ is increasing in $u$

**Theorem 2.18.** *If $X_{i:n} \leqslant_{disp} Y_{i:n}$ then $X_{i:n} \leqslant_{st} Y_{i:n}$.*

*Proof.* Using the definition of dispersive ordering, we have $Q_Y(u) - Q_X(u)$ is increasing in $u$, that is $q_Y(u) - q_X(u) \geqslant 0$, implies that $H_{X_{i:n}}(u) \geqslant H_{Y_{i:n}}(u)$, which is equivalent to $Q_{X_{i:n}}(u) \leqslant Q_{Y_{i:n}}(u)$. Hence the proof. □

# Chapter 3

# Quantile-based entropy of order statistics

## 3.1 Introduction

Quantile-based study of entropy measures found greater interest among researchers as an alternative method of measuring the uncertainty of a random variable. The quantile-based entropy measures possess some unique properties than its distribution function approach. Motivated by this, in the present Chapter, we introduce a quantile-based entropy of order statistics and study its properties. We also propose a quantile-based residual entropy of order statistics, an alternative method to measure the uncertainty of ordered observations for used items. Unlike the distribution function approach, the quantile approach provides an explicit relationship between the quantile density function and quantile-based residual entropy of order statistics.

The present chapter is organized as follows. In Section 3.2, we introduce a quantile-based entropy of order statistics and study its important properties. Section 3.3 introduces the quantile-based residual entropy of order statistics and proves that it uniquely determines the quantile density function. Various

[0]Results in this chapter have been published as entitled "Quantile-based entropy of order statistics" in "Journal of the Indian Society for Probability and Statistics" (See Sunoj et al. (2017)).

bounds of quantile-based entropy of order statistics and its residual version are obtained.

## 3.2 Quantile-based Shannon entropy of order statistics

Recently, Sunoj & Sankaran (2012) introduced a quantile-based Shannon entropy function, given by

$$\xi = -E(\log f(Q(u))) = \int_0^1 \log q(u) du, \tag{3.1}$$

where $q(u) = \frac{d}{du}Q(u)$ denotes the quantile density function. Sunoj & Sankaran (2012) further extended $\xi$ in (3.1) to the residual random variable in the quantile setup and studied its properties. The quantile-based residual entropy is given by,

$$\xi(u) = \log(1-u) + \frac{1}{(1-u)} \int_u^1 \log q(p) dp. \tag{3.2}$$

Sunoj & Sankaran (2012) have shown that the residual quantile entropy function determines the quantile density function uniquely. An extension of $\xi$ to the past lifetime is discussed in Sunoj et al. (2013) while a quantile-based residual Renyi's entropy is studied by Nanda et al. (2014). Also, a study on the quantile-based cumulative entropy measures for residual and past lifetime random variables is available in Sankaran & Sunoj (2017). Recently, a quantile Kullback-Leibler divergence is proposed by Sankaran et al. (2016), using the quantile function of the relative inverse.

From (2.52), Shannon entropy of $X_{i:n}$ is given by

$$\eta_{i:n}(x) = -E(\log f_{i:n}(X)) = -\int_0^\infty (\log f_{i:n}(x))\, f_{i:n}(x)dx.$$

Now using (2.52), the corresponding quantile-based Shannon entropy of $X_{i:n}$ is defined as

$$\xi_{X_{i:n}} = \eta_{i:n}(Q(u)) = -\int_0^1 (\log f_{i:n}(Q(u)) f_{i:n}(Q(u) dQ(u). \tag{3.3}$$

From (2.51), we have $FQ(u) = u$, then (2.53) becomes

$$\begin{aligned} f_{i:n}(u) = f_{i:n}(Q(u)) &= \frac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i} f(Q(u)) \\ &= \frac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i} \frac{1}{q(u)}. \end{aligned} \tag{3.4}$$

Denoting $g_i(u) = \dfrac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i}$ as the pdf of beta distribution with parameters $(i, n-i+1)$, (3.4) becomes $f_{i:n}(u) = \frac{g_i(u)}{q(u)}$ and hence $\xi_{X_{i:n}}$ in (3.3) reduces to

$$\begin{aligned} \xi_{X_{i:n}} &= -\int_0^1 \left(\log \frac{g_i(u)}{q(u)}\right) \frac{g_i(u)}{q(u)} q(u) du \\ &= -\int_0^1 (\log g_i(u)) g_i(u) du + \int_0^1 (\log q(u))\, g_i(u) du \\ &= \eta_{g_i} + E_{g_i}(\log q(U)), \end{aligned} \tag{3.5}$$

where

$$\eta_{g_i} = -\int_0^1 (\log g_i(u))g_i(u)du,$$

is the Shannon entropy of beta distribution with parameters $(i, (n-i+1))$ (beta entropy) and

$$E_{g_i}(\log q(U)) = \int_0^1 (\log q(u))\, g_i(u)du. \tag{3.6}$$

From (3.5), it is clear that $\xi_{X_{i:n}}$ is sum the Shannon entropy of beta distribution with parameters $(i, n-i+1)$ and expectation of $\log q(U)$ of beta distribution with parameters $(i, n-i+1)$. Also, $\xi_{X_{i:n}}$ can be expressed in terms of the hazard quantile function $H(u)$ by

$$\begin{aligned} \xi_{X_{i:n}} &= \eta_{g_i} + \int_0^1 \left(\log((1-u)H(u))^{-1}\right) g_i(u)du \\ &= \eta_{g_i} - \int_0^1 (\log(1-u))\, g_i(u)du - \int_0^1 (\log H(u))\, g_i(u)du. \end{aligned} \tag{3.7}$$

In terms of the reversed hazard quantile function $\Lambda(u)$, $\xi_{X_{i:n}}$ in (3.5) can be expressed as

$$\xi_{X_{i:n}} = \eta_{g_i} - \int_0^1 (\log \Lambda(u))g_i(u)du - \int_0^1 (\log u)g_i(u)du. \tag{3.8}$$

$\eta_{g_i}$ can be expressed as a function of digamma function defined as $\Psi(z) = \frac{d}{dz}(\log \Gamma(z))$, given by (Ebrahimi et al. (2004))

$$\begin{aligned} \eta_{g_i} = \log B(i, n-i+1) &- (i-1)[\Psi(i) - \Psi(n+1)] \\ &- (n-i)[\Psi(n-i+1) - \Psi(n+1)]. \end{aligned} \tag{3.9}$$

Table 3.1 provides quantile-based entropy of order statistics for some important lifetime distributions. In the following example we obtain $\xi_{X_{i:n}}$ for Govindarajulu distribution (Nair et al. (2013)), where the distribution function has no closed form.

**Example 3.1.** Let $X$ be a random variable having the Govindarajulu distribution with the quantile function $Q(u) = \theta + \sigma\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\}, 0 < u < 1, \sigma, \beta > 0$, so that $q(u) = \sigma\beta(\beta+1)u^{\beta-1}(1-u)$. Then

$$\begin{aligned} E_{g_i}(\log q(U)) &= E_{g_i}(\log \sigma\beta(\beta+1)U^{\beta-1}(1-U)) \\ &= \log \sigma\beta(\beta+1) + (\beta-1)E(\log U) + E(\log(1-U)) \\ &= \log \sigma\beta(\beta+1) + (\beta-1)(\psi(i) - \psi(n+1)) + \psi(n-i+1) - \psi(n+1). \end{aligned}$$

Using (3.5), $\xi_{X_{i:n}}$ becomes

$$\begin{aligned} \xi_{X_{i:n}} = \log B(i, n-i+1) - (i-1)[\psi(i) - \psi(n+1)] - (n-i)[\psi(n-i+1) - \psi(n+1)] \\ + \log \sigma\beta(\beta+1) + (\beta-1)(\psi(i) - \psi(n+1)) + \psi(n-i+1) - \psi(n+1). \end{aligned}$$

Figure 3.1 shows that the quantile-based entropy of order statistics $\xi_{X_{i:n}}$ is maximum at the median for the Govindarajulu distribution. Figure 3.2 indicates that for Govindarajulu distribution as $\beta$ increases the entropy of first-order statistics $\xi_{X_{1:n}}$ also increases linearly.

**Example 3.2.** Let $X$ be a random variable having generalized Pareto distribution with quantile function

$$Q(u) = \frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], a > -1, b > 0. \tag{3.10}$$

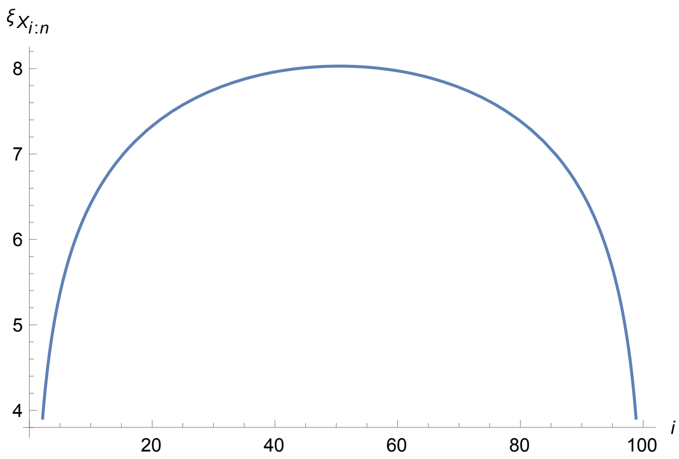



FIGURE 3.1: Plot of $\xi_{X_{i:n}}$ for Govindarajulu distribution with $\beta > 0, \sigma = 1$.

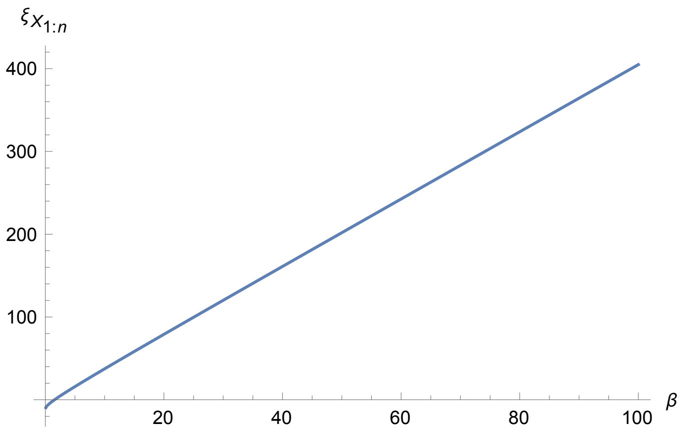



FIGURE 3.2: Plot of $\xi_{X_{1:n}}$ of Govindarajulu distribution for different values of $\beta$ and $\sigma = 1$.

Then the quantile version of entropy of order statistics $\xi_{X_{i:n}}$ is given by

$$\begin{aligned}
\xi_{X_{i:n}} &= \eta_{g_i} + E_{g_i}(\log q(U)) \\
&= \eta_{g_i} + \int_0^1 (\log q(u)) g_i(u) du \\
&= \eta_{g_i} + \int_0^1 \left( \log \left( \frac{b}{a+1} \right) (1-u)^{-\frac{a}{a+1}-1} \right) g_i(u) du \\
&= \eta_{g_i} + \log \left( \frac{b}{a+1} \right) - \left( \frac{2a+1}{a+1} \right) E_{g_i}(\log(1-U))
\end{aligned}$$

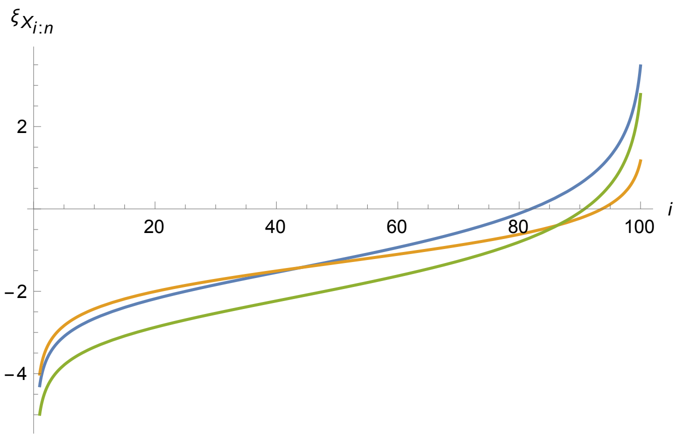



FIGURE 3.3: Plot of $\xi_{X_{i:n}}$ for generalized Pareto distribution

TABLE 3.1: Quantile function and quantile-based entropy of order statistics for lifetime distributions.

| Distribution | $Q(u)$ | $\xi_{X_{i:n}}$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\eta_{g_i} - \log \lambda - [\psi(n-i+1) - \psi(n+1)]$ |
| Pareto II | $\beta\left((1-u)^{-1/c} - 1\right), c, \beta > 0$ | $\eta_{g_i} + \log \frac{\beta}{c} - (\frac{c+1}{c})[\psi(n-i+1) - \psi(n+1)]$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), R, c > 0$ | $\eta_{g_i} + \log \frac{R}{c} + \frac{1-c}{c}[\psi(n-i+1) - \psi(n+1)]$ |
| Power | $\alpha u^{\frac{1}{\beta}}, \alpha, \beta > 0$ | $\eta_{g_i} + \log \frac{\alpha}{\beta} + \frac{1-\beta}{\beta}[\psi(i) - \psi(n+1)]$ |
| Pareto I | $\sigma(1-u)^{-\frac{1}{\alpha}}, \sigma, \alpha > 0$ | $\eta_{g_i} + \log \frac{\sigma}{\alpha} - (\frac{\alpha+1}{\alpha})[\psi(n-i+1) - \psi(n+1)]$ |
| Log logistic | $\alpha^{-1}(\frac{u}{1-u})^{\frac{1}{\beta}}, \alpha, \beta > 0$ | $\eta_{g_i} - \log \alpha\beta + \frac{1-\beta}{\beta}[\psi(i) - \psi(n+1)] - \frac{1+\beta}{\beta}[\psi(n-i+1) - \psi(n+1)]$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], b > 0, a > -1$ | $\eta_{g_i} + \log \frac{b}{a+1} - (\frac{2a+1}{a+1})[\Psi(n-i+1) - \Psi(n+1)]$ |
| Govindarajulu | $\theta + \sigma\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\}, \theta, \sigma\beta > 0$ | $\eta_{g_i} + \log \sigma\beta(\beta+1) + (\beta-1)(\psi(i) - \psi(n+1)) + \psi(n-i+1) - \psi(n+1)$ |

where $E_{g_i}(\log(1-U)) = \Psi(n-i+1) - \Psi(n+1)$. Then $\xi_{X_{i:n}}$ becomes

$$\xi_{X_{i:n}} = \eta_{g_i} + \log\left(\frac{b}{a+1}\right) - \left(\frac{2a+1}{a+1}\right)[\Psi(n-i+1) - \Psi(n+1)]. \qquad (3.11)$$

Figure 3.3 indicates that $\xi_{X_{i:n}}$ shows an increasing tendency when the observations are in increasing order at different values of *a and b*.

**Theorem 3.1.** *For the random variable X, $\xi_{X_{i:n}}$ possess the following bounds.*

(a) *Assume that $\xi < \infty$, then*

$$\xi_{X_{i;n}} \leq \eta_{g_i} + nB_i\xi, \tag{3.12}$$

*where $B_i$ is the $i^{th}$ term of binomial distribution with parameters $(n-1, p_i)$ and $p_i = \frac{i-1}{n-1}$ is the mode of the beta distribution.*

(b) *Let the quantile density function $q(m) = \frac{1}{M} < \infty$, where $m = Sup\{u : q(u) \geqslant \frac{1}{M}\}$ is the mode of the distribution of the random variable X. Then,*

$$\xi_{X_{i;n}} \geq \eta_{g_i} - \log M. \tag{3.13}$$

*Proof.* (a) Since $p_i = \frac{i-1}{n-1}$ is the mode, the maximum value of $g_i(u)$ will be attained at the mode. Thus,

$$g_i(u) \leq g_i(p_i) = n \begin{pmatrix} n-1 \\ i-1 \end{pmatrix} p_i^{i-1}(1-p_i)^{n-i} = nB_i \tag{3.14}$$

Using (3.5), we obtain

$$\begin{aligned} \xi_{X_{i;n}} - \eta_{g_i} &= E_{g_i}(\log q(U)) \\ &= \int_0^1 (\log q(u))g_i(u)du \\ &\leq \int_0^1 (\log q(u))nB_i du \\ &\leq nB_i\xi, \end{aligned} \tag{3.15}$$

where $\xi$ is the quantile-based entropy defined in (3.1).

(b) From (3.5), we have

$$\begin{aligned}
\xi_{X_{i;n}} &= \eta_{g_i} + \int_0^1 g_i(u) \log q(u) du \\
&= \eta_{g_i} - \int_0^1 g_i(u) \log(q(u))^{-1} du \\
&\geq \eta_{g_i} - \int_0^1 g_i(u) \log M du \\
&= \eta_{g_i} - \log M, \qquad (3.16)
\end{aligned}$$

completes the proof. □

These bounds are useful when the quantile function does not have a closed form or $\xi_{X_{i:n}}$ is difficult to compute.
In the next theorem, we derive the monotone behaviour of $\bar{\xi}_{X_{i:n}}$ with respect to the monotone nature of $q(u)$.

**Theorem 3.2.** *Let $X$ be a random variable and let $X_{i:n}, i = 1, 2, ...n$ denote the $i^{th}$ order statistics and $\xi_{X_{i:n}}$ be the quantile-based entropy of the order statistics.*
*a. If $q(u)$ is nondecreasing in $u$, then $\xi_{X_{i:n}}$ is increasing in $i$ for $i < \frac{n}{2}$.*
*b. If $q(u)$ is nonincreasing in $u$, then $\xi_{X_{i:n}}$ is decreasing in $i$ for $i > \frac{n}{2}$.*

*Proof.* Using (3.5), we can write

$$\xi_{X_{i+1:n}} - \xi_{X_{i:n}} = \eta_{g_{i+1}} - \eta_{g_i} + E_{g_{i+1}}(\log q(u)) - E_{g_i}(\log q(u)). \qquad (3.17)$$

Let

$$\Delta_n(i) = \eta_{g_{i+1}} - \eta_{g_i} = (\log(n-i) - \psi(n-i)) - (\log i - \psi(i)). \qquad (3.18)$$

Then $\Delta_n(i) < 0$ for $i < \dfrac{n}{2}$, $\Delta_n(i) > 0$, for $i > \dfrac{n}{2}$, and for an even $n, \Delta_n(\dfrac{n}{2}) = 0$. (see Ebrahimi et al. (2004))

$$\xi_{X_{i+1:n}} - \xi_{X_{i:n}} = -\Delta_n(i) + E_{g_{i+1}}(\log q(u)) - E_{g_i}(\log q(u)). \tag{3.19}$$

Since the order statistics are stochastically ordered,

$$U_i \leq^{st} U_{i+1} \Rightarrow E_{g_i}(\phi(U_i)) \leq E_{g_{i+1}}(\phi(U_{i+1})) \tag{3.20}$$

Thus, (3.19) reduces to

$$\xi_{X_{i+1:n}} - \xi_{X_{i:n}} \geq 0, i < \frac{n}{2} \tag{3.21}$$

Thus $\xi_{X_{i:n}}$ is increasing in $i$ for $i < \frac{n}{2}$, completes the first part of the proof. The second part can be proved in a similar manner and hence omitted. □

The following theorem provides the stochastic comparison of two random variables with respect to quantile-based entropy of order statistics.

**Theorem 3.3.** *Let $X$ and $Y$ be two random variables with quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. Then*

$$X \leq_{st} Y \Leftrightarrow \xi_{X_{i:n}} \leq \xi_{Y_{i:n}}, \tag{3.22}$$

*and*

$$X \leq_{hr} Y \Leftrightarrow \xi_{X_{i:n}} \leq \xi_{Y_{i:n}}. \tag{3.23}$$

*Proof.* If $X \leq_{st} Y$, then $Q_X(u) \leq Q_Y(u)$. From (3.5), we have

$$\begin{aligned} \xi_{X_{i:n}} &= \eta_{g_i} + \int_0^1 (\log q_X(u)) g_i(u) du \\ &\leq \eta_{g_i} + \int_0^1 (\log q_Y(u)) g_i(u) du \\ &= \xi_{Y_{i:n}}. \end{aligned}$$

By retracing the above inequality we get the converse part of (3.22). Similarly, one can prove (3.23) using the definition of hazard quantile order and (3.7). □

## 3.3 Quantile-based residual entropy of order statistics

In reliability and life testing, the data are generally truncated and in such situations $\xi_{X_{i:n}}$ is not an appropriate measure. Assume that the component $X$ has survived $t$ units time, then the residual lifetime of the components, say $X_t$ is of interest. Based on this, the residual entropy of the $i^{th}$ order statistic is defined by

$$\xi(X_{i:n}, t) = -\int_t^\infty \left(\log \frac{f_{i:n}(x)}{\bar{F}_{i:n}(t)}\right) \frac{f_{i:n}(x)}{\bar{F}_{i:n}(t)} dx. \tag{3.24}$$

Since $\bar{F}_{i:n}(t) = \dfrac{\bar{B}_{F(t)}(i, n-i+1)}{B(i, n-i+1)}$, the quantile version of the residual entropy of the $i^{th}$ order statistic is defined as,

$$\begin{aligned}\xi_{X_{i:n}}(u) = &-\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\int_u^1 (\log g_i(p))g_i(p)dp \\ &+\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\int_u^1 (\log q(p)g_i(p)dp) + \log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}\end{aligned} \quad (3.25)$$

where $\dfrac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}$ is the quantile form of $\bar{F}_{i:n}(t)$ with $\bar{B}_u(i,n-i+1) = \int_u^1 u^{i-1}(1-u)^{n-i}du$, the incomplete beta function. From (3.24), the first-order residual entropy is given by

$$\xi(X_{1:n}, t) = -\int_t^\infty \left(\log\frac{n(1-F(x))^{n-1}f(x)}{\bar{F}(t)^n}\right)\left(\frac{n(1-F(x))^{n-1}f(x)}{\bar{F}(t)^n}\right)dx. \quad (3.26)$$

and the $n^{th}$ order residual entropy is

$$\xi(X_{n:n}, t) = -\int_t^\infty \left(\log\frac{n(F(x))^{n-1}f(x)}{\bar{F}(t)^n}\right)\left(\frac{n(F(x))^{n-1}f(x)}{\bar{F}(t)^n}\right)dx. \quad (3.27)$$

The corresponding quantile residual entropy of the $1^{st}$ and $n^{th}$ order statistics obtained respectively as

$$\xi_{X_{1:n}}(u) = -\log n + \log(1-u) + \frac{n-1}{n} + n(1-u)^{-n}\int_u^1 (\log q(p))(1-p)^{n-1}dp \quad (3.28)$$

$$\xi_{X_{n:n}}(u) = -\log n + \frac{(n-1)u^n\log u}{(1-u)^n} + \frac{n-1}{n} - \log(1-u^n) - \int_u^1 (\log q(p))p^{n-1}dp \quad (3.29)$$

An equivalent representation of (3.25) is of the form

$$\xi_{X_{i:n}}(u) = -\frac{1}{\bar{B}_u(i,n-i+1)}\int_u^1 (\log g_i(p))p^{i-1}(1-p)^{n-i}dp \\ +\frac{1}{\bar{B}_u(i,n-i+1)}\int_u^1 (\log q(p))p^{i-1}(1-p)^{n-i}dp \\ +\log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}. \tag{3.30}$$

Order statistics play an important role in system reliability. For example, first-order statistics represents the lifetime of a series system while the $n$th order statistics measure the lifetime of a parallel system. Hence for measuring a quantile-based uncertainty of series and parallel systems with i.i.d components, $\xi_{X_{1:n}}(u)$ in (3.28) and $\xi_{X_{n:n}}(u)$ in (3.29) are of useful. We consider $\xi_{X_{1:n}}(u)$ to derive different properties, the results of $\xi_{X_{n:n}}(u)$ can be obtained in a similar way.

**Example 3.3.** For the Cox proportional hazards model, defined by $h_Y(x) = \theta h_X(x)$, $\theta > 0$, the equivalent quantile function is $Q_Y(u) = Q_X\left(1-(1-u)^{\frac{1}{\theta}}\right)$, and quantile density function $q_Y(u) = \frac{1}{\theta}(1-u)^{\frac{1}{\theta}-1}q_X\left(1-(1-u)^{\frac{1}{\theta}}\right)$. Then quantile-based residual entropy of first-order statistics is given by

$$\begin{aligned}
\xi_{Y_{1:n}}(u) &= -\log n + \log(1-u) + \frac{n-1}{n} \\
&\quad +n(1-u)^{-n}\int_u^1 \left(\log q_X(1-(1-p)^{\frac{1}{\theta}})\frac{1}{\theta}(1-p)^{\frac{1}{\theta}-1}\right)(1-p)^{n-1}dp, \\
&= -\log n + \log(1-u) + \frac{n-1}{n} \\
&\quad +n(1-u)^{-n}\int_u^1 \left(\log q_X(1-(1-p)^{\frac{1}{\theta}})\right)(1-p)^{n-1}dp \\
&\quad -\log\theta + \frac{n(1-\theta)}{\theta}(n\log(1-u)-1).
\end{aligned}$$

Table 3.2 provides the first-order quantile entropy of order statistics $\xi_{X_{1:n}}(u)$ for some important life distributions where the quantile function $Q(u)$ are of closed form expressions. However, in some cases only the quantile density function $q(u)$ has a closed form expression. Therefore, in the following example, we obtain $\xi_{X_{1:n}}(u)$ for a family of distributions that can be represented only through $q(u)$.

**Example 3.4.** Suppose $X$ is distributed with quantile density function given by

$$q(u) = Ku^{\alpha}(1-u)^{-(A+\alpha)},$$

where $K, \alpha$ and $A$ are real constants. It is to be noted that some of the members of this quantile density have non-monotone hazard quantile functions while some others have monotone hazard quantile functions. Further, it contains several well-known distributions which include the exponential $(\alpha = 0, A = 1)$, Pareto $(\alpha = 0, A < 1)$, rescaled beta $(\alpha = 0, A > 1)$, the log-logistic distribution $(\alpha = \lambda - 1, A = 2)$ and Govindarajulu distribution $(\alpha = \beta - 1, A = -\beta)$ given in Example 3.1. Then the residual quantile entropy of first-order statistics $\xi_{X_{1:n}}(u)$ is obtained as

$$\begin{aligned}\xi_{X_{1:n}}(u) = -\log n + \log(1-u) + \frac{n-1}{n} + \log k - \frac{(A+\alpha)}{n}(-1 + n\log(1-u)) \\ + n\alpha(1-u)^{-n}\int_u^1 (\log p)(1-p)^{n-1}dp.\end{aligned}$$

TABLE 3.2: Quantile function and the quantile-based residual entropy of the first-order statistics.

| Distribution | $Q(u)$ | $\xi_{X_{1:n}}(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}$ | $1-\log n-\log\lambda$ |
| Pareto II | $\alpha\left((1-u)^{-1/c}-1\right)$ | $-\log n+\log(1-u)+\frac{n-1}{n}+\log\frac{\alpha}{c}-\frac{1+c}{c}log(1-u)+\frac{1+c}{cn}$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right)$ | $-\log n+\log(1-u)+\frac{n-1}{n}+\log\frac{R}{c}+\frac{1-c}{c}[\log(1-u)-\frac{1}{n}]$ |
| Pareto I | $\sigma(1-u)^{-\frac{1}{\alpha}}$ | $-\log n+1+\log\frac{\sigma}{\alpha}-\frac{1}{\alpha}\log(1-u)+\frac{\alpha+1}{n\alpha}$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1]$ | $-\log n-\frac{a}{a+1}\log(1-u)+\frac{n-1}{n}+\log\frac{b}{a+1}+\frac{2a+1}{n(a+1)}$ |

Differentiating the equation (3.30) with respect to $u$, we get

$$\xi'_{X_{i:n}}(u)=\frac{u^{i-1}(1-u)^{n-i}}{\bar{B}_u(i,n-i+1)}\left[\xi_{X_{i:n}}(u)-\log q(u)-1-\log\frac{u^{i-1}(1-u)^{n-i}}{\bar{B}_u(i,n-i+1)}\right]. \tag{3.31}$$

Equation (3.31) is equivalent to

$$q(u)=\frac{u^{i-1}(1-u)^{n-i}}{\bar{B}_u(i,n-i+1)}\exp\left[\xi_{X_{i:n}}(u)-1-\frac{\xi'_{X_{i:n}}(u)\bar{B}_u(i,n-i+1)}{u^{i-1}(1-u)^{n-i}}\right]. \tag{3.32}$$

Equation (3.32) shows that the quantile-based residual entropy of order statistics uniquely determines the quantile density function.

**Example 3.5.** Suppose that $X$ follows generalized Pareto Distribution with quantile function

$$Q(u)=\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1],a>-1,b>0. \tag{3.33}$$

Then the first-order quantile entropy of residual life is given by

$$\xi_{X_{1:n}}(u)=-logn-\frac{a}{a+1}log(1-u)+\frac{n-1}{n}+log\frac{b}{a+1}+\frac{2a+1}{n(a+1)}. \tag{3.34}$$

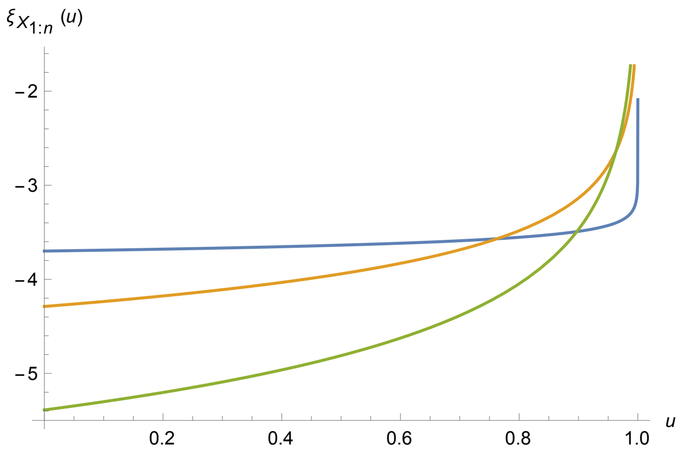



FIGURE 3.4: Plot of $\xi_{X_{1:n}}(u)$ of generalized Pareto.

Conversely, assume that (3.33) holds. Then using (3.32), we obtain

$$q(u) \;=\; exp\{\xi_{X_{1:n}}(u) - 1 + \log\frac{(1-u)^{n-1}}{B(1,n)} - \frac{\xi'_{X_{1:n}}(u)\bar{B}(1,n)}{(1-u)^{n-1}} - \log\frac{\bar{B}_u(1,n)}{B(1,n)}\},$$

which is equivalent to

$$q(u) = exp\{-\log n - \frac{a}{a+1}log(1-u) + \frac{n-1}{n} + \log\frac{b}{a+1} + \frac{2a+1}{n(a+1)} - 1 + \log n(1-u)^{n-1} - \log(1-u)^n - \frac{a}{n(a+1)}\}.$$

On simplification, we have

$$q(u) = \frac{b}{(a+1)}(1-u)^{\frac{a}{a+1}-1},$$

the quantile density function of generalized Pareto with quantile function (3.33). Figure 3.5 provides the plot of $\xi_{X_{1:n}}(u)$ of generalized Pareto distribution by fixing $b = 1$ and for different values of $a$ and $u$.

Figure 3.5 indicates that when the component lifetimes follow generalized Pareto distribution, the quantile-based residual entropy of a series system (first-order statistic) shows an increasing tendency as $u$ increases.

**Theorem 3.4.** *The relationship*

$$\xi_{X_{1:n}}(u) = \xi_{X_{1:n}} + c\log(1-u) \tag{3.35}$$

*holds if and only if X follows generalized Pareto with quantile function* (3.33)

*Proof.* Assume that $X$ follows generalized Pareto with quantile function (3.33). Using (3.5) we obtain

$$\xi_{X_{1:n}} = \frac{n-1}{n} - \log n + \log\frac{b}{a+1} + \frac{2a+1}{n(a+1)}. \tag{3.36}$$

Comparing (3.34) and (3.36), we get the relation (3.35). For proving the converse part, assume the relationship (3.35) holds. From (3.31), we get

$$\xi'_{X_{1:n}}(u) = \frac{n}{1-u}\left(\xi_{X_{1:n}}(u) - \log q(u) - 1 - \log n + \log(1-u)\right). \tag{3.37}$$

Differentiating (3.35) with respect to $u$, we get $\xi_{X_{1:n}} = \frac{-c}{1-u}$. Then (3.37) becomes

$$\xi'_{X_{1:n}}(u) = \frac{-c}{n} + \log q(u) - \log(1-u) + 1 + \log n.$$

Now, using the relationship (3.35) the above equation becomes

$$\xi_{X_{1:n}} = (1 - \frac{c}{n} + \log n) + \log q(u) - (c+1)\log(1-u). \tag{3.38}$$

Taking derivative on both sides of (3.38), we get

$$\frac{q'(u)}{q(u)} = -\frac{c+1}{1-u},$$

on integration we get $q(u) = (1-u)^{c+1}$. Thus the proof. □

**Remark 3.1.** For $c = 0, (c < 0, c > 0)$ it characterizes exponential distribution, (Pareto II, Rescaled Beta). The characterization of $\xi_{X_{1:n}}(u)$ for different distributions can be obtained from Table: 3.2.

**Theorem 3.5.** *The quantile-based residual entropy of a series system $\xi_{X_{1:n}}(u) = C$, a constant if and only if $X$ follows an exponential distribution.*

*Proof.* The proof is straight forward from (3.32). □

**Theorem 3.6.** *The relationship $\xi_{X_{1:n}}(u) = a + bu$, where $a, b$ are real constants, holds if and only if the quantile density of the form $q(u) = \frac{\theta}{(1-u)} e^{\lambda u}$, where $\theta, \lambda$ are real constants.*

*Proof.* The 'if' part can be proved easily from (3.28) and the 'only if' part can be proved from (3.32), which reduces to $q(u) = \frac{n}{(1-u)} exp\left[\xi_{X_{1:n}}(u) - 1 - \frac{(1-u)\xi'_{X_{1:n}}(u)}{n}\right]$, in turn, provides the required $q(u)$. □

**Definition 3.7.** $X$ is said to have increasing (decreasing) residual quantile entropy of order statistics (IRQEO (DRQEO)) if $\xi_{X_{i:n}}(u)$ is increasing (decreasing) in $u \geq 0$.

Now it is easy to show from the relationship (3.31) that if $X$ is IRQEO (DRQEO), then

$$\xi_{X_{i:n}}(u) \geq (\leq) 1 + \log \frac{q(u) u^{i-1} (1-u)^{n-i}}{\bar{B}_u(i, n-i+1)},$$

or equivalently

$$\xi_{X_{i:n}}(u) \geq (\leq) 1 + \log \frac{u^{i-1}(1-u)^{n-i-1}}{H(u)\bar{B}_u(i, n-i+1)},$$

gives the upper (lower) bound for $\xi_{X_{i:n}}(u)$. However for the exponential distribution of first-order statistics, $\xi_{X_{1:n}}(u) = 1 + \log \frac{q(u)(1-u)}{n} = 1 - \log H(u)n = 1 - \log n - \log \lambda = C$, a constant. Hence exponential distribution belongs to both increasing (decreasing) residual entropy of first-order statistics classes. For Pareto II distribution (Table 3.2), $\xi'_{X_{1:n}}(u) = \frac{1}{c(1-u)} > 0$, so that Pareto II random variable belong to increasing residual entropy of first-order statistics class. In the case of rescaled beta density (Table 3.2), $\xi'_{X_{1:n}}(u) = -\frac{1}{c(1-u)} < 0$, therefore belongs to the decreasing residual entropy of first-order statistics class. It is to be noted that the hazard rate ordering do not preserve quantile residual entropy of first-order statistics. For example, Pareto II is decreasing while $\xi_{X_{1:n}}(u)$ is increasing in $u$.

The ageing behaviours of life distributions is fundamental in reliability theory and practice and the hazard and mean residual quantile functions are the two basic concepts in this connection. We say that $X$ is increasing (decreasing) hazard rate (IHR(DHR)) if $H(u)$ is increasing (decreasing) in $u$. The monotonicity of $h(x)$ and $H(u)$ are identical. We can see that the residual quantile entropy of first-order statistics does not preserve the IHR (DHR) property. That is, IHR (DHR) property does not imply increasing (decreasing) residual quantile entropy of first-order statistics. For example, in the case of Pareto II with quantile function $Q(u) = \alpha \left[(1-u)^{-\frac{1}{c}} - 1\right]$, we have $H(u) = \frac{c}{a\alpha}(1-u)^{\frac{1}{c}}$ which is DHR while $\xi'_{X_{1:n}}(u) = \frac{1+c}{c(1-u)} > 0$ is increasing residual quantile entropy of first-order statistics.

**Theorem 3.8.** *If $X$ is increasing residual quantile entropy order(IRQEO) and if $\phi(.)$ is non-negative, increasing and convex, then $\phi(X)$ is also IRQEO.*

*Proof.* Let $Y = \phi(X)$ be a non-negative, increasing and convex function. Then

$$g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))} = \frac{1}{\phi'(Q(u))q(u)}.$$

Using (3.30), the quantile entropy of order $Y_{i:n}$ is given by

$$\xi_{Y_{i:n}}(u) = -\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\int_u^1 g_i(p)(\log g_i(p))dp + \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\int_u^1 g_i(p)(\log q_Y(p))dp + \log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}dp. \quad (3.39)$$

Equation (3.39) equivalent to

$$\begin{aligned}\xi_{Y_{i:n}}(u) &= \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\left(\int_u^1 g_i(p)(\log q(p))dp - \int_u^1 g_i(p)(\log g_i(p))dp\right) \\ &\quad + \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\int_u^1 g_i(p)(\log\phi'(Q(p)))dp + \log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}dp \\ &= \xi_{X_{i:n}}(u) + \int_u^1 \frac{p^{i-1}(1-p)^{n-i}}{\bar{B}_u(i,n-i+1)}(\log\phi'(Q(p)))dp.\end{aligned}$$

Since $\xi_{X_{i:n}}(u)$ is IRQEO and $\phi$ is non-negative, increasing function, $Y = \phi(X)$ is also IRQEO, which completes the proof. □

**Example 3.6.** Let $X$ be a random variable with Pareto II distribution with quantile function $Q(u) = \alpha[(1-u)^{-\frac{1}{c}} - 1], \alpha, c > 0$, and let $Y = X^\beta, \beta > 0$. Then $Y$ has Burr type XII distribution with $Q(u) = \alpha^\beta[(1-u)^{-\frac{1}{c}} - 1]^\beta$. The non-negative increasing function $\phi(X) = X^{\beta}$, is convex. Then by Theorem 3.8, the Burr type XII distribution is IRQEO.

Now we find the upper bound of residual entropy of order statistics based on the quantile function $\xi_{X_{i:n}}(u)$ in terms of the entropy of order statistics based on the quantile function $\xi_{X_{i:n}}$.
From (3.30), we have

$$\xi_{X_{i:n}}(u) = -\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\left(\int_u^1 g_i(p)(\log g_i(p))dp - \int_u^1 g_i(p)(\log q(p))dp\right) + \log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)},$$

$$\begin{aligned} &= \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\xi_{X_{i:n}} + \int_0^u \frac{p^{i-1}(1-p)^{n-i}}{\bar{B}_u(i,n-i+1)}\left(\log g_i(p)(q(p))^{-1}\right)dp \\ &\quad + \log\frac{\bar{B}_u(i,n-i+1)}{B(i,n-i+1)}, \\ &\leq \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\xi_{X_{i:n}}. \end{aligned}$$

Equivalently

$$\xi_{X_{i:n}}(u) \leq \frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\xi_{X_{i:n}}. \tag{3.40}$$

These bounds are useful when the quantile density has no closed form or the computation of $\xi_{X_{i:n}}(u)$ is difficult.

# Chapter 4

# Cumulative residual Tsallis entropy measures- A quantile approach

## 4.1 Introduction

Rao et al. (2004) proposed an alternative measure of Shannon entropy $H(X)$ known as cumulative residual entropy (CRE), received a great attention among researchers as it possesses certain unique properties and applications in comparison with Shannon entropy (see Asadi & Zohrevand (2007), Navarro et al. (2010) and Toomaj et al. (2017)). Motivated with these, Sati & Gupta (2015) introduced a cumulative Tsallis entropy (CTE) measure, defined by

$$\eta_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^\infty (\bar{F}(x))^\alpha dx\right), \alpha > 0, \alpha \neq 1, \tag{4.1}$$

obtained by replacing the pdf $f(.)$ in $S_\alpha(X)$ by the survival function $\bar{F}(.)$, and studied various properties of its residual form. Recently, Rajesh & Sunoj (2016) proposed an alternative form of $\eta_\alpha(X)$, deriving some new results based on it.

[0]Results in this chapter have been published as entitled "A quantile-based study of cumulative residual Tsallis entropy measures" in the Journal "Physica A: Statistical Mechanics and its Applications" (See Sunoj et al. (2018)).

As many quantile functions used in applied works do not have tractable distribution functions, the statistical study of the properties of $\eta_\alpha(X)$ using (4.1) is difficult. Thus a formulation of the definition and properties of $\eta_\alpha(X)$ in terms of quantile functions is required. The quantile approach gives an alternative methodology in the study of cumulative Tsallis entropy and its dynamic (residual) measure. There are certain properties of quantile functions that are not shared by the distribution function approach. For instance, the sum of two quantile functions is again a quantile function; the product of two positive quantile functions is a quantile function and if $T(x)$ is a non-decreasing function of $x$, then $T(Q(u))$ is a quantile function (for more properties, see Nair et al. (2013)). Also, unlike the distribution function approach the quantile approach provides an explicit relationship between quantile-based cumulative residual Tsallis entropy and the quantile density function, that uniquely determines the distribution.

The organization of the chapter is given as follows. In Section 4.2, we introduce the quantile-based cumulative Tsallis entropy and its dynamic version and obtain certain characterization results and bounds based on it. In section 4.3, we extend the quantile-based cumulative residual Tsallis entropy in the context of order statistics and study its properties. In Section 4.4, we propose a nonparametric estimator for the quantile-based cumulative residual Tsallis entropy and a simulation study is carried out to illustrate the performance of the estimator. We also investigate the usefulness of the estimator for the real data set.

## 4.2 Quantile-based cumulative Tsallis entropy and its dynamic version

Then the quantile-based cumulative Tsallis entropy (QCTE) based on (4.1) is defined as

$$\tau^{\alpha}(X) = \frac{1}{\alpha-1}\left(1-\int_0^1(1-p)^{\alpha}q(p)dp\right), \alpha>0,\ \alpha\neq 1. \tag{4.2}$$

In terms of $H(u)$, $\tau^{\alpha}(X)$ becomes

$$\tau^{\alpha}(X) = \frac{1}{\alpha-1}\left(1-\int_0^1\frac{(1-p)^{\alpha-1}}{H(p)}dp\right),\ \alpha>0,\ \alpha\neq 1.$$

Thus by knowing either $Q(u)$ (or its quantile density $q(u)$) or $H(u)$ we can easily compute $\tau^{\alpha}(X)$. Table 4.1 provides the expressions of $\tau^{\alpha}(X)$ for different distributions. As pointed out in Chapter 1, there are some models that do not have a tractable distribution function while the quantile function exists. Govindarajulu and generalized lambda distributions given in Table 4.1 are examples of models that do not have distribution function whereas the quantile function $Q(.)$ exists, illustrating the importance of $\tau^{\alpha}(X)$ in (4.2). Further, for some models only the quantile density function $q(.)$ exists with no closed form for its distribution function (see Nair et al. (2013)). Accordingly, in the following example, we obtain $\tau^{\alpha}(X)$ for which only $q(.)$ exists.

**Example 4.1.** Suppose $X$ is distributed with quantile density function given by

$$q(u) = (1-u)^{-A}(-\log(1-u))^{-M},$$

TABLE 4.1: Quantile function and the cumulative Tsallis entropy of some distributions.

| Distribution | $Q(u)$ | $\tau^{\alpha}(X)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\alpha\lambda}\right)$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}}-1), \gamma, c > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{-1+\alpha c}\right)$ |
| Rescaled beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{R}{1+\alpha c}\right)$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b}{a(\alpha-1)+\alpha}\right)$ |
| Govindarajulu | $\theta+\sigma\{(\beta+1)u^{\beta}-\beta u^{\beta+1}\}, \theta, \sigma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\sigma\beta(\beta+1)\left(\frac{\Gamma(2+\alpha)\Gamma(\beta)}{\Gamma(2+\alpha+\beta)}\right)\right)$ |
| Generalized lambda | $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \epsilon R, \lambda_3 \epsilon Z^{+}$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda_2}\left(\frac{\lambda_4}{\lambda_4+\alpha}+\frac{\Gamma(1+\lambda_3)\Gamma(1+\alpha)}{\Gamma(1+\lambda_3+\alpha)}\right)\right)$ |

where $M$ and $A$ are real constants. Further, it contains several distributions which include Weibull when $A = 1, M = \frac{\lambda-1}{\lambda}$ with shape parameter $\sigma = k\lambda$, uniform when $A = 0, M = 0$, Pareto when $A > 1, M = 0$, and rescaled beta when $A < 1, M = 0$. Then the quantile-based cumulative Tsallis entropy is obtained as,

$$\tau^{\alpha}(X) = \frac{1}{(\alpha-1)}(1-(1-A+\alpha)^{M-1})\Gamma(1-M), M < 1$$

where $\Gamma(.)$ represents the Gamma function. It is to be noted that $\tau^{\alpha}(X) > 0 (< 0)$, for $\alpha > 1 (0 < \alpha < 1)$.

### 4.2.1 Quantile-based cumulative residual Tsallis entropy

For truncated data, measuring uncertainty using $\tau^{\alpha}(X)$ is not appropriate and a modified version of $\tau^{\alpha}(X)$ is essential for such residual random variable, $X_t = (X - t|X > t)$. In the distribution approach, Sati & Gupta (2015) proposed a cumulative residual Tsallis entropy for $X_t$ as

$$\eta_\alpha(X;t) = \frac{1}{\alpha-1}\left(1 - \int_t^\infty \left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^\alpha dx\right), \alpha > 0,\ \alpha \neq 1. \tag{4.3}$$

Using (4.3), quantile-based cumulative residual Tsallis entropy (QCRTE) is obtained as

$$\tau^\alpha(u) = \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^\alpha}\int_u^1 (1-p)^\alpha q(p)dp\right), \alpha > 0,\ \alpha \neq 1. \tag{4.4}$$

In terms of the mean residual quantile function $M(u)$, (4.4) becomes

$$\tau^\alpha(u) = \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^\alpha}\int_u^1 (1-p)^{\alpha-1}\left(M(p) - (1-p)M'(p)\right)dp\right),$$

or equivalently

$$\begin{aligned}\tau^\alpha(u) = &\frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^\alpha}\int_u^1 (1-p)^{\alpha-1}M(p)dp\right)\\ &+\frac{1}{\alpha-1}\left(\frac{1}{(1-u)^\alpha}\int_u^1 (1-p)^\alpha dM(p)\right).\end{aligned} \tag{4.5}$$

Applying integration by parts on the third term of (4.5), we obtain

$$\tau^\alpha(u) = \frac{1}{\alpha-1}\left(1 - M(u) + \frac{\alpha-1}{(1-u)^\alpha}\int_u^1 (1-p)^{\alpha-1}M(p)dp\right). \tag{4.6}$$

Now differentiating (4.6) with respect to $u$ and using (2.33) simplifies to

$$q(u) = (\alpha-1)\tau^{\alpha\prime}(u) + \frac{\alpha}{1-u}\left(1 - (\alpha-1)\tau^\alpha(u)\right), \tag{4.7}$$

TABLE 4.2: Quantile functions and quantile-based cumulative residual Tsallis entropy of certain distributions

| Distribution | $Q(u)$ | $\tau^{\alpha}(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\alpha\lambda}\right)$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}}-1), \gamma, c > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{-1+\alpha c}\right)(1-u)^{\frac{-1}{c}}$ |
| Rescaled beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{R}{1+\alpha c}\right)(1-u)^{\frac{1}{c}}$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b}{a(\alpha-1)+\alpha}\right)(1-u)^{\frac{-a}{a+1}}$ |
| Govindarajulu | $\theta+\sigma\{(\beta+1)u^{\beta}-\beta u^{\beta+1}\}, \theta, \sigma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\sigma\beta(\beta+1)}{(1-u)^{\alpha}}(-\beta_u(\beta, 2+\alpha)+\beta(2+\alpha, \beta)\right)$ |
| Generalized lambda | $\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}+(1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \epsilon R, \lambda_3 \epsilon Z^{+}$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda_2(1-u)^{\alpha}}\left(\frac{\lambda_4(1-u)^{\lambda_4+\alpha}}{\lambda_4+\alpha}+\frac{\Gamma(1+\lambda_3)\Gamma(1+\alpha)}{\Gamma(1+\lambda_3+\alpha)}\right)\right)$ $-\frac{u^{\lambda_3}}{\alpha-1}\Sigma_{k=0}^{\infty}\frac{(\lambda_3)_k(-\alpha)_k}{(1+\lambda_3)_k}\frac{u^k}{k!}$ |

where $\tau^{\alpha\prime}(u)=\frac{d}{du}\tau^{\alpha}(u)$. The relationship (4.7) determines the quantile density function from the quantile-based cumulative residual Tsallis entropy $\tau^{\alpha}(u)$. The relationship (4.7) is a unique characteristic of $\tau^{\alpha}(u)$ unlike the cumulative residual Tsallis entropy $\eta_{\alpha}(X;t)$ in (4.3), where no such explicit relationship exists between $\eta_{\alpha}(X;t)$ and the distribution function $F(.)$. Table 4.2 provides quantile functions of some important probability models and the corresponding $\tau^{\alpha}(u)$.

**Example 4.2.** When $X$ is distributed with quantile density function given by

$$q(u)=Ku^{\delta}(1-u)^{-(A+\delta)},$$

where $K, \delta$ and $A$ are real constants. Some of the members of this quantile density have non-monotone hazard quantile functions while some others have monotone hazard quantile functions. Further, it contains several well known distributions which include the exponential $\delta=0; A=1$ , Pareto $(\delta=0; A<1)$, rescaled beta $(\delta=0; A>1)$, the log-logistic distribution $(\delta=\lambda-1; A=2)$ and Govindarajulu distribution $(\delta=\beta-1; A=-\beta)$ with quantile function

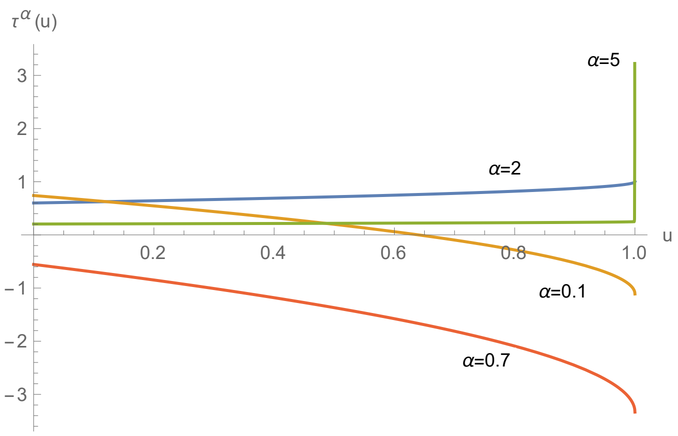



FIGURE 4.1: Plots of $\tau^{\alpha}(u)$ against $u$ with $\delta = 0, A = 0.5$

$(\theta + \sigma\{(\beta+1)u^{\beta} - \beta u^{\beta+1}\})$. Then $\tau^{\alpha}(u)$ becomes

$$\begin{aligned}\tau^{\alpha}(u) &= \frac{1}{\alpha-1}\left(1 - \int_u^1 \frac{(1-p)^{\alpha} K p^{\delta}(1-p)^{-(A+\delta)}}{(1-u)^{\alpha}} dp\right), \\ &= \frac{1}{\alpha-1}\left(1 - \frac{K\Gamma(1-A+\alpha-\delta)\Gamma(1+\delta)}{(1-u)^{\alpha}\Gamma(2-A+\alpha)} + \frac{KB_u(1+\delta, 1-A+\alpha-\delta)}{(1-u)^{\alpha}}\right),\end{aligned}$$

which is equivalent to

$$\tau^{\alpha}(u) = \frac{1}{\alpha-1}\left(1 - \frac{K}{(1-u)^{\alpha}}\left(-B_u(1+\delta, 1-A+\alpha-\delta) + \frac{\Gamma(1-A+\alpha-\delta)\Gamma(1+\delta)}{\Gamma(2-A+\alpha)}\right)\right).$$

Figure 4.1 provides the plots of $\tau^{\alpha}(u)$ and $u$ for $\alpha > 1$ and $0 < \alpha < 1$ respectively. Figure 4.1 indicates that at $\alpha > 1$ , $\tau^{\alpha}(u)$ increases as $u$ increases and at $0 < \alpha < 1$, $\tau^{\alpha}(u)$ decreases as $u$ increases.

**Definition 4.1.** *$X$ is said to have increasing (decreasing ) quantile-based cumulative residual Tsallis entropy (IQCRTE (DQCRTE)) if $\tau^{\alpha}(u)$ is increasing (decreasing) in $u \geq 0$.*

Now, we have obtained lower (upper) bounds to $\tau^{\alpha}(u)$ based on IQCRTE (DQCRTE). If $X$ is IQCRTE (DQCRTE), then

$$\tau^{\alpha}(u) \geq (\leq) \frac{1}{\alpha-1}\left(1-\frac{\alpha-1}{\alpha H(u)}\right),\ \alpha > 1\ (0<\alpha<1).$$

When $X$ is exponential with survival function $\bar{F}(x) = e^{-\lambda x}, x \geq 0, \lambda > 0$, then $\tau^{\alpha}(u) = \frac{\alpha\lambda-1}{\alpha\lambda(\alpha-1)} = C$, a constant. Thus exponential distribution belongs to both IQCRTE and DQCRTE classes or boundary of these two classes.

**Theorem 4.2.** *$\tau^{\alpha}(u) = C$, a constant iff $X$ is exponentially distributed.*

*Proof.* The proof directly follows from (4.7). □

**Theorem 4.3.** *If $X$ has increasing (decreasing) failure rate (IFR (DFR)) property, then $X$ is DQCRTE (IQCRTE) for $0<\alpha<1$ $(\alpha>1)$. But the converse need not be true.*

*Proof.* It is known that IFR (DFR) implies increasing (decreasing) mean residual life (DMRL), which is equivalent to increasing (decreasing) $H(u)$. Therefore $\tau_X^{\alpha}(u)$ is increasing (decreasing) in $u$ if $\alpha > 1$ $(0<\alpha<1)$.

On the other hand for rescaled beta (using Table 4.2), we have $\tau^{\alpha}(u) = \frac{1}{\alpha-1}(1-\frac{R}{1+\alpha c})(1-u)^{\frac{1}{c}}$, which is DQCRTE, while $H(u) = \frac{c}{R}(1-u)^{\frac{-1}{c}}$ is decreasing in $u$, completes the proof. □

**Theorem 4.4.** *The relationship $\tau^{\alpha}(u) = a + bu, a, b > 0$ holds iff $X$ follows a family of distributions with quantile function*

$$Q(u) = A(\alpha^2-1)u + \alpha(B(\alpha-1)-1)(\log(1-u)), A, B > 0.$$

*Proof.* The proof is similar to Theorem 4.2. □

**Theorem 4.5.** *Let $X$ and $Y$ be two non-negative absolutely continuous random variables with quantile functions $Q_X(u)$ and $Q_Y(u)$, hazard quantile functions $H_X(u)$ and $H_Y(u)$ respectively. If $X \leqslant_{QHR} Y$, then*

$$\tau_X^{\alpha}(u) \geqslant (\leqslant) \tau_Y^{\alpha}(u),$$

*for $\alpha > 1 (0 < \alpha < 1)$.*

*Proof.* When $X \leqslant_{QHR} Y$, we have $H_X(u) \geqslant H_Y(u)$, implies that for $\alpha > 1 (0 < \alpha < 1)$

$$-\int_u^1 \frac{(1-p)^{\alpha-1}}{H_X(u)} \geqslant (\leqslant) -\int_u^1 \frac{(1-p)^{\alpha-1}}{H_Y(u)}. \tag{4.8}$$

Therefore (4.8) becomes

$$\frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{\alpha}}\int_u^1 \frac{(1-p)^{\alpha-1}}{H_X(p)}dp\right) \geqslant (\leqslant) \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{\alpha}}\int_u^1 \frac{(1-p)^{\alpha-1}}{H_Y(p)}\right)dp \tag{4.9}$$

Hence the proof. □

**Theorem 4.6.** *If $X$ is increasing QCRTE and if $\phi(.)$ is non-negative, increasing and convex function, then $\phi(X)$ is also IQCRTE.*

*Proof.* Let $Y = \phi(X)$ be a non-negative, increasing and convex function. Then $g(y) = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))} = \frac{1}{\phi'(Q_X(u))q_X(u)}$.

So

$$\begin{aligned}\tau_Y^{\alpha}(u) &= \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha} q_Y(p)dp\right)\\ &= \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha}\phi'(Q_X(p))q_X(p)dp\right).\end{aligned}$$

Since $\tau_Y^{\alpha}(u)$ is IQCRTE and $\phi(.)$ is non-negative, increasing function, $Y=\phi(X)$ is also IQCRTE, which completes the proof. □

**Example 4.3.** Let $X$ be a random variable with Pareto II distribution with quantile function $Q(u)=\alpha[(1-u)^{-\frac{1}{c}}-1], \alpha, c>0$, and let $Y=X^{\beta}, \beta>0$. Then $Y$ has Burr type XII distribution with $Q(u)=\alpha^{\beta}[(1-u)^{-\frac{1}{c}}-1]^{\beta}$. The non-negative increasing function $\phi(X)=X^{\beta}$, is convex. Then by Theorem 4.6, the Burr type XII distribution is IQCRTE.

### 4.2.2 An alternate form of QCRTE

Recently, Rajesh & Sunoj (2016) proposed an alternative measure of cumulative Tsallis entropy as,

$$\zeta^{\alpha}(X)=\frac{1}{(\alpha-1)}\left(\mu-\int_0^{\infty}(\bar{F}(x))^{\alpha}\right)dx,\ \alpha>0,\ \alpha\neq 1, \tag{4.10}$$

where $\mu=E(X)$. Rajesh & Sunoj (2016) also studied $\zeta^{\alpha}(X)$ in (4.10) for the residual random variable $X_t; t>0$, given by

$$\zeta^{\alpha}(t)=\frac{1}{\alpha-1}\left(r(t)-\int_t^{\infty}\left(\frac{\bar{F}(x)}{\bar{F}(t)}\right)^{\alpha}dx\right),\ \alpha>0,\ \alpha\neq 1, \tag{4.11}$$

where $r(t)$ is the mean residual life function. Rajesh & Sunoj (2016) showed that unlike the cumulative residual Tsallis entropy introduced by Sati & Gupta Sati & Gupta (2015), $\zeta^{\alpha}(t)$ in (4.11) has some additional properties, such as simple relationships with other information and reliability measures. Motivated by this, in the present section we define the quantile version of cumulative Tsallis entropy based on $\zeta^{\alpha}(t)$. Using (4.10), an alternate form of quantile-based cumulative Tsallis entropy is defined as

$$\xi^{\alpha}(X) = \frac{1}{(\alpha-1)}\left(\mu - \int_0^1 (1-p)^{\alpha} q(p)dp\right), \ \alpha > 0, \ \alpha \neq 1.$$

If $X$ and $Y$ are independent, the two-dimensional version of $\xi^{\alpha}(X)$ becomes

$$\xi^{\alpha}(X,Y) = \mu_1 \xi^{\alpha}(Y) + \mu_2 \xi^{\alpha}(X) - (\alpha-1)\mu_1\mu_1 \xi^{\alpha}(X)\xi^{\alpha}(Y),$$

where $\mu_1 = E(X)$, $\mu_2 = E(Y)$, shows that $\xi^{\alpha}(X)$ is non-additive. Now the quantile-based cumulative residual Tsallis entropy based on $\xi^{\alpha}(X)$ becomes

$$\begin{aligned} \xi^{\alpha}(u) &= \frac{1}{\alpha-1}\left(M(u) - \frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha} q(p)dp\right), \ \alpha > 0, \ \alpha \neq 1 \quad (4.12) \\ &= \frac{1}{\alpha-1}\left(M(u) - 1 + (\alpha-1)\tau^{\alpha}(u)\right). \end{aligned}$$

The following theorem provides a simple relationship for finding the quantile-based cumulative residual Tsallis entropy $\xi^{\alpha}(u)$ using quantile mean residual life.

**Theorem 4.7.** *Let $X$ be a random variable with quantile MRLF $M(u)$, $0 < u < 1$. Then*

$$\xi^{\alpha}(u) = \frac{1}{(1-u)^{\alpha}} \int_u^1 (1-p)^{\alpha-1} M(p)dp.$$

*Proof.* Differentiating (2.30) with respect to $u$ we get

$$\frac{d}{du}((1-u)M(u)) = -(1-u)q(u).$$

Then quantile-based cumulative residual Tsallis entropy (4.12) reduces to,

$$\begin{aligned}
\xi^{\alpha}(u) &= \frac{1}{\alpha-1}\left(M(u) - \frac{1}{(1-u)^{\alpha}} \int_u^1 ((1-p)q(p))\,(1-p)^{\alpha-1}dp\right), \\
&= \frac{1}{\alpha-1}\left(M(u) + \frac{1}{(1-u)^{\alpha}} \int_u^1 \left(\frac{d}{dp}(1-p)M(p)\right)(1-p)^{\alpha-1}dp\right), \\
&= \frac{1}{(1-u)^{\alpha}} \int_u^1 (1-p)^{\alpha-1} M(p)dp.
\end{aligned}$$

Thus the proof. □

**Definition 4.8.** $X$ is said to have increasing (decreasing) cumulative residual Tsallis entropy if $\xi^{\alpha}(u)$ is increasing (decreasing) in $u$.

From (4.12) we have

$$(\alpha-1)\xi'^{\alpha}(u) = \left(\frac{M(u)}{1-u} - \frac{\alpha}{(1-u)^{\alpha+1}} \int_u^1 (1-p)^{\alpha} q(p)dp\right), \tag{4.13}$$

it is easy to show that if $X$ is increasing(decreasing) cumulative residual Tsallis entropy (IQCRTE) then $\eta_X^{\alpha}(u) \geq (\leq) \frac{M_X(u)}{\alpha}$. For exponential distribution

$\eta_X^{\alpha}(u) = \frac{1}{\lambda\alpha}$. Thus exponential distribution is the boundary of IQCRTE and DQCRTE classes.

**Theorem 4.9.** *The relationship $\xi^{\alpha}(u) = CM(u), C > 0$ holds iff X is distributed as Pareto II, exponential or rescaled beta according as $C\alpha \overset{\geq}{\underset{<}{=}} 1$.*

*Proof.* The "if "part of the theorem is straight forward. To prove the "only if "part let $\xi^{\alpha}(u) = CM(u), C > 0$ hold. From (4.12) we can write

$$CM(u) = \frac{1}{\alpha-1}(M(u) - \frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha}q(p)dp). \tag{4.14}$$

Taking derivatives on both sides of (4.14), we get

$$\begin{aligned} CM_X'(u) &= \frac{1}{\alpha-1}\left(M_X'(u) + q_X(u) - \frac{\alpha}{1-u}\int_u^1 \frac{(1-p)^{\alpha}q(p)}{(1-u)^{\alpha}}dp\right), \\ &= \frac{\alpha\eta_X(u) - M_X(u)}{1-u}. \end{aligned}$$

Again substitute $\eta_X^{\alpha}(u) = CM_X(u)$, we get

$$M_X(u)H_X(u) = \frac{C}{C-\alpha C+1}.$$

One can obtain the characterization easily using the relation

$$[H_X(u)]^{-1} = M_X(u) - (1-u)M_X'(u). \tag{4.15}$$

□

**Definition 4.10.** The random variable $X$ is said to have less quantile-based cumulative residual Tsallis entropy than the random variable $Y$ if $\xi_X^{\alpha}(u) \leqslant \xi_Y^{\alpha}(u)$. We denote $X \leqslant_{LQCRTE} Y$.

**Theorem 4.11.** *If $X \geq_{QHR} Y$ then $X \leqslant_{LQCRTE} Y$.*

*Proof.* From (4.12), we have

$$\begin{aligned}(\alpha-1)(\xi_X^{\alpha}(u)-\xi_Y^{\alpha}(u)) &= M_X(u)-M_Y(u)\\ &\quad+\frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha}\left(q_Y(p)-q_X(p)\right)dp.\end{aligned}$$

Since $X \geq_{QHR} Y$ implies $X \leqslant_{QMR} Y$ and $q_Y(u) \leqslant q_X(u)$, we obtain $\xi_X^{\alpha}(u) \leqslant \xi_Y^{\alpha}(u)$, completes the proof. □

## 4.3 Quantile-based cumulative residual Tsallis entropy of order statistics

Fashandi & Ahmadi (2012) have derived certain characterizations for symmetric distributions based on Renyi entropy of order statistics, $k$-record statistics and the FGM family of bivariate distributions. Gupta et al. (2014) proved some characterization results based on dynamic entropy of order statistics. Baratpour & Khammar (2016) defined Tsallis generalized entropy of order $\alpha$ of $i^{th}$ order statistic $X_{i:n}$ as

$$\begin{aligned} S_\alpha(X_{i:n}) &= \frac{1}{\alpha-1}(1-\int_0^\infty (f_{i:n}(x))^\alpha dx), \alpha>0, \alpha\neq 1, \\ &= \frac{1}{\alpha-1}(1-\int_0^\infty \frac{1}{\beta(i,n-i+1)}F(x)^{\alpha(i-1)}(1-F(x))^{\alpha(n-i)}f(x)^\alpha dx) \end{aligned} \tag{4.16}$$

However, the study of entropy of order statistics using quantile functions is of recent interest. Sunoj et al. (2017) introduced a quantile-based entropy of order statistics and study its properties. The quantile-based Tsallis entropy of $i^{th}$ order statistic is defined as,

$$T^\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left(1-\int_0^1 \left(\frac{1}{B(i,n-i+1)}\right)^\alpha p^{\alpha(i-1)}(1-p)^{\alpha(n-i)}(q(p))^{1-\alpha}dp\right) \tag{4.17}$$

Unlike (4.16), $T^\alpha(X_{i:n})$ in (4.17) will be more useful in cases we do not have a tractable distribution function but have a closed quantile function.

The cumulative Tsallis entropy of $i^{th}$ order statistic is defined as

$$\begin{aligned} \eta_\alpha(X_{i:n}) &= \frac{1}{\alpha-1}\left(1-\int_0^\infty (\bar{F}_{i:n}(x))^\alpha dx\right), \\ &= \frac{1}{\alpha-1}\left(1-\int_0^\infty \left(\frac{\bar{B}_{F(x)}(i,n-i+1)}{B(i,n-i+1)}\right)^\alpha\right) dx, \end{aligned} \tag{4.18}$$

where $\bar{B}_u(i,n-i+1) = \int_u^1 u^{i-1}(1-u)^{n-i}du$, is the incomplete beta function. The corresponding quantile version of the cumulative Tsallis entropy of order statistics (4.18) becomes,

$$\tau^\alpha(X_{i:n}) = \frac{1}{\alpha-1}\left(1-\int_0^1 \left(\frac{\bar{B}_p(i,n-i+1)}{B(i,n-i+1)}\right)^\alpha\right) q(p)dp. \tag{4.19}$$

In the case of a series system, the sample minimum $X_{1:n}$ is of importance. In

TABLE 4.3: Quantile function and quantile-based cumulative Tsallis entropy of first order statistic

| Distribution | $Q(u)$ | $\tau^{\alpha}_{X_{1:n}}$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{n\alpha\lambda}\right)$ |
| Pareto I | $\sigma(1-u)^{-\frac{1}{\beta}}, \sigma > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\sigma}{-1+n\alpha\beta}\right)$ |
| Rescaled beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{R}{-1+n\alpha c}\right)$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b}{n\alpha(a+1)-a}\right)$ |
| Govindarajulu | $\theta+\sigma\{(\beta+1)u^{\beta}-\beta u^{\beta+1}\}, \theta, \sigma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\sigma\beta(\beta+1)\left(\frac{\Gamma(2+n\alpha)\Gamma(\beta)}{\Gamma(2+n\alpha+\beta)}\right)\right)$ |
| Generalized lambda | $\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}+(1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \epsilon R, \lambda_3 \epsilon Z^+$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda_2}\left(\frac{\lambda_4}{\lambda_4+n\alpha}+\frac{\Gamma(1+\lambda_3)\Gamma(1+n\alpha)}{\Gamma(1+\lambda_3+n\alpha)}\right)\right)$ |

this case the quantile-based cumulative Tsallis entropy of first order statistic is of relevance, given by

$$\tau^{\alpha}(X_{1:n}) = \frac{1}{\alpha-1}\left(1-\int_0^1 (1-p)^{n\alpha} q(p)dp\right). \tag{4.20}$$

For the parallel systems with sample maximum $X_{n:n}$, then (4.19) turns to be

$$\tau^{\alpha}(X_{n:n}) = \frac{1}{\alpha-1}\left(1-\int_0^1 (1-p^n)^{\alpha} q(p)dp\right). \tag{4.21}$$

Table 4.3 provides some important quantile functions and the corresponding quantile-based cumulative Tsallis entropy of first order statistic (series system) $\tau^{\alpha}(X_{1:n})$.

### 4.3.1 Quantile-based cumulative residual Tsallis entropy of order statistics (CRTEO)

In the case of truncated data $\tau^{\alpha}(X_{i:n})$ is not useful for measuring the uncertainty. So we extend the same to the residual random variable $X_t$ as in Section 4.2. The corresponding cumulative residual Tsallis entropy of $i^{th}$ order statistic for $X_t$ is given by,

$$\eta_{\alpha}(X_{i:n};t) = \frac{1}{\alpha-1}\left(1-\int_t^{\infty}\left(\frac{\bar{F}_{i:n}(x)}{\bar{F}_{i:n}(t)}\right)^{\alpha}\right)dx. \tag{4.22}$$

In the quantile set up, (4.22) becomes

$$\tau^{\alpha}_{X_{i:n}}(u) = \frac{1}{\alpha-1}\left(1-\int_u^1\left(\frac{\bar{B}_p(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\right)^{\alpha} q(p)dp\right). \tag{4.23}$$

For the series systems, (4.23) reduces to

$$\tau^{\alpha}_{X_{1:n}}(u) = \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha}q(p)dp\right). \tag{4.24}$$

Differentiating (4.24) with respect to $u$ on both sides, we get

$$(1-u)q(u) = n\alpha\left(1-(\alpha-1)\tau_{X_{1:n}}(u)\right)+(\alpha-1)(1-u)\tau'_{X_{1:n}}(u),$$

which implies that

$$q(u) = \frac{n\alpha}{(1-u)}\left(1-(\alpha-1)\tau_{X_{1:n}}(u)\right)+(\alpha-1)\tau'_{X_{1:n}}(u). \tag{4.25}$$

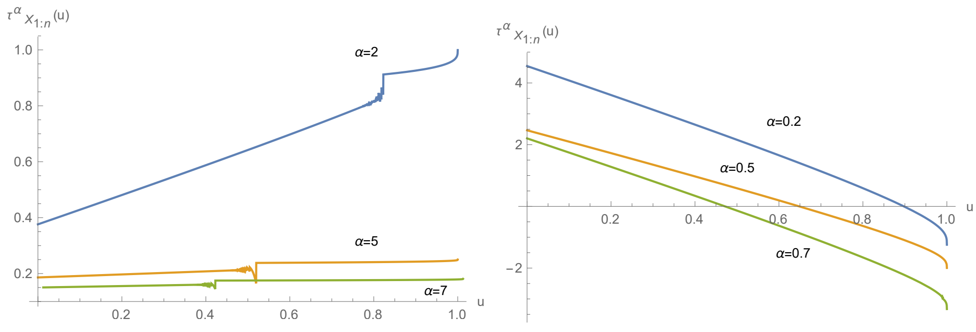
TABLE 4.4: Quantile function and quantile-based cumulative residual Tsallis entropy of first order statistic.

| Distribution | $Q(u)$ | $\tau^{\alpha}_{X_{1:n}}(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}(1-\frac{1}{n\alpha\lambda})$ |
| Pareto II | $\beta\left((1-u)^{-1/c}-1\right), \beta, c > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\beta(1-u)^{\frac{-1}{c}}}{n\alpha c-1}\right)$ |
| Rescaled beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), R, c > 0$ | $\frac{1}{\alpha-1}\left(1+\frac{R(1-u)^{\frac{1}{c}}}{n\alpha c+1}\right)$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b(1-u)^{\frac{-a}{a+1}}}{a(-1+n\alpha)+n\alpha}\right)$ |
| Govindarajulu | $\theta+\sigma\{(\beta+1)u^{\beta}-\beta u^{\beta+1}\}, \theta, \sigma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\sigma\beta(\beta+1)}{(1-u)^{n\alpha}}\left[-\beta_u(\beta, 2+n\alpha)+\frac{\Gamma(2+n\alpha)\Gamma(\beta)}{\Gamma(2+n\alpha+\beta)}\right]\right)$ |
| Generalized lambda | $\frac{1}{\lambda_2}\left(\lambda_3 u^{\lambda_3-1}+\lambda_4(1-u)^{\lambda_4-1}\right), \lambda_1, \lambda_2, \lambda_4 \epsilon R, \lambda_3 \epsilon Z^{+}$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda_2(1-u)^{n\alpha}}\left(\frac{\lambda_4(1-u)^{\lambda_4+n\alpha}}{\lambda_4+n\alpha}-\lambda_3\beta_u(\lambda_3, 1+n\alpha)\right)\right)$ $-\frac{1}{\lambda_2(\alpha-1)(1-u)^{n\alpha}}\frac{\Gamma(1+n\alpha)\Gamma(1+\lambda_3)}{\Gamma(1+\lambda_3+n\alpha)}$ |

Equation (4.25) shows that the quantile-based cumulative residual entropy of first order statistic uniquely determines the underlying distribution function. Table 4.4 gives different quantile functions and its corresponding cumulative residual Tsallis entropy of the first-order statistic (series system) $\tau^{\alpha}_{X_{1:n}}(u)$.

It is to be noted that the generalized lambda family does not have a closed form distribution function, while only the quantile function (see Table 4.4) exists. Figure 4.2 provides the plot of $\tau^{\alpha}_{X_{1:n}}(u)$ of generalized lambda family and $u$ for $\alpha > 1$ and $0 < \alpha < 1$ respectively. Figure 4.2(a) indicates that as $u$ increases $\tau^{\alpha}_{X_{1:n}}(u)$ also increases for $\alpha > 1$. However, Figure 4.2(b) explains the non-monotone nature of $\tau^{\alpha}_{X_{1:n}}(u)$.

**Example 4.4.** A random variable $X$ is distributed with quantile density function $q(u) = Ku^{\delta}(1-u)^{-(A+\delta)}$, where $K, \delta,$ *and* $A$ are real constants. Then the quantile-based cumulative residual Tsallis entropy of first order statistic is given




(a) $\alpha > 1$ (b) $0 < \alpha < 1$

FIGURE 4.2: Quantile-based cumulative residual Tsallis entropy of first order statistic against $u$.

by

$$\tau^{\alpha}_{X_{1:n}}(u) = \frac{1}{\alpha-1}\left(1+\frac{K}{(1-u)^{n\alpha}}\beta_u(1+\delta, 1-A+n\alpha-\delta)\right) + \frac{1}{\alpha-1}\left(\frac{\Gamma(1-A+n\alpha-\delta)\Gamma(1+\delta)}{\Gamma(2-A+n\alpha)}\right).$$

**Theorem 4.12.** *If X is increasing quantile cumulative residual Tsallis entropy IQCRTE (DQCRTE) and $\phi(.)$ is non negative, increasing and convex function, then $\phi(X)$ is also IQCRTE if $(\alpha > 1)$ and DQCRTE if $0 < \alpha < 1$.*

*Proof.* Let $Y = \phi(X)$, then $q_Y(u) = \phi^{'}(Q_X(u))q_X(u)$. We have

$$\begin{aligned}
\tau^{\alpha}_{Y_{1:n}}(u) &= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} q_Y(p)dp\right), \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} \phi^{'}(Q_X(p))q_X(p)dp\right), \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} (1+\phi^{'}(Q_X(p))-1)q_X(p)dp\right), \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} q_X(p)dp\right) \\
&\quad - \frac{1}{(1-u)^{n\alpha}(\alpha-1)}\int_u^1 (1-p)^{n\alpha} q_X(p)(\phi^{'}(Q_X(p)-1)dp \\
&= \tau^{\alpha}_{X_{1:n}}(u) + \frac{1}{(1-u)^{n\alpha}(\alpha-1)}\int_u^1 (1-p)^{n\alpha} q_X(p)(1-\phi^{'}(Q_X(p))dp.
\end{aligned}$$

Since $\tau^{\alpha}_{Y_{1:n}}(u)$ is IQCRTE and $\phi(.)$ is non-negative, increasing function, $Y = \phi(X)$ is also IQCRTE. Thus the proof. □

**Example 4.5.** Let $X$ be the exponentially distributed random variable with failure rate $\lambda$ and let $Y = X^{\frac{1}{\beta}}$, $\beta > 0$. Then $Y$ has the Weibull distribution with $Q(u) = \lambda^{\frac{-1}{\beta}}(-\log(1-u))^{\frac{1}{\beta}}$. The non-negative increasing function $\phi(X) = X^{\frac{1}{\beta}}, X > 0, \beta > 0$ is convex(concave) if $0 < \beta < 1 (\beta > 0)$. Hence by the Theorem 4.12 Weibull distribution is IQCRTEO if $0 < \beta < 1$ and $\alpha > 1$.

We now derive a relationship between cumulative residual Tsallis entropy of first order statistic and cumulative Tsallis entropy of first order statistic as

$$\begin{aligned}
\tau^{\alpha}_{X_{1:n}}(u) &= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} q(p)dp\right), \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\left(\int_0^1 (1-p)^{n\alpha} q(p)dp - \int_0^u (1-p)^{n\alpha} q(p)dp\right)\right), \\
&= \frac{1}{\alpha-1}\left(1 + \frac{1}{(1-u)^{n\alpha}}\left[(\alpha-1)\tau^{\alpha}_{X_{1:n}} - 1 + \int_0^u (1-p)^{n\alpha} q(p)dp\right]\right).
\end{aligned}$$

The constancy of quantile-based residual entropy of first order statistic is a characteristic property of exponential distribution (see Sunoj et al. (2017)). In the following theorem we prove that the exponential distribution holds the same property for the quantile-based cumulative residual Tsallis entropy of first order statistic.

**Theorem 4.13.** $\tau^{\alpha}_{X_{1:n}}(u) = c$*, a constant if and only if X is exponentially distributed.*

*Proof.* The "if"part is direct from Table 4.4 and the "only if"part follows from (4.25). □

Another simple form is a linear function. Hence, we prove a characterization for which $\tau^{\alpha}_{X_{1:n}}(u)$ assumes a linear form.

**Theorem 4.14.** *Let X be a random variable with quantile function $Q(u)$. For $\tau^{\alpha}_{X_{1:n}}(u) = a + bu, a, b > 0$ holds if and only if X follows a family of distributions with quantile function $Q(u) = ((\alpha-1)(b-n\alpha a)+n\alpha)(-\log(1-u)) - b(1-\alpha)(1+n\alpha)(u + \log(1-u))$.*

*Proof.* From (4.25), we get $H(u) = \frac{1}{(\alpha-1)(b-n\alpha a)+n\alpha-b(\alpha-1)(1+n\alpha)u}$, and using $Q(u) = \int_0^u \frac{1}{(1-p)H(p)}dp$, we obtain $Q(u) = ((\alpha-1)(b-n\alpha a)+n\alpha)(-\log(1-u)) + b(1-\alpha)(1+n\alpha)(-u-\log(1-u))$. The "if" part is direct from (4.24). □

Now we find bounds for quantile-based cumulative residual Tsallis entropy of first order statistic based on the hazard quantile function $H(u)$. These bounds are useful when the quantile density has no closed form or $\tau^{\alpha}_{X_{1:n}}(u)$ is difficult to compute.

**Theorem 4.15.** *Let $X$ be a continuous random variable with quantile function $Q(u)$ and hazard quantile function $H(u)$. If the quantile-based cumulative residual Tsallis entropy of first order statistic $\tau^{\alpha}_{X_{1:n}}(u)$ is increasing(decreasing) in $u$, then*

$$\tau^{\alpha}_{X_{1:n}}(u) \geqslant \frac{1}{\alpha-1}\left(1-\frac{1}{n\alpha H(u)}\right),$$

*when $\alpha > 1$ and*

$$\tau^{\alpha}_{X_{1:n}}(u) \leqslant \frac{1}{\alpha-1}\left(1-\frac{1}{n\alpha H(u)}\right),$$

*when $0 < \alpha < 1$.*

*Proof.* Assume that $\tau^{\alpha}_{X_{1:n}}(u)$ is increasing. So that the first derivative, $\tau^{\alpha'}_{X_{1:n}}(u) \geqslant 0$. Thus we obtained the bounds as

$$\tau^{\alpha}_{X_{1:n}}(u) \geqslant \frac{1}{\alpha-1}\left(1-\frac{1}{n\alpha H(u)}\right),$$

when $\alpha > 1$ and

$$\tau^{\alpha}_{X_{1:n}}(u) \leqslant \frac{1}{\alpha-1}\left(1-\frac{1}{n\alpha H(u)}\right),$$

when $0 < \alpha < 1$. □

**Remark 4.1.** For the Cox proportional hazards model, defined by $h_Y(x) = \theta h_X(x)$, $\theta > 0$, the quantile-based cumulative residual Tsallis entropy of first order statistic,

$$\begin{aligned}\tau^{\alpha}_{Y_{1:n}}(u) &= \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha} q_Y(p)dp\right),\\ &= \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-p)^{n\alpha}\frac{(1-p)^{\frac{1}{\theta}-1}}{\theta} q_X(1-(1-p)^{\frac{1}{\theta}})dp\right)\end{aligned} \tag{4.26}$$

Taking $v=(1-(1-p)^{\frac{1}{\theta}})$, (4.26) becomes

$$\begin{aligned}\tau^{\alpha}_{Y_{1:n}}(u) &= \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{n\alpha}}\int_{1-(1-u)^{\frac{1}{\theta}}}^1 (1-v)^{n\alpha\theta} q_X(v)dv\right)\\ &\leqslant \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{n\alpha}}\int_u^1 (1-v)^{n\alpha} q_X(v)dv\right)\\ &\leqslant \tau^{\alpha}_{X_{1:n}}(u).\end{aligned}$$

## 4.4 An estimator of quantile-based cumulative residual Tsallis entropy

In this section, we propose a non-parametric estimator for the quantile-based cumulative residual Tsallis entropy. Let $X_1, X_2, ..., X_n$ be random samples. We define the integral estimate of quantile-based cumulative residual Tsallis entropy as

$$\hat{\tau}(u) = \frac{1}{\alpha-1}\left(1-\frac{1}{(1-u)^{\alpha}}\int_u^1 (1-p)^{\alpha}\hat{q}(p)dp\right), \tag{4.27}$$

where $\hat{q}(u) = n(X_{(j)} - X_{(j-1)})$ for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, 2..., n$, (see Parzen (1979)).

TABLE 4.5: Bias, mean square error and the estimates of the empirical estimator $\hat{\tau}(u)$ with $\alpha = 5$.

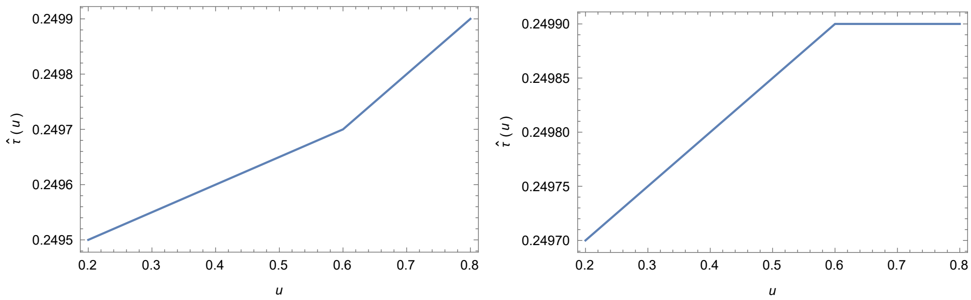
| | u | $\hat{\tau}(u)$ | Bias | MSE |
|---|---|---|---|---|
| n=100 | 0.2 | 0.2495 | 0.0224 | $5.0028 * 10^{-6}$ |
| | 0.4 | 0.2496 | 0.0125 | $1.5626 * 10^{-6}$ |
| | 0.6 | 0.2497 | 0.0052 | $2.9523 * 10^{-6}$ |
| | 0.8 | 0.2499 | 0.0013 | $1.6047 * 10^{-8}$ |
| n=500 | 0.2 | 0.2497 | 0.0227 | $1.0286 * 10^{-6}$ |
| | 0.4 | 0.2498 | 0.0127 | $3.2428 * 10^{-7}$ |
| | 0.6 | 0.2499 | 0.0056 | $6.2474 * 10^{-8}$ |
| | 0.8 | 0.2499 | 0.0013 | $3.6153 * 10^{-9}$ |

### 4.4.1 Simulation study

To study the performance of the estimator we carried out a series of 1000 simulations each of size $n(n = 100 \text{ and } 500)$ with different values of $u = 0.2, 0.4, 0.6, 0.8$ from Govindarajulu distribution with the quantile function in (2.50) for $\sigma = 1, \beta = 1$.

The simulation study shows that the estimate has small bias and negligible MSE. From Table 4.5 it is clear that MSE of the proposed empirical estimator decreases with increasing sample sizes.

Figure 4.3 indicates that $\hat{\tau}(u)$ increases as $u$ increases.




(a) $n = 100$ (b) $n = 500$

FIGURE 4.3: Plot of $\hat{\tau}(u)$

### 4.4.2 Application to real data

To illustrate the performance of the proposed estimator we use the data on failure times of 50 devices (Aarset data given in Aarset (1987)) arranged in order of magnitude. Nair et al. (2012) fitted Govindarajulu distribution to this Aarset data and the corresponding estimates of the parameters using $L$ moments are $\hat{\beta} = 2.0915$ and $\hat{\sigma} = 93.463$ respectively. To test the adequacy of the model we divide the data into 10 observations each by taking $u_i = \frac{i}{5}, i = 1, 2, ..., 5$ the corresponding $x$ values were computed using (2.50) with the $L$ moment estimates given above. The observed frequencies are $11, 8, 8, 13$ and 10 against the expected frequency of 10 in each class. The chi-square value is obtained as 1.8 which does not reject the hypothesis that the given data follows Govindarajulu distribution. $Q - Q$ plot also gives the adequacy of the model which is represented in Figure 4.4. Figure 4.5 indicates that the value of the estimator $\hat{\tau}(u)$ increases for $\alpha = 5$ and decreases for $\alpha = 0.5$.

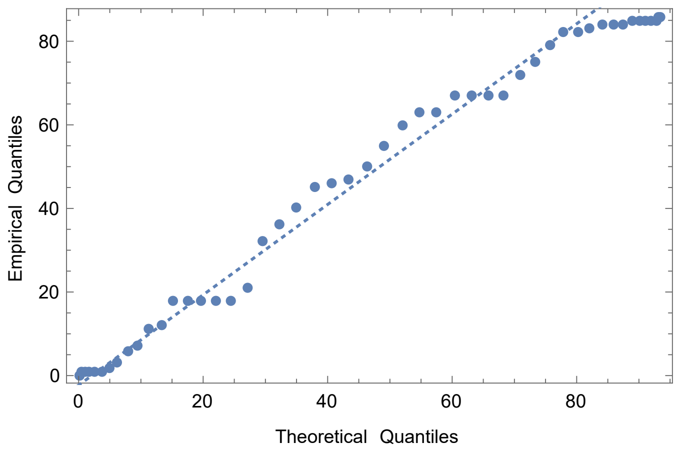
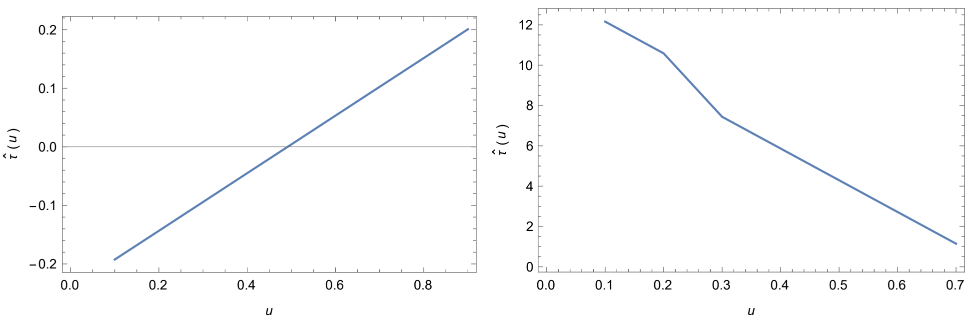



FIGURE 4.4: Q-Q plot for Aarset(1987) data




(a) $\alpha = 5$ (b) $\alpha = 0.5$

FIGURE 4.5: Plot of $\hat{\tau}(u)$ for real data

# Chapter 5

# Quantile-based reliability aspects of cumulative Tsallis entropy in past lifetime

## 5.1 Introduction

Shannon entropy (see Shannon (1948)) that plays an important role in measuring the average uncertainty of a random variable. However, in certain situations the Shannon entropy may not be suitable where some generalized versions are of importance. Various generalized entropy measures are available in the literature, which possesses many important properties such as smoothness, large dynamic range with respect to certain conditions, etc. that make them more flexible in practice. One popular generalization is the Tsallis entropy of order $\alpha$ given by Tsallis (1988), derived as a generalization of Boltzmann-Gibbs entropy, as discussed in Chapter 4. In the study of statistical mechanics, Tsallis entropy provides for a much broader view of how disorder arises in macroscopic systems.

[0]Results in this chapter have been published as entitled "Quantile-based reliability aspects of cumulative Tsallis entropy in past lifetime" in "Metrika-International Journal for Theoretical and Applied Statistics" (See Krishnan et al. (2019)).

Recently, Rao et al. (2004) introduced an alternative measure to the differential entropy known as the cumulative residual entropy (CRE), based on the survival function $\bar{F}(x) = P(X > x)$ instead of the probability density function $f(x)$ used in $S(X)$. CRE is considered to be more stable as survival function is more regular than the probability density function and possess more mathematical properties. Further, the distribution function exists even if the probability density function does not exists. Motivated by these, Sati & Gupta (2015) introduced a cumulative residual Tsallis entropy of order $\alpha$ and extended it to its dynamic form based on the residual lifetime and studied its properties. Rajesh & Sunoj (2016) introduced an alternative form of $\varphi_\alpha(X)$ and proved certain results useful in reliability modelling. Kumar (2017) obtained some characterization results based on the dynamic cumulative residual Tsallis entropy.

A wide variety of works on different entropy measures in context with past lifetime distributions have been studied extensively in the literature (see Di Crescenzo & Longobardi (2002), Di Crescenzo & Longobardi (2004),Di Crescenzo & Longobardi (2009), Sachlas & Papaioannou (2014), Di Crescenzo & Toomaj (2015), etc). Further our work facilitates the extension of the domain of application of cumulative Tsallis entropy in past lifetime to many flexible quantile functions that serve as useful lifetime models, that possesses no tractable distribution function. Accordingly, in the present Chapter, we further study on the cumulative Tsallis entropy in past lifetime using quantile functions.

This chapter is organized as follows. In Section 5.2, we propose the cumulative

Tsallis entropy and its dynamic form using quantile function and obtain certain characterizations and bounds of it. In Section 5.3, we study the proposed measures in the context of order statistics. In Section 5.4, we introduced a non-parametric estimator for the quantile-based cumulative Tsallis entropy in the past lifetime and carried out a simulation study to illustrate the performance of the estimator. The usefulness of the estimator for the real data set is also investigated.

## 5.2 Quantile-based cumulative Tsallis entropy in past lifetime

Let $t_X = [t - X | X < t]$ be a random variable describes the past lifetime of a system at age $t$. The two relevant ageing functions related to the past lifetime random variable $t_X$ are the reversed hazard rate function and the mean past lifetime or mean inactivity time (Ebrahimi & Pellerey (1995)), defined respectively as in (2.24) and (2.27). The quantile-based reversed hazard rate function (Nair & Sankaran (2009)) defined in (2.35) and (2.37)

The cumulative Tsallis entropy can also be defined using the distribution function (Sati & Gupta (2015)) is given by,

$$\bar{\eta}_\alpha(X) = \frac{1}{\alpha - 1}\left(1 - \int_0^\infty F^\alpha(x)dx\right), \alpha > 0, \alpha \neq 1. \tag{5.1}$$

TABLE 5.1: Quantile function and the quantile-based cumulative Tsallis entropy of some distributions.

| Distribution | $Q(u)$ | $\bar{\tau}_\alpha(X)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda(\alpha+1)(\alpha+2)}\right)$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}}-1), \gamma, c > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{c}B(\alpha+1,-1/c)\right)$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{R}{c}B(\alpha+1,1/c)\right)$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b}{a+1}B(\alpha+1,\frac{-a}{a+1})\right)$ |
| Power | $\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{1+\alpha\beta}\right)$ |
| Uniform | $a+(b-a)u, -\infty < a < b < \infty$ | $\frac{1}{\alpha-1}\left(1-\frac{(b-a)}{\alpha+1}\right)$ |
| Davies | $\frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, c > 0; \lambda_1, \lambda_2 > 0$ | $\frac{1}{\alpha-1}\left(1-c\lambda_{1B(\alpha+\lambda_1,1-\lambda_2)-c\lambda_2B(\alpha+\lambda_1+1,-\lambda_2)}\right)$ |
| Skew lambda | $\delta u^\lambda-(1-u)^\lambda, \delta, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\delta\lambda}{\alpha+\lambda}-\lambda B(\lambda,\alpha+1)\right)$ |
| Generalized lambda | $\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \in R, \lambda_3 \in Z^+$ | $\lambda_1+\frac{1}{\lambda_2}\left(\lambda_3B(\alpha+1,\lambda_3)+1\right)$ |
| Govindarajulu | $\sigma\left((\beta+1)u^\beta-\beta u^{\beta+1}\right), \sigma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\sigma\beta(\beta+1)}{(\sigma+\beta)(\sigma+\beta+1)}\right)$ |

Following Sankaran & Sunoj (2017), the corresponding quantile-based cumulative Tsallis entropy based on (5.1) becomes

$$\bar{\tau}_\alpha(X) = \frac{1}{\alpha-1}\left(1-\int_0^1 p^\alpha q(p)dp\right), \alpha > 0, \alpha \neq 1. \tag{5.2}$$

Table 5.1 provides some important quantile functions of distributions and its corresponding $\bar{\tau}_\alpha(X)$.

### 5.2.1 Quantile-based cumulative Tsallis entropy in past lifetime (QCTEP)

The cumulative Tsallis entropy function for the past lifetime $t_X$ is given by,

$$\bar{\eta}_\alpha(X,t) = \frac{1}{\alpha-1}\left(1-\int_0^t \left(\frac{F(x)}{F(t)}\right)^\alpha dx\right), \alpha > 0, \alpha \neq 1. \tag{5.3}$$

TABLE 5.2: Quantile function and the quantile-based cumulative Tsallis entropy in past lifetime of some distributions.

| Distribution | $Q(u)$ | $\bar{\tau}_\alpha(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{1}{\lambda u^\alpha B_u(\alpha+1,0)}\right)$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}}-1), \gamma, c > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{c}\frac{B_u(\alpha+1,\frac{-1}{c})}{u^\alpha}\right)$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{R}{c}\frac{B_u(\alpha+1,\frac{1}{c})}{u^\alpha}\right)$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$ | $\frac{1}{\alpha-1}\left(1-\frac{b}{a+1}\frac{B_u(1+\alpha)}{\frac{-a}{a+1}}\right)$ |
| Power | $\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\gamma}{1+\alpha\beta}u^{\frac{1}{\beta}}\right)$ |
| Uniform | $a+(b-a)u, -\infty < a < b < \infty$ | $\frac{1}{\alpha-1}\left(1-\frac{(b-a)u}{\alpha+1}\right)$ |
| Davies | $\frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, \lambda_1, \lambda_2 > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{c\lambda_1 B_u(\alpha+\lambda_1,1-\lambda_2)}{u^\alpha}-\frac{c\lambda_2 B_u(\alpha+\lambda_1+1,-\lambda_2)}{u^\alpha}\right)$ |
| Inverse Weibull | $(-\log u)^{-\lambda}, \lambda > 0$ | $\frac{1}{\alpha-1}(1-\frac{\lambda}{u^\alpha}(\alpha^\lambda\Gamma(-\lambda)+(-\log(u))^{-\lambda}(-\alpha\log(u))^\lambda(\Gamma(-\lambda,-\alpha\log(u))-\Gamma(-\lambda))))$ |
| Skew lambda | $\delta u^\lambda-(1-u)^\lambda, \delta, \lambda > 0$ | $\frac{1}{\alpha-1}\left(1-\frac{\lambda}{u^\alpha}(\frac{\alpha u^{\alpha+\lambda}}{\alpha+\lambda}+B_u(\alpha+1,\lambda))\right)$ |
| Generalized lambda | $\lambda_1+\frac{1}{\lambda_2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \in R, \lambda_3 \in Z^+$ | $\frac{1}{\alpha-1}\left(1-\frac{\frac{\lambda_3 u^{\alpha+\lambda_3}}{\alpha+\lambda_3}+\lambda_4 B_u(\alpha+1,\lambda_4)}{\lambda_2 u^\alpha}\right)$ |

Then the corresponding quantile-based cumulative Tsallis entropy function in the past lifetime using (5.3) is given by,

$$\bar{\tau}_\alpha(u) = \frac{1}{\alpha-1}\left(1-\int_0^u \left(\frac{p}{u}\right)^\alpha q(p)dp\right), \alpha > 0, \alpha \neq 1,\ 0 < u < 1. \tag{5.4}$$

Differentiating both sides of (5.4) with respect to $u$, we get

$$q(u) = \frac{\alpha}{u}\left(1-(\alpha-1)\bar{\tau}_\alpha(u)\right)-(\alpha-1)\bar{\tau}_\alpha^{'}(u), \tag{5.5}$$

where $\bar{\tau}_\alpha^{'}(u) = \frac{d}{du}\bar{\tau}_\alpha(u)$. Thus $\bar{\tau}_\alpha(u)$ uniquely determines the distribution function. Unlike $\bar{\eta}_\alpha(X,t)$, distributions which have no explicit distribution function can be easily modelled through the relationship (5.5). Table 5.2 provides the quantile functions and the corresponding $\bar{\tau}^\alpha(u)$ of some important distributions.

In terms of reversed hazard quantile function, $\bar{\tau}_\alpha(u)$ reduces to

$$\bar{\tau}_\alpha(u) = \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}(\Lambda(p))^{-1}dp\right).$$

Further, $\bar{\tau}_\alpha(u)$ can be expressed in terms of the reversed mean residual quantile function by,

$$\begin{aligned}\bar{\tau}_\alpha(u) &= \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}(R(p) + pR'(p))dp\right) \\ &= \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}R(p)dp - \frac{1}{u^\alpha}\int_0^u p^\alpha dR(p)\right). \end{aligned} \tag{5.6}$$

Applying integration by parts on the third term of (5.6), we get

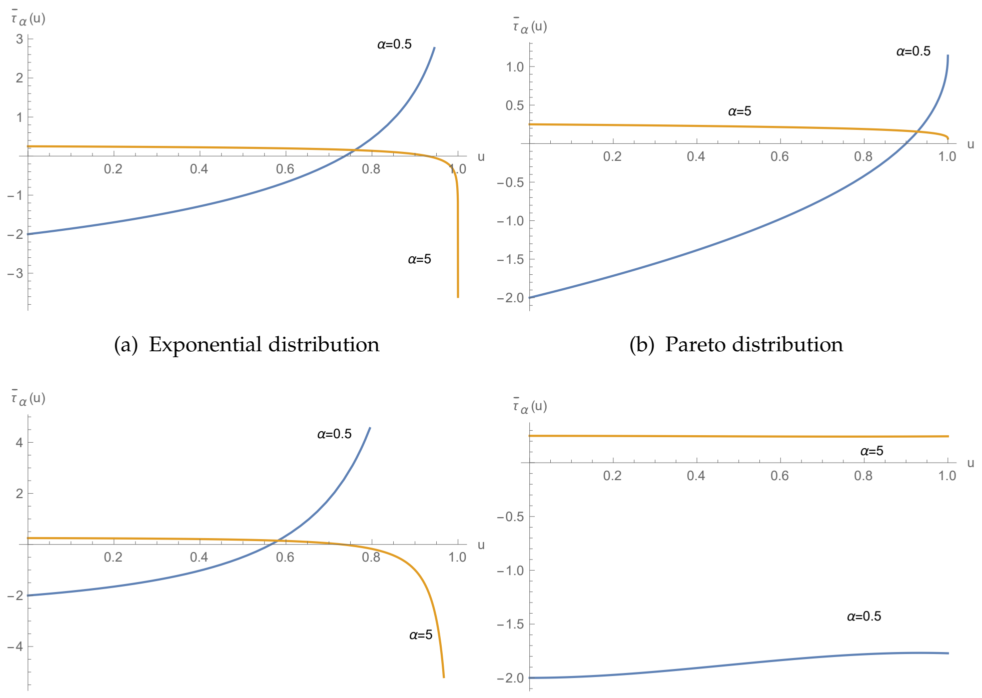
$$\bar{\tau}_\alpha(u) = \left(\frac{1 - R(u)}{\alpha - 1}\right) + \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}R(p)dp.$$

**Remark 5.1.** The quantile-based cumulative Tsallis entropy in past lifetime $\bar{\tau}_\alpha(u)$ is related to quantile-based cumulative Tsallis entropy $\tau_\alpha(X)$ and its residual form $\tau_\alpha(u)$ (Sunoj et al. (2018)) as follows:

$$\bar{\tau}_\alpha(u) = \frac{1}{\alpha - 1} + \frac{\tau_\alpha(X) - \tau_\alpha(u)}{u^\alpha},$$

where $\tau_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_0^1 (1-p)^\alpha q(p)dp\right)$ and $\tau_\alpha(u) = \frac{1}{\alpha-1}\left(1 - \frac{1}{(1-u)^\alpha}\int_u^1 (1-p)^\alpha q(p)dp\right)$.

The following characterization theorem considers a probability model that does not have a closed expression for its quantile function.




(a) Exponential distribution (b) Pareto distribution

(c) Rescaled beta distribution (d) Govindarajulu distribution

Figure 5.1: Plot of $\bar{\tau}_\alpha(u)$ against $u$ for different distributions.

**Theorem 5.1.** *The relationship* $\bar{\tau}_\alpha(u) = \frac{1}{\alpha-1}\left(1 - \frac{k}{u^\alpha}B_u(\alpha+\delta+1, -A-\delta+1)\right), 0 < u < 1$ *if and only if*

$$q(u) = ku^\delta(1-u)^{-(A+\delta)}, k > 0, \tag{5.7}$$

*where $A$ and $\delta$ are real constants.*

*Proof.* The proof directly follows from (5.5). □

**Remark 5.2.** The plots of $\bar{\tau}_\alpha(u)$ for different members of the model (5.7) are depicted in Figure 5.1.

It is often useful to identify a different class of probability models based on the monotone behaviours of uncertainty measures. Accordingly, we define the following non-parametric classes based on $\bar{\tau}_\alpha(u)$.

**Definition 5.2.** $X$ is said to have increasing (decreasing) quantile-based cumulative Tsallis entropy in past lifetime (IQCTEP (DQCTEP)) if $\bar{\tau}_\alpha(u)$ is increasing (decreasing) in $u$.

**Theorem 5.3.** *If $X$ is IQCTEP (DQCTEP), then for all $u \geq 0$,*

$$\bar{\tau}_\alpha(u) \leq (\geq) \frac{1}{\alpha - 1}\left(1 - \frac{1}{\alpha\Lambda(u)}\right), \alpha > 1 \ (0 < \alpha < 1),$$

*which provides upper (lower) bounds to $\bar{\tau}_\alpha(u)$ with respect to IQCTEP (DQCTEP).*

The next theorem proves the characterization of exponential distribution with support $(-\infty, 0)$ based on the constancy of $\bar{\tau}_\alpha(u)$.

**Theorem 5.4.** *$\bar{\tau}_\alpha(u) = c$, a constant if and only if $X$ has exponential distribution with negative support (see Block et al. (1998)).*

*Proof.* Assume that $\bar{\tau}_\alpha(u) = c$. Using (5.5), we obtain $Q(u) = \frac{1}{\alpha(1-c(\alpha-1))} \log u = \frac{1}{\lambda} \log u$, where $\lambda = \alpha(1 - c(\alpha - 1))$. Conversely, assume that $X$ follows exponential distribution with support $(-\infty, 0)$ having quantile function $Q(u) = \frac{1}{\lambda} \log u$. Now from (5.4), we get $\bar{\tau}_\alpha(u) = \frac{1}{\alpha-1}\left(1 - \frac{1}{\lambda\alpha}\right)$, a constant. □

In the following theorem, we prove that the IQCTEP property is closed under monotonic increasing convex transformation.

**Theorem 5.5.** *If $X$ is IQCTEP and if $\phi(.)$ is non-negative, increasing and convex function, then $\phi(X)$ is also IQCTEP for $0 < \alpha < 1$.*

*Proof.* Let $Y = \phi(X)$ be a non-negative, increasing and convex function. Then the pdf of $Y = \phi(X)$ is $g_Y(y) = \frac{f(x)}{\phi'(x)} = \frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))} = \frac{1}{\phi'(Q_X(u))q_X(u)}$, so that

$$\begin{aligned}
\bar{\tau}_\alpha^Y(u) &= \frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^\alpha q_Y(p)\right), \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^\alpha \phi'(Q_X(p))q_X(p)dp\right) \\
&= \frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^\alpha (\phi'(Q_X(p)) - 1 + 1)q_X(p)dp\right) \\
&= \bar{\tau}_\alpha^X(u) + \frac{1}{(\alpha-1)u^\alpha}\int_0^u p^\alpha(1 - \phi'(Q_X(p)))q_X(p)dp.
\end{aligned}$$

Since $\phi(.)$ is a convex, $\phi'(Q_X(p)) < \phi'(Q_X(u)), 0 < p < u$. Therefore

$$\bar{\tau}_\alpha^Y(u) \geq \frac{(1 - \phi'(Q_X(u)))}{\alpha - 1} + \phi'(Q_X(u))\bar{\tau}_\alpha^X(u),$$

which completes the proof. □

**Example 5.1.** Let $X$ have Pareto II distribution with $Q_X(u) = \gamma[(1-u)^{-1/c} - 1]$. When $Y = X^\beta, \beta > 0$ the distribution of $Y$ is Burr type XII distribution with quantile function $Q_Y(u) = \gamma^\beta[(1-u)^{-1/c} - 1]^\beta$. Using the above theorem, if $X$ is IQCTEP then $Y$ is also IQCTEP for $0 < \alpha < 1$.

**Theorem 5.6.** *Let $X$ be a non-negative absolutely continuous random variable such that $\bar{\tau}_\alpha(u) = cR(u)$. Then*

$$Q(u) = \frac{\alpha}{1 + c(\alpha-1)\alpha} + 2u + ck(\alpha-1)^{-1-c\alpha(\alpha-1)}(1 + \log u),$$

*where $c > 0$ is a constant.*

*Proof.* Assume that $\bar{\tau}_\alpha(u) = cR(u)$. Using (5.5), we get $c(\alpha-1)R'(u) = -(\Lambda(u))^{-1} + \alpha(1-(\alpha-1))\bar{\tau}^\alpha(u)$ which on simplification resolve to

$$R(u) = \frac{\alpha}{1+c(\alpha-1)\alpha} + \left(u + c(\alpha-1)^{-1-ck\alpha(\alpha-1)}\right).$$

Substituting it in (2.38) yields to $Q(u) = \frac{\alpha}{1+c(\alpha-1)\alpha} + 2u + ck(\alpha-1)^{-1-c\alpha(\alpha-1)}(1+\log u)$, the required form. □

In the next theorem, we show that the power function distribution can be characterized using $\bar{\tau}_\alpha(u)$.

**Theorem 5.7.** *Let $X$ be a non-negative continuous random variable with quantile function $Q(.)$ and reversed mean residual quantile function $R(.)$. Then for $\alpha > 1$,*

$$\bar{\tau}_\alpha(u) = \frac{1}{\alpha-1}\left(1 - cR(u)\right) \tag{5.8}$$

*where $c > \frac{1}{\alpha}$ if and only if $X$ has power function distribution.*

*Proof.* When $X$ follows power function distribution with $Q(u) = \gamma u^{\frac{1}{\beta}}, \beta, \gamma > 0$, we have $R(u) = \frac{\gamma}{\beta+1}u^{\frac{1}{\beta}}$ and,

$$\begin{aligned}\bar{\tau}_\alpha(u) &= \frac{1}{\alpha-1}\left(1 - \frac{\gamma}{1+\alpha\beta}u^{\frac{1}{\beta}}\right)\\ &= \frac{1}{\alpha-1}\left(1 - \frac{(\beta+1)R(u)}{1+\alpha\beta}\right).\end{aligned}$$

To prove the only part, assume that (5.8) holds. Then from (5.4), we obtain

$$\frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^\alpha q(p)dp\right) = \frac{1}{\alpha-1}(1 - cR(u)),$$

equivalently

$$\int_0^u p^\alpha q(p)dp = cu^\alpha R(u). \tag{5.9}$$

Differentiating (5.9) with respect to $u$,

$$u^\alpha q(u) = c\left(u^\alpha R'(u) + \alpha u^{\alpha-1} R(u)\right). \tag{5.10}$$

Using the relationships (2.25), (2.39) and (5.10), we have

$$\Lambda(u)R(u) = \frac{1-c}{c(\alpha-1)} = k,$$

where $k < 1$, a constant. Now, taking the derivative with respect to $u$ on both sides we get

$$\Lambda'(u)R(u) = -\Lambda(u)R'(u). \tag{5.11}$$

From (2.39) and $\Lambda(u)R(u) = k$, we have

$$\Lambda(u)R'(u) = \frac{1-k}{u}.$$

Then (5.11) becomes

$$\Lambda'(u)R(u) = \frac{k-1}{u}. \tag{5.12}$$

Differentiating $uq(u)\Lambda(u) = 1$ with respect to $u$, we get

$$\Lambda'(u) = -\Lambda(u)\left(\frac{1}{u} + \frac{q'(u)}{q(u)}\right),$$

so that (5.12) reduces to,

$$-\Lambda(u)\left(\frac{1}{u} + \frac{q'(u)}{q(u)}\right)R(u) = \frac{k-1}{u}.$$

On simplification we get

$$\log q(u) = k_2 \log u + \log k_1,$$

where $k_2 = \frac{1-2k}{k}$ and $k_1$ is the constant of integration. Equivalently,

$$q(u) = k_1 u^{k_2},$$

which is the quantile density function of power distribution. This completes the proof. □

Now, we extend the Theorem 5.7 to a general case by taking $c$ as a function of $u$.

**Theorem 5.8.** *If $X$ is a continuous random variable such that*

$$\bar{\tau}_\alpha(u) = \frac{1}{\alpha - 1}(1 - c(u)R(u)),$$

*then*

$$R(u) = \frac{1}{1-c(u)} e^{\int_0^u \frac{(\alpha c(p)-1)dp}{p(1-c(p))}}.$$

*Proof.* Assume that $\bar{\tau}_\alpha(u) = \frac{1}{\alpha-1}(1 - c(u)R(u))$ holds. It is equivalent to,

$$\frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u p^\alpha q(p)dp\right) = \frac{1}{\alpha-1}(1 - c(u)R(u)),$$

or

$$\int_0^u p^\alpha q(p)dp = c(u)R(u)u^\alpha.$$

Differentiating with respect to $u$ we obtain

$$q(u) - \frac{\alpha}{u}c(u)R(u) = c(u)R'(u) + R(u)c'(u),$$

using (2.39) we may write

$$R'(u)(u - uc(u)) + R(u)(1 - \alpha c(u) - uc'(u)) = 0,$$

which is a first-order differential equation in $R(u)$, on solving we get the required form. This completes the proof. □

Various stochastic orders are generally used to compare two random variables. We now consider the reversed hazard quantile function order (Nair et al. (2013)) to compare two random variables based on $\bar{\tau}_\alpha(u)$.

**Theorem 5.9.** *If $X \leq_{rhq} Y$ then $\bar{\tau}^X_\alpha(u) \leqslant (\geq)\bar{\tau}^Y_\alpha(u)$ for $\alpha > 1 (0 < \alpha < 1)$.*

*Proof.* Assume that $X \leq_{rhq} Y$, it implies $\Lambda_X(u) \leq \Lambda_Y(u)$. This is equivalent to, $\frac{-u^{\alpha-1}}{\Lambda_X(u)} \leq (\geq)\frac{-u^{\alpha-1}}{\Lambda_Y(u)}$, $\alpha > 1$ $(0 < \alpha < 1)$. Then

$$\frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u \frac{p^{\alpha-1}}{\Lambda_X(p)}dp\right) \leq (\geq)\frac{1}{\alpha-1}\left(1 - \frac{1}{u^\alpha}\int_0^u \frac{p^{\alpha-1}}{\Lambda_Y(p)}dp\right), \ \alpha > 1 \ (0 < \alpha < 1).$$

Thus $\bar{\tau}_{\alpha}^{X}(u) \leq (\geq) \bar{\tau}_{\alpha}^{Y}(u)$, $\alpha > 1 (0 < \alpha < 1)$. □

**Example 5.2.** Let $X$ and $Y$ be two random variables with quantile functions $Q_X(u) = \frac{u}{(1-u)^2}$ and $Q_Y(u) = 2u - u^2$, which has no closed form distribution function. $Q_X(u)$ is a special case of power-Pareto distributions due to Hankin & Lee (2006) and $Q_Y(u)$ is a special case of Govindarajulu distribution (Govindarajulu (1977)). The reversed hazard quantile functions of $X$ and $Y$ are given by $\Lambda_X(u) = \frac{(1-u)^3}{u(1-u)}$ and $\Lambda_Y(u) = \frac{1}{2u(1-u)}$. Then,

$$\frac{d}{du}\left(\frac{\Lambda_X(u)}{\Lambda_Y(u)}\right) = \frac{-(1-p)^3(5+3u)}{(1+u)^2} < 0,$$

which means $\frac{\Lambda_X(u)}{\Lambda_Y(u)}$ is a decreasing function of $u$. At $u = 0$, $\frac{\Lambda_X(u)}{\Lambda_Y(u)} = 1$, so that $\frac{\Lambda_X(u)}{\Lambda_Y(u)} \leq 1$. Thus $X \underset{rhq}{\leq} Y$. We also obtain

$$\begin{aligned} \bar{\tau}_{\alpha}^{X}(u) &= \frac{1}{\alpha-1}\left(1 - \frac{1}{u^{\alpha}}\int_0^u p^{\alpha}\left(\frac{(1-p)^2 + 2p(1-p)}{(1-p)^4}\right)dp\right) \\ &= \frac{1}{\alpha-1}\left(1 - \frac{u}{1-u} + \frac{\alpha}{u^{\alpha}}B_u(1+\alpha,0) + \frac{u^2(-1+\alpha-u\alpha)}{(u-1)^2} - \frac{\alpha(\alpha+1)B_u(2+\alpha,0)}{u^{\alpha}}\right), \end{aligned}$$

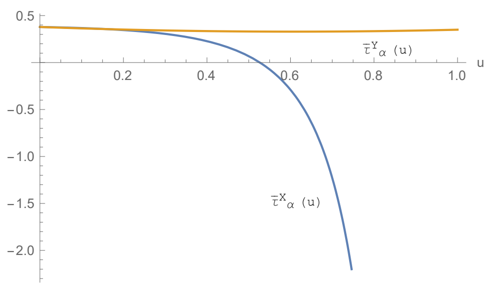
and

$$\bar{\tau}_{\alpha}^{Y}(u) = \frac{1}{\alpha-1}\left(1 - \frac{1}{u^{\alpha}}\int_0^u p^{\alpha}(2-2p)dp\right) = \frac{1}{\alpha-1}\left(1 - \frac{2}{u^{\alpha}}B_u(\alpha+1,2)\right).$$

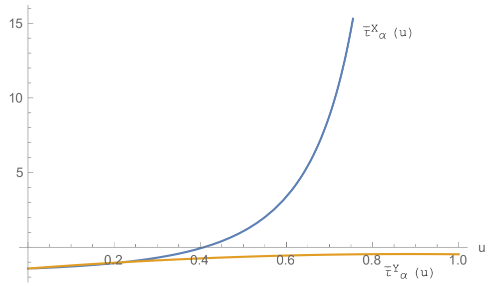
From Figure 5.2, it is clear that $\bar{\tau}_{\alpha}^{X}(u) \leq (\geq) \bar{\tau}_{\alpha}^{Y}(u),\ for\ \alpha > 1\ (0 < \alpha < 1)$.

**Definition 5.10.** $X$ is said to be smaller than $Y$ in QCTEP (written as $X \leq_{QCTEP} Y$), if $\bar{\tau}_{\alpha}^{X}(u) \leq \bar{\tau}_{\alpha}^{Y}(u)$ for all $u \in (0,1)$.

We use the following lemma by Nanda et al. (2014) to prove the next theorem.




(a) $\alpha > 1$




(b) $0 < \alpha < 1$

FIGURE 5.2: Plots of $\bar{\tau}_{\alpha}^{X}(u)$ and $\bar{\tau}_{\alpha}^{Y}(u)$ against $u$

**Lemma 5.11.** *Let $f(u,x) : (0,\infty) \times (0,\infty) \longrightarrow (0,\infty)$ and $g : (0,\infty) \longrightarrow (0,\infty)$ be any two functions. If $\int_u^{\infty} f(u,x)dx$ is increasing and $g(u)$ is increasing in $u$, then $\int_u^{\infty} f(u,x)g(x)dx$ is increasing in $u$, provided the integrals exist.*

**Theorem 5.12.** *Let $X$ and $Y$ be two random variables such that $X \leq_{QCTEP} Y$. Then for a non-negative convex function $\phi(.)$, $\phi(X) \leq_{QCTEP} \phi(Y)$.*

*Proof.* It is enough to show that

$$\int_0^u p^{\alpha}\phi'(Q_X(p))q_X(p)dp \geqslant \int_0^u p^{\alpha}\phi'(Q_Y(p))q_Y(p)dp. \tag{5.13}$$

Since $X \leq_{QCTEP} Y$, we have

$$\bar{\tau}_{\alpha}^{X}(u) \leqslant \bar{\tau}_{\alpha}^{Y}(u),$$

which is equivalent to

$$\int_0^u p^{\alpha}q_X(p)dp \geqslant \int_0^u p^{\alpha}q_Y(p)dp. \tag{5.14}$$

Thus, $q_X(u) \geqslant q_Y(u)$. Since $\phi(.)$ is increasing convex function, $\phi'(Q_X(u)) \geqslant \phi'(Q_Y(u))$. Now applying Lemma 1 on $\phi'(Q_X(u)) \geqslant \phi'(Q_Y(u))$ and (5.14), we get $\phi(X) \leqslant \phi(Y)$. □

Recently, Rajesh & Sunoj (2016) proposed an alternative measure of cumulative residual Tsallis entropy as in (4.10) and (4.11). Sunoj et al. (2018) studied the quantile versions of (4.10) and (4.11). Unlike the dynamic form of $\varphi_\alpha(X)$ due to Sati & Gupta (2015), $\zeta_\alpha(X,t)$ is a function of mean residual life function and hence more reliability properties were derived based on it. Motivated with this, an analogous cumulative Tsallis entropy in the past lifetime was recently proposed by Calì et al. (2017), defined by

$$\zeta_{\alpha*}(X,t) = \frac{1}{\alpha-1}\left(r(t) - \int_0^t \left(\frac{F(x)}{F(t)}\right)^\alpha dx\right), \alpha > 0, \alpha \neq 1. \tag{5.15}$$

Following (5.15), we define a quantile-based cumulative Tsallis entropy as an alternative to (5.1) and (5.3), obtained as

$$\bar{\bar{\xi}}_\alpha(X) = \frac{1}{\alpha-1}\left(\mu - \int_0^1 p^\alpha q(p)dp\right), \tag{5.16}$$

and for the past lifetime, (5.16) reduces to

$$\begin{aligned}\bar{\bar{\xi}}_\alpha(u) &= \frac{1}{\alpha-1}\int_0^u \left(\frac{p}{u} - \frac{p^\alpha}{u^\alpha}\right) q(p)dp \\ &= \frac{1}{\alpha-1}\left(R(u) - \frac{1}{u^\alpha}\int_0^u p^\alpha q(p)dp\right), \quad 0 \le u \le 1. \end{aligned} \tag{5.17}$$

Differentiating (5.17) with respect to $u$, we obtain

$$(\alpha-1)\bar{\xi}'_{\alpha}(u)=R'(u)-q(u)-\frac{\alpha}{u}\left((\alpha-1)\bar{\xi}_{\alpha}(u)-R(u)\right), \tag{5.18}$$

(5.18) uniquely determines the quantile function. The following theorem provides the upper (lower) bounds for $\bar{\xi}_{\alpha}(u)$ in terms of $R(u)$.

**Theorem 5.13.** *Let $X$ be a non-negative absolutely continuous random variable with quantile function $Q(.)$. Then $\bar{\xi}_{\alpha}(u)$ is increasing (decreasing), if and only if*

$$\bar{\xi}_{\alpha}(u)\leq(\geq)\frac{R(u)}{\alpha}, \textit{for all } 0\leq u\leq 1 \textit{ and } \alpha>1 (0<\alpha<1).$$

*Proof.* Assume that $\bar{\xi}_{\alpha}(u)$ is increasing (decreasing) in $u$, or $\bar{\xi}'_{\alpha}(u)\geq 0$. Using (5.18),

$$R'(u)-q(u)-\frac{\alpha}{u}\left((\alpha-1)\bar{\xi}_{\alpha}(u)-R(u)\right)\geq(\leq)0$$

Equivalently,

$$\begin{aligned}
\bar{\xi}_{\alpha}(u) &\leq(\geq) \frac{1}{\alpha-1}\left(\frac{uR'(u)-uq(u)}{\alpha}+R(u)\right)\\
&= \frac{1}{\alpha-1}\left(\frac{uR'(u)-(R(u)+uR'(u))}{\alpha}+R(u)\right)\\
&= \frac{R(u)}{\alpha},
\end{aligned}$$

proves the result. □

The next result will be useful for computing $\bar{\xi}_{\alpha}(u)$ when the functional form of $R(u)$ is known to us.

**Theorem 5.14.** *Let $X$ be a non-negative absolutely continuous random variable with quantile mean inactivity time $R(u)$, then*

$$\bar{\xi}_\alpha(u) = \frac{1}{u^\alpha} \int_0^u p^{\alpha-1} R(p) dp.$$

*Proof.* We have

$$uR(u) = \int_0^u pq(p)dp. \tag{5.19}$$

Differentiating both sides of (5.19) with respect to $u$ we get,

$$\frac{d}{du}(uR(u)) = \frac{d}{du} \int_0^u pq(p)dp = uq(u).$$

Using (5.17) and (5.19) we obtain

$$\begin{aligned}
\bar{\xi}_\alpha(u) &= \frac{1}{\alpha - 1}\left(R(u) - \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}(pq(p))dp\right) \\
&= \frac{1}{\alpha - 1}\left(R(u) - \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}\frac{d}{dp}(pR(p))dp\right) \\
&= \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}R(p)dp.
\end{aligned}$$

Hence the proof. □

**Example 5.3.** Let $X$ and $Y$ be non-negative absolutely continuous random variables with quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. Then $X$ and $Y$ satisfy proportional reversed hazards model (PRHM) when $Q_Y(u) = Q_X(u^{\frac{1}{\theta}}), \theta > 0$ or equivalently $q_Y(u) = \frac{1}{\theta} u^{\frac{1}{\theta}-1} q_X\left(u^{\frac{1}{\theta}}\right)$. Then $R_Y(u) = \frac{1}{\theta u} \int_0^u p^{\frac{1}{\theta}-1} q_X\left(p^{\frac{1}{\theta}}\right) dp$.

Assume that $X$ follows Govindarajulu distribution with quantile function

$$Q_X(u) = \alpha + \sigma\left((\beta+1)u^{\beta} - \beta u^{\beta+1}\right), \ \alpha, \sigma, \beta > 0, \ 0 \leq u \leq 1.$$

Then

$$\begin{aligned} R_Y(u) &= \frac{1}{\theta u}\int_0^u p^{\frac{1}{\theta}}\left(\sigma\beta(\beta+1)p^{\frac{\beta-1}{\theta}}(1-p^{\frac{1}{\theta}})\right)dp, \\ &= \sigma\beta(\beta+1)\left(\frac{u^{\frac{\beta}{\theta}}}{\beta+\theta} - \frac{u^{\frac{\beta+1}{\theta}}}{1+\beta+\theta}\right). \end{aligned}$$

Now using Theorem 5.14, we have

$$\begin{aligned} \bar{\xi}^{Y}_{\alpha}(u) &= \frac{1}{u^{\alpha}}\int_0^u p^{\alpha-1}R_Y(p)dp \\ &= \sigma\beta(\beta+1)\theta\left(\frac{u^{\frac{\beta}{\theta}}}{(\beta+\theta)(\beta+\alpha\theta)} - \frac{u^{\frac{\beta+1}{\theta}}}{(1+\beta+\theta)(1+\beta+\alpha\theta)}\right). \end{aligned}$$

**Remark 5.3.** If $\bar{\xi}_{\alpha}(u) = cR(u)$ then

$$Q(u) = exp[-\frac{1}{c}(1+\alpha(c(\alpha-1)-1)u)] - \frac{1}{c}\left(\log u + \alpha(c(\alpha-1)-1)\right).$$

**Remark 5.4.** If $\bar{\xi}_{\alpha}(u) = c(u)R(u)$ then $R(u) = exp[\int_0^u \frac{\alpha - pc'(p) - 1 - \alpha(\alpha-1)c(p)}{pc(p)}dp]$.

**Theorem 5.15.** *If $X$ is decreasing reversed hazard (DRHR) then $X$ is increasing (decreasing) quantile dynamic cumulative Tsallis entropy in past lifetime with respect to $\bar{\xi}_{\alpha}(u)$ for $0 < \alpha < 1$ $(\alpha > 1)$.*

*Proof.* *DRHR* $\implies$ *IRMR* (see Nanda et al. (2003)), which is equivalent to increasing $R(u)$. Therefore $\bar{\xi}_{\alpha}(u)$ is increasing(decreasing) in $u > 0$ for $0 < \alpha < 1(\alpha > 1)$. □

**Theorem 5.16.** *If $R(u) = c$, a constant then $\bar{\bar{\xi}}_\alpha(u)$ is also a constant.*

*Proof.* Let $R(u) = c$. Using Theorem 5.14 we obtain $\bar{\bar{\xi}}_\alpha(u) = \frac{c}{\alpha}$.

Hence the proof. □

**Theorem 5.17.** *If $R(u) = a + bu$, a linear function then $\bar{\bar{\xi}}_\alpha(u)$ is also linear.*

*Proof.* From Theorem 5.14, we have $\bar{\bar{\xi}}_\alpha(u) = \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}R(p)dp = \frac{1}{u^\alpha}\int_0^u p^{\alpha-1}(a + bp)dp = m + nu$, where $m = \frac{a}{\alpha}$ and $n = \frac{b}{\alpha+1}$. Thus the proof.

□

## 5.3 Quantile-based cumulative Tsallis entropy of order statistics in past lifetime

For past lifetime random variables, the reversed hazard rates are more important than the usual hazard rates and to study the failure pattern of reliability systems, it is more appropriate in studying the failure behaviour of parallel systems.

In analogy with (5.1), the cumulative Tsallis entropy of $i^{th}$ order statistic is defined as

$$\begin{aligned} \bar{\eta}_\alpha(X_{i:n}) &= \frac{1}{\alpha-1}\left(1 - \int_0^\infty F_{i:n}^\alpha(x)dx\right) \\ &= \frac{1}{\alpha-1}\left(1 - \int_0^\infty \left(\frac{B_{F(x)}(i, n-i+1)}{B(i, n-i+1)}\right)^\alpha dx\right). \end{aligned} \tag{5.20}$$

Then the corresponding quantile-based measure of (5.20) becomes (see also (5.2))

$$\bar{\tau}_\alpha(X_{i:n}) = \frac{1}{\alpha - 1}\left(1 - \int_0^1 \left(\frac{B_p(i, n-i+1)}{B(i, n-i+1)}\right)^\alpha q(p)dp\right). \tag{5.21}$$

For the sample minimum $X_{1:n}$,

$$\bar{\tau}_\alpha(X_{1:n}) = \frac{1}{\alpha - 1}\left(1 - \int_0^1 (-1)^\alpha (1-p)^{n\alpha} q(p)dp\right),$$

and for the sample maximum

$$\bar{\tau}_\alpha(X_{n:n}) = \frac{1}{\alpha - 1}\left(1 - \int_0^1 (-1)^\alpha p^{n\alpha} q(p)dp\right).$$

Since reversed hazard rates has an affinity to parallel system, we consider mainly the cumulative Tsallis entropy of $n^{th}$ order statistic. The reversed hazard quantile function and reversed mean residual quantile function (quantile-based mean inactivity time) for the $n^{th}$ order statistic are $\Lambda_{n:n}(u) = n\Lambda(u)$ and $R_{n:n}(u) = \frac{1}{u^n}\int_0^u p^n q(p)dp$ respectively. Then

$$uR'_{n:n}(u) + nR_{n:n}(u) = (\Lambda(u))^{-1}. \tag{5.22}$$

Hence the cumulative Tsallis entropy of $i^{th}$ order statistic in past lifetime is given by

$$\begin{aligned}\bar{\eta}_\alpha(X_{i:n}, t) &= \frac{1}{\alpha - 1}\left(1 - \int_0^t \frac{F^\alpha_{i:n}(x)}{F^\alpha_{i:n}(t)}dx\right) \\ &= \frac{1}{\alpha - 1}\left(1 - \int_0^t \left(\frac{B_{F(x)}(i, n-i+1)}{B_{F(t)}(i, n-i+1)}\right)^\alpha dx\right).\end{aligned} \tag{5.23}$$

In terms of quantile function, (5.23) becomes

$$\bar{\tau}_{\alpha}^{X_{i:n}}(u) = \frac{1}{\alpha - 1}\left(1 - \int_0^u \left(\frac{B_p(i, n-i+1)}{B_u(i, n-i+1)}\right)^{\alpha} q(p)dp\right). \tag{5.24}$$

For convenience, we may denote $\bar{\tau}_{\alpha}^{X_{i:n}}(u) = \bar{\chi}_{i:n}(u)$. Using (5.24), the cumulative Tsallis entropy in past lifetime for series and parallel systems are obtained respectively as

$$\bar{\chi}_{1:n}(u) = \frac{1}{\alpha - 1}\left(1 - \frac{1}{(1-u)^{n\alpha}}\int_0^u (1-p)^{n\alpha} q(p)dp\right), \tag{5.25}$$

and

$$\bar{\chi}_{n:n}(u) = \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{n\alpha}}\int_0^u p^{n\alpha} q(p)dp\right). \tag{5.26}$$

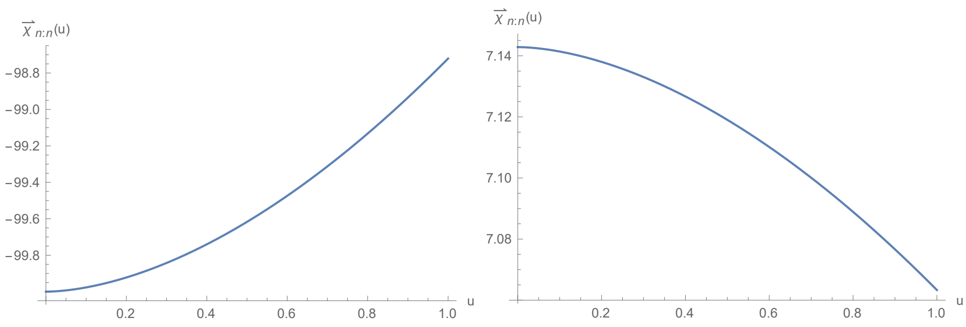
Differentiating both sides of (5.26) with respect to u, we get

$$(\alpha - 1)\frac{d}{du}(\bar{\chi}_{n:n}(u)) + q(u) = \frac{n\alpha}{u}(1 - (\alpha - 1)\bar{\chi}_{n:n}(u)). \tag{5.27}$$

Thus (5.27) uniquely determines the quantile function for a parallel system. The following example illustrate this.

**Example 5.4.** Let $X$ be a the generalized lambda distribution with quantile function

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2}\left(u^{\lambda_3} - (1-u)^{\lambda_4}\right), \lambda_1, \lambda_2, \lambda_4 \in R, \lambda_3 \in Z^{+}.$$




(a) $0 < \alpha < 1$ (b) $\alpha > 1$

FIGURE 5.3: Plot of $\bar{\chi}_{n:n}(u)$ against $u$.

Then the quantile-based cumulative Tsallis entropy in past lifetime for the parallel system is given by

$$\bar{\chi}_{n:n}(u) = \frac{1}{\alpha - 1}\left(1 - \frac{\frac{\lambda_3 u^{n\alpha+\lambda_3}}{n\alpha+\lambda_3} + \lambda_4 B_u\left(n\alpha+1, \lambda_4\right)}{\lambda_2 u^{n\alpha}}\right),$$

and displayed in Figure 5.3.

The following theorem gives the bounds for $\bar{\chi}_{n:n}(u)$, which may be useful when the computation of $\bar{\chi}_{n:n}(u)$ is difficult.

**Theorem 5.18.** *If $\psi_n^{\alpha}(u)$ is increasing then*

$$\bar{\chi}_{n:n}(u) \leq (\geq) \frac{1}{\alpha - 1}\left(1 - \frac{1}{\alpha \Lambda_{n:n}(u)}\right), \ for\ \alpha > 1\ (0 < \alpha < 1).$$

**Theorem 5.19.** *Let $X$ be a non-negative continuous random variable with quantile function $Q(.)$ and the quantile-based mean inactivity time for $n^{th}$ order statistic $R_{n:n}(.)$, then*

$$\bar{\chi}_{n:n}(u) = \frac{1}{\alpha - 1}\left(1 - cR_{n:n}(u)\right) \tag{5.28}$$

*if and only if X has power distribution.*

Next remark can be used for finding the quantile-based cumulative past Tsallis entropy of a parallel system using $\bar{\tau}_{\alpha}^{X}(u)$.

**Remark 5.5.** Quantile-based cumulative Tsallis entropy in the past lifetime of $n^{th}$ order statistic can be obtained using the following relation:

$$\bar{\chi}_{n:n}(u) = \left( \frac{n\alpha - 1}{\alpha - 1} \right) \bar{\tau}_{n\alpha}^{X}(u). \tag{5.29}$$

**Definition 5.20.** $X_{i:n}$ is smaller than $Y_{i:n}$ in dispersive order, $X_{i:n} \leq_{disp} Y_{i:n}$ if $F_{Y_{i:n}}^{-1} F_{X_{i:n}}(x) - x$ is increasing in $x$.

Since $F_{Y_{i:n}}^{-1} F_{X_{i:n}}(x) = F_{Y}^{-1} F_{X}(x)$ (see Barlow & Proschan (1975) ). We say that $X_{i:n} \leq_{disp} Y_{i:n}$ if $Q_Y(u) - Q_X(u)$ is increasing in $u$.

**Theorem 5.21.** *If* $X_{i:n} \leq_{disp} Y_{i:n}$ *then* $\bar{\chi}_{n:n}^{X}(u) \geq (\leq) \bar{\chi}_{n:n}^{Y}(u),\ for\ \alpha > 1\ (0 < \alpha < 1)$.

*Proof.* Assume that $X_{i:n} \leq_{disp} Y_{i:n}$. This implies that $q_Y(u) \geq q_X(u)$, from (5.26),

$$\begin{aligned} \bar{\chi}_{n:n}^{X}(u) &= \frac{1}{\alpha - 1} \left( 1 - \frac{1}{u^{n\alpha}} \int_0^1 p^{n\alpha} q_X(p) dp \right) \\ &\geq \frac{1}{\alpha - 1} \left( 1 - \frac{1}{u^{n\alpha}} \int_0^1 p^{n\alpha} q_Y(p) dp \right),\ for\ \alpha > 1. \end{aligned}$$

Hence the proof. For $0 < \alpha < 1$ the proof is similar by inverting the inequalities. □

**Theorem 5.22.** $\bar{\chi}_{n:n}(u) = C$ *if and only X has an exponential distribution with support* $(-\infty, 0)$.

*Proof.* Assume that $\bar{\chi}_{n:n}(u) = C$, a constant. This implies that

$$\frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{n\alpha}}\int_0^u p^{n\alpha} q(p)dp\right) = C,$$

equivalently

$$\int_0^u p^{n\alpha} q(p)dp = K u^{n\alpha}, \tag{5.30}$$

where $K = C(\alpha - 1) + 1$. Differentiating both sides (5.30), we get $\Lambda(u) = K^*$, a constant, thus exponential distribution with support $(-\infty, 0)$ (see Sunoj et al. (2013)). □

**Theorem 5.23.** *If X is increasing quantile-based cumulative Tsallis entropy of nth order statistics in past lifetime and $\phi(.)$ is non-negative, increasing and convex function, then $\phi(X)$ is also increasing quantile-based cumulative Tsallis entropy of nth order statistics in past lifetime for $0 < \alpha < 1$.*

*Proof.*

$$\begin{aligned}
\bar{\chi}^Y_{n:n}(u) &= \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{n\alpha}}\int_0^u p^{n\alpha} q_Y(p)dp\right) \\
&= \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{n\alpha}}\int_0^u p^{n\alpha} \phi'(Q_X(p))q_X(p)dp\right) \\
&= \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{n\alpha}}\int_0^u p^{n\alpha} (1 + \phi'(Q_X(p)) - 1)q_X(p)dp\right) \\
&= \bar{\chi}^X_{n:n}(u) - \frac{1}{(\alpha - 1)u^{n\alpha}}\int_0^u p^{n\alpha}(1 - \phi'(Q_X(p)))q_X(p)dp. \quad (5.31)
\end{aligned}$$

Since $X$ is increasing quantile-based cumulative Tsallis entropy of $n$th order statistics in past lifetime and $\phi(.)$ is increasing, from (5.31) we can say that $\phi(X)$

is also increasing quantile-based cumulative Tsallis entropy of $n^{th}$ order statistics in past lifetime for $0 < \alpha < 1$. □

## 5.4 Estimation of quantile-based cumulative Tsallis entropy in past lifetime

In this section, we present a non-parametric estimator for the quantile-based cumulative Tsallis entropy in past lifetime. We define the integral estimate as,

$$\hat{\hat{\tau}}(u) = \frac{1}{\alpha - 1}\left(1 - \frac{1}{u^{\alpha}}\int_0^u p^{\alpha}\hat{q}(p)dp\right), \tag{5.32}$$

where $\hat{q}(u) = n(X_{(j)} - X_{(j-1)})$ for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, 2, ..., n$.

### 5.4.1 Simulation study

To asses the performance of the proposed measure we perform a simulation study of 1000 samples of sizes $n = 100, 500$ from 2.50 with $\sigma = 1, \beta = 1$ for values of $u = 0.2, 0.4, 0.6, 0.8$. We present the simulation results in Table 5.3.

### 5.4.2 Data Analysis

To illustrate the estimation procedure, we consider the data set of the first external leakage of 32 centrifugal pumps. We fitted Govindarajulu distribution to this data. For estimating the parameters we have used the method of $L-$ moments

TABLE 5.3: Bias, mean square error and the estimates of the empirical estimator $\hat{\bar{\tau}}(u)$ with $\alpha = 5$.

| | u | $\hat{\bar{\tau}}(u)$ | Bias | MSE |
|---|---|---|---|---|
| n=100 | 0.2 | 0.2499 | 0.0137 | $1.8678 * 10^{-6}$ |
| | 0.4 | 0.2497 | 0.0126 | $4.6801 * 10^{-6}$ |
| | 0.6 | 0.2496 | 0.0239 | $5.7122 * 10^{-6}$ |
| | 0.8 | 0.2495 | 0.0205 | $4.1889 * 10^{-6}$ |
| n=500 | 0.2 | 0.2499 | 0.0137 | $3.7782 * 10^{-7}$ |
| | 0.4 | 0.2499 | 0.0218 | $9.4952 * 10^{-7}$ |
| | 0.6 | 0.2498 | 0.0241 | $1.1649 * 10^{-6}$ |
| | 0.8 | 0.2498 | 0.0208 | $8.6345 * 10^{-7}$ |

by equating the first two $L-$moments of the distribution with the sample counterparts. The first two sample $L-$ moments are $l_1 = 5024.72$ and $l_2 = 1644.68$. Using the equations

$$l_1 = \frac{2\sigma}{\beta + 2}$$

$$l_2 = \frac{2\sigma\beta}{(\beta + 2)(\beta + 3)}$$

we obtain $\hat{\sigma} = 8692.16, \hat{\beta} = 1.4598$. To test the adequacy of the model $\bar{\tau}_0(u)$, we use the chi-square test of goodness of fit. Dividing the data into 4 groups of 8 observations each and taking

$$u_i = \frac{i}{4}, i = 1, 2, ..., 4$$

the corresponding $x$ values were computed using (2.50) with the estimates given above. The observed frequencies in the 4 classes were $6, 13, 8$ and 5 against the expected frequency of 8 in each class. Thus the chi-square value of 4.75 obtained here does not reject the hypothesis that the data fits Govindarajulu distribution.

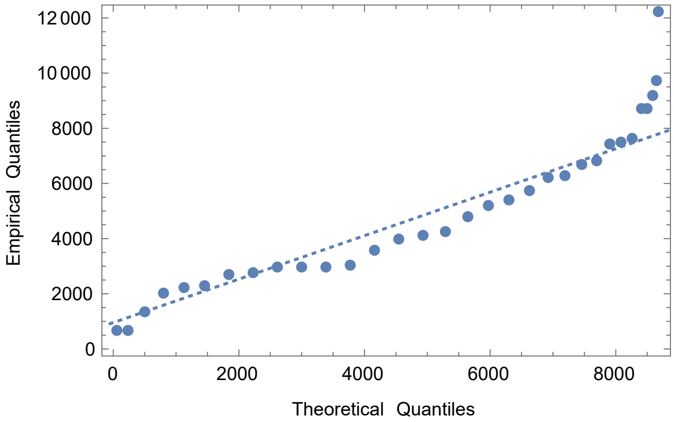
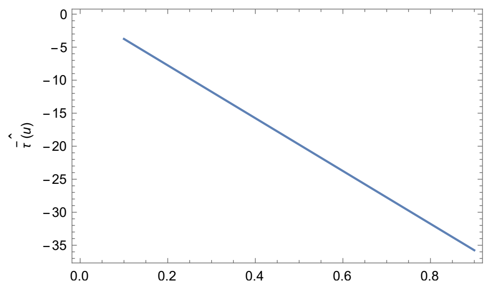
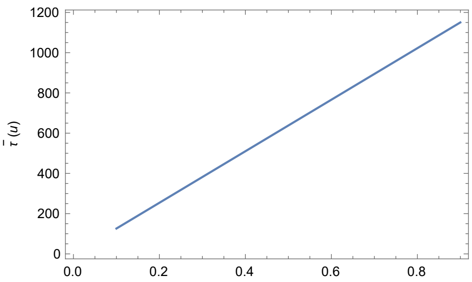



FIGURE 5.4: Q-Q plot for the first external leakage of 32 centrifugal pumps



(a) $\alpha = 5$



(b) $\alpha = .5$

FIGURE 5.5: Plot of $\hat{\bar{\tau}}(u)$ for real data

Figure 5.4 gives the $Q - Q$ plot which also shows the adequacy of the model.

Figure 5.5 indicates that the estimates of quantile-based cumulative Tsallis entropy in past lifetime show a decreasing trend for $\alpha = 5$ and an increasing trend for $\alpha = 0.5$.

# Chapter 6

# Reliability properties of extropy for residual and past lifetime random variable

## 6.1 Introduction

Extropy is a recent addition to the family of information measures proposed by Ayres & Martinas (1995) as the complementary dual of the Shannon's entropy $H_X = -\sum_{i=1}^{n} p_i \log p_i$, in the case of discrete random variable $X$ with $p_i = P(X = x_i)$. Lad et al. (2015) point out that as a measure of uncertainty extropy is invariant under permutations and monotonic transformations, maximum extropy distribution is uniform and satisfy Shannon's first and second axioms. As $X$ increases in such a way that $\sum_{i=1}^{n} p_i$ decreases to zero, $\sum_{i=1}^{n}(1-p_i)\log(1-p_i)$ is well approximated by

$$-\sum_{i=1}^{n}(1-p_i)(-p_i) = 1 - \sum_{i=1}^{n} p_i^2.$$

Also $\lim_{\Delta x \longrightarrow 0} \left(\frac{J_X - 1}{\Delta x}\right) = -\frac{1}{2}\int f^2(x)dx$. In statistical analysis, the discrete approximation has been used much earlier as repeat rate of a distribution (Good (1979))

[0]Results in this chapter have been accepted for publication entitled "Some reliability properties of extropy for residual and past lifetime random variables" in "Journal of the Korean Statistical Society".

and a Gini index of homogeneity is that widely used in database applications. It is also an indicator of the degree of skewness of the data.

The role of information measures relating to the residual and past lifetime in reliability modelling has been extensively investigated during the last two decades, starting from the works of Muliere et al. (1993), Ebrahimi (1996) and Di Crescenzo & Longobardi (2002) with reference to Shannon's entropy. In the meantime the variant approaches of employing the quantile version of various entropies were also introduced with the objectives of providing alternative methodology, new results and different methods of stochastic comparisons. For details, we refer to Sunoj & Sankaran (2012), Yu & Wang (2013), Nanda et al. (2014), Sunoj et al. (2017) and Qiu (2019).

The objective of the present work is to investigate some new aspects of residual and past extropy functions using the distribution function and quantile function approaches. It is observed that that the extropy being always negative, it is not valid for the entire parameter space. Further, we need to examine whether the two extropy functions determine the life distribution uniquely, a problem that has not been addressed earlier. The quantile function approach, enables us to establish some results which are difficult to obtain using the definition of residual and past extropies in terms of the distribution function of $X$. We also illustrate certain quantile function models for which the distribution function approach does not apply. Stochastic orders needed to compare the extropy functions of two lifetimes in the quantile framework which turns out to be distinct from the orderings conceived in the traditional settings, are also discussed in our work.

The chapter is organized as follows. In Section 6.2, we discuss extropy in quantile set up. Section 6.3 and Section 6.4 addresses the residual extropy function in the distribution function and the quantile framework respectively. We obtain their properties and some monotone properties and characterization results. In Section 6.5 we discuss the extropy in past lifetime and the characteristics associated with it. Section 6.6 presents extropy of order statistics using quantile function. We extended the measure based on survival function known as cumulative residual extropy and study its properties in Section 6.7. In Section 6.8 we obtain a non-parametric estimator for quantile-based extropy function and applying the method to a real data set.

## 6.2 Extropy

Let $X$ be a non-negative random variable with absolutely continuous distribution function $F(.)$ and probability density function $f(.)$. Then the differential extropy is defined as (Lad et al. (2015))

$$J_X = -\frac{1}{2}\int_0^{\infty} f^2(x)dx. \tag{6.1}$$

Qiu (2017) has obtained several results on the extropy of order statistics and record values based on copies of $X$ and also of coherent systems. Qiu & Jia (2018a) have obtained two estimators of extropy using spacings. Qiu & Jia (2018b) further studied the residual extropy of $X$, proved characterizations of exponential, Pareto and finite range distributions and discussed various properties of the order statistics. For more recent works on extropy, one can also refer to Alizadeh Noughabi & Jarrahiferiz (2018), Yang et al. (2018), Jose & Sathar (2019) and the references therein.

Now the quantile-based extropy based on (6.1) is given by

$$\begin{aligned} J(X) &= -\frac{1}{2}\int_0^1 f^2(Q(p))dQ(p) \\ &= -\frac{1}{2}\int_0^1 (q(p))^{-1}dp \\ &= -\frac{1}{2}\int_0^1 (1-p)H(p)dp. \end{aligned} \tag{6.2}$$

$J(X)$ provides a quantile version of the extropy, that measures the uncertainty of $X$, using either quantile density function or hazard quantile function.

**Example 6.1.** Consider the quantile function (Midhu et al. (2013))

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \mu > 0; -\mu \leq c < \mu,$$

corresponding to the linear MRLF (see Sankaran & Nair (2009))

$$M(u) = cu + \mu,\ \mu > 0,\ -\mu < c < \mu,\ 0 \leq u \leq 1. \tag{6.3}$$

Then $J(X)$ for (6.3) is obtained as

$$J(X) = -\frac{1}{2}\left(\frac{-2c - (\mu+c)\log(\mu-c) + (\mu+c)\log(\mu+c)}{4c^2}\right).$$

TABLE 6.1: Quantile function and the quantile-based extropy of some distributions.

| Distribution | $Q(u)$ | $J(X)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{-\lambda}{4}$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}} - 1), \gamma, c > 0$ | $-\frac{c^2}{2(2c+1)\gamma}$ |
| Rescaled Beta | $R\left(1 - (1-u)^{\frac{1}{c}}\right), c, R > 0$ | $-\frac{c^2}{2(2c-1)R}$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], b > 0, a > -1$ | $-\frac{(a+1)^2}{2(3a+2)b}$ |
| Power | $\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$ | $-\frac{\beta^2}{2(2\beta-1)\gamma}$ |
| Uniform | $a + (b-a)u, \ -\infty < a < b < \infty$ | $\frac{-1}{2(b-a)}$ |
| Davies | $\frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, c > 0; \lambda_1, \lambda_2 > 0$ | $-\frac{{}_2\tilde{F}_1\left(1,2-\lambda_1;-\lambda_1+\lambda_2+4;1-\frac{\lambda_2}{\lambda_1}\right)\Gamma(2-\lambda_1)\Gamma(\lambda_2+2)}{2(c\lambda_1)}$ |

$J(X)$ is not useful for a system that has survived for some units of time $t$.

## 6.3 Residual-extropy

Since $J_X$ is not applicable to a system that has survived for some units of time Qiu & Jia (2018b) considered the residual extropy of $X$

$$J_F(t) = -\frac{1}{2\bar{F}^2(t)} \int_t^{\infty} f^2(x)dx \tag{6.4}$$

and established the identity

$$J_F'(t) = \frac{1}{2}\left(h^2(t) + 4h(t)J_F(t)\right), \tag{6.5}$$

where $h(t)$ is the hazard rate function of $X$. A main limitation of residual extropy is that it is always negative so that it is not defined for certain regions of the parametric space. For example when $X$ has rescaled beta distribution with survival function

$$\bar{F}(x) = \left(1 - \frac{x}{R}\right)^c, 0 \le x \le R, R, c > 0.$$

We have,

$$J_F(t) = -\frac{c^2}{(4c-2)(R-t)},$$

so that $J_F(t)$ is not defined for $c < \frac{1}{2}$. Similarly for the power distribution $F(x) = \left(\frac{x}{\beta}\right)^\alpha, 0 \le x \le \beta, \alpha, \beta > 0$,

$$J_F(t) = \frac{\alpha^2(1 - (\frac{t}{\beta})^{2\alpha-1})}{(2-4\alpha)\left(1 - (\frac{t}{\beta})^\alpha\right)^2},$$

is valid only for $\alpha > \frac{1}{2}$. Thus the parameter values become a crucial aspect when discussing properties of residual extropy. Qiu & Jia (2018a) (their Theorem 2.9) proves the result for $k > \frac{1}{4}$ by taking the distribution of $X$ as $\bar{F}(x) = (1-x)^c, c > 1$. One can consider a more flexible range for the parameter as seen from the next theorem which modifies their result.

**Theorem 6.1.** *The relationship $J_F(t) = -kh(t)$ where $k$ is a non-negative constant holds for all $t > 0$ if and only if $X$ has*

*(i) rescaled beta distribution $\bar{F}(x) = (1 - \frac{x}{R})^c, 0 \le x \le R, R > 0, c > \frac{1}{2}$ if $k > \frac{1}{4}$*

*(ii) exponential law $\bar{F}(x) = e^{-\lambda x}, x > 0; \lambda > 0$ if $k = \frac{1}{4}$ and*

*(iii) Pareto II distribution $\bar{F}(x) = (1 + \frac{x}{\alpha})^{-c}, x > 0; c, \alpha > 0$ if $k < \frac{1}{4}$.*

*Proof.* When $X$ has the given distribution $J_F(t) = -\frac{c^2}{(4c-2)(R-t)}$ and $h(t) = \frac{c}{R-t}$ provides $J_F(t) = -kh(t), k = \frac{c}{4c-2}$. Thus $k > \frac{1}{4}$ when $c > \frac{1}{2}$. If $X$ follows the exponential distribution given in $(ii)$, then $J_F(t) =$ and $h(t) = \lambda$ provides the relation $J_F(t) = -kh(t)$. When $X$ follows Pareto II given in $(iii)$, $J_F(t) = -\frac{c^2}{(-4c-2)(t+\alpha)}$ and $h(t) = \frac{c}{t+\alpha}$ provides the given relation between $J_F(t)$ and $h(t)$.

Conversely suppose that $J_F(t) = -kh(t)$. Taking derivative on both sides and using the relation (6.5), we obtain

$$\frac{h'(t)}{h^2(t)} = \frac{4k-1}{2k}.$$

By solving the above differential equation we get

$$h(t) = (pt+d)^{-1}, p = \frac{1-4k}{2k}, d > 0$$

when $k < \frac{1}{4}$, $p$ is negative and hence $\bar{F}(x) = (1+\frac{pt}{d})^{-\frac{1}{p}}$ which is rescaled beta with $c = -\frac{1}{p}$ and $R = -\frac{p}{d} > 0$. □

We now examine whether $J_F(t)$ determines $F(x)$ uniquely. For this, we write (6.5) as a quadratic in $h(t)$,

$$h^2(t) + 4h(t)J(t) - 2J_F'(t) = 0. \tag{6.6}$$

It is seen that (6.6) has only one positive root when $J_F'(t) > 0$ or when $J_F'(t) = 0$ and two positive roots when $J_F'(t) < 0$. In the first two cases, $J_F(t)$ determines $h(t)$ and the distribution of $X$ uniquely. However the last case the two roots can be hazard rate function and hence, in general, $J_F(t)$ does not characterize the distribution if we adopt (6.6) as the basis of argument. To see this, from (6.6), when there are two roots, they are given by

$$h(t) = -2J_F(t) \pm \left(4J_F^2(t) + 2J_F'(t)\right)^{\frac{1}{2}}. \tag{6.7}$$

**Example 6.2.** Let $X$ follow the power distribution $F(x) = x^2, 0 \leq x \leq 1$, then

$$J_F(t) = -\frac{2}{3}\frac{(1-t^3)}{(1-t^2)^2}$$

and

$$J_F'(t) = -\frac{2}{3}\frac{t(1-t)(t^2+t+4)}{(1-t^2)^3}$$

showing that both $J_F(t)$ and $J_F'(t)$ are less than zero. After some algebra we get

$$4J_F^2(t) + 2J_F'(t) = \frac{16(1-t)^4(t+2)^2}{36(1-t^2)^4}.$$

Thus using (6.7) we have two solutions for $h(t)$, given by $h_1(t) = \frac{2}{(1-t)^2}$ and $h_2(t) = \frac{2(t^2+t+4)}{3(1-t)(1+t)^2}$. Writing $h_2(t)$ as

$$h_2(t) = \frac{3}{2(1-t)} + \frac{1}{2(1+t)} + \frac{2}{(1+t)^2},$$

it is easy to see that $h_2(t) \geq 0$ and $\int_0^1 h_2(t)dt$ diverges to $\infty$ proving it to be a hazard rate. Also, $h_1(t)$ is the hazard rate of the power distribution mentioned at the beginning of this example and $h_2(t)$ corresponds to

$$\bar{F}(x) = (1-x)^{\frac{3}{2}}(1+x)^{-\frac{1}{2}}e^{-\frac{2x}{1+x}}, 0 \leq x \leq 1.$$

However, $J_F(t)$ can also characterize some distributions. For example, if $X$ assumes uniform distribution in $[0, b]$, still there are two positive roots, but they are equal to $(b-t)^{-1}$. Hence the uniform distribution is characterized by its residual extropy function.

The random variable $X$ is said to have increasing (decreasing) residual extropy, IRE (DRE) if $J_F(t)$ is increasing (decreasing) for all $t > 0$. Thus the IRE (DRE) class is defined by the property $J_F(t) \geq (\leq) - \frac{h(t)}{4}$ for all $t$. From (6.6), the equality $J_F(t) = -\frac{h(t)}{4}$ holds for all $t > 0$ if and only if $X$ is exponential. Since rescaled beta is DRE and Pareto II is IRE the classes are not empty. There is a clear distinction between monotonic entropy and extropy. While the monotonicity of the hazard rate implies monotone residual entropy, the magnitude of the hazard rate is the determining factor for residual extropy. The rescaled beta is DRE where $h(t)$ is increasing, power distribution is DRE where $\frac{1}{2} < a < 1$, $h(t)$ is bathtub shaped.

## 6.4 Quantile-based residual extropy

Within the framework of quantile functions, we can write the residual quantile extropy as

$$J_Q(u) = J_F(Q(u)) = -\frac{1}{2(1-u)^2} \int_u^1 \frac{dp}{q(p)}. \tag{6.8}$$

Differentiation of (6.8) yields

$$q(u) = [2(1-u)^2 J_Q'(u) - 4(1-u) J_Q(u)]^{-1}. \tag{6.9}$$

The utility of the representation (6.9) is two-fold. Firstly it shows that the distribution of $X$ is characterized in terms of $J_Q(u)$ and secondly it helps to generate new quantile functions based on assumed functional forms of $J_Q(u)$. Also

$$H(u) = 2(1-u)J_Q'(u) - 4J_Q(u). \tag{6.10}$$

Accordingly, $J_Q(u)$ determines $H(u)$ and the distribution through $Q(u)$. Equations (6.9) and (6.10) exhibit the advantage of quantile approach over the distribution function counterpart in the sense the former gives a unique representation of the distribution where as it could not be accomplished in the latter as was shown in the above example. The difference between (6.7) and (6.10) is that the former gives scope for two values for $h(t)$, the latter is only a linear function providing unique value.

**Example 6.3.** Consider the linear mean residual quantile function family of distribution (Midhu et al. (2013)) specified by

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \mu > 0; -\mu \leq c < \mu, \tag{6.11}$$

which contains the exponential and uniform distributions and closely approximates several continuous distributions. It does not have a tractable distribution to study the properties of $J_F(t)$ using $F$. In this case, the quantile residual extropy is

$$J_Q(u) = \frac{1}{2(1-u)} - \frac{c+\mu}{8c^2(1-u)^2}\log\frac{\mu+c}{\mu-c+2u}.$$

Note that the hazard quantile function of (6.11) is $H(u) = (\mu - c + 2cu)^{-1}$.

TABLE 6.2: Quantile function and the quantile-based dynamic extropy of some distributions.

| Distribution | $Q(u)$ | $J_Q(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{-\lambda}{4}$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}} - 1), \gamma, c > 0$ | $-\frac{c^2(1-u)^{\frac{1}{c}+2}}{(2(1-u)^2)((2c+1)\gamma)}$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $-\frac{c^2(1-u)^{-1/c}}{2(2c-1)R}$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], b > 0, a > -1$ | $-\frac{(a+1)^2(1-u)^{1-\frac{1}{a+1}}}{2(3a+2)b}$ |
| Power | $\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$ | $\frac{\beta\left(\beta-\beta u^{2-\frac{1}{\beta}}\right)}{2(1-2\beta)\gamma(1-u)^2}$ |
| Uniform | $a + (b-a)u, -\infty < a < b < \infty$ | $-\frac{1}{2(1-u)(b-a)}$ |
| Davies | $\frac{cu}{(1-u)}, c > 0.$ | $\frac{1-u}{c\left(\frac{u}{(1-u)^2} + \frac{1}{1-u}\right)}$ |

**Example 6.4.** This example shows how to construct new quantile functions that confirms to known functional forms of $J_Q(u)$. Let $J_Q(u) = -e^{bu}$. Then from (6.8), $\int_u^1 \frac{dp}{q(p)} = 2(1-u)^2 e^{bu}, 0 < b < 2$ or

$$q(u) = [2(1-u)e^{bu}(2 - b(1-u))]^{-1},$$

which is a quantile density function.

**Theorem 6.2.** *Let $X$ be an absolutely continuous non-negative random variable. Then $J_Q(u)$ is linear of the form $J_Q(u) = A + Bu, A < 0, A + B < 0, B > 2A$ if and only if $H(u)$ is linear.*

*Proof.* Then from (6.10)

$$H(u) = 2B - 4A - 6Bu = a + bu, a = 2B - 4A > 0, b = -6B,$$

which is a hazard quantile function whenever $B > 2A$. Conversely let $H(u) = a + bu, a > 0, a + b > 0$, (see Midhu et al. (2014)). Then

$$a + bu = -4J_Q(u) + 2(1-u)J'_Q(u) \tag{6.12}$$

which reduces to a linear differential equation

$$J'_Q(u) - \frac{2J_Q(u)}{(1-u)} = \frac{(a+bu)}{2(1-u)}$$

with integrating factor $(1-u)^2$. Thus

$$\frac{d}{du}(1-u)^2 J_Q(u) = \frac{(a+bu)(1-u)}{2}$$

giving the general solution

$$J_Q(u) = \frac{-3a - b + 2bu}{12} + \frac{C}{(1-u)^2}.$$

It satisfies (6.12) only when $C = 0$. Thus

$$J_Q(u) = \frac{-3a - b + 2bu}{12} = (A + Bu).$$

□

**Remark 6.1.** Distribution with linear hazard quantile function has been discussed in Nair et al. (2013) and Midhu et al. (2013) where its properties and applications are given. It subsumes the exponential, half-logistic, exponential-geometric and Marshall-Olkin type distributions. Thus Theorem 6.2 provides a characterization of the linear hazard quantile function family in terms of linear residual extropy quantile function.

With the aid of (6.10), we can derive some ageing properties of $X$ by means of $J_Q(u)$. Differentiating (6.10), we get

$$H(u) = -6J_Q'(u) + 2(1-u)J_Q''(u).$$

Consequently, $X$ is IFR if $J_Q(u)$ is increasing and concave, while $X$ is DFR when $J_Q(u)$ is decreasing and convex. Also, we have some stochastic orders connecting two lifetimes $X$ and $Y$ with residual extropies $J_{Q_X}(u)$and $J_{Q_Y}(u)$.

**Definition 6.3.** We say that $X$ has less residual quantile extropy than $Y$, denoted by $X \leq_{RQE} Y$ if $J_{Q_X}(u) \leq J_{Q_Y}(u)$ for all $0 < u < 1$.

**Theorem 6.4.**

$$X \leq_{HQ} Y \Rightarrow X \leq_{RQE} Y$$

*where $\leq_{HQ}$ is the hazard quantile function order.*

*Proof.*

$$X \leq_{HQ} Y \Leftrightarrow \frac{1}{(1-u)q_X(u)} \geq \frac{1}{(1-u)q_Y(u)}$$

$$\Rightarrow \int_u^1 \frac{dp}{q_X(p)} \geq \int_u^1 \frac{dp}{q_Y(p)} \Rightarrow -\frac{1}{2(1-u)^2}\int_u^1 \frac{dp}{q_X(p)} \leq -\frac{1}{2(1-u)^2}\int_u^1 \frac{dp}{q_Y(p)} \Leftrightarrow X \leq_{RQE} Y.$$

□

The implication of this theorem is that when $X \leq_{HQ} Y$, the device with lifetime $X$ is less reliable than that with lifetime $Y$ which also means that the information content in the residual life distribution of $X$ is smaller than the content in $Y$.

**Remark 6.2.** Since $X \leq_{HQ} Y \Leftrightarrow X \geq_{RHQ} Y$ we have $X \geq_{RHQ} Y \Leftrightarrow X \leq_{RQE} Y$.

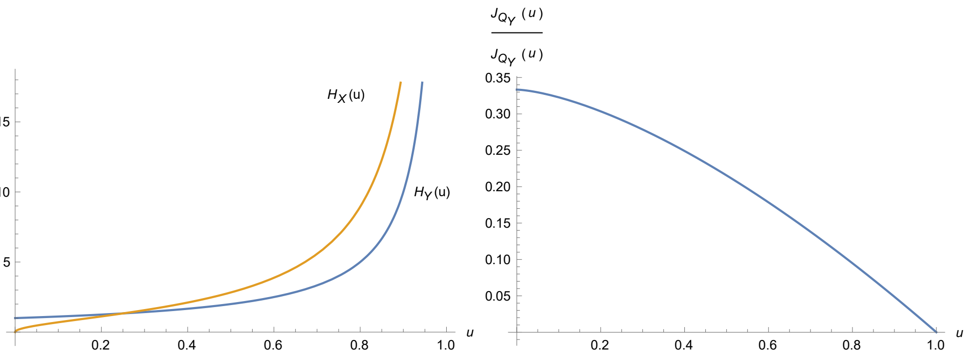
It may be noted in this connection that the residual extropy (6.4) can also be ordered. We say that $X$ is less than $Y$ in residual extropy, denoted by $X \leq_{RE} Y$, if $J_{F_X}(t) \leq J_{F_Y}(t)$ for all $t > 0$. However, $X \leq_{RE} Y$ and $X \leq_{RQE} Y$ are not identical. Also, it appears that implications of $\leq_{RE}$ with other stochastic orders are difficult to find in contrast to $\leq_{RQE}$ .

Some comments about the above result seem to be in order. Since $X \leq_{HQ} Y$ is equivalent to the dispersive order $X \leq_{disp} Y$ all properties of the dispersive order given in Shaked & Shanthikumar (2007), Section 3B holds good for the residual quantile extropy order. Two important results in this connection are when $X$ and $Y$ have zero as the lower endpoint of their supports, from Theorem 3.4 $X \leq_{RQE} Y \Rightarrow X \geq_{st} Y \Leftrightarrow Q_X(u) \geq Q_Y(u)$ for all $u$ or $F_X(x) \leq F_Y(x)$ for all $x$. The other result is $X \leq_{RQE} Y \Rightarrow V(X) \leq V(Y)$ where $V$ stands for the variance. While extropy accounts for the uncertainty due to the individual probabilities involved in the distributions, variance speaks about the variation about the mean and have apparently these two notions have conceptually no connection. The above result brings an implication between the two. The converse of Theorem 6.4 may not be true and therefore the additional requirements that ensure the converse is stated in the next theorem.

**Theorem 6.5.** *If* $\frac{J_{Q_Y}(u)}{J_{Q_X}(u)}$ *is increasing in* $u$*, then* $X \leq_{RQE} Y \Rightarrow X \leq_{HQ} Y$.

*Proof.*

$$\frac{J_{Q_Y}(u)}{J_{Q_X}(u)} \text{ is increasing} \Leftrightarrow \frac{\int_u^1 \frac{dp}{q_Y(p)}}{\int_u^1 \frac{dp}{q_X(p)}} \text{ is increasing.}$$




(a) Plot of $H(u)$ against $u$.

(b) Plot pf $\frac{J_{Q_Y}(u)}{J_{Q_X}(u)}$ against $u$.

FIGURE 6.1

$$\Leftrightarrow \frac{1}{q_X(u)}\int_u^1 \frac{dp}{q_Y(p)} - \frac{1}{q_Y(u)}\int_u^1 \frac{dp}{q_X(p)} \geq 0$$

$$\Leftrightarrow \frac{q_Y}{q_X} \geq \frac{\int_u^1 \frac{dp}{q_Y(p)}}{\int_u^1 \frac{dp}{q_X(p)}} \geq 1 \Leftrightarrow (1-u)q_Y \geq (1-u)q_X$$

$$\Leftrightarrow H_X(u) \geq H_Y(u) \Rightarrow X \leq_{HQ} Y.$$

Since $X \leq_{HQ} Y \Leftrightarrow X \leq_{RHQ} Y$ we also have $X \leq_{RHQ} Y \Rightarrow X \leq_{RQE} Y$. □

**Example 6.5.** For the uniform distribution $Q_X(u) = u$, $J_{Q_X}(u) = -\frac{2}{1-u}$ and for $Q_Y(u) = u^{\frac{1}{2}}$, $J_{Q_Y}(u) = -\frac{4}{6}\frac{(1-u^{\frac{3}{2}})}{(1-u)^2}$ $0 < u < 1$, so that $X \geq_{RQE} Y$. From Figure 6.1(a) it is clear that $X$ and $Y$ are not in hazard hazard quantile order. Figure 6.1(b) shows that $\frac{J_{Q_Y}(u)}{J_{Q_X}(u)}$ is decreasing in $u$. So that the condition given in Theorem 6.5 can not be relaxed for satisfying the implication.

**Theorem 6.6.** *Let $X$ be a random variable with quantile function $Q(u)$ and hazard quantile function $H(u)$ for all $u \in (0,1)$. The relationship $J_Q(u) = -kH(u)$, where $k$ is a non-negative constant holds for all $u$ if and only if $X$ has*

*(i) rescaled beta distribution $Q(u) = R\left(1-(1-u)^{\frac{1}{c}}\right), 0 \leq u \leq 1, R > 0, c > \frac{1}{2}$ if $k > \frac{1}{4}$*

(ii) *exponential law* $Q(u) = \frac{log(1-u)}{\lambda}, 0 \leq u \leq 1; \lambda > 0$ *if* $k = \frac{1}{4}$ *and*

(iii) *Pareto II distribution* $Q(u) = \gamma((1-u)^{\frac{-1}{c}} - 1), 0 \leq u \leq 1; c, \alpha > 0$ *if* $k < \frac{1}{4}$.

*Proof.* Assume that $J_Q(u) = -kH(u)$. Differentiating both sides and using (6.9), we get

$$H(u) = -2k(1-u)H'(u) + 4kH(u),$$

implies,

$$\frac{H'(u)}{H(u)} = \frac{4k-1}{2k(1-u)}$$

Now,

$$\log H(u) = \frac{4k-1}{2k}(-\log(1-u)) + \log c,$$

equivalently,

$$H(u) = \frac{c_1}{(1-u)^{\frac{4k-1}{2k}}}.$$

When $k = \frac{1}{4}, H(u) = 1$, a constant which characterizes exponential distribution. For Pareto II $H(u) = \frac{c}{\gamma}(1-u)^{\frac{1}{c}}$ and for rescaled beta $H(u) = \frac{c}{R}(1-u)^{\frac{-1}{c}}$. So that $k < \frac{1}{4}$ characterizes Pareto II and $k > \frac{1}{4}$ characterizes rescaled beta.

□

**Theorem 6.7.** *Let X and Y be two non-negative random variables with quantile densities* $q_X(.)$ *and* $q_Y(.)$ *respectively such that* $q_X(0) \leq q_Y(0)$. *If* $X \leq_c Y$, *then* $J_{Q_X}(u) \leq J_{Q_Y}(u)$.

*Proof.* If $X \in IFR$ then $H(u)$ is increasing in $u$. We have $X \leq_c Y$, iff $\frac{q_Y(u)}{q_X(u)}$ is increasing in $0 \leq u \leq 1$. That is,

$$\frac{q'_X(u)}{q_X(u)} \leq \frac{q'_Y(u)}{q_Y(u)},$$

equivalently,

$$q_X(u) \leq q_Y(u).$$

Thus

$$J_{Q_X}(u) = \frac{-1}{2(1-u)^2} \int_u^1 (q_X(p))^{-1} dp \leq \frac{-1}{2(1-u)^2} \int_u^1 (q_Y(p))^{-1} dp,$$

$$J_{Q_X}(u) \leq J_{Q_Y}(u).$$

□

The following theorem provides the upper bound of extropy.

**Theorem 6.8.** *If X is said to have an increasing failure rate (IFR (IFRA, NBU)) then* $J_{Q_X}(u) \leq J_{Q_Y}(u)$, *where Y has exponential distribution.*

*Proof.* $X \in IFR(IFRA, NBU)$ if and only if $X \leq_c (\leq_*, \leq_{su})Y$, where $Y$ has the exponential distribution with mean $\frac{1}{\lambda}$ (see Shaked & Shanthikumar (2007)). Since $J_{Q_Y}(u) = \frac{-\lambda}{4}$, we obtain the upper bound of extropy of IFR(IFRA, NBU) classes by Theorem 6.7 □

**Theorem 6.9.** *Let $X$ be a non-negative continuous random variable with quantile function $Q(.)$ and mean residual quantile function $M(.)$. Then*

$$J_Q(u)M(u) = c, \tag{6.13}$$

*where $c \in R$ if and only if $X$ has exponential distribution.*

*Proof.* Let $X$ has exponential distribution with quantile function $Q(u) = -\frac{1}{\lambda}\log(1-u), \lambda > 0$. We have $M(u) = \frac{1}{\lambda}$. Now,

$$\begin{aligned} J_Q(u) &= -\frac{\lambda}{4} \\ &= -\frac{1}{4M(u)}, \end{aligned}$$

that is $J_Q(u)M(u) = c$, a constant.

Let (6.13) holds. Taking derivative on both sides of (6.13) with respect to $u$ we get

$$J'_Q(u)M(u) + J_Q(u)M'(u). \tag{6.14}$$

Using the relation (see Nair et al. (2013))

$$(H(u))^{-1} = M(u) - (1-u)M'(u),$$

(6.14) becomes

$$(H(u)M(u))^2 + 6c(H(u)M(u)) - 2c = 0.$$

By solving the above differential equantion we get

$$H(u)M(u) = c,$$

which characterizes exponential distribution. □

## 6.5 Extropy of order statistics

Extropy of $i^{th}$ order statistic based on (6.1) is given by

$$J_{X_{i:n}} = \frac{-1}{2}\int_0^\infty f_{i:n}^2(x)dx, \tag{6.15}$$

where $f_{i:n}(x) = \frac{1}{B(i,n-i+1)}F^{i-1}(x)(1-F(x))^{n-i}f(x)$.

Within the framework of quantile functions we obtain the quantile-based extropy based on (6.15) as

$$\begin{aligned} J(X_{i:n}) &= \frac{-1}{2}\int_0^1 \left(\frac{1}{B(i,n-i+1)}p^{i-1}(1-p)^{n-i}(q(p))^{-1}\right)^2 dQ(p) \\ &= \frac{-1}{2}\int_0^1 \left(\frac{p^{i-1}(1-p)^{n-i}}{B(i,n-i+1)}\right)^2 (q(p))^{-1}dp. \end{aligned} \tag{6.16}$$

In system reliability, first-order statistic represents the lifetime of a series system while the $n^{th}$ order statistic measures the lifetime of a parallel system. For a series system $(i = 1)$,

$$J(X_{1:n}) = \frac{-1}{2}\int_0^1 \left(n(1-p)^{n-1}\right)^2 (q(p))^{-1}dp. \tag{6.17}$$

For a parallel system $(i = n)$,

$$J(X_{n:n}) = \frac{-1}{2} \int_0^1 (np^{n-1})^2 (q(p))^{-1} dp. \tag{6.18}$$

The following theorem provides some interesting properties of quantile-based extropy of order statistics when the pdf of the underlying iid random variables are symmetric.

**Theorem 6.10.** *Let $X_1, X_2, ..., X_n$ be iid samples whose distribution is symmetric about mean $\mu$. Then*

*(a)* $J(X_{i:n}) = J(X_{n-i+1:n})$

*(b)* $\Delta J(X_{i:n}) = -\Delta J(X_{n-i:n}), \forall i = 1, 2, ..., n-1$ *where* $\Delta J(X_{i:n}) = J(X_{i+1:n}) - J(X_{i:n})$.

*(c) If* $Y = \frac{X-\mu}{a}$ *then* $J(Y_{i:n}) = aJ(X_{i:n})$.

*Proof.* For a symmetric random variable $X_{i:n} = -X_{n-i+1}$, equivalently $f_{i:n}(\mu + x) = f_{n-i+1:n}(\mu - x)$. That is

$$\frac{u^{i-1}(1-u)^{n-i}}{B(i, n-i+1)} = \frac{u^{n-i}(1-u)^{i-1}}{B(n-i+1, i)}.$$

Therefore

(a)

$$J(X_{n-i+1:n}) = \frac{-1}{2}\int_0^1 \left(\frac{p^{n-1}(1-p)^{i-1}}{B(n-i+1,i)}\right)^2 (q(p))^{-1}dp$$
$$= \frac{-1}{2}\int_0^1 \left(\frac{p^{i-1}(1-p)^{n-i}}{B(i,n-i+1)}\right)^2 (q(p))^{-1}dp$$
$$= J(X_{i:n}).$$

(b) We have, $\Delta J(X_{i:n}) = J(X_{i+1:n}) - J(X_{i:n})$.

Now, $\Delta J(X_{n-i:n}) = J(X_{n-i+1:n}) - J(X_{n-i} : n) = -\Delta J(X_{i:n})$.

(c) Let $Y = \frac{X-\mu}{a}$. $Q_Y(u) = \frac{Q_X(u)-\mu}{a}$.

Then

$$J(Y_{i:n}) = \frac{-1}{2}\int_0^1 \left(\frac{p^{i-1}(1-p)^{n-i}}{B(i,n-i+1)}\right)^2 \left(\frac{q_X(p)}{a}\right)^{-1} dp$$
$$= aJ(X_{i:n}).$$

□

Recently, Baratpour et al. (2007), Baratpour et al. (2008) showed that Shannon entropy and Renyi entropy of the $i^{th}$ order statistic can characterize the underlying distribution uniquely. Motivated by these we propose the quantile-based residual extropy of $i^{th}$ order statistic as

$$J_{Q_{X_{i:n}}}(u) = \frac{-1}{2(\bar{B}_u(i,n-i+1)^2)}\int_u^1 p^{2(i-1)}(1-p)^{2(n-i)}(q(p))^{-1}dp. \quad (6.19)$$

For series system $(i = 1)$, the quantile-based residual extropy is given by

$$J_{Q_{X_{1:n}}}(u) = \frac{-n^2}{2(1-u)^{2n}} \int_u^1 (1-p)^{2(n-1)}(q(p))^{-1}dp. \tag{6.20}$$

For a parallel system $(i = n)$, the quantile-based residual extropy is given by

$$J_{Q_{X_{n:n}}}(u) = \frac{-n^2}{u^{2n}} \int_u^1 p^{2(n-1)}(q(p))^{-1}dp. \tag{6.21}$$

Here we consider series system. Differentiating (6.20) with respect to $u$, we get

$$J'_{Q_{X_{1:n}}}(u) = \frac{n^2 H(u)}{2(1-u)} + \frac{2n}{(1-u)} J_{X_{1:n}}(u),$$

equivalently

$$q(u) = \frac{n^2}{2}\left((1-u)^2 J'_{Q_{X_{1:n}}}(u) - 2nJ_{X_{1:n}}(u)\right)^{-1}, \tag{6.22}$$

(6.22) shows that the quantile-based residual extropy of the first-order statistic can characterize the underlying distribution uniquely.

**Theorem 6.11.** *Let $X$ be a continuous random variable with quantile function $Q(u)$ and hazard quantile function $H(u)$. If the quantile-based dynamic extropy of first-order statistics is increasing (decreasing) in $u$, then $J_{Q_{X_{1:n}}}(u) \geq \frac{-nH(u)}{4}$.*

**Theorem 6.12.** *Let $X$ be a random variable with hazard quantile function $H(u)$. If $J_{Q_{X_{1:n}}}(u) = -kH(u)$, for all $u \in (0,1)$*

*i a rescaled beta distribution, if and only if $k > \frac{n}{4}$ and $c > \frac{1}{2n}$;*

*ii an exponential distribution if and only if $k = \frac{n}{4}$;*

TABLE 6.3: Quantile function and the quantile-based extropy of first-order statistic for some distributions.

| Distribution | $Q(u)$ | $J_{Q_{X_{1:n}}}(u)$ |
|---|---|---|
| Exponential | $-\frac{\log(1-u)}{\lambda}, \lambda > 0$ | $\frac{-n\lambda}{4}$ |
| Pareto II | $\gamma((1-u)^{\frac{-1}{c}} - 1), \gamma, c > 0$ | $\frac{c^2 n^2 (1-u)^{1/c}}{2\gamma(2cn+1)}$ |
| Rescaled Beta | $R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$ | $-\frac{c^2 n^2 (1-u)^{-1/c}}{2R(2cn-1)}$ |
| Generalized Pareto | $\frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], b > 0, a > -1$ | $-\frac{(a+1)^2 n^2 (1-u)^{\frac{a}{a+1}}}{2b(2(a+1)n+a)}$ |
| Power | $\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$ | $-\frac{\beta n^2 (1-u)^{-2n}\left(\frac{\Gamma\left(2-\frac{1}{\beta}\right)\Gamma(2n-1)}{\Gamma\left(2n-\frac{1}{\beta}+1\right)} - B_u\left(2-\frac{1}{\beta}, 2n-1\right)\right)}{2\gamma}$ |
| Uniform | $a+(b-a)u, -\infty < a < b < \infty$ | $-\frac{n^2}{2(2n-1)(u-1)(a-b)}$ |
| Davies | $\frac{cu}{(1-u)}, c > 0.$ | $-\frac{n^2(u-1)^2}{2c(2n-1)(1-u)}$ |

*iii a Pareto distribution if and only if* $k < \frac{n}{4}$.

*Proof.* The proof is similar to Theorem 6.6. □

**Theorem 6.13.** *The exponential distribution with quantile function* $Q(u) = \frac{-1}{\lambda}\log(1-u)$ *can be characterized by* $J(X_{i:n}) = nJ(X)$.

The next theorem implicates the monotonic property of quantile-based extropy of first-order statistic.

**Theorem 6.14.** *If X has a decreasing density quantile function* $f(Q(.))$ *then* $J_{Q_{X_{1:n}}}(u)$ *is decreasing.*

*Proof.* For $0 < u_1 < u_2 < 1$, $f(Q(u_2)) \leq f(Q(u_1))$. From (6.20), we write

$$\begin{aligned} J_{Q_{X_{1:n}}}(u_1) &= \frac{-n^2}{2(1-u_1)^{2n}} \int_{u_1}^{1} (1-p_1)^{2(n-1)} f(Q(p_1))dp_1 \\ &> \frac{-n^2}{2(1-u_2)^{2n}} \int_{u_2}^{1} (1-p_2)^{2(n-1)} f(Q(p_2))dp_2 \\ &= J_{Q_{X_{1:n}}}(u_2). \end{aligned}$$

Thus the proof. □

The following counterexample shows that the above theorem is applicable for first-order statistics only. The theorem violates for $i > 1$.

**Example 6.6.** *Consider power-Pareto* $(c = \lambda_1 = \lambda_2 = 1)$ *distribution with quantile function* $Q(u) = \dfrac{u}{1-u}$. *It is clear that the density quantile function* $f(Q(u)) = (1-u)^2$ *is decreasing in* $u$. *Now, we obtain*

$$\begin{aligned} J_{X_{2:2}}(u) &= \frac{-2}{(1-u^2)^2}\int_u^1 p^2 (q(p))^{-1} dp \\ &= \frac{-2}{(1-u^2)^2}\int_u^1 p^2 (1-p)^2 dp. \end{aligned} \tag{6.23}$$

*From* (6.23), *we get* $J_{X_{2:2}}(\frac{1}{4}) = -0.068$ *and* $J_{X_{2:2}}(\frac{1}{2}) = -0.059$. *That is* $J_{X_{2:2}}(\frac{1}{4}) < J_{X_{2:2}}(\frac{1}{2})$, *which is not decreasing in* $u$.

## 6.6 Past extropy

It is reasonable to presume that in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. The extropy of past lifetime $[X|X \leq t]$ is defined as

$$\bar{J}_F(t) = -\frac{1}{2}\int_0^t \left(\frac{f(x)}{F(t)}\right)^2 dx. \tag{6.24}$$

TABLE 6.4: Quantile function and the quantile-based extropy of some distributions.

| Distribution | $F(x)$ | $\lambda(t)$ | $\bar{J}_F(t)$ |
|---|---|---|---|
| Uniform | $x, 0 \leq x \leq 1$ | $\frac{1}{t}$ | $-\frac{1}{2t}$ |
| Power | $\left(\frac{x}{\alpha}\right)^{\beta}, 0 \leq x \leq \alpha; \beta > 0$ | $\frac{\beta}{t}$ | $-\frac{\beta^2}{(4\beta-2)t}, \beta > \frac{1}{2}$ |
| Reciprocal exponential | $exp(-\frac{\lambda}{x})$ | $\frac{\lambda}{t^2}$ | $-\frac{(t^2+2\lambda t+2\lambda^2)}{8\lambda t^2}$ |
| Generalized exponential | $(1-e^{\lambda x})^{\theta}, x > 0; \theta, \lambda > 0$ | $\frac{\theta\lambda}{e^{\lambda t}-1}$ | $-\frac{\lambda\theta^2}{2(2\theta-1)}\left(\frac{1}{e^{\lambda t}-1} - \frac{1}{2\theta}\right)$ |
| Lindely distribution | $\frac{\beta^2}{1+\beta}(1+x)e^{-\beta x, x>0; \beta>0}$ | $\frac{\beta^2(1+t)}{1+\beta+\beta t}$ | $\frac{\beta^3}{2(1+\beta+\beta t)^2}\left(\frac{1+\beta}{2\beta} - \frac{(1+t)}{\beta} - (1+t)^2\right)$ |
| Reversed exponential | $e^{a(x-b)}, 0 \leq x \leq b; a, b > 0$ | $a$ | $-\frac{1}{4a}$ |
| Reversed generalized Pareto | $\left(\frac{ax+c}{ab+c}\right)^{\frac{a+1}{a}}, a > -1, c > 0, 0 \leq x \leq b$ | $\frac{a+1}{c+at}$ | $-\frac{(a+1)^2}{(2a+4)(at+c)}, 0 < x \leq b$ |

The reversed hazard rate function related to the past lifetime random variable, defined as $\lambda(t) = \frac{f(t)}{F(t)}$. By differentiation with respect to $t$, (6.24) becomes

$$\lambda^2(t) + 4\lambda(t)\bar{J}_F(t) + 2\bar{J}'_F(t) = 0, \tag{6.25}$$

as a relationship between reversed hazard rate $\lambda(t)$ and past extropy. Some examples of $\lambda(t)$ and $\bar{J}_F(t)$ are given in Table 6.4. Since $\bar{J}_F(t)$ lies in $(-\infty, 0)$, the analysis of (6.25) depends on whether $\bar{J}'_F(t)$ is zero, positive or negative. When $\bar{J}'_F(t) > 0$ there can be two roots and in the other cases there is only one positive root for (6.25). The roots are given by

$$\lambda(t) = -2\bar{J}_F(t) \pm \sqrt{4\bar{J}^2_F(t) - \bar{J}'_F(t)}.$$

In the case of uniform distribution $\bar{J}'_F(t) = \frac{1}{2t^2} > 0$, both the roots are identical. Hence $\bar{J}_F(t) = -\frac{1}{2t}$ characterizes the uniform law. All distributions with decreasing past extropy are characterized by the form of $\bar{J}_F(t)$. But when $\bar{J}_F(t)$ is increasing both the solutions of (6.25) can be reversed hazard rates. Thus in general, $\bar{J}_F(t)$ does not determine $F$ uniquely.

**Example 6.7.** In Table 6.4, $\bar{J}_F(t) = -\frac{\beta^2}{(4\beta-2)t}, \beta > \frac{1}{2}$ for the power distribution. Substituting in (6.25) and solving

$$\lambda(t) = \frac{2\beta^2}{(4\beta-2)t} \pm \frac{2\beta(\beta-1)}{(4\beta-2)t}.$$

Thus the two solutions are $\lambda_1(t) = \frac{\beta}{t}$ giving the original power distribution and $\lambda_2(t) = \frac{2\beta}{(4\beta-2)t}$ giving another power distribution $F(t) = \left(\frac{t}{\beta}\right)^{\frac{2\beta}{4\beta-2}}, \beta > \frac{1}{2}$. Thus $\bar{J}_F(t)$ does not uniquely determine $F$.

Further, (6.25) reveals that $\bar{J}_F(t)$ is strictly increasing (decreasing) in $t$ according as $\bar{J}_F(t) < (>) -\frac{\lambda(t)}{4}$. Examples of increasing (decreasing) past extropy, IPE (DPE) and some characterizations by relationships reversed hazard rates are seen in the next two theorems.

**Theorem 6.15.** *Let X be a non-negative random variable with support* $[0, b]$. *Then the property* $\bar{J}_F(t) = -k\lambda(t)$ *where k is some non-negative constant is satisfied by all* $t \in (0, b]$ *if and only if X follows reversed generalized Pareto with*

$$F(x) = \left(\frac{ax+c}{ab+c}\right)^{\frac{a+1}{a}}, a > -1, c > 0, x \in (0, b]. \tag{6.26}$$

*Proof.* The necessary part follows from Table 6.4, where $k = \frac{a+1}{4a+2} > 0$. From the given property $\frac{\lambda'(t)}{\lambda^2(t)} = \frac{1-4k}{1-2k}$ and on integration $\lambda(t)(p+qt)^{-1}$ with $p > 0$ and $q = \frac{4k-1}{1-2k}$. Thus

$$F(x) = exp[-\int_x^{\infty} \lambda(t)dt] = \left(\frac{p+qx}{p+qb}\right)^{\frac{1}{q}}, 0 < x \leq b. \tag{6.27}$$

By setting $k = \frac{2a+1}{2(3a+2)}, a = 4k-1, c = p(1-2k)$, (6.27) leads to (6.26). □

**Remark 6.3.** As $a \longrightarrow 0$ in (6.26), we have

$$F(x) = exp[c^{-1}(x-b)], 0 \leq x \leq b$$

the reversed exponential distribution. In this case $k = \frac{1}{4}$ and $\bar{J}_F(t) = -\frac{1}{4c}$. Two other members of the family (6.26) are of the form

$$F(x) = \left(\frac{1+\frac{ax}{c}}{1+\frac{ab}{c}}\right)^{\frac{a+1}{a}}, a, c > 0$$

and

$$F(x) = \left(\frac{1-\frac{px}{c}}{1-\frac{pb}{c}}\right), p, c > 0,$$

obtained when $k < \frac{1}{4}$ and $k > \frac{1}{4}$ respectively.

**Remark 6.4.** $X$ is IPE when $a > 0$, DPE when $-1 < a < 0$ and $X$ has constant $\bar{J}_F(t)$ when $a = 0$. Thus $\bar{J}_F(t)$ can be increasing, decreasing or constant, the last two cases holding only when zero is a point of discontinuity of $F$.

**Theorem 6.16.** *If $F(x)$ is absolutely continuous satisfying $F(0) = 0$ then $\bar{J}_F(t) = -k\lambda(t)$ for $k > 0$ and all $t$ if and only if $F(x) = \left(\frac{x}{\alpha}\right)^{\beta}, 0 \leq x \leq \alpha, \beta > \frac{1}{2}$.*

*Proof.* The 'if' part follows from Table 6.4. Further to prove the converse, proceeding as in the previous theorem we have

$$\bar{F}(x) = \left(\frac{qx+p}{q\alpha+p}\right)^{\frac{1}{q}}, 0 \leq x \leq \alpha.$$

Since$F(0) = 0$ we have $p = 0$ and taking $q = \beta^{-1}$, we have the required $F(x)$. □

Much more general and powerful results can be found by considering the past quantile extropy

$$\bar{J}_Q(u) = \bar{J}_F(Q(u)) = -\frac{1}{2u^2}\int_0^u \frac{1}{q(p)}dp. \tag{6.28}$$

A major benefit of (6.28) is that unlike $\bar{J}_F(t)$ the past quantile extropy function $\bar{J}_Q(u)$ determines the distribution of $X$ uniquely through the inversion formula

$$q(u) = \left(-2(2u\bar{J}_Q(u) + u^2\bar{J}'_Q(u))\right)^{-1}. \tag{6.29}$$

This enables construction of distributions based on past quantile extropy function. We demonstrate the utility of (6.29) in model building. Consider the rational function

$$\bar{J}_Q(u) = -\left(a - bu + \frac{c}{u}\right), a^2 + 4bc \leq 0.$$

By (6.29) we can write

$$q(u) = (2c + 4au - 6bu^2)^{-1}, a^2 \leq min(-4bc, 3bc). \tag{6.30}$$

Equation (6.30) defines a distribution with hazard quantile function

$$H(u) = \frac{2c + 4au + 6bu^2}{1-u}.$$

When $2a - 3b + c = 0, c \neq 0, H(u) = 6bu - 4a + 6b$, giving a linear hazard quantile distribution discussed earlier for $X$. When $c = 0, 2a = 3b$ and $J_Q(u) = bu - a$, a linear function of $u$ with $a > b$. Also when $b = 0$, we have $J_Q(u)$ as a homographic function with $a > 0, c > 0$. Finally $b = c = 0$ leads to the exponential model. Thus (6.30) comprises of a new class of distributions with exponential, half-logistic, exponential-geometric etc., as special cases.

The residual and past extropies can be related to one another in the quantile framework. Thus the knowledge of one of these is enough to determine the other which saves the computational work involved in finding it from first principles.

**Theorem 6.17.**

$$J_Q(u) = (1-u)^{-2}\left(\bar{J}_Q(1) - u^2\bar{J}_Q(u)\right)$$

$$\bar{J}_Q(u) = u^{-2}\left(J_Q(0) - (1-u)^2 J_Q(u)\right), \bar{J}_Q(1) = J_Q(0) = J_X.$$

*Proof.* We have

$$q(u) = \left(2\frac{d}{du}(1-u)^2 J_Q(u)\right)^{-1}$$

and

$$q(u) = \left(-2\frac{d}{du}u^2\bar{J}_Q(u)\right)^{-1}.$$

Equating these expressions and integrating

$$(1-u)^2 J_Q(u) = -u^2\bar{J}_Q(u) + k.$$

As $u \to 0, k = J(0)$ and as $u \to 1, k = \bar{J}_Q(1)$. The identities in the Theorem now follow. □

With reference to reversed hazard quantile function, we find

$$\lambda_Q(u) = -\left(4\bar{J}_Q(u) + 2u\bar{J}'_Q(u)\right). \tag{6.31}$$

Hence for an absolutely continuous $F$, $\lambda_Q(u)$ is decreasing whenever $\bar{J}_Q(u)$ is increasing and convex. As seen earlier $\lambda_Q(u)$ can also be increasing or non-monotone when $X$ has support $[0, b], b < \infty$ and zero is a point of discontinuity. It can also be concluded that $\bar{J}_Q(u)$ determine $\lambda_Q(u)$ uniquely. Since equation (6.31) being a differential equation in $\bar{J}_Q(u)$, it can be solved to find

$$\bar{J}_Q(u) = u^{-2}\left(\int -\frac{u\lambda_Q(u)}{2} + k\right),$$

where $k$ is determined such that $\lim_{u\to 1} \bar{J}_Q(u) = \bar{J}_X$, the extropy of $X$. The last equation shows that $\bar{J}_Q(u)$ is completely specified by $\lambda_Q(u)$. Comparison of past extropies of two non-negative and absolutely continuous random variables $X$ and $Y$ can be accomplished in terms of stochastic orderings.

**Definition 6.18.** (i) The random variable $X$ is less than $Y$ in past extropy denoted by $X \leq_{PEX} Y$ if $\bar{J}_X(t) \leq \bar{J}_Y(t)$ for all $t > 0$.

(ii) We say that $X$ is less than $Y$ in past quantile extropy, $X \leq_{PQEX} Y$ if $\bar{J}_{Q_X}(u) \leq \bar{J}_{Q_Y}(u)$ for all $u \in (0, 1)$.

The following are the some important properties of the two orderings $\leq_{PEX}$ and $\leq_{PQEX}$.

(i) The stochastic order $\leq_{PEX}$ does not imply $\leq_{PQEX}$ .

**Example 6.8.** Let $F_X(x) = x, 0 \leq x \leq 1$ and $F_Y(x) = x^2, 0 \leq x \leq 1$. Then $\bar{J}_X(t) = -\frac{1}{2t}$ and $\bar{J}_Y(t) = -\frac{2}{3t}$ so that $X \geq_{PEX} Y$. However, $Q_X(u) = u$ and $Q_Y(u) = u^{\frac{1}{2}}$ leads to $\bar{J}_{Q_X}(u) = -\frac{1}{2u}$ and $\bar{J}_{Q_Y}(u) = -\frac{2}{3u^{\frac{1}{2}}}$. Now, $\bar{J}_{Q_X}(u)$ and $\bar{J}_{Q_Y}(u)$ cross each other at $u = \frac{9}{16}$ and hence not ordered.

(ii) Also, $\leq_{PQEX}$ does not imply $\leq_{PEX}$ .

**Example 6.9.** Let $F(x;\lambda) = e^{-\frac{\lambda}{x}}, x > 0; \lambda > 0$. Then $\bar{J}_X(t;\lambda) = -\frac{t^2+2\lambda t+2\lambda^2}{8\lambda t^2}$, and it is easy to see that $\bar{J}_X(t;1)$ and $\bar{J}_X(t;2)$ cross at $t = 2$. Hence $\bar{J}_X(t;1)$ and $\bar{J}_X(t;2)$ are not ordered by $\leq_{PEX}$ . On the other hand, $Q_X(u) = -\lambda(\log u)^{-1}$ gives

$$\bar{J}_Q(u,\lambda) = -\frac{1}{\lambda u^2}\int_0^u p(\log p)^2 dp.$$

Hence $\bar{J}_Q(u;1) \leq_{PQEX} \bar{J}_Q(u;2)$.

(iii) $X \leq_{RHQ} Y \Leftrightarrow X \geq_{HQ} Y \Rightarrow X \leq_{PQEX} Y$. Conversely if $\frac{\bar{J}_Y(u)}{\bar{J}_X(u)}$ is increasing in $u$ then $X \leq_{PQEX} Y \Rightarrow X \leq_{RHQ} Y \Rightarrow X \geq_{HQ}$ . The proof is similar to that of Theorem 4.3.

(iv) Although $\leq_{PEX}$ and $\leq_{PQEX}$ are not mutually implied, they can do so under certain conditions. If $\bar{J}_Y(t)$ be increasing and $X \leq_{hr} Y$, then $X \leq_{PEX} Y \Rightarrow X \leq_{PQEX} Y$.

*Proof.* $X \leq_{PEX} Y \Leftrightarrow \bar{J}_X(t) \leq \bar{J}_Y(t) \Leftrightarrow \bar{J}_X(Q_X(u)) \leq \bar{J}_Y(Q_Y(u))$. Since $\bar{J}$ is increasing and $X \leq_{hr} Y \Rightarrow Q_X(u) \leq Q_Y(u)$. The last inequality is $\bar{J}_{Q_X}(u) \leq \bar{J}_{Q_Y}(u)$ and the result follows. □

**Remark 6.5.** In the above result $X \leq_{hr} Y$ can be replaced by $X \leq_{rhr} Y$. Also one can use $X \leq_{HQ} Y$ or $X \leq_{RHQ} Y$.

There are occasions where one has to compare the reliabilities of two devices, e.g. those with the same specifications and use but produced by different manufacturing processes. In such cases, we identify those with less residual uncertainty as more reliable. Let $X$ and $Y$ be non-negative random variables with zero as the left extremity of the support.

**Definition 6.19.** The random variable $X$ is smaller than $Y$ in decreasing residual extropy, denoted by $X \leq_{DRE} Y$, if if $\frac{J_{Q_Y}(u)}{J_{Q_X}(u)}$ is increasing in $u \in [0,1]$. This is equivalent to saying that $X \leq_{DRE} Y \Leftrightarrow \frac{\int_u^1 \frac{dp}{q_Y(p)}}{\int_u^1 \frac{dp}{q_X(p)}}$ is increasing in $u$ for all $u \in [0,1]$.

Similarly $X$ is smaller than decreasing past extropy if $\frac{\bar{J}_{Q_Y}(u)}{\bar{J}_{Q_X}(u)}$ is increasing in $u$ and is written as $X \leq_{DPE} Y$. Obviously $X \leq_{DPE} Y$ is equivalent to $\frac{\int_0^u \frac{dp}{q_Y(p)}}{\int_0^u \frac{dp}{q_X(p)}}$ is increasing in $u$. Thus

$$X \leq_{DPE} Y \Leftrightarrow \frac{\int_0^u \frac{dp}{q_Y(p)}}{\int_0^u \frac{dp}{q_X(p)}} \geq \frac{q_X(u)}{q_Y(u)}$$

$$\Leftrightarrow \frac{\bar{J}_{Q_X}(u)}{\bar{J}_{Q_Y}(u)} \geq \frac{H_Y(u)}{H_X(u)}. \tag{6.32}$$

In cases where it is difficult to establish the monotonicity of $\frac{\bar{J}_{Q_X}(u)}{\bar{J}_{Q_Y}(u)}$, inequality (6.32) gives a graphical procedure. When the graph of the ratio $\frac{\bar{J}_{Q_X}(u)}{\bar{J}_{Q_Y}(u)}$ lies below that of $\frac{H_Y(u)}{H_X(u)}$, we conclude that $X \leq_{DPE} Y$. Likewise we have the bound for

$X \leq_{DRE} Y$ as

$$X \leq_{DRE} Y \Leftrightarrow \frac{J_{Q_X}(u)}{J_{Q_Y}(u)} \leq \frac{H_X(u)}{H_Y(u)}. \tag{6.33}$$

Inequalities (6.32) and (6.33) are necessary and sufficient conditions.

The two orders $\leq_{DRE}$ and $\leq_{DPE}$ are also useful in defining the DREx and DPEx classes. In fact we have

$$X \leq_{DPEx} E \Leftrightarrow X \text{ } is \text{ } DREx$$

and

$$X \leq_{DPEx} E^* \Leftrightarrow X \text{ } is \text{ } DPEx$$

where $E$ is the exponential random variable and $E^*$ is the reversed exponential random variable. From Theorem 6.17 we see that for two non-negative random variables $X$ and $Y$

$$(J_{Q_X}(u) - J_{Q_Y}(u))) = (1-u)^{-2}\left(\bar{J}_{Q_X}(1) - \bar{J}_{Q_Y}(1) - u^2(\bar{J}_{Q_X}(u) - \bar{J}_{Q_Y}(u))\right).$$

In lifetime models for which $\bar{J}_{Q_X}(1) = \bar{J}_{Q_Y}(1)$, that is $X$ and $Y$ have the same extropy measure $J_X = J_Y$, it is easy to see that

$$J_{Q_X}(u) \geq J_{Q_Y}(u) \Leftrightarrow \bar{J}_{Q_X}(u) \leq \bar{J}_{Q_Y}(u)$$

or $X \geq (\leq) DREx$ is equivalent to $X \leq (\geq) DPEx$. The condition $J_X = J_Y$ is not a trivial case as can be seen in the next example.

**Example 6.10.** Assume that $F_X(x) = x^2, 0 \leq x \leq 1$ and $F_Y(x) = 1-(1-x)^2, 0 \leq x \leq 1$. Then $J_X = J_Y = -\frac{2}{3}$. Notice that $Q_X(u) = u^{\frac{1}{2}}$ and $Q_Y(u) = 1-(1-u)^{\frac{1}{2}}$. We have $J_{Q_X}(u) = -\frac{2(1-u^{\frac{3}{2}})}{3(1-u)^2}$ and $J_{Q_Y}(u) = -\frac{2(1-u)^{-\frac{3}{2}}}{3(1-u)^2}$ so that $J_{Q_X}(u) \geq J_{Q_Y}(u)$. Also $\bar{J}_{Q_X}(u) = -\frac{2}{3u^{\frac{1}{2}}}$ and $\bar{J}_{Q_Y}(u) = -\frac{2((1-u)^{\frac{3}{2}}-1)}{3u^2}$ so that $\bar{J}_{Q_X}(u) \leq \bar{J}_{Q_Y}(u)$.

Past extropy is defined as,

$$\bar{J}_t(X) = -\frac{1}{2F^2(t)} \int_0^t f^2(x)dx. \tag{6.34}$$

**Theorem 6.20.** *Let X be a random variable having generalized-Pareto distribution with quantile function* $Q(u) = \frac{b}{a}\left((1-u)^{\frac{-a}{a+1}} - 1\right), a > -1, b > 0$ *if and only if*

$$\bar{J}_Q(u) = \frac{c_1}{u^2}\left((1-u)(q(u))^{-1} - c_2\right). \tag{6.35}$$

*Proof.* Suppose that (6.35) holds. Then using (6.28), we have

$$\frac{-1}{2u^2}\int_0^u (q(p))^{-1}dp = \frac{c_1}{u^2}\left((1-u)(q(u))^{-1} - c_2\right). \tag{6.36}$$

Differentiating both sides of (6.36) with respect to $u$, we get

$$\frac{q'(u)}{q(u)}(1-u) = \frac{2c_1 - 1}{2c_1}.$$

Integrating both sides we get the $q(u)$ of the generalized-Pareto distribution. Thus the proof. □

**Theorem 6.21.** $\bar{J}_Q(u) = c$, *a constant if and only if X has exponential distribution with negative support.*

*Proof.* Let $\bar{J}_Q(u) = c$. Using (6.29), we have $\Lambda(u) = -4c$, which characterizes exponential distribution with negative support. Conversely suppose that $X$ has exponential distribution with support $(-\infty, 0)$ having quantile function $Q(u) = \frac{1}{\lambda}\log u$. Using (6.28), we get $\bar{J}_Q(u) = \frac{-\lambda}{4}$, a constant. □

**Theorem 6.22.** *Let $X$ be a non-negative continuous random variable with quantile function $Q(.)$ and reversed mean residual quantile function $R(.)$. Then*

$$\bar{J}_Q(u)R(u) = c, \tag{6.37}$$

*where $c > 0$ if and only if $X$ has power function distribution.*

*Proof.* Let $X$ has power distribution with quantile function $Q(u) = \gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$. We have $R(u) = \frac{\gamma}{\beta+1}u^{\frac{1}{\beta}}$. Now, using (6.28)

$$\begin{aligned}\bar{J}_Q(u) &= \frac{-\beta^2}{2\gamma(2\beta-1)}u^{\frac{-1}{\beta}} \\ &= \frac{-\beta^2}{2(\beta+1)(2\beta-1)}(R(u))^{-1}.\end{aligned}$$

Conversely suppose that (6.37) holds. Differentiating (6.37) with respect to $u$, we get

$$\bar{J}_Q(u)R'(u) = \frac{\Lambda(u)R(u) + 4c}{2u}.$$

Using the relationship (2.39) and (6.37), we have obtain

$$(\Lambda(u)R(u))^2 + 6c\Lambda(u)R(u) - 2c = 0. \tag{6.38}$$

The solution of (6.38) is obtained as $\Lambda(u)R(u) = c$, a constant which characterizes power distribution. This completes the proof. □

**Remark 6.6.** Quantile-based dynamic past extropy can be expressed in terms of quantile-based dynamic extropy and quantile-based extropy as,

$$\bar{J}_Q(u) = \frac{-1}{2u^2}\left(J(X) + (1-u)^2 J_Q(u)\right).$$

## 6.7 Cumulative residual extropy

In this section, we introduce a cumulative residual extropy using distribution function and quantile approaches. Cumulative extropy of a non-negative continuous random variable $X$ can be defined by

$$\mathcal{C}E(X) = -\frac{1}{2}\int_0^\infty \bar{F}^2(x)dx. \tag{6.39}$$

$\mathcal{C}E(X)$ is obtained by replacing the probability density function $f(.)$ in (6.1) by the survival function $\bar{F}(.)$. Unlike (6.1), $\mathcal{C}E(X)$ is more stable as the cumulative distribution function $F(.)$ or $\bar{F}(.)$ always exists. For the residual random variable $X_t$, (6.39) modified to

$$\mathcal{C}E(X_t) = -\frac{1}{2}\int_t^\infty \left(\frac{\bar{F}^2(x)}{\bar{F}^2(t)}\right)dx, \tag{6.40}$$

can be termed as cumulative residual extropy. Based on (6.39), the quantile-based cumulative residual extropy can be defined as

$$\Phi(X) = -\frac{1}{2}\int_0^1 (1-p)^2 q(p)dp. \tag{6.41}$$

The corresponding quantile-based cumulative residual extropy function using (6.40) becomes

$$\Phi_Q(u) = -\frac{1}{2}\int_u^1 \left(\frac{(1-p)^2}{(1-u)^2}\right) q(p)dp. \tag{6.42}$$

Differentiating both sides of (6.42) with respect to $u$, we get

$$q(u) = \left(\frac{-4\Phi_Q(u)}{(1-u)} + 2\Phi_Q'(u)\right). \tag{6.43}$$

The identity (6.43) uniquely determines the quantile density function.

Equation (6.42) can be also expressed in terms of the hazard quantile function by

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 \left(\frac{(1-p)}{H(p)}\right) dp. \tag{6.44}$$

Using the interrelationship between hazard quantile function and mean residual quantile function, given by $(H(u))^{-1} = M(u) - (1-u)M'(u)$, (6.44) becomes

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)M(p)dp + \frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 dM(p). \tag{6.45}$$

Applying integration by-parts on the second term of (6.45), yield

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)M(p)dp - \frac{M(u)}{2}. \tag{6.46}$$

Equation (6.46) represents the quantile-based cumulative residual extropy in terms of mean residual quantile function $M(u)$.

**Example 6.11.** *For proportional hazard quantile function model $Q_Y(u) = Q_X(1-(1-u)^{\frac{1}{\theta}})$,*

$$\Phi_{Q_Y}(u) = -\frac{1}{2\theta(1-u)^2}\int_u^1 (1-p)^{1+\frac{1}{\theta}} q_X(1-(1-p)^{\frac{1}{\theta}})dp.$$

*Taking $v = 1-(1-p)^{\frac{1}{\theta}}$, then*

$$\Phi_{Q_Y}(u) = -\frac{1}{2\theta(1-u)^2}\int_{1-(1-u)^{\frac{1}{\theta}}}^1 (1-v)^{2\theta} q_X(v)dv.$$

**Theorem 6.23.** *For a non-negative continuous random variable $X$ with $\Phi_Q(u) = c$, where $c > 0$ is a constant. Then $H(u)$ is a constant, which characterizes exponential distribution.*

*Proof.* The proof directly follows from (6.43). □

**Definition 6.24.** We say that $X$ has less cumulative residual quantile extropy than $Y$, denoted by $X \leq_{CRQE} Y$ if $\Phi_{Q_X}(u) \leq \Phi_{Q_Y}(u)$, for all $0 < u < 1$.

**Theorem 6.25.** *If $X \leq_{HQ} Y$ then $X \geq_{CRQE} Y$.*

*Proof.* Let $X \leq_{HQ} Y$. So that $q_X(u) \leq q_Y(u)$ implies that

$$\int_u^1 (1-p)^2 q_X(p)dp \leq \int_u^1 (1-p)^2 q_Y(p)dp,$$

equivalently

$$-\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_X(p)dp \geq -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_Y(p)dp.$$

Thus $\Phi_{Q_X}(u) \geq \Phi_{Q_Y}(u)$. □

**Theorem 6.26.** $X \leq_{CRQE} Y \nRightarrow X \leq_{HQ} Y$.

The following counter example illustrates the above theorem.

**Example 6.12.** Let $Q_X(u) = u^2$ and $Q_Y(u) = 2u - u^2$, both does not have a tractable distribution function. We have $\Phi_{Q_X}(u) = \frac{(3u+1)(u-1)}{12}$ and $\Phi_{Q_Y}(u) = -\frac{(u-1)^2}{4}$ holds $X \leq_{CRQE} Y$. But the hazard quantile functions$H_X(u) = \frac{1}{2u(1-u)}$ and $H_Y(u) = \frac{1}{2(1-u)^2}$ has the property $H_X(u) > H_Y(u)$ for $u = \frac{1}{3}$ and $H_X(u) < H_Y(u)$ for $u = \frac{2}{3}$. Thus $X \leq_{CRQE} Y$ does not imply $X \leq_{HQ} Y$.

**Definition 6.27.** $X$ is said to have increasing (decreasing) cumulative residual quantile extropy (ICRQE (DCRQE)) if $\Phi_Q(u)$ is increasing in $u$.

**Theorem 6.28.** *If X is ICRQE (DCRQE) then* $\Phi_Q(u) \leq (\geq)\frac{1}{4}\left((1-u)M'(u) - M(u)\right)$.

For exponential distribution with $Q(u) = -\frac{1}{\lambda}\log(1-u)$, $\Phi_Q(u) = -\frac{1}{4\lambda}$. The exponential distribution is the boundary class of ICRQE and DCRQE classes.

**Theorem 6.29.** *Let $X$ be a random variable with quantile function $Q(u)$ and mean residual quantile function $M(u)$ for all $u \in (0,1)$. The relationship $\Phi_Q(u) = cM(u)$, where $c$ is a non-negative constant holds for all $u$ if and only if $X$ is distributed as Pareto II, exponential or rescaled beta when $c \gtreqless \frac{1}{4}$.*

*Proof.* Assume that

$$\Phi_Q(u) = cM(u), \tag{6.47}$$

holds. Differentiating (6.47) with respect to $u$, we get

$$\Phi'_Q(u) = cM(u),$$

and using (6.43), we obtain

$$2cM'(u) = q(u) + \frac{4cM(u)}{1-u},$$

equivalently

$$\frac{M'(u)}{M(u)} = \left(\frac{4c-1}{2c+1}\right)\left(\frac{1}{1-u}\right),$$

$$\frac{d}{du}\log M(u) = \left(\frac{-4c+1}{2c+1}\right)\frac{d}{du}(-\log(1-u)),$$

implies $M(u) = c_1(1-u)^{\frac{1-4c}{1+2c}}$. □

**Theorem 6.30.** *If $Y = aX + b$, with $a > 0$ and $b > 0$, then $\Phi_{Q_Y}(u) = a\Phi_{Q_X}(u)$.*

*Proof.* Let $Y = aX + b$, with $a > 0$ and $b \geq 0$. Then

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = F_X(\frac{y-b}{a}).$$

By setting $F_X\left(\frac{y-b}{a}\right) = u$, we get $Q_Y(u) = aQ_X(u) + b$, we have

$$\Phi_{Q_Y}(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_Y(p)dp = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 a q_X(p)dp = a\Phi_{Q_X}(u).$$

□

**Theorem 6.31.** *Let X be a random variable with quantile density function*

$$q(u) = ku^{\delta}(1-u)^{-(A+\delta)}, k > 0, A, \delta \in R,\ u \in (0,1)$$

*if and only if*

$$\Phi_Q(u) = \frac{1}{(1-u)^2}\left(\Phi(X) + \frac{c}{2}B_u(1+\delta, 3-\delta-A)\right).$$

## 6.8 Application of quantile-based extropy

To apply the quantile-based extropy in real life situation, we need to develop a non-parametric estimator

### 6.8.1 Non-parametric estimation and simulation studies

To propose a non-parametric estimator for quantile-based extropy function, we consider a random sample $X_1, X_2, ..., X_n$. We compute empirical distribution function $F_{X:n}$. The empirical quantile function is given by (Parzen (1979))

$$\hat{Q}(u) = n\left(\frac{j}{n} - u\right) X_{(j-1)} + n\left(u - \frac{j-1}{n}\right) X_{(j)}, \tag{6.48}$$

for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, ..., n$. The corresponding quantile density function $\hat{q}(u) = \hat{Q}'(u)$ is given by $\hat{q}(u) = n(X_{(j)} - X_{(j-1)})$ for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, ..., n$. We consider estimator for quantile-based extropy given by,

$$\hat{J}(X) = \frac{-1}{2} \int_0^1 (\hat{q(p)})^{-1} dp,$$

where $\hat{q(p)} = n(X_{(j)} - X_{(j-1)})$ is the empirical estimator of quantile density (see Parzen (1979)). The estimator can be written in the form

$$\hat{J}(X) = \frac{-1}{2n} \sum_{j=1}^{n} \left( n(X_{(j)} - X_{(j-1)}) \right)^{-1}. \tag{6.49}$$

To assess the efficiency of the estimator (6.49), we conduct a Monte Carlo simulation study with various sample sizes $n = 20, 100, 300, 600, 900$ and 1000. The data are generated from Davies (power-Pareto) distribution given in Table 6.1 with parameters $c = 1, \lambda_1 = 1$ and $\lambda_2 = 7$. The true value of $J(X)$ for the Davies distribution is $-0.038$. The bias and MSE are computed for each of these sample sizes and are given in Table 6.5. It is evident from Figure 6.2 that the MSE decreases as sample size increases.

TABLE 6.5

| $n$ | 20 | 100 | 300 | 600 | 900 | 1000 |
|---|---|---|---|---|---|---|
| Bias | 0.4782 | 0.4345 | 0.4393 | 0.4288 | 0.4008 | 0.3759 |
| MSE | 0.3429 | 0.0426 | 0.0075 | 0.0066 | 0.0020 | 0.0008 |
| $J(\hat{X})$ | -0.5162 | -0.4725 | -0.4774 | -0.4668 | -0.4389 | -0.4139 |

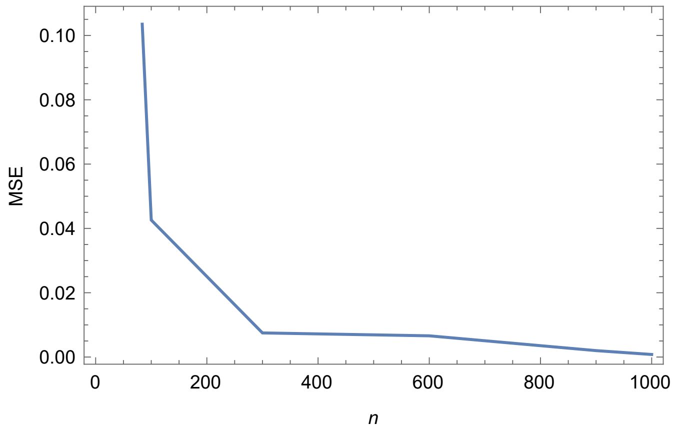



FIGURE 6.2: Mean square error of the estimator

### 6.8.2 Data Analysis

For establishing the usefulness of the proposed quantile-based extropy, we apply the above empirical estimator to real-life data set. The data consists of the times (in months) to first failure of 20 small electric carts (Zimmer et al. (1998)). Now, we fit the data using Davies distribution. To estimate the parameters, we use the method of $L$- moments, which are the competing alternatives to the conventional moments (see Hosking (1992)). Since Davies distribution contains three parameters, we take three sample $L$- moments which are given respectively by,

$$l_1 = \binom{n}{1}^{-1} \sum_{i=1}^{n} X_{(i)} = 14.675$$

$$l_2 = \frac{1}{2}\binom{n}{2}^{-1} \sum_{i=1}^{n} \left( \binom{i-1}{1} - \binom{n-i}{1} \right) X_{(i)} = 7.33447$$

$$l_3 = \frac{1}{3}\binom{n}{3}^{-1} \sum_{i=1}^{n} \left( \binom{i-1}{2} - 2\binom{i-1}{1}\binom{n-i}{1} + \binom{n-i}{2} \right) X_{(i)} = 2.4678,$$

where $X_{(i)}$ is the $i^{th}$ order statistic. The corresponding population $L-$ moments are given by

$$L_1 = \mu = cB(\lambda_1 + 1, 1 - \lambda_2),$$

$$L_2 = \frac{c(\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2 + 2} B(\lambda_1 + 1, 1 - \lambda_2),$$

and

$$L_3 = \frac{c(\lambda_1^2 + \lambda_2^2 + 4\lambda_1\lambda_2 + \lambda_2 - \lambda_1)B(\lambda_1 + 1, 1 - \lambda_2)}{(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 3)}.$$

We equate sample $L$- moments to population $L$- moments given by,

$$l_r = L_r, r = 1, 2, 3. \tag{6.50}$$

Solutions of set of equations (6.50) give the estimates of $c, \lambda_1$ and $\lambda_2$. The estimates using $L$- moments are

$$\hat{c} = 18.6139, \hat{\lambda_1} = 1.12554, \hat{\lambda_2} = 0.291096.$$

Thus the non-parametric estimate of extropy is $-0.020254$ and the parametric

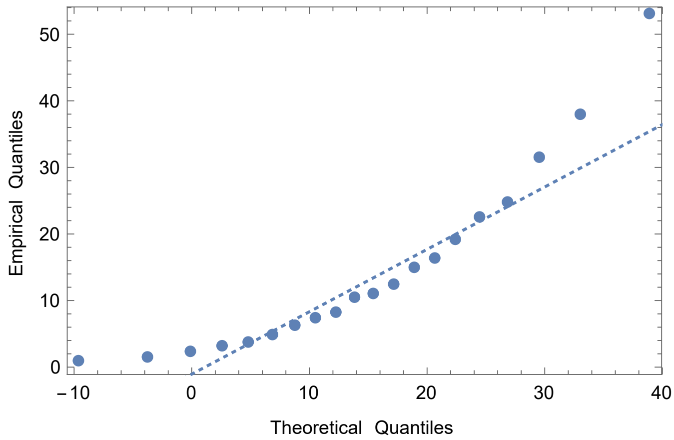



FIGURE 6.3: Q-Q plot for the data set

estimate of the extropy measure for the same family of distribution is $-0.01735$. To check the goodness of fit, we use $Q - Q$ plot which is represented in Figure: 6.3, shows the adequacy of the model since the number of observations is small to accommodate the chi-square test.

# Chapter 7

# Summary and Future Work

Information theory provides a unification of known results and leads to natural generalizations and the derivation of new results. Recently, quantile-based studies are of special interest among many researchers while it has some special features which are not shared with the distribution function approach. Quantile function exists for a few distributions which do not have a tractable distribution function. Motivated by the special features of the quantile function, we studied some information measures within the framework of the quantile function. Chapter 1 provides an introduction to the thesis which contains a brief outline of the work we have carried out. In Chapter 2, we gave a brief review of the literature and basic concepts regarding quantile functions and some information measures. We also conducted a quantile-based study for order statistics along with redefined ageing concepts. The order relations can be used for the comparison of the characteristics of two distributions. We studied the order relations of order statistics in terms of quantile functions.

In Chapter 3, we have introduced entropy and residual entropy of order statistics in terms of the quantile function. It has been shown that the quantile-based residual entropy of order statistics determines the quantile density function uniquely using a simple relationship between quantile density function and quantile-based residual entropy of order statistics. We have derived some upper bounds for the entropy of order statistics in quantile set up, which may be useful when the quantile density has no closed form or the computation is difficult.

Chapter 4 derived a quantile version of the cumulative residual Tsallis entropy due to Sati & Gupta (2015) and Rajesh & Sunoj (2016) respectively and studied its different properties. We have illustrated the usefulness of these measures in modelling certain distributions using quantile-based reliability functions. We obtained certain bounds to these quantile-based cumulative residual Tsallis entropy measures. We also extended these quantile measures in the context of order statistics and studied its properties.

In Chapter 5, we have introduced quantile-based cumulative Tsallis entropy and its dynamic version for past lifetime random variables. It is shown that the quantile-based cumulative Tsallis entropy in past lifetime determines the distribution uniquely through an explicit expression. Our approach gives an alternative method for finding the cumulative Tsallis entropy in past lifetime and useful for the probability models which do not have a closed distribution function. We have also studied different properties of these quantile-based measures in the context of order statistics. In particular, we have examined the properties of quantile-based cumulative Tsallis entropy of $n$th order statistic.

Chapter 6 established some new monotone properties, characterizations, ageing properties and orderings of residual and past lifetime extropy measures. We also studied these extropy functions using quantile functions and found that the quantile function approach gives several new results that are not achieved when the definition using distribution function is employed. We have discussed the quantile-based extropy of order statistics and obtained some properties. We derived the cumulative extropy and its residual form based on quantile function and obtained some characteristic results. Finally, we have proposed a non-parametric empirical estimator for quantile-based extropy and illustrated its performance using simulated and real data sets.

In the previous chapters, we have seen more results and findings in entropy and extropy using quantile functions and order statistics. We identify the following problems which require further investigation.

- Chapter 3 devotes to entropy of order statistics. Analogues results for the entropy of record values using quantile function is an open problem.
- The problem of estimating various information measures using quantile functions is yet another problem to be examined.
- Only very little work seems to have been done on the extropy measure. One can study the generalized extropy measures and its quantile approach.
- Another future work is the study on the divergence measure of extropy using quantile functions.

# List of Published/Accepted Papers

1. Sunoj, S M, Aswathy S Krishnan, and Sankaran P G. (2017). Quantile-based entropy of order statistics, *Journal of the Indian Society for Probability and Statistics*, 18 (2017) 1-17.

2. Sunoj, S M, Aswathy S Krishnan, and Sankaran P G. (2018). A quantile-based study of cumulative residual Tsallis entropy measures, *Physica A: Statistical Mechanics and its Applications* 494: 410-421.

3. Aswathy S Krishnan, Sunoj, S M, and Sankaran P G. (2019). Quantile-based reliability aspects of cumulative Tsallis entropy in past lifetime, *Metrika*, 82:17-38.

4. Aswathy S Krishnan, S. M. Sunoj, and N. Unnikrishnan Nair. Some reliability properties of extropy for residual and past lifetime random variables, *Journal of the Korean Statistical Society (accepted).*

# Papers Communicated

1. Aswathy S Krishnan, S. M. Sunoj, and P. G. Sankaran. Some reliability properties of extropy and its related measures using quantile function.

# Bibliography

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, *36*(1), 106–108.

Abbasnejad, M. & Arghami, N. R. (2010). Renyi entropy properties of order statistics. *Communications in Statistics-Theory and Methods*, *40*(1), 40–52.

Abouammoh, A., Ahmad, R., & Khalique, A. (2000). On new renewal better than used classes of life distributions. *Statistics & Probability Letters*, *48*(2), 189–194.

Ahmad, I. & Kayid, M. (2005). Characterizations of the RHR and MIT orderings and the DRHR and IMIT classes of life distributions. *Probability in the Engineering and Informational Sciences*, *19*(4), 447–461.

Alizadeh Noughabi, H. & Jarrahiferiz, J. (2018). On the estimation of extropy. *Journal of Nonparametric Statistics*, *31*(1), 88–99.

Arimitsu, T. & Arimitsu, N. (2002). Pdf of velocity fluctuation in turbulence by a statistics based on generalized entropy. *Physica A: Statistical Mechanics and its Applications*, *305*(1-2), 218–226.

Arnold, B. C. (1977). Recurrence relations between expectations of functions of order statistics. *Scandinavian Actuarial Journal*, *1977*(3), 169–174.

Asadi, M. & Ebrahimi, N. (2000). Residual entropy and its characterizations in terms of hazard function and mean residual life function. *Statistics & Probability Letters*, *49*(3), 263–269.

Asadi, M. & Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference*, *137*(6), 1931–1941.

Ayres, R. U. & Martinas, K. (1995). Waste potential entropy: The ultimate ecotoxic. *Economie Appliquee, France*, *XLVIII*(2), 95–120.

Balakrishnan, N. (1990). Improving the Hartley-David-Gumbel bound for the mean of extreme order statistics. *Statistics and Probability Letters*, *9*(4), 291–294.

Balakrishnan, N. & Malik, H. J. (1986). A note on moments of order statistics. *The American Statistician*, *40*(2), 147–148.

Balakrishnan, N. & Rao, C. R. (1998a). *Order statistics: Applications*. Elsevier, Amsterdam.

Balakrishnan, N. & Rao, C. R. (1998b). *Order statistics: Theory and Methods*. Elsevier, Amsterdam.

Baratpour, S., Ahmadi, J., & Arghami, N. R. (2007). Some characterizations based on entropy of order statistics and record values. *Communications in Statistics-Theory and Methods*, *36*(1), 47–57.

Baratpour, S., Ahmadi, J., & Arghami, N. R. (2008). Characterizations based on Rényi entropy of order statistics and record values. *Journal of Statistical Planning and Inference*, *138*(8), 2544–2551.

Baratpour, S. & Khammar, A. H. (2016). Results on Tsallis entropy of order statistics and record values. *İSTATİSTİK, Journal of the Turkish Statistical Association*, *8*(3), 60–73.

Barlow, R. & Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart and Winston: New York.

Barlow, R. & Proschan, F. (1981). Statistical Theory of Reliability and Life Testing: Probability Models. *To Begin With, Silver Springs, Maryland*.

Barlow, R. E., Marshall, A. W., & Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. *The Annals of Mathematical Statistics*, *34*(2), 375–389.

Barlow, R. E. & Proschan, F. (1996). *Mathematical Theory of Reliability*. Society for Industrial and Applied Mathematics, Philadelphia.

Belzunce, F., Navarro, J., Ruiz, J. M., & del Aguila, Y. (2004). Some results on residual entropy function. *Metrika*, *59*(2), 147–161.

Block, H. W., Savits, T. H., & Singh, H. (1998). The reversed hazard rate function. *Probability in the Engineering and Informational Sciences*, *12*(1), 69–90.

Boland, P. J., El-Neweihi, E., & Proschan, F. (1994). Applications of the hazard rate ordering in reliability and order statistics. *Journal of Applied Probability*, *31*(01), 180–192.

Calì, C., Longobardi, M., & Ahmadi, J. (2017). Some properties of cumulative Tsallis entropy. *Physica A: Statistical Mechanics and its Applications*, *486*, 1012–1021.

Cartwright, J. (2014). Roll over, Boltzmann. *Physics World*, *27*(05), 31–35.

Cole, R. H. (1951). Relations between moments of order statistics. *The Annals of Mathematical Statistics*, *22*, 308–310.

Cover, T. M. & Thomas, J. A. (2012). *Elements of Information Theory*. John Wiley & Sons, New York.

David, H. (1986). Inequalities for ordered sums. *Annals of the Institute of Statistical Mathematics*, *38*(1), 551–555.

Davis, H. T. (1962). *Introduction to nonlinear differential and integral equations*. Dover Publications, INC, New York.

Deshpande, J. V., Kochar, S. C., & Singh, H. (1986). Aspects of positive ageing. *Journal of Applied Probability*, *23*(3), 748–758.

Di Crescenzo, A. & Longobardi, M. (2002). Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability*, *39*(2), 434–440.

Di Crescenzo, A. & Longobardi, M. (2004). A measure of discrimination between past lifetime distributions. *Statistics & probability letters*, *67*(2), 173–182.

Di Crescenzo, A. & Longobardi, M. (2009). On cumulative entropies. *Journal of Statistical Planning and Inference*, *139*(12), 4072–4087.

Di Crescenzo, A. & Toomaj, A. (2015). Extension of the past lifetime and its connection to the cumulative entropy. *Journal of Applied Probability*, *52*(4), 1156–1174.

Ebrahimi, N. (1996). How to measure uncertainty in the residual life time distribution. *Sankhyā: The Indian Journal of Statistics, Series A*, 48–56.

Ebrahimi, N. & Pellerey, F. (1995). New partial ordering of survival functions based on the notion of uncertainty. *Journal of Applied Probability*, 202–211.

Ebrahimi, N., Soofi, E. S., & Zahedi, H. (2004). Information properties of order statistics and spacings. *IEEE Transactions on Information Theory*, *50*(1), 177–183.

Fashandi, M. & Ahmadi, J. (2012). Characterizations of symmetric distributions based on rényi entropy. *Statistics & Probability Letters*, *82*(4), 798–804.

Freimer, M., Kollia, G., Mudholkar, G. S., & Lin, C. T. (1988). A study of the generalized tukey lambda family. *Communications in Statistics-Theory and Methods*, *17*(10), 3547–3567.

Galambos, J. & Kotz, S. (1978). Characterization of probability distributions. *Spring-Verlag, New York*.

Gandotra, N., Bajaj, R. K., & Gupta, N. (2011). On some reliability properties of mean inactivity time under weighing. *International Journal of Computer Applications*, *30*(3), 28–32.

Gilchrist, W. (2000). *Statistical modelling with quantile functions*. Chapman and Hall/CRC, Boca Raton, FL.

Good, I. J. (1979). Studies in the history of probability and statistics. xxxvii am turing's statistical work in world war ii. *Biometrika*, 393–396.

Govindarajulu, Z. (1977). *A class of distributions useful in lifetesting and reliability with applications to nonparametric testing. In: C P Tsokos and I. N. Shimi (eds.) Theory and Applications of Reliability, Vol. 1,*. Academic Press, New York.

Gumbel, E. (1954). The maxima of the mean largest value and of the range. *The Annals of Mathematical Statistics*, *25*, 76–84.

Gupta, R. C. (2016). Mean residual life function for additive and multiplicative hazard rate models. *Probability in the Engineering and Informational Sciences*, *30*(2), 281–297.

Gupta, R. C., Taneja, H., & Thapliyal, R. (2014). Stochastic comparisons of residual entropy of order statistics and some characterization results. *Journal of Statistical Theory and Applications*, *13*(1), 27–37.

Gupta, R. D. & Nanda, A. K. (2001). Some results on reversed hazard rate ordering. *Communications in Statistics-theory and Methods*, *30*(11), 2447–2457.

Hall, W., Wellner, J., Csörgö, M., Dawson, D., Rao, J., & Saleh, A. (1981). Statistics and related topics. *M. Csorgo, DA Dawson, JNKR and Saleh, AKME, editors, Mean residual life*, 169–184.

Hankin, R. K. & Lee, A. (2006). A new family of non-negative distributions. *Australian & New Zealand Journal of Statistics*, *48*(1), 67–78.

Hartley, H. O. & David, H. A. (1954). Universal bounds for mean range and extreme observation. *The Annals of Mathematical Statistics*, *25*(1), 85–99.

Hastings Jr, C., Mosteller, F., Tukey, J. W., & Winsor, C. P. (1947). Low moments for small samples: a comparative study of order statistics. *The Annals of Mathematical Statistics*, 413–426.

Hosking, J. (2006). On the characterization of distributions by their L-moments. *Journal of Statistical Planning and Inference*, *136*(1), 193–198.

Hosking, J. R. (1990). L-moments: analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society. Series B (Methodological)*, 105–124.

Hosking, J. R. (1992). Moments or L moments? an example comparing two measures of distributional shape. *The American Statistician*, *46*(3), 186–189.

Hosking, J. R. M. (1989). *Some theoretical results concerning L-moments*. IBM Thomas J. Watson Research Division.

Hoskins, J. & Wallis, J. (1997). *Regional frequency analysis: An approach based on L-moments*. Cambridge University.

Hua, H., Bagkavos, D., & Patil, P. N. (2018). An L1 analysis of a kernel-based hazard rate estimator. *Australian & New Zealand Journal of Statistics*, *60*(1), 43–64.

Jones, M. (2004). On some expressions for variance, covariance, skewness and L-moments. *Journal of Statistical Planning and Inference*, *126*(1), 97–106.

Jose, J. & Sathar, E. A. (2019). Residual extropy of k-record values. *Statistics & Probability Letters*, *146*, 1–6.

Kayid, M. & Ahmad, I. (2004). On the mean inactivity time ordering with reliability applications. *Probability in the Engineering and Informational Sciences*, *18*(3), 395–409.

Krishnan, A. S., Sunoj, S. M., & Sankaran, P. G. (2019). Quantile-based reliability aspects of cumulative Tsallis entropy in past lifetime. *Metrika*, *82*(1), 17–38.

Kumar, V. (2016). Some results on Tsallis entropy measure and k-record values. *Physica A: Statistical Mechanics and its Applications*, *462*, 667–673.

Kumar, V. (2017). Characterization results based on dynamic Tsallis cumulative residual entropy. *Communications in Statistics-Theory and Methods*, *46*(17), 8343–8354.

Kundu, C. & Ghosh, A. (2017). Inequalities involving expectations of selected functions in reliability theory to characterize distributions. *Communications in Statistics-Theory and Methods*, *46*(17), 8468–8478.

Lad, F., Sanfilippo, G., & Agro, G. (2015). Extropy: complementary dual of entropy. *Statistical Science*, *30*(1), 40–58.

Lai, C. D. & Xie, M. (2006). *Stochastic ageing and dependence for reliability*. Springer Science & Business Media.

Martinas, K. (1997). Entropy and information. *World Futures: Journal of General Evolution*, *50*(1-4), 483–493.

Martinas, K. & Frankowicz, M. (2000). Extropy-reformulation of the entropy principle. *Periodica Polytechnica Chemical Engineering*, *44*(1), 29–38.

Mathai, A. & Haubold, H. J. (2007). Pathway model, superstatistics, Tsallis statistics, and a generalized measure of entropy. *Physica A: Statistical Mechanics and its Applications*, *375*(1), 110–122.

Midhu, N. N., Sankaran, P. G., & Nair, N. U. (2013). A class of distributions with the linear mean residual quantile function and it's generalizations. *Statistical Methodology*, *15*, 1–24.

Midhu, N. N., Sankaran, P. G., & Nair, N. U. (2014). A class of distributions with linear hazard quantile function. *Communications in Statistics-Theory and Methods*, *43*(17), 3674–3689.

Moors, J. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, *37*(1), 25–32.

Moriguti, S. (1951). Extremal properties of extreme value distributions. *The Annals of Mathematical Statistics*, *22*(4), 523–536.

Moriguti, S. (1953). A modification of Schwarz's inequality with applications to distributions. *The Annals of Mathematical Statistics*, *24*(1), 107–113.

Moriguti, S. (1954). Bounds for second moments of the sample range. *Reports of Statistical Application Research , JUSE*, *3*, 57–64.

Mukherjee, S. & Roy, D. (1986). Some characterizations of the exponential and related life distributions. *Calcutta Statistical Association Bulletin*, *35*(3-4), 189–198.

Muliere, P., Parmigiani, G., & Polson, N. G. (1993). A note on the residual entropy function. *Probability in the Engineering and Informational Sciences*, *7*(3), 413–420.

Nagaraja, H. N. (1990). Some reliability properties of order statistics. *Communications in Statistics-Theory and Methods*, *19*(1), 307–316.

Nair, N. & Asha, G. (2004). Characterizations using failure and reversed failure rates. *Journal of the Indian Society for Probability and Statistics*, *8*, 45–56.

Nair, N. U. & Sankaran, P. G. (2009). Quantile-based reliability analysis. *Communications in Statistics-Theory and Methods*, *38*(2), 222–232.

Nair, N. U., Sankaran, P. G., & Balakrishnan, N. (2013). *Quantile-Based Reliability Analysis*. Springer Science & Business Media, New York.

Nair, N. U., Sankaran, P. G., & Vineshkumar, B. (2012). The govindarajulu distribution: Some properties and applications. *Communications in Statistics-Theory and Methods*, *41*(24), 4391–4406.

Nair, N. U. & Vineshkumar, B. (2010). L-moments of residual life. *Journal of Statistical Planning and Inference*, *140*(9), 2618–2631.

Nair, N. U. & Vineshkumar, B. (2011). Ageing concepts: An approach based on quantile function. *Statistics & Probability Letters*, *81*(12), 2016–2025.

Nakamichi, A., Joichi, I., Iguchi, O., & Morikawa, M. (2002). Non-extensive galaxy distributions–Tsallis statistical mechanics. *Chaos, Solitons & Fractals*, *13*(3), 595–601.

Nanda, A. K., Sankaran, P. G., & Sunoj, S. M. (2014). Renyi's residual entropy: A quantile approach. *Statistics & Probability Letters*, *85*, 114–121.

Nanda, A. K. & Shaked, M. (2001). The hazard rate and the reversed hazard rate orders, with applications to order statistics. *Annals of the Institute of Statistical Mathematics*, *53*(4), 853–864.

Nanda, A. K., Singh, H., Misra, N., & Paul, P. (2003). Reliability properties of reversed residual lifetime. *Communications in Statistics-Theory and Methods*, *32*(10), 2031–2042.

Navarro, J., del Aguila, Y., & Asadi, M. (2010). Some new results on the cumulative residual entropy. *Journal of Statistical Planning and Inference*, *140*(1), 310–322.

Noughabi, M. S., Borzadaran, G. M., & Roknabadi, A. R. (2013). On the reliability properties of some weighted models of bathtub shaped hazard rate distributions. *Probability in the Engineering and Informational Sciences*, *27*(1), 125–140.

Park, S. (1995). The entropy of consecutive order statistics. *IEEE Transactions on Information Theory*, *41*(6), 2003–2007.

Parzen, E. (1979). Nonparametric statistical data modeling. *Journal of the American Statistical Association*, *74*(365), 105–121.

Preda, V., Dedu, S., & Gheorghe, C. (2015). New classes of Lorenz curves by maximizing tsallis entropy under mean and gini equality and inequality constraints. *Physica A: Statistical Mechanics and its Applications*, *436*, 925–932.

Qiu, G. (2017). The extropy of order statistics and record values. *Statistics & Probability Letters*, *120*, 52–60.

Qiu, G. (2019). Further results on quantile entropy in the past lifetime. *Probability in the Engineering and Informational Sciences*, *33*(1), 146–159.

Qiu, G. & Jia, K. (2018a). Extropy estimators with applications in testing uniformity. *Journal of Nonparametric Statistics*, *30*(1), 182–196.

Qiu, G. & Jia, K. (2018b). The residual extropy of order statistics. *Statistics & Probability Letters*, *133*, 15–22.

Rajesh, G. & Sunoj, S. M. (2016). Some properties of cumulative Tsallis entropy of order $\alpha$. *Statistical Papers*, *DOI: 10.1007/s00362-016- 0855-7*, 1–11.

Ramberg, J. S. & Schmeiser, B. W. (1972). An approximate method for generating symmetric random variables. *Communications of the ACM*, *15*(11), 987–990.

Ramberg, J. S. & Schmeiser, B. W. (1974). An approximate method for generating asymmetric random variables. *Communications of the ACM*, *17*(2), 78–82.

Rao, M., Chen, Y., Vemuri, B. C., & Wang, F. (2004). Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory*, *50*(6), 1220–1228.

Renyi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California.

Sachlas, A. & Papaioannou, T. (2014). Residual and past entropy in actuarial science and survival models. *Methodology and Computing in Applied Probability*, *16*(1), 79–99.

Sankaran, P., Gleeja, V., & Jacob, T. (2007). Nonparametric estimation of reversed hazard rate. *Calcutta Statistical Association Bulletin*, *59*(1-2), 55–68.

Sankaran, P. G. & Dileep Kumar, M. D. (2018). A class of distributions with the quadratic mean residual quantile function. *Communications in Statistics-Theory and Methods*, 1–22.

Sankaran, P. G. & Kumar, D. (2018). Pareto-weibull qauntile function. *Journal of Applied Probability*, *13*(1), 81–95.

Sankaran, P. G. & Nair, N. U. (2009). Nonparametric estimation of hazard quantile function. *Journal of Nonparametric Statistics*, *21*(6), 757–767.

Sankaran, P. G., Nair, N. U., & Sreedevi, E. P. (2010). A quantile based test for comparing cumulative incidence functions of competing risks models. *Statistics & Probability Letters*, *80*(9), 886–891.

Sankaran, P. G. & Sunoj, S. M. (2017). Quantile-based cumulative entropies. *Communications in Statistics-Theory and Methods*, *46*(2), 805–814.

Sankaran, P. G., Sunoj, S. M., & Nair, N. U. (2016). Kullback-leibler divergence: A quantile approach. *Statistics & Probability Letters*, *111*, 72–79.

Sati, M. M. & Gupta, N. (2015). Some characterization results on dynamic cumulative residual Tsallis entropy. *Journal of Probability and Statistics*, *2015((Article ID 694203)*, 8. http://dx.doi.org/10.1155/2015/694203.

Shaked, M. & Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer Science & Business Media, New York.

Shaked, M., Shanthikumar, J. G., & Tong, Y. (1995). Stochastic orders and their applications. *Society for Industrial and Applied Mathematics*, *37*(3), 477–478.

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, *27*, 379–423.

Sreelakshmi, N., Kattumannil, S. K., & Asha, G. (2018). Quantile based tests for exponentiality against DMRQ and NBUE alternatives. *Journal of the Korean Statistical Society*, *47*, 185–200.

Sunoj, S. M., Krishnan, A. S., & Sankaran, P. G. (2017). Quantile-based entropy of order statistics. *Journal of the Indian Society for Probability and Statistics*, *18*(1), 1–17.

Sunoj, S. M., Krishnan, A. S., & Sankaran, P. G. (2018). A quantile-based study of cumulative residual Tsallis entropy measures. *Physica A: Statistical Mechanics and its Applications*, *494*, 410–421.

Sunoj, S. M. & Maya, S. S. (2006). Some properties of weighted distributions in the context of repairable systems. *Communications in Statistics-Theory and Methods*, *35*(2), 223–228.

Sunoj, S. M. & Sankaran, P. G. (2012). Quantile based entropy function. *Statistics & Probability Letters*, *82*(6), 1049–1053.

Sunoj, S. M., Sankaran, P. G., & Maya, S. S. (2009). Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables. *Communications in Statistics-Theory and Methods*, *38*(9), 1441–1452.

Sunoj, S. M., Sankaran, P. G., & Nanda, A. K. (2013). Quantile based entropy function in past lifetime. *Statistics & Probability Letters*, *83*(1), 366–372.

Takahasi, K. (1988). A note on hazard rates of order statistics. *Communications in Statistics-Theory and Methods*, *17*(12), 4133–4136.

Taruya, A. & Sakagami, M.-a. (2003). Gravothermal catastrophe and Tsallis' generalized entropy of self-gravitating systems.(ii). thermodynamic properties of stellar polytrope. *Physica A: Statistical Mechanics and its Applications*, *318*(3-4), 387–413.

Thapliyal, R. & Taneja, H. C. (2012). Generalized entropy of order statistics. *Applied Mathematics*, *3*(12), 1977.

Tong, S., Bezerianos, A., Paul, J., Zhu, Y., & Thakor, N. (2002). Nonextensive entropy measure of eeg following brain injury from cardiac arrest. *Physica A: Statistical Mechanics and its Applications*, *305*(3-4), 619–628.

Toomaj, A., Sunoj, S. M., & Navarro, J. (2017). Some properties of the cumulative residual entropy of coherent and mixed systems. *Journal of Applied Probability*, *54*(2), 379–393.

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, *52*(1-2), 479–487.

Tukey, J. W. (1962). The future of data analysis. *The Annals of Mathematical Statistics*, *33*(1), 1–67.

van Staden, P. J. & Loots, M. (2009). Method of l-moment estimation for the generalized lambda distribution. In *Proceedings of the Third Annual ASEARC Conference*.

Weili, S., Yu, M., Zhanfang, C., & Hongbiao, Z. (2009). Research of automatic medical image segmentation algorithm based on Tsallis entropy and improved pcnn. In *Mechatronics and Automation, 2009. ICMA 2009. International Conference on*, (pp. 1004–1008). IEEE.

Wong, K. M. & Chen, S. (1990). The entropy of ordered sequences and order statistics. *IEEE Transactions on Information Theory*, *36*(2), 276–284.

Yang, J., Xia, W., & Hu, T. (2018). Bounds on eextropy with variational distance constraint. *Probability in the Engineering and Informational Sciences*, 1–19. doi:10.1017/S0269964818000098.

Yitzhaki, S. (2003). Gini's mean difference: A superior measure of variability for non-normal distributions. *Metron*, *61*(2), 285–316.

Yu, H.-L. & Wang, C.-H. (2013). Quantile-based bayesian maximum entropy approach for spatiotemporal modeling of ambient air quality levels. *Environmental Science & Technology*, *47*(3), 1416–1424.

Zarezadeh, S. & Asadi, M. (2010). Results on residual Renyi entropy of order statistics and record values. *Information Sciences*, *180*(21), 4195–4206.

Zimmer, W. J., Keats, J. B., & Wang, F. (1998). The Burr XII distribution in reliability analysis. *Journal of Quality Technology*, *30*(4), 386–394.